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# On parsimonious edge-colouring of graphs with maximum degree three

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### Abstract

In a graph G of maximum degree  $\Delta$  let  $\gamma$  denote the largest fraction of edges that can be  $\Delta$  edge-coloured. Albertson and Haas showned that  $\gamma \geq \frac{13}{15}$  when G is cubic [1]. We show here that this result can be extended to graphs with maximum degree 3 with the exception of a graph on 5 vertices. Moreover, there are exactly two graphs with maximum degree 3 (one being obviously The Petersen graph) for which  $\gamma = \frac{13}{15}$ . This extends a result given in [11]. These results are obtained in giving structural properties of the so called  $\delta$ -minimum edge colourings for graphs with maximum degree 3.

Key words: Cubic graph; Edge-colouring;

### 1 Introduction

Throughout this paper, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing's theorem [12] that these graphs can be edge-coloured with 4 colours. Let  $\phi E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$  be a proper edge-colouring of G. It is often of interest to try tu use one colour (say  $\delta$ ) as few as possible. When an edge colouring is optimal, following this constraints, we shall say that  $\phi$  is  $\delta - minimum$ . In [3] we gave without proof (unfortunately in french) results on  $\delta - minimum$  edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [10,11]. We give in section 2 a translation of these results and their proofs, they are contained in [4] (excepted for lemma 13 and Proposition 4).

An edge colouring of G with  $\{\alpha, \beta, \gamma, \delta\}$  is said to be  $\delta$ -improper whenever we only allow edges coloured with  $\delta$  to be incident. It must be clear that a proper edge colouring (and hence a  $\delta$ -minimum edge-colouring ) of G is a particular  $\delta$ -improper edge colouring. For a proper or  $\delta$ -improper edge colouring  $\phi$  of G,

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it will be convenient to denote  $E_{\phi}(x)$   $(x \in \{\alpha, \beta, \gamma, \delta\}$  the set of edges coloured with x by  $\phi$ . For  $x, y \in \{\alpha, \beta, \gamma, \delta\} \phi(x, y)$  is the partial subgraph of G spanned by this two colours, that is  $E_{\phi}(x) \cup E_{\phi}(y)$  (this subgraph being an union of paths and even cycles where the colours x and y alternate). Since any two  $\delta$ -minimum edge-colouring of G have the same number of edges coloured  $\delta$ we shall denote by s(G) this number (the *colour number* as defined in [10]).

As usually, for any undirected graph G, we denote by V(G) the set of its vertices and by E(G) the set of its edges and we consider that |V(G)| = n and |E(G)| = m. A strong matching C in a graph G is a matching C such that there is no edge of E(G) connecting any two edges of C, or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set V(C).

### 2 On $\delta$ -minimum edge-colouring

Our goal, in this section, is to give some structural properties of  $\delta$ -improper colourings and  $\delta$ -minimum edge-colourings. Graphs considered in the following series of lemmas will have maximum degree 3.

**Lemma 1** Assume that G can be provide with a perfect matching then any 2-factor of G contains at least s(G) disjoint odd cycles.

**Proof** Assume that we can find a 2-factor of G with k < s(G) odd cycles. Then let us colour the edges of this two factor with  $\alpha$  and  $\beta$ , excepted one edge (coloured  $\delta$ ) on each odd cycle of our 2-factor and let us colour the remaining edges by  $\gamma$  (the edges of the 2-factor). We get hence a new edge colouring  $\phi$  with  $E_{\phi}(\delta) < s(G)$ , impossible.

**Lemma 2** Let  $\phi$  be a  $\delta$ -minimum edge-colouring of G. Any edge in  $E_{\phi}(\delta)$  is incident to  $\alpha$ ,  $\beta$  and  $\gamma$ . Moreover this edge has one end of degree 2 and the other of degree 3 or the two ends of degree 3.

**Proof** Any edge of  $E_{\phi}(\delta)$  is certainly incident to  $\alpha, \beta$  and  $\gamma$ . Otherwise this edge could be coloured with the missing colour and we should obtain an edge colouring  $\phi'$  with  $|E_{\phi'}(\delta)| < |E_{\phi}(\delta)|$ .

**Lemma 3** Let  $\phi$  be  $\delta$ -improper colouring of G then there exists a proper colouring of  $G \phi'$  such that  $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$ 

**Proof** Let  $\phi$  be a  $\delta$ -improper edge colouring of G. If  $\phi$  is a proper colouring, we are done. Hence, assume that uv and uw are coloured  $\delta$ . If d(u) = 2 we can

change the colour of uv in  $\alpha, \beta$  or  $\gamma$  since v is incident to at most two colours in this set.

If d(u) = 3 assume that the third edge uz incident to u is also coloured  $\delta$ , then we can change the colour of uv for the same reason as above.

If uz is coloured with  $\alpha, \beta$  or  $\gamma$ , then v and w are incident to the two other colours otherwise one of them can be recoloured with the missing colour. W.l.o.g., consider that uz is coloured  $\alpha$  then v and w are incident to  $\beta$  and  $\gamma$ . The path P of  $\phi(\alpha, \beta)$  containing uz ends eventually in v or w (since these vertices have degree 1 in  $\phi(\alpha, \beta)$ ). We can thus assume that v or w (say v) is not the other end vertex of P. Exchanging  $\alpha$  and  $\beta$  along P does not change the colours incident to v. But now uz is coloured  $\beta \alpha$  and we can change the colour of uv with  $\beta \beta$ .

In each case, we get hence a new  $\delta$ -improper edge colouring  $\phi_1$  with  $E_{\phi_1}(\delta) \subseteq E_{\phi}(\delta)$ . Repeating this process leads us to construct a proper edge colouring of G with  $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$  as claimed.  $\Box$ 

**Proposition 4** Let  $v_1, v_2, \ldots, v_k \in V(G)$  such that  $G - \{v_1, v_2, \ldots, v_k\}$  is 3-edge colourable. Then  $s(G) \leq k$ .

