



HAL
open science

TOPOLOGY OF MANIFOLDS WITH ASYMPTOTICALLY NONNEGATIVE RICCI CURVATURE

Bazanfare Mahaman

► **To cite this version:**

Bazanfare Mahaman. TOPOLOGY OF MANIFOLDS WITH ASYMPTOTICALLY NONNEGATIVE RICCI CURVATURE. African Diaspora Journal of Mathematics, 2015, 18 (2), pp.11-17. hal-00324963

HAL Id: hal-00324963

<https://hal.science/hal-00324963>

Submitted on 25 Sep 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

TOPOLOGY OF MANIFOLDS WITH ASYMPTOTICALLY NONNEGATIVE RICCI CURVATURE

BAZANFARÉ MAHAMAN

ABSTRACT. In this paper, we study the topology of complete noncompact Riemannian manifolds with asymptotically nonnegative Ricci curvature. We show that a complete noncompact manifold with asymptotically nonnegative Ricci curvature and sectional curvature $K_M(x) \geq -\frac{C}{d_p(x)^\alpha}$ is diffeomorphic to a Euclidean n -space \mathbb{R}^n under some conditions on the density of rays starting from the base point p or on the volume growth of geodesic balls in M .

1. INTRODUCTION

One of most important problems in Riemannian geometry is to find conditions under which manifold is of finite topological type: A manifold is said to have finite topological type if there exists a compact domain Ω with boundary such that $M \setminus \Omega$ is homeomorphic to $\partial\Omega \times [0, \infty[$. The fundamental notion involved in such a finite topological type result is that of the critical point of a distance function introduced by Grove and Shiohama [8]. Let p a fix point and set $d_p(x) = d(p, x)$ A point $x \neq p$ is called critical point of d_p if for any v in the tangent space $T_x M$ there is minimal geodesic γ from x to p forming an angle less or equal to $\pi/2$ with $\gamma'(0)$ (see [8]).

In several papers it has been proved results for manifolds with nonnegative curvature. By isotopy lemma (see below), the absence of critical point assumed that the manifold is diffeomorphic to the euclidean space \mathbb{R}^n .

X. Menguy in [11] and J. Sha and D. Yang in [13] constructed manifolds with nonnegative Ricci curvature and infinite topological type. Hence a natural question is under what additional conditions are manifolds with nonnegative Ricci curvature of finite topological type? Are those manifolds diffeomorphic to the unit sphere or the euclidean space? Under volume growth, diameter or density of rays conditions, some results were obtained on the geometry and topology of open manifolds with nonnegative Ricci curvature. See [2], [3],[6],[10],[12],[13], [14],[15], [17],[18]. . .

Let K denotes the sectional curvature of M and fix a point $p \in M$. For $r > 0$ let

$$k_p(r) = \inf_{M \setminus B(p,r)} K$$

where $B(p, r)$ is the open geodesic ball around with radius r and the infimum is taken over all the sections at points on $M \setminus B(p, r)$. If (M, g) is a complete noncompact Riemannian manifold, we say M has sectional curvature decay at most quadratic if $k_p(r) \geq -\frac{C}{r^\alpha}$ for some $C > 0$, $\alpha \in [0, 2]$ and all $r > 0$.

In this paper we see the case of manifolds with asymptotically nonnegative Ricci curvature and with sectional curvature decay almost quadratically.

A complete noncompact Riemannian manifold is said to have an asymptotically nonnegative sectional curvature (Ricci curvature) if there exists a point p , called base point, and a monotone decreasing positive function λ such that $\int_0^{+\infty} s\lambda(s)ds = b_0 < +\infty$ and for any point x in M we have

$$K(x) \geq -\lambda(d_p(x)) \text{ (resp. } Ric(x) \geq -(n-1)\lambda(d_p(x)))$$

where d_p is the distance to p . Let $B(x, r)$ denote the metric ball of radius r and centre x in M and $B(\bar{x}, r)$ denote the similar metric ball in the simply connected noncompact complete manifold with sectional curvature $-\lambda(d_{\bar{p}}(\bar{x}))$ at the point \bar{x} where $d_p(x) = d(p, x)$ is the distance from p to x .

The volume comparison theorem proved in [9]] says that the function $r \mapsto \frac{volB(x, r)}{volB(\bar{x}, r)}$ is monotone decreasing. Set

$$\alpha_x = \lim_{r \rightarrow +\infty} \frac{volB(x, r)}{volB(\bar{x}, r)} \text{ and } \alpha_M = \inf_{x \in M} \alpha_x.$$

We say M is large volume growth if $\alpha_M > 0$.

