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On parsimonious edge-colouring of graphs with maximum degree three

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Abstract

In a graph G of maximum degree Δ let γ denote the largest fraction of edges that can be Δ edge-coloured. Albertson and Haas showed that $\gamma \geq \frac{13}{15}$ when G is cubic [1]. We show here that this result can be extended to graphs with maximum degree 3 with the exception of a graph on 5 vertices. Moreover, there are exactly two graphs with maximum degree 3 (one being obviously Petersen's graph) for which $\gamma = \frac{13}{15}$. This extends a result given in [11]. These results are obtained in giving structural properties of the so called δ -minimum edge colourings for graphs with maximum degree 3.

Key words: Cubic graph; Edge-colouring;

1 Introduction

Throughout this paper, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing's theorem [12] that these graphs can be edge-coloured with 4 colours. Let $\phi: E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-colouring of G . It is often of interest to try to use one colour (say δ) as few as possible. When an edge colouring is optimal, following this constraints, we shall say that ϕ is δ -*minimum*. In [3] we gave without proof (unfortunately in french) results on δ -*minimum* edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [10,11]. We give in section 2 a translation of these results and their proofs, they are contained in [4] (excepted for lemma 13 and Proposition 4).

An edge colouring of G with $\{\alpha, \beta, \gamma, \delta\}$ is said to be δ -*improper* whenever we only allow edges coloured with δ to be incident. It must be clear that a proper edge colouring (and hence a δ -minimum edge-colouring) of G is a particular δ -improper edge colouring. For a proper or δ -improper edge colouring ϕ of G ,

it will be convenient to denote $E_\phi(x)$ ($x \in \{\alpha, \beta, \gamma, \delta\}$) the set of edges coloured with x by ϕ . For $x, y \in \{\alpha, \beta, \gamma, \delta\}$ $\phi(x, y)$ is the partial subgraph of G spanned by this two colours, that is $E_\phi(x) \cup E_\phi(y)$ (this subgraph being an union of paths and even cycles where the colours x and y alternate). Since any two δ -minimum edge-colouring of G have the same number of edges coloured δ we shall denote by $s(G)$ this number (the *colour number* as defined in [10]).

As usually, for any undirected graph G , we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges and we consider that $|V(G)| = n$ and $|E(G)| = m$. A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C , or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set $V(C)$.

2 On δ -minimum edge-colouring

Our goal, in this section, is to give some structural properties of δ -improper colourings and δ -minimum edge-colourings. Graphs considered in the following serie of lemmas will have maximum degree 3.

Lemma 1 *Assume that G can be provide with a perfect matching then any 2-factor of G contains at least $s(G)$ disjoint odd cycles.*

Proof Assume that we can find a 2-factor of G with $k < s(G)$ odd cycles. Then let us colour the edges of this two factor with α and β , excepted one edge (coloured δ) on each odd cycle of our 2-factor and let us colour the remaining edges by γ (the edges of the 2-factor). We get hence a new edge colouring ϕ with $E_\phi(\delta) < s(G)$, impossible. \square

Lemma 2 *Let ϕ be a δ -minimum edge-colouring of G . Any edge in $E_\phi(\delta)$ is incident to α, β and γ . Moreover this edge has one end of degree 2 and the other of degree 3 or the two ends of degree 3.*

Proof Any edge of $E_\phi(\delta)$ is certainly incident to α, β and γ . Otherwise this edge could be coloured with the missing colour and we should obtain an edge colouring ϕ' with $|E_{\phi'}(\delta)| < |E_\phi(\delta)|$. \square

Lemma 3 *Let ϕ be δ -improper colouring of G then there exists a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$*

Proof Let ϕ be a δ -improper edge colouring of G . If ϕ is a proper colouring, we are done. Hence, assume that uv and uw are coloured δ . If $d(u) = 2$ we can

change the colour of uv in α, β or γ since v is incident to at most two colours in this set.

If $d(u) = 3$ assume that the third edge uz incident to u is also coloured δ , then we can change the colour of uv for the same reason as above.

If uz is coloured with α, β or γ , then v and w are incident to the two other colours otherwise one of them can be recoloured with the missing colour. W.l.o.g., consider that uz is coloured α then v and w are incident to β and γ . The path P of $\phi(\alpha, \beta)$ containing uz ends eventually in v or w (since these vertices have degree 1 in $\phi(\alpha, \beta)$). We can thus assume that v or w (say v) is not the other end vertex of P . Exchanging α and β along P does not change the colours incident to v . But now uz is coloured β and we can change the colour of uv with α .

In each case, we get hence a new δ -improper edge colouring ϕ_1 with $E_{\phi_1}(\delta) \subseteq E_{\phi}(\delta)$. Repeating this process leads us to construct a proper edge colouring of G with $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$ as claimed. \square

Proposition 4 *Let $v_1, v_2, \dots, v_k \in V(G)$ such that $G - \{v_1, v_2, \dots, v_k\}$ is 3-edge colourable. Then $s(G) \leq k$.*

Proof Let us consider a 3-edge colouring of $G - \{v_1, v_2, \dots, v_k\}$ with α, β and γ and let us colour the edges incident to v_1, v_2, \dots, v_k with δ . We get a δ -improper edge colouring ϕ of G . Lemma 3 gives a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$. Hence ϕ' has at most k edges coloured with δ and $s(G) \leq k$ \square

Proposition 4 above has been obtained by Steffen [10] for cubic graphs.