**Proof** Let us consider a 3-edge colouring of  $G - \{v_1, v_2, \ldots, v_k\}$  with  $\alpha, \beta$ and  $\gamma$  and let us colour the edges incident to  $v_1, v_2, \ldots, v_k$  with  $\delta$ . We get a  $\delta$ -improper edge colouring  $\phi$  of G. Lemma 3 gives a proper colouring of  $G \phi'$ such that  $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$ . Hence  $\phi'$  has at most k edges coloured with  $\delta$  and  $s(G) \leq k$ 

Proposition 4 above has been obtained by Steffen [10] for cubic graphs.

**Lemma 5** Let  $\phi$  be  $\delta$ -improper colouring of G then  $|E_{\phi}(\delta)| \geq s(G)$ 

**Proof** Applying Lemma 3, let  $\phi'$  be a proper edge colouring of G such that  $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$ . We clearly have  $|E_{\phi}(\delta)| \ge |E_{\phi'}(\delta)| \ge s(G)$ 

**Lemma 6** Let  $\phi$  be a  $\delta$ -minimum edge-colouring of G. For any edge  $e = uv \in E_{\phi}(\delta)$  there are two colours x and y in  $\{\alpha, \beta, \gamma\}$  such that the connected component of  $\phi(x, y)$  containing the two ends of e is an even path joining this two ends.

**Proof** Without loss of generality assume that u is incident to  $\alpha$  and  $\beta$  and v is incident to  $\gamma$  (see Lemma 2). In any case (v has degree 3 or degree 2) u and v are contained in paths of  $\phi(\alpha, \gamma)$  or  $\phi(\beta, \gamma)$ . Assume that they are contained in paths of  $\phi(\alpha, \gamma)$ . If these paths are disjoint then we can exchange the two colours on the path containing u, e will be incident hence to only two colours  $\beta$  and  $\gamma$  in this new edge-colouring and e could be recoloured with  $\alpha$ , a contradiction since we consider a  $\delta$ -minimum edge-colouring.

An edge of  $E_{\phi}(\delta)$  is in  $A_{\phi}$  when its ends can be connected by a path of  $\phi(\alpha, \beta)$ ,  $B_{\phi}$  by a path of  $\phi(\beta, \gamma)$  and  $C_{\phi}$  by a path of  $\phi(\alpha, \gamma)$ .

**Lemma 7** If G is a cubic graph then  $|A_{\Phi}| \equiv |B_{\Phi}| \equiv |C_{\Phi}| \equiv s(G) \pmod{2}$ .

**Proof**  $\phi(\alpha, \beta)$  contains  $2|A_{\Phi}| + |B_{\Phi}| + |C_{\Phi}|$  vertices of degree 1 and must be even. Hence we get  $|B_{\Phi}| \equiv |C_{\Phi}| \pmod{2}$ . In the same way we get  $|A_{\Phi}| \equiv |B_{\Phi}| \pmod{2}$  leading to  $|A_{\Phi}| \equiv |B_{\Phi}| \equiv |C_{\Phi}| \equiv C(G) \pmod{2}$ 

**Remark 8** It must be clear that  $A_{\phi}$ ,  $B_{\phi}$  and  $C_{\phi}$  are not necessarily pairwise disjoint since an edge of  $E_{\phi}(\delta)$  with one end of degree 2 is contained into 2 such sets. Assume indeed that  $e = uv \in E_{\phi}(\delta)$  with d(u) = 3 and d(v) = 2then, if u is incident to  $\alpha$  and  $\beta$  and v is incident to  $\gamma$  we have an alternating path whose ends are u and v in  $\phi(\alpha, \gamma)$  as well as in  $\phi(\beta, \gamma)$ . Hence e is in  $A_{\phi} \cap B_{\phi}$ .

When  $e \in A_{\phi}$  we can associate to e the odd cycle  $C_{A_{\phi}}(e)$  obtained in considering the path of  $\phi(\alpha, \beta)$  together with e. We define in the same way  $C_{B_{\phi}}(e)$  and  $C_{C_{\phi}}(e)$  when e is in  $B_{\phi}$  or  $C_{\phi}$ . In the following lemma we consider en edge in  $A_{\phi}$ , an analogous result holds true whenever we consider edges in  $B_{\phi}$  or  $C_{\phi}$  as well.

**Lemma 9** Let  $\phi$  be a  $\delta$ -minimum edge-colouring of G and let e be an edge in  $A_{\phi}$  ( $B_{\phi}$ ,  $C_{\phi}$ ) then for any edge  $e' \in C_{A_{\phi}}(e)$  there is a  $\delta$ -minimum edgecolouring  $\phi'$  such that  $E_{\phi'}(\delta) = E_{\phi}(\delta) - e + e'$ ,  $e' \in A_{\phi'}$  and  $C_{A_{\phi}}(e) = C_{A_{\phi'}}(e')$ . Moreover, each edge outside  $C_{A_{\phi}}(e)$  but incident with this cycle is coloured  $\gamma$ .

**Proof** In exchanging colours  $\delta$  and  $\alpha$  and  $\delta$  and  $\beta$  successively along the cycle  $C_{A_{\phi}}(e)$ , we are sure to obtain an edge colouring preserving the number of edges coloured  $\delta$ . Since we have supposed that  $\phi$  is  $\delta$ -minimum, at each step, the resulting edge colouring is proper and  $\delta$ -minimum (Lemma 3). Hence, there is no edge coloured  $\delta$  incident with  $C_{A_{\phi}}(e)$ , which means that every such edge is coloured with  $\gamma$ .

We can perform these exchanges until e' is coloured  $\delta$ . In the  $\delta$ -minimum edge-colouring  $\phi'$  hence obtained, the two ends of e' are joined by a path of  $\phi(\alpha, \beta)$ . Which means that e' is in  $A_{\phi}$  and  $C_{A_{\phi}}(e) = C_{A_{\phi'}}(e')$ .

For each edge  $e \in E_{\phi}(\delta)$  (where  $\phi$  is a  $\delta$ -minimum edge-colouring of G) and we can associate one or two odd cycles following the fact that e is in one or two sets among  $A_{\phi}$ ,  $B_{\phi}$  or  $C_{\phi}$ . Let  $\mathcal{C}$  be the set of odd cycles associated to edges in  $E_{\phi}(\delta)$ .