In [1] U. Abresch proved that asymptotically nonnegative sectional curvature have finite topological type.

Let R_p denotes the set of all ray issuing from p and $S(p, r)$ the geodesic ball of radius r and the center p . Set $H(p, r) = \max_{x \in S(p, r)} d(x, R_p)$. By definition, we have $H(p, r) \leq r$. Some results have been obtained by geometers on manifolds with nonnegative Ricci curvature by using the density of the rays. For manifolds with quadratic sectional curvature decay, Q. Wang and C. Xia proved that there exists a constant δ such that if $H(p, r) < \delta r$ then they are diffeomorphic to \mathbb{R}^n .

In this paper we prove the following theorem:

Theorem 1.1. *Given $c > 0$ and $\alpha \in [0, 2]$; suppose that M is an n -dimensional complete noncompact Riemannian manifold with $Ricci_M(x) \geq -(n-1)\lambda(d_p(x))$ and $K(x) \geq -\frac{C}{d_p(x)^\alpha}$, $Crit_p \geq r_0$ then there exists a positive constant $\delta_0 > 0$ such that if $H(p, r) < \delta_0 r^{\beta/2}$ then M is diffeomorphic to \mathbb{R}^n where $\beta = \frac{2}{n} + \alpha(1 - \frac{1}{n})$.*

Remark 1.2. (i) Theorem1.1 is an improvement of theorem1.1 [16] where nonnegative Ricci curvature was assumed and sectional curvature $K_p(r) \geq -\frac{C}{(1+r)^\alpha}$.

(ii) For $\alpha = 0$ theorem1.1 is a generalisation of lemma 3.1 [18].

In [16] Q. Wang and C. Xia proved the following theorem (Theorem 1.3)

Theorem 1.3. *Given $\alpha \in [0, 2]$, positive numbers r_0 and C , and an integer $n \geq 2$, there is an $\epsilon = \epsilon(n, r_0, C, \alpha) > 0$ such that any complete Riemannian n -manifold M with Ricci curvature $\text{Ric}_M \geq 0$, $\alpha_M > 0$, $\text{crit}_p \geq r_0$ and*

$$K(x) \geq -\frac{C}{(1 + d_p(x))^\alpha}, \quad \frac{\text{vol}B(p, r)}{\omega_n r^n} \leq \left(1 + \frac{\epsilon}{r^{(n-2\frac{1}{n})(1-\frac{\alpha}{2})}}\right) \alpha_M$$

for some $p \in M$ and all $r \geq r_0$ is diffeomorphic to \mathbb{R}^n .

In this paper we prove a more general result:

Theorem 1.4. *Given $c > 0$ and $\alpha \in [0, 2]$; suppose that M is an n -dimensional complete noncompact Riemannian manifold with $\text{Ric}_M(x) \geq -(n-1)\lambda(d_p(x))$ and $K(x) \geq -\frac{C}{d_p(x)^\alpha}$, $\text{Crit}_p \geq r_0$ then there exists a positive constant $\epsilon = \epsilon(C, \alpha, r_0)$ such that if*

$$(1.1) \quad \frac{\text{vol}B(p, r)}{\text{vol}B(\bar{p}, r)} \leq \left(1 + \frac{\epsilon}{r^{(n-2+\frac{1}{n})(1-\frac{\alpha}{2})}}\right) \alpha_p$$

then M is diffeomorphic to \mathbb{R}^n .

2. PRELIMANARIES

To prove our results we need some lemmas.

The following one is proved in [8]

Lemma 2.1. *(Isotopy Lemma).*

Let $0 \leq r_1 \leq r_2 \leq \infty$. If a connected component C of $B(p, r_2) \setminus B(p, r_1)$ is free of critical points of p , then C is homeomorphic to $C_1 \times [r_1, r_2]$, where C_1 is a topological submanifold without boundary.

If $r_1 = 0$ and $r_2 = \infty$ then the homeomorphism becomes diffeomorphism (see for example [7].)