Lemma 5 *Let ϕ be δ -improper colouring of G then $|E_{\phi}(\delta)| \geq s(G)$*

Proof Applying Lemma 3, let ϕ' be a proper edge colouring of G such that $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$. We clearly have $|E_{\phi}(\delta)| \geq |E_{\phi'}(\delta)| \geq s(G)$ \square

Lemma 6 *Let ϕ be a δ -minimum edge-colouring of G . For any edge $e = uv \in E_{\phi}(\delta)$ there are two colours x and y in $\{\alpha, \beta, \gamma\}$ such that the connected component of $\phi(x, y)$ containing the two ends of e is an even path joining this two ends.*

Proof W.l.o.g. Assume that u is incident to α and β and v is incident to γ (see Lemma 2). In any case (v has degree 3 or degree 2) u and v are contained

in paths of $\phi(\alpha, \gamma)$ or $\phi(\beta, \gamma)$. Assume that they are contained in paths of $\phi(\alpha, \gamma)$. If these paths are disjoint then we can exchange the two colours on the path containing u , e will be incident hence to only two colours β and γ in this new edge-colouring and e could be recoloured with α , a contradiction since we consider a δ -minimum edge-colouring. \square

An edge of $E_\phi(\delta)$ is in A_ϕ when its ends can be connected by a path of $\phi(\alpha, \beta)$, B_ϕ by a path of $\phi(\beta, \gamma)$ and C_ϕ by a path of $\phi(\alpha, \gamma)$.

Lemma 7 *If G is a cubic graph then*

$$|A_\phi| \equiv |B_\phi| \equiv |C_\phi| \equiv s(G) \pmod{2}$$

Proof $\phi(\alpha, \beta)$ contains $2|A_\phi| + |B_\phi| + |C_\phi|$ vertices of degree 1 and must be even. Hence we get $|B_\phi| \equiv |C_\phi| \pmod{2}$. In the same way we get $|A_\phi| \equiv |B_\phi| \pmod{2}$ leading to $|A_\phi| \equiv |B_\phi| \equiv |C_\phi| \equiv C(G) \pmod{2}$ \square

Remark 8 It must be clear that A_ϕ , B_ϕ and C_ϕ are not necessarily pairwise disjoint since an edge of $E_\phi(\delta)$ with one end of degree 2 is contained into 2 such sets. Assume indeed that $e = uv \in E_\phi(\delta)$ with $d(u) = 3$ and $d(v) = 2$ then, if u is incident to α and β and v is incident to γ we have an alternating path whose ends are u and v in $\phi(\alpha, \gamma)$ as well as in $\phi(\beta, \gamma)$. Hence e is in $A_\phi \cap B_\phi$.

When $e \in A_\phi$ we can associate to e the odd cycle $C_{A_\phi}(e)$ obtained in considering the path of $\phi(\alpha, \beta)$ together with e . We define in the same way $C_{B_\phi}(e)$ and $C_{C_\phi}(e)$ when e is in B_ϕ or C_ϕ . In the following lemma we consider an edge in A_ϕ , an analogous result holds true whenever we consider edges in B_ϕ or C_ϕ as well.

Lemma 9 *Let ϕ be a δ -minimum edge-colouring of G and let e be an edge in A_ϕ (B_ϕ , C_ϕ) then for any edge $e' \in C_{A_\phi}(e)$ there is a δ -minimum edge-colouring ϕ' such that $E_{\phi'}(\delta) = E_\phi(\delta) - e + e'$, $e' \in A_{\phi'}$ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. Moreover, each edge outside $C_{A_\phi}(e)$ but incident with this cycle is coloured γ .*

Proof In exchanging colours δ and α and δ and β successively along the cycle $C_{A_\phi}(e)$, we are sure to obtain an edge colouring preserving the number of edges coloured δ . Since we have supposed that ϕ is δ -minimum, at each step, the resulting edge colouring is proper and δ -minimum (Lemma 3). Hence, there is no edge coloured δ incident with $C_{A_\phi}(e)$, which means that every such edge is coloured with γ .

We can perform these exchanges until e' is coloured δ . In the δ -minimum edge-colouring ϕ' hence obtained, the two ends of e' are joined by a path of $\phi(\alpha, \beta)$. Which means that e' is in A_{ϕ} and $C_{A_{\phi}}(e) = C_{A_{\phi'}}(e')$. \square

For each edge $e \in E_{\phi}(\delta)$ (where ϕ is a δ -minimum edge-colouring of G) and we can associate one or two odd cycles following the fact that e is in one or two sets among A_{ϕ} , B_{ϕ} or C_{ϕ} . Let \mathcal{C} be the set of odd cycles associated to edges in $E_{\phi}(\delta)$.

Remark 10 Since each edge coloured with δ in a δ -minimum edge-colouring is contained in a cycle. These edges cannot be bridges of G .