**Remark 10** Since each edge coloured with  $\delta$  in a  $\delta$ -minimum edge-colouring is contained in a cycle. These edges cannot be bridges of G.

**Lemma 11** For each cycle  $C \in C$ , there are no two consecutive vertices with degree two.

**Proof** Otherwise, we exchange colours along C in order to put the colour  $\delta$  on the corresponding edge and, by Lemma 2, this is impossible in a  $\delta$ -minimum edge-colouring.

**Lemma 12** Let  $e_1, e_2 \in E_{\phi}(\delta)$  and let  $C_1, C_2 \in \mathcal{C}$  such that  $e_1 \in E(C_1)$  and  $e_2 \in E(C_2)$  then  $C_1$  and  $C_2$  are disjoint.

**Proof** If  $e_1$  and  $e_2$  are contained in the same set  $A_{\phi}$ ,  $B_{\phi}$  or  $C_{\phi}$ , we are done since their respective ends are joined by an alternating path of  $\phi(x, y)$  for some two colours x and y in  $\{\alpha, \beta, \gamma\}$ .

Without loss of generality assume that  $e_1 \in A_{\phi}$  and  $e_2 \in B_{\phi}$ . Assume moreover that there exists an edge e such that  $e \in C_1 \cap C_2$ . We have hence an edge  $f \in C_1$  with exactly one end on  $C_2$ . We can exchange colours on  $C_1$  in order to put the colour  $\delta$  on f. Which is impossible by Lemma 9.

By Lemma 9 any two cycles in  $\mathcal{C}$  corresponding to edges in distinct sets  $A_{\phi}$ ,  $B_{\phi}$ or  $C_{\phi}$  are at distance at least 2. Assume that  $C_1 = C_{A_{\phi}}(e_1)$  and  $C_2 = C_{A_{\phi}}(e_2)$ for some edges  $e_1$  and  $e_2$  in  $A_{\phi}$ . Can we say something about the subgraph of G whose vertex set is  $V(C_1) \cup V(C_2)$ ? In general, we have no answer to this problem. However, when G is cubic and any vertex of G lies on some cycle of  $\mathcal{C}$ (we shall say that  $\mathcal{C}$  is *spanning*), we have a property which will be useful after. Let us remark first that whenever  $\mathcal{C}$  is spanning, we can consider that G is edge-coloured in such a way that the edges of the cycles of  $\mathcal{C}$  are alternatively coloured with  $\alpha$  and  $\beta$  (excepted one edge coloured  $\delta$ ) and the remaining perfect matching is coloured with  $\gamma$ . For this  $\delta$ -minimum edge-colouring of G we have  $B_{\phi} = \emptyset$  as well as  $C_{\phi} = \emptyset$ . **Lemma 13** Assume that G is cubic and C is spanning. Let  $e_1, e_2 \in A_{\phi}$  and let  $C_1, C_2 \in \mathcal{C}$  such that  $C_1 = C_{A_{\phi}}(e_1)$  and  $C_2 = C_{A_{\phi}}(e_2)$ . Then at least one of the followings is true

(i) C<sub>1</sub> and C<sub>2</sub> are at distance at least 2
(ii) C<sub>1</sub> and C<sub>2</sub> are joined by at least 3 edges
(iii) C<sub>1</sub> and C<sub>2</sub> have at least two chords each

**Proof** Since  $e_1, e_2 \in A_{\phi}$  and  $\mathcal{C}$  is spanning we have  $B_{\phi} = C_{\phi} = \emptyset$ . Let  $C_1 = v_0v_1 \dots v_{2k_1}$  and  $C_2 = w_0w_1 \dots w_{2k_2}$ . Assume that  $C_1$  and  $C_2$  are joined by the edge  $v_0w_0$ . By lemma 6, we can consider a  $\delta$ -minimum edge-colouring  $\phi$  such that  $\phi(v_0v_1) = \phi(w_0w_1) = \delta$ ,  $\phi(v_1v_2) = \phi(w_1w_2) = \beta$  and  $\phi(v_0v_{2k_1}) = \phi(w_0w_{2k_2}) = \alpha$ . Moreover each edge of G (in particular  $v_0w_0$ ) incident with these cycles is coloured  $\gamma$ . We can change the colour of  $v_0w_0$  in  $\beta$ . We obtain thus a new  $\delta$ -minimum edge-colouring  $\phi'$ . Performing that exchange of colours on  $v_0w_0$  transforms the edges coloured  $\delta v_0v_1$  and  $w_0w_1$  in two edges of  $C_{\phi'}$  lying on odd cycles  $C'_1$  and  $C'_2$  respectively. We get hence a new set  $\mathcal{C}' = \mathcal{C} - \{C_1, C_2\} + \{C'_1, C'_2\}$  of odd cycles associated to  $\delta$ -coloured edges in  $\phi'$ .

From Lemma 9  $C'_1$  ( $C'_2$  respectively) is at distance at least 2 from any cycle in  $\mathcal{C} - \{C_1, C_2\}$ . Hence  $V(C'_1) \cup V(C'_2) \subseteq V(C_1) \cup V(C_2)$ . It is an easy task to check now that (ii) or (iii) above must be verified.  $\Box$ 

**Lemma 14** Let  $e_1 = uv_1$  be an edge of  $E_{\phi}(\delta)$  such that  $v_1$  has degree 2 in G. Then  $v_1$  is the only vertex in N(u) of degree 2.