Let p and q be two points of a complete Riemannian manifold M . The excess function e_{pq} is defined by: $e_{pq}(x) = d_p(x) + d_q(x) - d(p, q)$. In [2] U. Abresch and D. Gromoll gave an explicit upper bound of the excess function in manifolds with curvature bounded below. They proved the following lemma:

Lemma 2.2. *(Proposition 3.1 [2]) Let M be an n -dimensional complete Riemannian manifold ($n \geq 3$ and let γ be a minimal geodesic joining the base point p and another point $q \in M$, $x \in M$ is a third point and the excess function $e_{pq}(x) = d_p(x) + d_q(x) - d(p, q)$. Suppose $d(p, q) \geq 2d_p(x)$ and, moreover, that there exists a nonincreasing function $\lambda : [0, +\infty[\rightarrow [0, +\infty[$ such that $b_0 = \int_0^\infty r\lambda(r)dr$ converges and $\text{Ric} \geq -(n-1)\lambda(d_p(x))$ at all points $x \in M$. Then the height of the triangles can be bounded from below in terms of $d_p(x)$ and excess $e_{pq}(x)$. More precisely,*

$$(2.1) \quad s \geq \min \left\{ \frac{1}{6}d_p(x), \frac{d_p(x)}{(1 + 8b_0)^{1/2}}, C_0 d_p(x)^{1/n} (2e_{pq}(x))^{1-\frac{1}{n}} \right\}$$

where $C_0 = \frac{4}{17} \frac{n-2}{n-1} \left(\frac{5}{1+8b_0}\right)^{1/n}$.

Lemma 2.3 (lemma [9]). *Let (M, g) be a complete noncompact Riemannian manifold with asymptotically nonnegative Ricci curvature with base point p . Then for all $x \in M$ and all numbers R', R with $0 < R' < R$ we have*

$$(2.2) \quad \frac{\text{vol}B(x, R)}{\text{vol}B(x, R')} \leq \frac{\text{vol}B(\bar{x}, R)}{\text{vol}B(\bar{x}, R')} \leq \begin{cases} e^{(n-1)b_0} \left(\frac{R}{R'}\right)^n & \text{if } 0 < R < r = d(p, x) \\ e^{(n-1)b_0} \left(\frac{R+r}{R'}\right)^n & \text{if } R \geq r \end{cases}$$

where $B(\bar{x}, s)$ is the ball in \bar{M} with center \bar{x} and radius s .

Let Σ_p be a closed subset of $U_p = \{u \in T_p M, \|u\| = 1\}$.

Set $\Sigma_p(r) = \{v \in \Sigma_p / \gamma(t) = \exp_p tv, \gamma \text{ is minimal on } [0, r]\}$ and

$$B_{\Sigma_p(r)}(p, r) = \{x \in B(p, r) / \exists \gamma : [0, s] \rightarrow M, \gamma(0) = p, \gamma(s) = x \text{ and } \gamma'(0) \in \Sigma_p\}.$$

Set $\Sigma_p(\infty) = \bigcap_{r>0} \Sigma_p(r)$.

The following two lemmas generalised the above one.

Lemma 2.4 (Lemma3.9 [10]). *Let (M, g) be a Riemannian complete noncompact manifold such that $\text{Ric}_M \geq -(n-1)\lambda(d_p(x))$ and Σ_p be a closed subset of U_p . Then the function $r \mapsto \frac{\text{vol}B_{\Sigma_p(r)}(p, r)}{\text{vol}B(\bar{p}, r)}$ is non increasing.*

Lemma 2.5 (Lemma 3.10 [10]). *Let (M, g) be a Riemannian complete noncompact manifold such that $\text{Ric}_M \geq -(n-1)\lambda(d_p(x))$ and Σ_p be a closed subset of U_p . Then $\frac{\text{vol}B_{\Sigma_p(r)}(p, r)}{\text{vol}B(\bar{p}, r)} \geq \alpha_p$.*

3. PROOFS

Proof of theorem1.1

To prove the theorem1.1, it suffices to show that d_p has no critical point other than p . Let x be a point of M . Set $r = d(p, x)$; $s = d(x, R_p)$. Since R_p is closed there exists a ray γ issuing from p such that $s = d(x, \gamma)$. Set $q = \gamma(t_0)$ for $t_0 \geq 2r$. Let σ_1 and σ_2 be geodesics joining x to p and q respectively.