Lemma 11 *For each cycle $C \in \mathcal{C}$, there are no two consecutive vertices with degree two.*

Proof Otherwise, we exchange colours along C in order to put the colour δ on the corresponding edge and, by Lemma 2, this is impossible in a δ -minimum edge-colouring. \square

Lemma 12 *Let $e_1, e_2 \in E_{\phi}(\delta)$ and let $C_1, C_2 \in \mathcal{C}$ such that $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$ then C_1 and C_2 are disjoint.*

Proof If e_1 and e_2 are contained in the same set A_{ϕ} , B_{ϕ} or C_{ϕ} , we are done since their respective ends are joined by an alternating path of $\phi(x, y)$ for some two colours x and y in $\{\alpha, \beta, \gamma\}$.

W.l.o.g. assume that $e_1 \in A_{\phi}$ and $e_2 \in B_{\phi}$. Assume moreover that there exists an edge e such that $e \in C_1 \cap C_2$. We have hence an edge $f \in C_1$ with exactly one end on C_2 . We can exchange colours on C_1 in order to put the colour δ on f . Which is impossible by Lemma 9. \square

By Lemma 9 any two cycles in \mathcal{C} corresponding to edges in distinct sets A_{ϕ} , B_{ϕ} or C_{ϕ} are at distance at least 2. Assume that $C_1 = C_{A_{\phi}}(e_1)$ and $C_2 = C_{A_{\phi}}(e_2)$ for some edges e_1 and e_2 in A_{ϕ} . Can we say something about the subgraph of G whose vertex set is $V(C_1) \cup V(C_2)$? In general, we have no answer to this problem. However, when G is cubic and any vertex of G lies on some cycle of \mathcal{C} (we shall say that \mathcal{C} is *spanning*), we have a property which will be useful after. Let us remark first that whenever \mathcal{C} is spanning, we can consider that G is edge-coloured in such a way that the edges of the cycles of \mathcal{C} are alternatively coloured with α and β (excepted one edge coloured δ) and the remaining perfect matching is coloured with γ . For this δ -minimum edge-colouring of G we have $B_{\phi} = \emptyset$ as well as $C_{\phi} = \emptyset$.

Lemma 13 *Assume that G is cubic and \mathcal{C} is spanning. Let $e_1, e_2 \in A_\phi$ and let $C_1, C_2 \in \mathcal{C}$ such that $C_1 = C_{A_\phi}(e_1)$ and $C_2 = C_{A_\phi}(e_2)$. Then at least one of the followings is true*

- (i) C_1 and C_2 are at distance at least 2
- (ii) C_1 and C_2 are joined by at least 3 edges
- (iii) C_1 and C_2 have at least two chords each

Proof Let $C_1 = v_0v_1 \dots v_{2k_1}$ and $C_2 = w_0w_1 \dots w_{2k_2}$. Assume that C_1 and C_2 are joined by the edge v_0w_0 . By lemma 6, we can consider a δ -minimum edge-colouring ϕ such that $\phi(v_0v_1) = \phi(w_0w_1) = \delta$, $\phi(v_1v_2) = \phi(w_1w_2) = \beta$ and $\phi(v_0v_{2k_1}) = \phi(w_0w_{2k_2}) = \alpha$. Moreover each edge of G (in particular v_0w_0) incident with these cycles is coloured γ . We can change the colour of v_0w_0 in β . We obtain thus a new δ -minimum edge-colouring ϕ' . Performing that exchange of colours on v_0w_0 transforms the edges coloured δ v_0v_1 and w_0w_1 in two edges of $C_{\phi'}$ lying on odd cycles C'_1 and C'_2 respectively. We get hence a new set $\mathcal{C}' = \mathcal{C} - \{C_1, C_2\} + \{C'_1, C'_2\}$ of odd cycles associated to δ -coloured edges in ϕ' .

From Lemma 12 C'_1 (C'_2 respectively) is at distance at least 2 from any cycle in $\mathcal{C} - \{C_1, C_2\}$. Hence $V(C'_1) \cup V(C'_2) \subseteq V(C_1) \cup V(C_2)$. It is an easy task to check now that (ii) or (iii) above must be verified. \square

Lemma 14 *Let $e_1 = uv_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G . Then v_1 is the only vertex in $N(u)$ of degree 2.*

Proof We have seen in Lemma 2 that uv has one end of degree 3 while the other has degree 2 or 3. Hence, we have $d(u) = 3$ and $d(v_1) = 2$. Let v_2 and v_3 the other neighbors of v . From Remark 8, we know that v_2 and v_3 are not pendent vertices. Assume that $d(v_2) = 2$ and uv_2 is coloured α , uv_3 is coloured β and, finally v_1 is incident to an edge coloured γ . The alternating path of $\phi(\beta, \gamma)$ using the edge uv_3 ends with the vertex v_1 (see Lemma 8), then, exchanging the colours along the component of $\phi(\beta, \gamma)$ containing v_2 allows us to colour uv_2 with γ and uv_1 with α . The new edge colouring ϕ' so obtained is such that $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, impossible. \square