**Proof** We have seen in Lemma 2 that  $uv_1$  has one end of degree 3 while the other has degree 2 or 3. Hence, we have d(u) = 3 and  $d(v_1) = 2$ . Let  $v_2$ and  $v_3$  the other neighbors of u. From Remark 8, we know that  $v_2$  and  $v_3$  are not pendent vertices. Assume that  $d(v_2) = 2$  and  $uv_2$  is coloured  $\alpha$ ,  $uv_3$  is coloured  $\beta$  and, finally  $v_1$  is incident to an edge coloured  $\gamma$ . The alternating path of  $\phi(\beta, \gamma)$  using the edge  $uv_3$  ends with the vertex  $v_1$  (see Lemma 6), then, exchanging the colours along the component of  $\phi(\beta, \gamma)$  containing  $v_2$ allows us to colour  $uv_2$  with  $\gamma$  and  $uv_1$  with  $\alpha$ . The new edge colouring  $\phi'$  so obtained is such that  $|E_{\phi'}(\delta)| \leq |E_{\phi}(\delta)| - 1$ , impossible.

**Lemma 15** Let  $e_1$  and  $e_2$  two edges of  $E_{\phi}(\delta)$  contained into two distinct subsets of  $A_{\phi}, B_{\phi}$  or  $C_{\phi}$ . Then  $\{e_1, e_2\}$  induces a  $2K_2$ .

**Proof** Without loss of generality assume that  $e_1 \in A_{\phi}$  and  $e_2 \in B_{\phi}$ . Since  $C_{e_1}(\phi)$  and  $C_{e_2}(\phi)$  are disjoint by Lemma 12, we can consider that  $e_1$  and

 $e_2$  have no common vertex. The edges having exactly one end in  $C_{e_1}(\phi)$  are coloured  $\gamma$  while those having exactly one end in  $C_{e_2}(\phi)$  are coloured  $\alpha$ . Hence there is no edge between  $e_1$  and  $e_2$  as claimed.

**Lemma 16** Let  $e_1$  and  $e_2$  two edges of  $E_{\phi}(\delta)$  contained into the same subset  $A_{\phi}, B_{\phi}$  or  $C_{\phi}$ . Then  $\{e_1, e_2\}$  are joined by at most one edge

**Proof** Without loss of generality assume that  $e_1 = u_1v_1, e_2 = u_2v_2 \in A_{\phi}$ . Since  $C_{e_1}(\phi)$  and  $C_{e_2}(\phi)$  are disjoint by Lemma 12, we can consider that  $e_1$  and  $e_2$  have no common vertex. The edges having exactly one end in  $C_{e_1}(\phi)$  (or  $C_{e_2}(\phi)$ ) are coloured  $\gamma$ . Assume that  $u_1u_2$  and  $v_1v_2$  are edges of G. In exchanging eventually colours  $\alpha$  and  $\beta$  on  $C_{e_1}(\phi)$  and  $C_{e_2}(\phi)$  we can consider that  $u_1$  and  $u_2$  are incident to  $\alpha$  while  $v_1$  and  $v_2$  are incident to  $\beta$ . We know that  $u_1u_2$  and  $v_1v_2$  are coloured  $\gamma$ . Let us colour  $e_1$  and  $e_2$  with  $\gamma$  and  $u_1u_2$  with  $\beta$  and  $v_1v_2$  with  $\alpha$ . We get a new edge colouring  $\phi'$  where  $|E_{\phi'}(\delta)| \leq |E_{\phi}(\delta)| - 2$ , contradiction since  $\phi$  is a  $\delta$ -minimum edge-colouring.

**Lemma 17** Let  $e_1, e_2$  and  $e_3$  three edges of  $E_{\phi}(\delta)$  contained into the same subset  $A_{\phi}, B_{\phi}$  or  $C_{\phi}$ . Then  $\{e_1, e_2, e_3\}$  induces a subgraph with at most four edges.

**Proof** Without loss of generality assume that  $e_1 = u_1v_1, e_2 = u_2v_2$  and  $e_3 = u_3v_3 \in A_{\phi}$ . From Lemma 16 we have just to suppose that (up to the names of vertices)  $u_1u_3 \in E(G)$  and  $v_1v_2 \in E(G)$ . In exchanging eventually the colours  $\alpha$  and  $\beta$  along the 3 disjoint paths of  $\phi(\alpha, \beta)$  joining the ends of each edge  $e_1, e_2$  and  $e_3$ , we can suppose that  $u_1$  and  $u_3$  are incident to  $\beta$  while  $v_1$  and  $v_2$  are incident to  $\alpha$ . Let  $\phi'$  be obtained from  $\phi$  when  $u_1u_3$  is coloured with  $\alpha$ ,  $v_1v_2$  with  $\beta$  and  $u_1v_1$  with  $\gamma$ . It is easy to check that  $\phi'$  is a proper edge-colouring with  $|E_{\phi'}(\delta)| \leq |E_{\phi}(\delta)| - 1$ , contradiction since  $\phi$  is a  $\delta$ -minimum edge-colouring.

**Lemma 18** Let  $e_1 = u_1v_1$  be an edge of  $E_{\phi}(\delta)$  such that  $v_1$  has degree 2 in G. Then for any edge  $e_2 = u_2v_2 \in E_{\phi}(\delta)$   $\{e_1, e_2\}$  induces a  $2K_2$ .

**Proof** From Lemma 15 we have to consider that  $e_1$  and  $e_2$  are contained in the same subset  $A_{\phi}$ ,  $B_{\phi}$  or  $C_{\phi}$ . Assume without loos of generality that they are contained in  $A_{\phi}$ . From Lemma 16 we have just to consider that there is a unique edge joining these two edges and we can suppose that  $u_1u_2 \in E(G)$ . In exchanging eventually the colours  $\alpha$  and  $\beta$  along the 2 disjoint paths of  $\phi(\alpha, \beta)$  joining the ends of each edge  $e_1, e_2$ , we can suppose that  $u_1 and u_2$ are incident to  $\beta$  while  $v_1$  and  $v_2$  are incident to  $\alpha$ . We know that  $u_1u_2$  is coloured  $\gamma$ . Let  $\phi'$  be obtained from  $\phi$  when  $u_1u_2$  is coloured with  $\alpha$  and  $u_1v_1$  is coloured with  $\gamma$ . It is easy to check that  $\phi'$  is a proper edge-colouring with  $|E_{\phi'}(\delta)| \leq |E_{\phi}(\delta)| - 1$ , contradiction since  $\phi$  is a  $\delta$ -minimum edge-colouring.  $\Box$ 

### 3 Applications and problems

#### 3.1 On a result by Payan

In [8] Payan showed that it is always possible to edge-colour a graph of maximum degree 3 with three maximal matchings (for the inclusion) and introduces henceforth a notion of *strong-edge colouring* where a strong edge-colouring means that one colour is a strong matching while the remaining colours are usual matchings. Payan conjectured that any d-regular graph has d pairwise disjoint maximal matchings and showed that this conjecture holds true for graphs with maximum degree 3.