Set $\tilde{p} = \sigma_1(\delta r^{\alpha/2})$; $\tilde{q} = \sigma_2(\delta r^{\alpha/2})$ with

$$(3.1) \quad \delta < \min \left\{ C_0^n, \frac{r_0^{1-\beta/2}}{20}, \frac{r_0^{1-\beta/2}}{\sqrt{1+8b_0}}, C_0^n r_0^{1-\beta/2} \right\}.$$

Consider the triangle $(x, \tilde{p}, \tilde{q})$; if y is a point on this triangle, then

$$d(p, y) \geq d(p, x) - d(x, y) \geq d(p, x) - d(\tilde{p}, x) - d(\tilde{p}, y) \geq d(p, x) - 2\delta r^{\alpha/2}.$$

Since $\beta \geq \alpha$. we have

$$(3.2) \quad d(p, y) \geq d(p, x) - 2\delta r^{\beta/2} \geq r(1 - 2\delta r^{\frac{\beta}{2}-1}) \geq r(1 - 2\delta r^{\frac{\beta}{2}-1}) \geq r/4.$$

Hence $y \in M \setminus B(p, r/4)$ and $K_M(y) \geq -\frac{4^\alpha C}{r^\alpha}$.

Thus the triangle $(x, \tilde{p}, \tilde{q}) \subset M \setminus B(p, \frac{r}{4})$. Set $\theta = \angle \sigma_1'(0), \sigma_2'(0)$.

Applying the Toponogov's theorem to the triangle $(x, \tilde{p}, \tilde{q})$ we have:

$$(3.3) \quad \cosh\left(\frac{2^\alpha C^{1/2}}{r^{\alpha/2}}d(\tilde{p}, \tilde{q})\right) \leq \cosh^2\left(\frac{2^\alpha C^{1/2}}{r^{\alpha/2}}d(\tilde{p}, x)\right) - \sinh^2\left(\frac{2^\alpha C^{1/2}}{r^{\alpha/2}}d(\tilde{p}, x)\right) \cos\theta$$

Since $s < \delta r^{\beta/2}$, we deduce from inequaties (2.1) and (3.1)

$$C_0 r^{1/n} (2e_{pq}(x))^{1-\frac{1}{n}} < \delta r^{\beta/2},$$

hence

$$(3.4) \quad e_{pq}(x) \leq \frac{\delta^{n/n-1}}{2C_0^{n/n-1}} r^{\alpha/2} \leq \frac{\delta}{2} r^{\alpha/2}.$$

By triangle inequality, we have

$$(3.5) \quad \begin{aligned} d(\tilde{p}, \tilde{q}) &\geq d(p, q) - d(p, \tilde{p}) - d(q, \tilde{q}) \\ &\geq d(p, q) - d(p, x) + d(\tilde{p}, x) - d(x, q) + d(\tilde{q}, x) \quad . \\ &\geq 2\delta r^{\alpha/2} - e_{pq}(x). \end{aligned}$$

Hence

$$(3.6) \quad d(\tilde{p}, \tilde{q}) \geq 2\delta r^{\alpha/2} - \frac{\delta}{2} r^{\alpha/2} \geq \frac{3}{2} \delta r^{\alpha/2}.$$

From inequalities (3.3) and (3.6) we deduce

$$\cosh\left(\frac{3}{2}C^{1/2}2^\alpha\delta\right) \leq \cosh^2\left(C^{1/2}2^\alpha\delta\right) - \sinh^2\left(C^{1/2}2^\alpha\delta\right) \cos\theta.$$

Therefore

$$\sinh^2\left(C^{1/2}2^\alpha\delta\right) \cos\theta \leq \cosh^2\left(C^{1/2}2^\alpha\delta\right) - \cosh\left(\frac{3}{2}C^{1/2}2^\alpha\delta\right)$$

Let X_0 be the solution of the equation $\cosh^2 2X - \cosh 3X = 0$. If $\delta_0 < \frac{X_0}{2^{\alpha-1}}$ then $\theta > \frac{\pi}{2}$ which means that x is not a critical point of d_p and the conclusion follows.