Lemma 15 *Let e_1 and e_2 two edges of $E_\phi(\delta)$ contained into two distinct subsets of A_ϕ, B_ϕ or C_ϕ . Then $\{e_1, e_2\}$ induces a $2K_2$.*

Proof W.l.o.g. assume that $e_1 \in A_\phi$ and $e_2 \in B_\phi$. Since $C_{e_1}(\phi)$ and $C_{e_2}(\phi)$ are disjoint by Lemma 12, we can consider that e_1 and e_2 have no common vertex. The edges having exactly one end in $C_{e_1}(\phi)$ are coloured γ while those

having exactly one end in $C_{e_2}(\phi)$ are coloured α . Hence there is no edge between e_1 and e_2 as claimed. \square

Lemma 16 *Let e_1 and e_2 two edges of $E_\phi(\delta)$ contained into the same subset A_ϕ, B_ϕ or C_ϕ . Then $\{e_1, e_2\}$ are joined by at most one edge*

Proof W.l.o.g. assume that $e_1 = u_1v_1, e_2 = u_2v_2 \in A_\phi$. Since $C_{e_1}(\phi)$ and $C_{e_2}(\phi)$ are disjoint by Lemma 12, we can consider that e_1 and e_2 have no common vertex. The edges having exactly one end in $C_{e_1}(\phi)$ (or $C_{e_2}(\phi)$) are coloured γ . Assume that u_1u_2 and v_1v_2 are edges of G . In exchanging eventually colours α and β along the path of $\phi(\alpha, \beta)$ joining the two ends of u_1u_2 we can consider that u_1 and u_2 are incident to α while v_1 and v_2 are incident to β . We know that u_1u_2 and v_1v_2 are coloured γ . Let us colour e_1 and e_2 with γ and u_1u_2 with β and v_1v_2 with α . We get a new edge colouring ϕ' where $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 2$, contradiction since ϕ is a δ -minimum edge-colouring. \square

Lemma 17 *Let e_1, e_2 and e_3 three edges of $E_\phi(\delta)$ contained into the same subset A_ϕ, B_ϕ or C_ϕ . Then $\{e_1, e_2, e_3\}$ contains at most one edge more.*

Proof W.l.o.g. assume that $e_1 = u_1v_1, e_2 = u_2v_2$ and $e_3 = u_3v_3 \in A_\phi$. From Lemma 16 we have just to suppose that (up to the names of vertices) $u_1u_3 \in E(G)$ and $v_1v_2 \in E(G)$. In exchanging eventually the colours α and β along the 3 disjoint paths of $\phi(\alpha, \beta)$ joining the ends of each edge e_1, e_2 and e_3 , we can suppose that u_1 and u_3 are incident to β while v_1 and v_2 are incident to α . Let ϕ' be obtained from ϕ when u_1u_3 is coloured with α , v_1v_2 with β and u_1v_1 with γ . It is easy to check that ϕ' is a proper edge-colouring with $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, contradiction since ϕ is a δ -minimum edge-colouring. \square

Lemma 18 *Let $e_1 = u_1v_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G . Then for any edge $e_2 = u_2v_2 \in E_\phi(\delta)$ $\{e_1, e_2\}$ induces a $2K_2$.*

Proof From Lemma 15 we have to consider that e_1 and e_2 are contained in the same subset A_ϕ, B_ϕ or C_ϕ . Assume w.l.o.g. that they are contained in A_ϕ . From Lemma 16 we have just to consider that there is a unique edge joining these two edges and we can suppose that $u_1u_2 \in E(G)$. In exchanging eventually the colours α and β along the 2 disjoint paths of $\phi(\alpha, \beta)$ joining the ends of each edge e_1, e_2 , we can suppose that u_1 and u_2 are incident to β while v_1 and v_2 are incident to α . We know that u_1u_2 is coloured γ . Let ϕ' be obtained from ϕ when u_1u_2 and u_2v_2 are coloured with α and u_1v_1 is coloured with γ . It is easy to check that ϕ' is a proper edge-colouring with $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, contradiction since ϕ is a δ -minimum edge-colouring. \square

3 Applications and problems

3.1 On a result by Payan

In [8] Payan showed that it is always possible to edge-colour a graph of maximum degree 3 with three maximal matchings (for the inclusion) and introduces henceforth a notion of *strong-edge colouring* where a strong edge-colouring means that one colour is a strong matching while the remaining colours are usual matchings. Payan conjectured that any d -regular graph has d pairwise disjoint maximal matchings and showed that this conjecture holds true for graphs with maximum degree 3.

Theorem 19 *Let G be a graph with maximum degree at most 3. Then G has δ -minimum edge-colouring ϕ where $E_\phi(\delta)$ is a strong matching and, moreover, any edge in $E_\phi(\delta)$ has its two ends of degree 3 in G .*

Proof Let ϕ be a δ -minimum edge-colouring of G . From Lemma 15, any two edges of $E_\phi(\delta)$ belonging to distinct subsets in A_ϕ, B_ϕ and C_ϕ induce a strong matching. Hence, we have to find a δ -minimum edge-colouring where each subset A_ϕ, B_ϕ or C_ϕ induces a strong matching (with the supplementary property that the end vertices of these edges have degree 3). That means that we can work on each subset A_ϕ, B_ϕ and C_ϕ independently. W.l.o.g., we only consider A_ϕ here.