**Theorem 19** Let G be a graph with maximum degree at most 3. Then G has  $\delta$ -minimum edge-colouring  $\phi$  where  $E_{\phi}(\delta)$  is a strong matching and, moreover, any edge in  $E_{\phi}(\delta)$  has its two ends of degree 3 in G.

**Proof** Let  $\phi$  be a  $\delta$ -minimum edge-colouring of G. From Lemma 15, any two edges of  $E_{\phi}(\delta)$  belonging to distinct subsets in  $A_{\phi}, B_{\phi}$  and  $C_{\phi}$  induce a strong matching. Hence, we have to find a  $\delta$ -minimum edge-colouring where each subset  $A_{\phi}, B_{\phi}$  or  $C_{\phi}$  induces a strong matching (with the supplementary property that the end vertices of these edges have degree 3). That means that we can work on each subset  $A_{\phi}, B_{\phi}$  and  $C_{\phi}$  independently. Without loss of generality, we only consider  $A_{\phi}$  here.

Assume that  $A_{\phi} = \{e_1, e_2, \dots e_k\}$  and  $A'_{\phi} = \{e_1, \dots e_i\}$   $(1 \leq i \leq k-1)$  is a strong matching and each edge of  $A'_{\phi}$  has its two ends with degree 3 in G. Consider the edge  $e_{i+1}$  and let  $C = C_{e_{i+1}}(\phi) = (u_0, u_1 \dots u_{2p})$  be the odd cycle associated to this edge (Lemma 6).

Let us mark any vertex v of degree 3 on C with a + whenever the edge of colour  $\gamma$  incident to this vertex has its other end which is a vertex incident to an edge of  $A'_{\phi}$  and let us mark v with - otherwise. By Lemma 11 a vertex of degree 2 on C has its two neighbors of degree 3 and by Lemma 18 these two vertices are marked with a -. By Lemma 17 we cannot have two consecutive vertices marked with a +. Hence, C must have two consecutive vertices of degree 3 marked with - whatever is the number of vertices of degree 2 on C.

Let  $u_j$  and  $u_{j+1}$  be two vertices of C of degree 3 marked with -(j being taken module 2p + 1). We can transform the edge colouring  $\phi$  by exchanging colours on C uniquely, in such a way that the edge of colour  $\delta$  of this cycle is  $u_j u_{j+1}$ . In the resulting edge colouring  $\phi_1$  we have  $A_{\phi_1} = A_{\phi} - e_{i+1} + u_j u_{j+1}$  and  $A'_{\phi_1} = A'_{\phi} + u_j u_{j+1}$  is a strong matching where each edge has its two ends of degree 3. Repeating this process we are left with a new  $\delta$ -minimum colouring  $\phi'$  where  $A_{\phi'}$  is a strong matching.

This result has been obtained first by Payan [8], but his technique does not exhibit explicitly the odd cycles associated to the edges of  $E_{\phi}(\delta)$  and their properties. It appears in [4] with this proof.

**Corollary 20** Let G be a graph with maximum degree 3 then there are s(G) vertices of degree 3 pairwise non-adjacent  $v_1 \ldots v_{s(G)}$  such that  $G - \{v_1 \ldots v_{s(G)}\}$  is 3-colourable.

**Proof** Pick a vertex on each edge coloured  $\delta$  in a  $\delta$ -minimum colouring  $\phi$  of G where  $E_{\phi}(\delta)$  is a strong matching (Theorem 19). We get a subset S of vertices satisfying our corollary.

Steffen [10] obtained Corollary 20 for bridgeless cubic graphs.

#### 3.2 Parsimonious edge colouring

Let  $\chi'(G)$  be the classical chromatic index of G. For convenience let

$$c(G) = max\{|E(H)| : H \subseteq G \ \chi'(H) = 3\}$$
$$\gamma(G) = \frac{c(G)}{E(G)}$$

Staton [9] (and independently Locke [7]) showed that whenever G is a cubic graph distinct from  $K_4$  then G contains a bipartite subgraph (and hence a 3-edge colourable graph, by König's theorem [6]) with at least  $\frac{7}{9}$  of the edges of G. Bondy and Locke [2] obtained  $\frac{4}{5}$  when considering graphs with maximum degree at most 3.

In [1] Albertson and Haas showed that whenever G is a cubic graph, we have  $\gamma(G) \geq \frac{13}{15}$  while for graphs with maximum degree 3 they obtained  $\gamma(G) \geq \frac{26}{31}$ . Our purpose here is to show that  $\frac{13}{15}$  is a lower bound for  $\gamma(G)$  when G has maximum degree 3, with the exception of the graph  $G_5$  depicted in Figure 1 below.



Fig. 1.  $G_5$ 

**Lemma 21** Let G be a graph with maximum degree 3 then  $\gamma(G) = 1 - \frac{s(G)}{m}$ .

**Proof** Let  $\phi$  be a  $\delta$ -minimum edge-colouring we need to colour s(G) edges of G with  $\delta$  by Lemma 12. We have thus m - s(G) edges coloured with  $\alpha, \beta$  and  $\gamma$ . The results follows.

**Lemma 22** [1] Let G be a graph with maximum degree 3. Assume that  $v \in V(G)$  is such that d(v) = 1 then  $\gamma(G) > \gamma(G - v)$ .

**Proof** We have certainly s(G - v) = s(G) since v is not incident to an edge coloured  $\delta$  in a  $\delta$ -minimum edge-colouring of G by Lemma 2. We have thus

$$\gamma(G) = 1 - \frac{s(G)}{m} > 1 - \frac{s(G-v)}{m-1} = \gamma(G-v)$$

as claimed.