Proof of theorem 1.4

If $y(t)$ denotes the function given by the Jacobi equation

$$y''(t) = \lambda(t)y(t)$$

in the simply connected manifold with sectional curvature $-\lambda(d(\bar{p}, \bar{x}))$ at the point \bar{x} then (see [9])

$$(3.7) \quad t \leq y(t) \leq e^{b_0 t}$$

and it follows that

$$(3.8) \quad \omega_n r^n \leq \text{vol}B(\bar{p}, r) \leq \omega_n e^{(n-1)b_0} r^n.$$

In one hand we have:

Let $x \in M$, $x \neq p$; set $s = d(x, R_p)$ and $\Sigma_p^c(\infty) = U_p \setminus \Sigma_p(\infty)$. Thus

$$B(x, \frac{s}{2}) \subset B_{\Sigma_p^c(\infty)}(p, r + \frac{s}{2}) \setminus B(p, r - \frac{s}{2}).$$

Hence

$$(3.9) \quad \text{vol}B(x, \frac{s}{2}) \leq \text{vol}B_{\Sigma_p^c(\infty)}(p, r + \frac{s}{2}) - \text{vol}B(p, r - \frac{s}{2})$$

$$(3.10) \quad \leq \text{vol}B_{\Sigma_p^c(\infty)}(p, r + \frac{s}{2}) - \text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{s}{2})$$

$$(3.11) \quad \leq \text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{r}{2}) \left(\frac{\text{vol}B_{\Sigma_p^c(\infty)}(p, r + \frac{s}{2})}{\text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{r}{2})} - 1 \right).$$

We deduce from lemma2.5

$$(3.12) \quad \text{vol}B(x, \frac{s}{2}) \leq \text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{s}{2}) \left(\frac{\text{vol}B(\bar{p}, r + \frac{s}{2})}{\text{vol}B(\bar{p}, r - \frac{s}{2})} - 1 \right)$$

$$(3.13) \quad \leq \frac{\text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{s}{2})}{\omega_n(r - \frac{s}{2})^n} \left(\int_{U_{\bar{p}}} \int_{r-s/2}^{r+s/2} J^{n-1}(t) dt \right)$$

where $J(t)$ denotes the exponential Jacobi in polar coordinates. Since the function J/y is nonincreasing (see [9]) and using the inequality (3.7 we have:

$$(3.14) \quad \text{vol}B(x, \frac{s}{2}) \leq \frac{\text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{s}{2})}{\omega_n(r - \frac{s}{2})^n} e^{(n-1)b_0} \left(\int_{U_{\bar{p}}} \int_{r-s/2}^{r+s/2} t^{n-1} dt \right)$$

$$(3.15) \quad \leq \frac{\text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{s}{2})}{(r - \frac{s}{2})^n} e^{(n-1)b_0} ((r + s/2)^n - (r - s/2)^n)$$

$$(3.16) \quad \leq \text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{s}{2}) e^{(n-1)b_0} \left(\left(\frac{r + s/2}{r - s/2} \right)^n - 1 \right)$$

$$(3.17) \quad \leq \text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{s}{2}) e^{(n-1)b_0} \left(\left(1 + \frac{2s}{r} \right)^n - 1 \right) \leq \text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{s}{2}) e^{(n-1)b_0} \cdot \frac{s}{r} (3^n - 1).$$

In other hand we have

$$\text{vol}B_{\Sigma_p^c(\infty)}(p, r - \frac{s}{2}) = \text{vol}B(p, r - s/2) - \text{vol}B_{\Sigma_p(\infty)}(p, r - s/2)$$

By (2.5) we have

$$(3.18) \quad \text{vol}B_{\Sigma_p(\infty)}(p, r - s/2) \geq \alpha_p \text{vol}B(\bar{p}, r - s/2).$$

From (3.17) and (3.18) we deduce

$$\text{vol}B(x, s/2) \leq [\text{vol}B(p, r - s/2) - \alpha_p \text{vol}B(\bar{p}, r - s/2)] \cdot e^{(n-1)b_0} \cdot \frac{s}{r} (3^n - 1).$$

By 1.1 we have

$$(3.19) \quad \text{vol}B(x, s/2) \leq \frac{\epsilon \alpha_p}{(r - s/2)^{(n-2+\frac{1}{n})(1-\frac{\alpha}{2})}} e^{(n-1)b_0} \frac{s}{r} 3^n \text{vol}B(\bar{p}, r - \frac{s}{2}).$$

From (3.8) and (3.19) we have

$$(3.20) \quad \text{vol}B(x, s/2) \leq \epsilon \alpha_p e^{2(n-1)b_0} s 3^n \omega_n r^{(n-1)(\frac{1}{n}+\frac{\alpha}{2}(1-\frac{1}{n}))}.$$

We claim that

$$(3.21) \quad \text{vol}B(x, s/2) \geq \frac{\omega_n \alpha_p}{6^n e^{(n-1)b_0}} s^n.$$