Assume that $A_\phi = \{e_1, e_2, \dots, e_k\}$ and $A'_\phi = \{e_1, \dots, e_i\}$ ($1 \leq i \leq k-1$) is a strong matching and each edge of A'_ϕ has its two ends with degree 3 in G . Consider the edge e_{i+1} and let $C = C_{e_{i+1}}(\phi) = (u_0, u_1 \dots u_{2p})$ be the odd cycle associated to this edge (Lemma 6).

Let us mark any vertex v of degree 3 on C with a $+$ whenever the edge of colour γ incident to this vertex has its other end which is a vertex incident to an edge of A'_ϕ and let us mark v with $-$ otherwise. By Lemma 11 a vertex of degree 2 on C has its two neighbors of degree 3 and by Lemma 18 these two vertices are marked with a $-$. By Lemma 17 we cannot have two consecutive vertices marked with a $+$. Hence, C must have two consecutive vertices of degree 3 marked with $-$ whatever is the number of vertices of degree 2 on C .

Let u_j and u_{j+1} be two vertices of C of degree 3 marked with $-$ (j being taken module $2p+1$). We can transform the edge colouring ϕ by exchanging colours on C uniquely, in such a way that the edge of colour δ of this cycle is $u_j u_{j+1}$. In the resulting edge colouring ϕ_1 we have $A_{\phi_1} = A_\phi - e_{i+1} + u_j u_{j+1}$ and $A'_{\phi_1} = A'_\phi + u_j u_{j+1}$ is a strong matching where each edge has its two ends of degree 3. Repeating this process we are left with a new δ -minimum colouring

ϕ' where $A_{\phi'}$ is a strong matching. □

This result has been obtained first by Payan [8], but his technique does not exhibit explicitly the odd cycles associated to the edges of $E_{\phi}(\delta)$ and their properties. It appears in [4] with this proof.

Corollary 20 *Let G be a graph with maximum degree 3 then there are $s(G)$ vertices of degree 3 pairwise non-adjacent $v_1 \dots v_{s(G)}$ such that $G - \{v_1 \dots v_{s(G)}\}$ is 3-colourable.*

Proof Pick a vertex on each edge coloured δ in a δ -minimum colouring ϕ of G where $E_{\phi}(\delta)$ is a strong matching (Theorem 19). We get a subset S of vertices satisfying our corollary. □

Steffen [10] obtained Corollary 20 for bridgeless cubic graphs.

3.2 Parsimonious edge colouring

Let $\chi'(G)$ be the classical chromatic index of G . For convenience let

$$c(G) = \max\{|E(H)| : H \subseteq G \chi'(H) = 3\}$$

$$\gamma(G) = \frac{c(G)}{E(G)}$$

Staton [9] (and independently Locke [7]) showed that whenever G is a cubic graph distinct from K_4 then G contains a bipartite subgraph (and hence a 3-edge colourable graph, by König's theorem [6]) with at least $\frac{7}{9}$ of the edges of G . Bondy and Locke [2] obtained $\frac{4}{5}$ when considering graphs with maximum degree at most 3.

In [1] Albertson and Haas showed that whenever G is a cubic graph, we have $\gamma(G) \geq \frac{13}{15}$ while for graphs with maximum degree 3 they obtained $\gamma(G) \geq \frac{26}{31}$. Our purpose here is to show that $\frac{13}{15}$ is a lower bound for $\gamma(G)$ when G has maximum degree 3, with the exception of the graph G_5 depicted in Figure 1 below.

Lemma 21 *Let G be a graph with maximum degree 3 then $\gamma(G) = 1 - \frac{s(G)}{m}$.*

Proof Let ϕ be a δ -minimum edge-colouring we need to colour $s(G)$ edges of G with δ by Lemma 12. We have thus $m - s(G)$ edges coloured with α, β and γ . The results follows. □

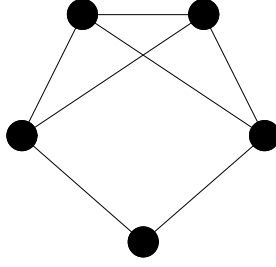


Fig. 1. G_5

Lemma 22 [1] *Let G be a graph with maximum degree 3. Assume that $v \in V(G)$ is such that $d(v) = 1$ then $\gamma(G) > \gamma(G - v)$.*

Proof We have certainly $s(G - v) = s(G)$ since v is not incident to an edge coloured δ in a δ -minimum edge-colouring of G by Lemma 2. We have thus

$$\gamma(G) = 1 - \frac{s(G)}{m} > 1 - \frac{s(G - v)}{m - 1} = \gamma(G - v)$$

as claimed. □

A triangle $T = \{a, b, c\}$ is said to be *reducible* whenever its neighbors are distinct. When T is a reducible triangle in G (G having maximum degree 3) we can obtain a new graph G' with maximum degree 3 in shrinking this triangle into a single vertex and joining this new vertex to the neighbors of T in G .