A triangle  $T = \{a, b, c\}$  is said to be *reducible* whenever its neighbors are distinct. When T is a reducible triangle in G (G having maximum degree 3) we can obtain a new graph G' with maximum degree 3 in shrinking this triangle into a single vertex and joining this new vertex to the neighbors of T in G.

**Lemma 23** [1] Let G be a graph with maximum degree 3. Assume that  $T = \{a, b, c\}$  is a reducible triangle and let G' be the graph obtained by reduction of this triangle. Then  $\gamma(G) > \gamma(G')$ .

**Proof** Assume that s(G) = s(G') then

$$\gamma(G) = 1 - \frac{s(G)}{m} > 1 - \frac{s(G')}{m-3} = \gamma(G')$$

and we are done.

Hence, we have just to prove that s(G) = s(G'). It is a classical task to transform an edge colouring of G' into an edge colouring of G without increasing the number of edges coloured  $\delta$ . We get immediately that  $s(G') \ge s(G)$ . Let  $\phi$ be a  $\delta$ -minimum edge-colouring of G. Using Theorem 19 we can suppose that we have at most one edge coloured  $\delta$  involved in T and the incident edges to T. When considering G', we get eventually a a  $\delta$ -improper colouring  $\phi_1$ , where v is eventually the only vertex incident to 2 or 3 edges coloured  $\delta$ . Leading to s(G), s(G) + 1 or s(G) + 2 edges in  $E_{\phi_1}(\delta)$ . This colouring can be transformed in a proper colouring  $\phi'$  with colouring  $\phi'$  with Lemma 3 in order to eliminate the colour  $\delta$  appearing on one or two edges incident to v if necessary. Since this operation does not modify the colour of the edges coloured  $\delta$  which are not incident with v, we have  $s(G) \ge s(G')$ .

**Lemma 24** Let G be a cubic graph which can be factored into s(G) cycles of length 5 and without reducible triangle. Then every 2-factor of G contains s(G) cycles of length 5.

**Proof** Since G has no reducible triangle, all cycles in a 2-factor have length at least 4. Let  $\mathcal{C}$  be any 2-factor of G. When  $n_5$  denotes the number of cycles of length 5 and  $n_>$  denotes the number of cycles on at least 6 vertices in  $\mathcal{C}$ , there is an edge-colouring of G which uses at most  $n_5 + n_>$  edges of colour  $\delta$ . In addition, since  $\mathcal{C}$  is a spanning subgraph we have  $5s(G) \ge 5n_5 + 6n_>$  and if  $n_> \ne 0$ ,  $s(G) > n_5 + n_>$ , a contradiction. Thus  $n_> = 0$  and  $n_5 = s(G)$ , the result follows.

**Theorem 25** Let G be a graph with maximum degree 3 and  $G \neq G_5$  then  $\gamma(G) \geq \frac{13}{15}$ .

**Proof** From Lemma 22 and Lemma 23JMV: Vrifier numro, we can consider that G has only vertices of degree 2 or 3 and that G contains no reducible triangle. Let  $V_2$  be the set of vertices with degree 2 in G and  $V_3$  those of degree 3.

Assume that we can associate a set  $P_e$  of at least 5 distinct vertices of  $V_3$  for each edge  $e \in E_{\phi}(\delta)$  in a  $\delta$ -minimum edge-colouring  $\phi$  of G. Assume moreover that

$$\forall e, e' \in E_{\phi}(\delta) \quad P_e \cap P'_e = \emptyset \tag{1}$$

Then

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{3}{2}|V_3| + |V_2|} \ge 1 - \frac{\frac{|V_3|}{5}}{\frac{3}{2}|V_3| + |V_2|}$$
$$\gamma(G) \ge 1 - \frac{\frac{2}{15}}{\frac{1}{5}|V_3|}$$

Hence

$$G) \ge 1 - \frac{\frac{1}{15}}{1 + \frac{2}{3}\frac{|V_2|}{|V_3|}}$$

Which leads to  $\gamma(G) \ge \frac{13}{15}$  as claimed.

It remains to see how to construct the sets  $P_e$  satisfying Property (1). Let  $\mathcal{C}$  be the set of odd cycles associated to edges in  $E_{\phi}(\delta)$  (see Lemma 12). Let  $e \in E_{\phi}(\delta)$ , assume that e is contained in a cycle  $C \in \mathcal{C}$  of length 3. By Lemma 12, the edges incident to that triangle have the same colour in  $\alpha, \beta$  or  $\gamma$ . This triangle is hence reducible, impossible. We can thus consider that each cycle of  $\mathcal{C}$  has length at least 5. By Lemma 2 and Lemma 14, we know that whenever such a cycle contains vertices of  $V_2$ , their distance on this cycle is at least 3. Which means that every cycle  $C \in \mathcal{C}$  contains at least 5 vertices in  $V_3$  as soon as C has length at least 7 or C has length 5 but does not contain a vertex of  $V_2$ . For each edge  $e \in E_{\phi}(\delta)$  contained in such a cycle we associate  $P_e$  as any set of 5 vertices of  $V_3$  contained in the cycle.

It remains eventually edges in  $E_{\phi}(\delta)$  which are contained in a  $C_5$  of  $\mathcal{C}$  using a (unique) vertex of  $V_2$ . Let  $C = a_1 a_2 a_3 a_4 a_5$  be such a cycle and assume that  $a_1 \in V_2$ . By Lemma 2 and Lemma 17,  $a_1$  is the only vertex of degree 2 and in exchanging colours along this cycle, we can suppose that  $a_1 a_2 \in E_{\phi}(\delta)$ . Since  $a_1 \in V_2$ ,  $e = a_1 a_2$  is contained into a second cycle C' of  $\mathcal{C}$  (see Remark 8). If C' contains a vertex  $x \in V_3$  distinct from  $a_2, a_3, a_4$  and  $a_5$  then we set  $P_e = \{a_2, a_3, a_4, a_5, x\}$ . Otherwise  $C' = a_1 a_2 a_4 a_3 a_5$  and G is isomorphic to  $G_5$ , impossible.

The sets  $\{P_e | e \in E_{\phi}(\delta)\}$  are pairwise disjoint since any two cycle of  $\mathcal{C}$  associated to distinct edges in  $E_{\phi}(\delta)$  are disjoint. Hence property 1 holds and the proof is complete.

Albertson and Haas [1] proved that  $\gamma(G) \geq \frac{26}{31}$  when G is a graph with maximum degree 3. The bound given in Theorem 25 is better.

**Corollary 26** Let G be a graph with maximum degree 3 such that  $\gamma(G) = \frac{13}{15}$ . Then G is a cubic graph which can be factored into s(G) cycles of length 5. Moreover every 2-factor of G has this property.

**Proof** The optimal for  $\gamma(G)$  in Theorem 25 is obtained whenever  $s(G) = \frac{|V_3|}{5}$  and  $|V_2| = 0$ . That is, G is a cubic graph together with a 2-factor of s(G) cycles of length 5. Moreover by Lemma 23 G has no reducible triangle, the result comes from Lemma 24.

As pointed out by Albertson and Haas [1], the Petersen graph with  $\gamma(G) = \frac{13}{15}$  supplies the extremal example for cubic graphs. Steffen [11] proved that the only cubic bridgeless graph with  $\gamma(G) = \frac{13}{15}$  is the Petersen graph. In fact, we can extend this result to graphs with maximum degree 3 where bridges are

allowed (excluding the graph  $G_5$ ). Let P' be the cubic graph on 10 vertices obtained from two copies of  $G_5$  (Figure 1) in joining by an edge the two vertices of degree 2.

**Theorem 27** Let G be a connected graph with maximum degree 3 such that  $\gamma(G) = \frac{13}{15}$ . Then G is isomorphic to the Petersen graph or to P'.

**Proof** Let G be a graph with maximum degree 3 such that  $\gamma(G) = \frac{13}{15}$ . From Corollary 26, we can consider that G is cubic and G has a 2-factor of cycles of length 5. Let  $\mathcal{C} = \{C_1 \dots C_{s(G)}\}$  be such a 2-factor ( $\mathcal{C}$  is spanning). Let  $\phi$ be a  $\delta$ -minimum edge-colouring of G induced by this 2-factor.

Without loss of generality consider two cycles in  $\mathcal{C}$ , namely  $C_1$  and  $C_2$ , and let us denote  $C_1 = v_1 v_2 v_3 v_4 v_5$  while  $C_2 = u_1 u_2 u_3 u_4 u_5$  and assume that  $v_1 u_1 \in G$ . From Lemma 13  $C_1$  and  $C_2$  are joined by at least 3 edges or  $\mathcal{M} \mathcal{M} \mathcal{M} \mathcal{M}$  each of them has two chords. If S(G) > 2 there is a cycle  $C_3 \in \mathcal{C}$ . Without loss of generality, G being connected, we can suppose that  $C_3$  is joined to  $C_1$  by an edge. Applying once more time Lemma 13,  $C_1$  and  $C_3$  have two chords or are joined by at least 3 edges, contradiction with the constraints imposed by  $C_1$  and  $C_2$ . Hence s(G) = 2 and G has 10 vertices, which leads to a graph isomorphic to P' or the Petersen graph as claimed.  $\Box$ 

We can construct cubic graphs with chromatic index 4 (*snarks* in the literrature) which are cyclically 4- edge connected and having a 2-factor of  $C_5$ 's.

Indeed, let G be a cubic cyclically 4-edge connected graph of size n and M be a perfect matching of G,  $M = \{x_i y_i | i = 1 \dots \frac{n}{2}\}$ . Let  $P_1 \dots P_{\frac{n}{2}}$  be  $\frac{n}{2}$  copies of the Petersen graph. For each  $P_i$   $(i = 1 \dots \frac{n}{2})$  we consider two non adjacent edges  $e_i^1$  and  $e_i^2$ . Let us observe that  $P_i - \{e_i^1, e_i^2\}$  contains a 2-factor of two  $C_5$ 's  $(C_i^1 \text{ and } C_i^2)$ .

We construct then a new cyclically 4-edge connected cubic graph H with index chromatic 4 by applying the well known operation dot-product on  $\{e_i^1, e_i^2\}$ and the edge  $x_i y_i$   $(i = 1 \dots \frac{n}{2})$ . We remark that the vertices of G vanish in the operation and the resulting graph H has a 2 factor of  $C_5$ , namely  $\{C_1^1, C_1^2, \dots C_i^1, C_i^2, \dots C_{\frac{n}{2}}^1, C_{\frac{n}{2}}^2\}$ .

We do not know example of cyclically 5-edge connected snarks (excepted the Petersen graph) with a 2-factor of induced cycles of length 5.

**Problem 28** Is there any 5-edge connected snark distinct from the Petersen graph with a 2-factor of  $C_5$ 's ?

As a first step towards the resolution of this Problem we propose :

**Theorem 29** Let G be a cubic graph which can be factored into s(G) induced cycles of length 5, then G is the Petersen graph.

**Proof** Let  $\mathcal{F}$  be a 2-factor of s(G) cycles of length 5 in G, every cycle of  $\mathcal{F}$  being an induced cycle of G. We consider the  $\delta$ -minimum edge-colouring  $\phi$  such that the edges of all cycles of  $\mathcal{F}$  are alternatively coloured  $\alpha$  and  $\beta$  excepted for exactly one edge per cycle which is coloured with  $\delta$ , all the remaining edges of G being coloured  $\gamma$ . By construction we have  $B_{\phi} = C_{\phi} = \emptyset$  and  $A_{\phi} = \mathcal{F}$ .

By Lemma 13 two cycles of  $\mathcal{F}$  which are connected must be connected with at least three edges. Consequently a cycle of  $\mathcal{F}$  cannot be connected by at least one edge to more than one other cycle of  $\mathcal{C}$ . It follows that  $\mathcal{C}$  contains exactly two induced cycles of length 5. The only graph with 10 vertices and s(G) = 2 with such a 2-factor is the Petersen graph.