Lemma 23 [1] *Let G be a graph with maximum degree 3. Assume that $T = \{a, b, c\}$ is a reducible triangle and let G' be the graph obtained by reduction of this triangle. Then $\gamma(G) > \gamma(G')$.*

Proof Assume that $s(G) = s(G')$ then

$$\gamma(G) = 1 - \frac{s(G)}{m} > 1 - \frac{s(G')}{m - 3} = \gamma(G')$$

and we are done.

Hence, we have just to prove that $s(G) = s(G')$. It is a classical task to transform an edge colouring of G' into an edge colouring of G without increasing the number of edges coloured δ . We get immediately that $s(G') \geq s(G)$. Let ϕ be a δ -minimum edge-colouring of G . Using Theorem 19 we can suppose that we have at most one edge coloured δ involved in T and the incident edges to T . When considering G' , we get eventually a δ -improper colouring ϕ_1 , where v is eventually the only vertex incident to 2 or 3 edges coloured δ . Leading to $s(G)$, $s(G) + 1$ or $s(G) + 2$ edges in $E_{\phi_1}(\delta)$. This colouring can be transformed

in a proper colouring ϕ' with colouring ϕ' with Lemma 3 in order to eliminate the colour δ appearing on one or two edges incident to v if necessary. Since this operation does not modify the colour of the edges coloured δ which are not incident with v , we have $s(G) \geq s(G')$. \square

Lemma 24 *Let G be a cubic graph which can be factored into $s(G)$ cycles of length 5 and without reducible triangle. The every 2-factor of G contains $s(G)$ cycles of length 5.*

Proof Since G has no reducible triangle, all cycles in a 2-factor have length at least 4. Let \mathcal{C} be any 2-factor of G . When n_5 denotes the number of cycles of length 5 and $n_{>}$ denotes the number of cycles on at least 6 vertices in \mathcal{C} , there is an edge-coloring of G which uses at most $n_5 + n_{>}$ edges of color δ . In addition, since \mathcal{C} is a spanning subgraph we have $5s(G) \geq 5n_5 + 6n_{>}$ and if $n_{>} \neq 0$, $s(G) > n_5 + n_{>}$, a contradiction. Thus $n_{>} = 0$ and $n_5 = s(G)$, the result follows. \square

Theorem 25 *Let G be a graph with maximum degree 3 and $G \neq G_5$ then $\gamma(G) \geq \frac{13}{15}$.*

Proof From Lemma 22 and Lemma 21, we can consider that G has only vertices of degree 2 or 3 and that G contains no reducible triangle. Let V_2 be the set of vertices with degree 2 in G and V_3 those of degree 3.

Assume that we can associate a set P_e of at least 5 distinct vertices of V_3 for each edge $e \in E_\phi(\delta)$ in a δ -minimum edge-colouring ϕ of G . Assume moreover that

$$\forall e, e' \in E_\phi(\delta) \quad P_e \cap P_{e'} = \emptyset \quad (1)$$

Then

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{3}{2}|V_3| + |V_2|} \geq 1 - \frac{\frac{|V_3|}{5}}{\frac{3}{2}|V_3| + |V_2|}$$

Hence

$$\gamma(G) \geq 1 - \frac{\frac{2}{15}}{1 + \frac{2|V_2|}{3|V_3|}}$$

Which leads to $\gamma(G) \geq \frac{13}{15}$ as claimed.

It remains to see how to construct the sets P_e satisfying property (2). Let \mathcal{C} be the set of odd cycles associated to edges in $E_\phi(\delta)$ (see Lemma 12). Let $e \in E_\phi(\delta)$, assume that e is contained in a cycle $C \in \mathcal{C}$ of length 3. By Lemma 12, the edges incident to that triangle have the same colour in α, β or γ . This triangle is hence reducible, impossible. We can thus consider that each cycle of \mathcal{C} has length at least 5. By Lemma 2 and Lemma 14, we know that whenever

such a cycle contains vertices of V_2 , their distance on this cycle is at least 3. Which means that every cycle $C \in \mathcal{C}$ contains at least 5 vertices in V_3 as soon as C has length at least 7 or C has length 5 but does not contain a vertex of V_2 . For each edge $e \in E_\phi(\delta)$ contained in such a cycle we associate P_e as any set of 5 vertices of V_3 contained in the cycle.

It remains eventually edges in $E_\phi(\delta)$ which are contained in a C_5 of \mathcal{C} using a (unique) vertex of V_2 . Let $C = a_1a_2a_3a_4a_5$ be such a cycle and assume that $a_1 \in V_2$. By Lemma 2 and Lemma 17, a_1 is the only vertex of degree 2 and in exchanging colours along this cycle, we can suppose that $a_1a_2 \in E_\phi(\delta)$. Since $a_1 \in V_2$, $e = a_1a_2$ is contained into a second cycle C' of \mathcal{C} (see Remark 8). If C' contains a vertex $x \in V_3$ distinct from a_2, a_3, a_4 and a_5 then we set $P_e = \{a_2, a_3, a_4, a_5, x\}$. Otherwise $C' = a_1a_2a_4a_3a_5$ and G is isomorphic to G_5 , impossible.