**Comments:** The index s(G) used here is certainly less than o(G) the oddness of G used by Huck and Kochol [5]. The oddness o(G) is the minimum number of odd cycles in any 2-factor of a cubic graphs (assuming that we consider graphs with that property). Obviously o(G) is an even number and it is an easy task to construct a cubic graph G with s(G) odd which ensures that 0 < s(G) < o(G). We can even construct cyclically-5-edge-connected cubic graphs with that property with s(G) = k for any integer  $k \neq 1$  (see [4] and [11]). It can be pointed out that, using a parity argument (see 7 or [10]) a graph with oddness at least 2 needs certainly to be edge coloured with at least 2 edges in the less popular colour. In other words,  $o(G) = 2 \Leftrightarrow s(G) = 2$ .

When G is a cubic bridgeless planar graph, we know from the 4 colours Theorem that G is 3-edge colourable and hence  $\gamma(G) = 1$ . Albertson and Haas [1] gave  $\gamma(G) \geq \frac{6}{7} - \frac{2}{35m}$  when G is a planar bridgeless graph with maximum degree 3. Our Theorem 25 improves this lower bound (allowing moreover bridges). On the other hand, they exhibit a family of planar graphs with maximum degree 3 (bridges are allowed) for which  $\gamma(G) = \frac{8}{9} - \frac{2}{9n}$ .

As in [11] we denote  $g(\mathcal{F}) = \min\{V(C) | C \text{ is an odd circuit of } \mathcal{F}\}$ and  $g^+(G) = \max\{g(\mathcal{F}) | \mathcal{F} \text{ is a } 2 - \text{factor of } G\}$ . We suppose that  $g^+(G)$  is defined, that is G has at least one 2-factor (when G is a cubic bridgeless graphs this condition is obviously fulfilled).

When G is cubic bridgeless, Steffen [11] showed that we have :

$$\gamma(G) \ge \max\{1 - \frac{2}{3g^+(G)}, \frac{11}{12}\}\$$

The difficult part being to show that  $\gamma(G) \geq \frac{11}{12}$ . We can remark that whenever

 $g^+(G) \ge 11$  we obtain the same result for graphs with maximum degree 3 and having at most  $\frac{n}{3}$  vertices of degree 1 or 2.

**Theorem 30** Let G be a graph with maximum degree 3 and having at least one 2-factor. Then  $\gamma(G) \geq 1 - \frac{2}{3g^+(G)}$ .

**Proof** We have  $\gamma(G) = 1 - \frac{s(G)}{m}$  where  $m = \frac{3n}{2}$  and, obviously,  $s(G) \leq \frac{n}{g^+(G)}$ . Hence  $\gamma(G) \geq 1 - \frac{2}{3g^+(G)}$ .

**Theorem 31** Let G be a graph with maximum degree 3 and having at least one 2-factor. Let  $V_i$  (i = 1..3) be the set of vertices of degree i. Assume that  $|V_2| \leq \frac{n}{3}$  and  $g^+(G) \geq 11$  then  $\gamma(G) \geq \max\{1 - \frac{2}{3g^+(G)}, \frac{11}{12}\}.$ 

**Proof** Since, by Lemma 22  $\gamma(G) > \gamma(G - v)$  we can assume that  $V_1 = \emptyset$ . From Theorem 30 we have just to prove that  $\gamma(G) \ge \frac{11}{12}$ . Following the proof of Theorem 25, we try to associate a set  $P_e$  of at least 8 distinct vertices of  $V_3$  for each edge  $e \in E_{\phi}(\delta)$  in a  $\delta$ -minimum edge-colouring  $\phi$  of G such that

$$\forall e, e' \in E_{\phi}(\delta) \quad P_e \cap P'_e = \emptyset \tag{2}$$

Indeed, let  $\mathcal{F}$  be a 2-factor of G where each odd cycle has length at least 11 and let  $C_1, C_2 \dots C_{2k}$  be its set of odd cycles. We have, obviously  $s(G) \leq 2k$ . Let  $V_3^{1\dots 2k}$  and  $V_2^{1\dots 2k}$  be the sets of vertices of degree 3 and 2 respectively contained in these odd cycles. As soon as  $|V_3^{1\dots 2k}| \geq 8s(G)$  we have

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{3}{2}|V_3| + |V_2|} \ge 1 - \frac{\frac{|V_3^{1...2k}|}{8}}{\frac{3}{2}|V_3| + |V_2|}$$
(3)

which leads to

$$\gamma(G) \ge 1 - \frac{\frac{2|V_3^{1...2k}|}{24|V_3|}}{1 + \frac{2}{3}\frac{|V_2|}{|V_3|}}$$

Since  $|V_3| \ge |V_3^{1\dots 2k}|$ , we have

$$\gamma(G) \geq 1 - \frac{\frac{2}{24}}{1 + \frac{2}{3}\frac{|V_2|}{|V_3|}}$$

and

$$\gamma(G) \ge \frac{11}{12}$$

as claimed.

It remains the case where  $|V_3^{1...2k}| < 8s(G)$ . Since each cycle has at least 11 vertices we have  $|V_2^{1...2k}| > 11 \times 2k - |V_3^{1...2k}| > 3s(G)$ .

$$\gamma(G) = \frac{m - s(G)}{m} \ge \frac{m - \frac{|V_2^{1\dots 2k}|}{3}}{m}$$

We have

$$\frac{m - \frac{|V_2^{1\dots 2k}|}{3}}{m} \ge \frac{11}{12}$$

when

$$m \ge 4|V_2^{1\dots 2k}| \tag{4}$$

Since  $|V_2| \leq \frac{n}{3}$  we have  $|V_3| \geq \frac{2n}{3}$  and

$$m = 3\frac{|V_3|}{2} + |V_2| = 3\frac{n - |V_2|}{2} + |V_2| = 3\frac{n}{2} - \frac{|V_2|}{2} \ge 4\frac{n}{3} \ge 4|V_2^{1\dots 2k}|$$
(5)

and the result holds.

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