The sets $\{P_e | e \in E_\phi(\delta)\}$ are pairwise disjoint since any two cycle of \mathcal{C} associated to distinct edges in $E_\phi(\delta)$ are disjoint. Hence property 2 holds and the proof is complete. \square

Albertson and Haas [1] proved that $\gamma(G) \geq \frac{26}{31}$ when G is a graph with maximum degree 3. The bound given in Theorem 25 is better.

Corollary 26 *Let G be a graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$. Then G is a cubic graph which can be factored into $s(G)$ cycles of length 5. Moreover every 2-factor of G has this property.*

Proof The optimal for $\gamma(G)$ in Theorem 25 is obtained whenever $s(G) = \frac{|V_3|}{5}$ and $|V_2| = 0$. That is, G is a cubic graph together with a 2-factor of $s(G)$ cycles of length 5. Moreover by Lemma 23 G has no reducible triangle, the result comes from Lemma 24. \square

As pointed out by Albertson and Haas [1], Petersen's graph with $\gamma(G) = \frac{13}{15}$ supplies the extremal example for cubic graphs. Steffen [11] proved that the only cubic bridgeless graph with $\gamma(G) = \frac{13}{15}$ is the Petersen's graph. In fact, we can extend this result to graphs with maximum degree 3 where bridges are allowed (excluding the graph G_5).

Let P' be the cubic graph on 10 vertices obtained from two copies of G_5 (Figure 1) in joining by an edge the two vertices of degree 2.

Theorem 27 *Let G be a connected graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$. Then G is isomorphic to Petersen's graph or to P' .*

Proof Let G be a graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$. From Corollary 26, we can consider that G is cubic and G has a 2-factor of cycles of length 5. Let $\mathcal{C} = \{C_1 \dots C_{s(G)}\}$ be such a 2-factor (\mathcal{C} is spanning). Let ϕ be a δ -minimum edge-colouring of G induced by this 2-factor.

W.l.o.g. consider two cycles in \mathcal{C} , namely C_1 and C_2 , and let us denote $C_1 = v_1v_2v_3v_4v_5$ while $C_2 = u_1u_2u_3u_4u_5$ and assume that $v_1u_1 \in G$. From Lemma 13 C_1 and C_2 are joined by at least 3 edges or have 2 chords. If $s(G) > 2$ there is a cycle $C_3 \in \mathcal{C}$. W.l.o.g., G being connected, we can suppose that C_3 is joined to C_1 by an edge. Applying once more time Lemma 13, C_1 and C_3 have two chords or are joined by at least 3 edges, contradiction with the constraints imposed by C_1 and C_2 . Hence $s(G) = 2$ and G has 10 vertices, which leads to a graph isomorphic to P' or Petersen's graph as claimed. \square

We do not know example of 3-connected cubic graphs (excepted Petersen's graph) with a 2-factor of induced cycles of length 5 with chromatic index 4 and we propose the following conjecture.

Conjecture 28 *Let G be a 3-connected cubic graph distinct from Petersen's graph having a 2-factor of cycles of length 5 which are induced cycles of G , then G is 3-edge colourable.*

As a first step towards the resolution of this Conjecture we propose :

Theorem 29 *Let G be a cubic graph which can be factored into $s(G)$ induced cycles of length 5, then G is the Petersen's graph.*

Proof Let \mathcal{F} be a 2-factor of $s(G)$ cycles of length 5 in G , every cycle of \mathcal{F} being an induced cycle of G . We consider the δ -minimum edge-colouring ϕ such that the edges of all cycles of \mathcal{F} are alternatively coloured α and β excepted for exactly one edge per cycle which is coloured with δ , all the remaining edges of G being coloured γ . By construction we have $B_\phi = C_\phi = \emptyset$ and $A_\phi = \mathcal{F}$.

By Lemma 13 two cycles of \mathcal{F} which are connected must be connected with at least three edges. Consequently a cycle of \mathcal{F} cannot be connected by at least one edge to more than one other cycle of \mathcal{C} . It follows that \mathcal{C} contains exactly two induced cycles of length 5. The only graph with 10 vertices and $s(G) = 2$ with such a 2-factor is the Petersen's graph. \square

Comments: The index $s(G)$ used here is certainly less than $o(G)$ the *oddness* of G used by Huck and Kochol [5]. $o(G)$ is the minimum number of odd cycles in any 2-factor of a cubic graphs (assuming that we consider graphs with that

property). Obviously $o(G)$ is an even number and it is an easy task to construct a cubic graph G with $s(G)$ odd which ensures that $0 < s(G) < o(G)$. We can even construct cyclically-5-edge-connected cubic graphs with that property with $s(G) = k$ for any integer $k \neq 1$ (see [4] and [11]). It can be pointed out that, using a parity argument (see 7, or [10]) a graph with oddness at least 2 needs certainly to be edge coloured with at least 2 edges in the less popular colour. In other words, $o(G) = 2 \Leftrightarrow s(G) = 2$.

When G is a cubic bridgeless planar graph, we know from the 4CT that G is 3-edge colourable and hence $\gamma(G) = 1$. Albertson and Haas [1] gave $\gamma(G) \geq \frac{6}{7} - \frac{2}{35m}$ when G is a planar bridgeless graph with maximum degree 3. Our Theorem 25 improves this lower bound (allowing moreover bridges). On the other hand, they exhibit a family of planar graphs with maximum degree 3 (bridges are allowed) for which $\gamma(G) = \frac{8}{9} - \frac{2}{9n}$.

As in [11] we denote $g(\mathcal{F}) = \min\{V(C) \mid C \text{ is an odd circuit of } \mathcal{F}\}$ and $g^+(G) = \max\{g(\mathcal{F}) \mid \mathcal{F} \text{ is a 2-factor of } G\}$. We suppose that $g^+(G)$ is defined, that is G has at least one 2-factor (when G is a cubic bridgeless graphs this condition is obviously fulfilled).

When G is cubic bridgeless, Steffen [11] showed that we have :

$$\gamma(G) \geq \max\left\{1 - \frac{2}{3g^+(G)}, \frac{11}{12}\right\}$$

The difficult part being to show that $\gamma(G) \geq \frac{11}{12}$. We can remark that whenever $g^+(G) \geq 11$ we obtain the same result for graphs with maximum degree 3 and having at most $\frac{n}{3}$ vertices of degree 1 or 2.

Theorem 30 *Let G be a graph with maximum degree 3 and having at least one 2-factor. Then $\gamma(G) \geq 1 - \frac{2}{3g^+(G)}$.*

Proof We have $\gamma(G) = 1 - \frac{s(G)}{m}$ where $m = \frac{3n}{2}$ and, obviously, $s(G) \leq \frac{n}{g^+(G)}$. Hence $\gamma(G) \geq 1 - \frac{2}{3g^+(G)}$. \square

Theorem 31 *Let G be a graph with maximum degree 3 and having at least one 2-factor. Let V_i ($i = 1..3$) be the set of vertices of degree i . Assume that $|V_2| \leq \frac{n}{3}$ and $g^+(G) \geq 11$ then $\gamma(G) \geq \max\left\{1 - \frac{2}{3g^+(G)}, \frac{11}{12}\right\}$.*

Proof Since, by Lemma 22 $\gamma(G) > \gamma(G - v)$ we can assume that $V_1 = \emptyset$. From Theorem 30 we have just to prove that $\gamma(G) \geq \frac{11}{12}$. Following the proof of Theorem 25, we try to associate a set P_e of at least 8 distinct vertices of V_3 for each edge $e \in E_\phi(\delta)$ in a δ -minimum edge-colouring ϕ of G such that

$$\forall e, e' \in E_\phi(\delta) \quad P_e \cap P_{e'} = \emptyset \quad (2)$$

Indeed, let \mathcal{F} be 2-factor of G where each odd cycle has length at least 11 and let $C_1, C_2 \dots C_{2k}$ be its set of odd cycles. We have, obviously $s(G) \leq 2k$. Let $V_3^{1\dots 2k}$ and $V_2^{1\dots 2k}$ be the sets of vertices of degree 3 and 2 respectively contained in the odd cycles. As soon as $|V_3^{1\dots 2k}| \geq 8s(G)$ we have

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{3}{2}|V_3| + |V_2|} \geq 1 - \frac{\frac{|V_3^{1\dots 2k}|}{8}}{\frac{3}{2}|V_3| + |V_2|} \quad (3)$$

which leads to

$$\gamma(G) \geq 1 - \frac{\frac{2|V_3^{1\dots 2k}|}{24|V_3|}}{1 + \frac{2}{3} \frac{|V_2|}{|V_3|}}$$

Since $|V_3| \geq |V_3^{1\dots 2k}|$, we have

$$\gamma(G) \geq 1 - \frac{\frac{2}{24}}{1 + \frac{2}{3} \frac{|V_2|}{|V_3|}}$$

and

$$\gamma(G) \geq \frac{11}{12}$$

as claimed.

It remains the case where $|V_3^{1\dots 2k}| < 8s(G)$. Since each cycle has at least 11 vertices we have $|V_2^{1\dots 2k}| > 11 \times 2k - |V_3^{1\dots 2k}| > 3s(G)$.

$$\gamma(G) = \frac{m - s(G)}{m} \geq \frac{m - \frac{|V_2^{1\dots 2k}|}{3}}{m}$$

We have

$$\frac{m - \frac{|V_2^{1\dots 2k}|}{3}}{m} \geq \frac{11}{12}$$

when

$$m \geq 4|V_2^{1\dots 2k}| \quad (4)$$

Since $|V_2| \leq \frac{n}{3}$ we have $|V_3| \geq \frac{2n}{3}$ and

$$m = 3\frac{|V_3|}{2} + |V_2| = 3\frac{n - |V_2|}{2} + |V_2| = 3\frac{n}{2} - \frac{|V_2|}{2} \geq 4\frac{n}{3} \geq 4|V_2^{1\dots 2k}| \quad (5)$$

and the result holds. \square

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