Indeed we have $B(p, r) \subset B(x, 2r)$, and by (2.3) we deduce

$$(3.22) \quad \frac{\text{vol}B(p, r)}{\text{vol}B(x, s/2)} \leq \frac{\text{vol}B(x, 2r)}{\text{vol}B(x, s/2)} \leq \frac{B(\bar{x}, 2r)}{\text{vol}B(\bar{x}, s/2)}$$

$$(3.23) \quad \leq e^{(n-1)b_0} \left(\frac{2r + s}{s/2} \right)^n \leq e^{(n-1)b_0} 6^n \left(\frac{r}{s} \right)^n.$$

Thus

$$(3.24) \quad \text{vol}B(x, s/2) \geq \frac{s^n \text{vol}B(p, r)}{6^n e^{(n-1)b_0} r^n}.$$

Hence from (3.8), lemma2.5 and (3.24) the conclusion follows.

Thus from (3.20) and the inequality (3.21) we have

$$s^{n-1} \leq \epsilon 18^n e^{3(n-1)b_0} r^{(n-1)(\frac{1}{n}+\frac{\alpha}{2}(1-\frac{1}{n}))}$$

which means that

$$s \leq \epsilon^{1/(n-1)} 18^{n/(n-1)} e^{3b_0} r^{\frac{1}{n}+\frac{\alpha}{2}(1-\frac{1}{n})}.$$

Then it suffices to take $\epsilon < \frac{\delta^{n-1}}{18^n e^{3(n-1)b_0}}$.

REFERENCES

- [1] U. ABRESCH, *Lower curvature bounds, Toponogovs theorem and bounded topology I*, Ann. Sci. Ecole Norm. Sup. (4) 18, 651-670 (1985). MR 839689 (87j:53058)
- [2] U. ABRESCH AND D. GROMOLL, *On complete manifolds with nonnegative Ricci curvature*, J. Amer. Math. Soc. 3, 355-374 (1990) .
- [3] M. ANDERSON, *On the topology of complete manifolds with nonnegative Ricci curvature*, Topology 29, 41-55 (1990).
- [4] J. CHEEGER, *Critical points of distance functions and applications to geometry*, Lecture Notes in Mathematics Vol. 1504 Springer-Verlag, New York, 1991, pp. 1-38.
- [5] J. CHEEGER AND T. COLDING, *On the structure of spaces with Ricci curvature bounded below*, J. Differential Geometry 46, 406-480 (1997).
- [6] M. DO CARMO AND C. Y. XIA, *Ricci curvature and the topology of open manifolds*, Math. Ann. 316, 391-400 (2000).
- [7] K. GROVE, *Critical point theory for distance functions*, Proc. Sympos. Pure Math. 54 (3), 357-385 (1993).
- [8] K. GROVE AND K. SHIOHAMA, *A generalized sphere theorem*, Ann. of Math. (2) 106, 201-211 (1977).
- [9] B. MAHAMAN, *A volume comparison theorem and number of ends for manifolds with asymptotically nonnegative Ricci curvature*, Rev.Mat.Complut. 13 (2) 399-409 (2000).

- [10] B. MAHAMAN, *Open manifolds with asymptotically nonnegative curvature*, Illinois J. Math. 49 (3) 705-717 (2005)
- [11] X. MENGUY, *Noncollapsing examples with positive Ricci curvature and infinite topological type*, Geom.funct. anal. 10, 600-627 (2000)
- [12] J. SHA AND Z. SHEN, *Complete manifolds with nonnegative Ricci curvature and quadratically nonnegatively curved infinity*, Amer. J. Math. 119, 1399-1404 (1997).
- [13] J. SHA AND D. YANG, *Examples of manifolds of positive Ricci curvature*, J. Diff. Geom. 29 95-103 (1989)
- [14] Z. SHEN, *On complete manifolds of nonnegative k th-Ricci curvature*, Trans. Amer. Math. Soc. 338, 289-310 (1993).
- [15] Z. SHEN, *Complete manifolds with nonnegative Ricci curvature and large volume growth*, Invent. Math. 125, 393-404 (1996).
- [16] Q. WANG AND C.Y.XIA, *Topological rigidity theorems for open Riemannian manifolds*, Math. Nachr. 279, No. 7, 805-811 (2006)
- [17] C. Y. XIA, *Open manifolds with sectional curvature bounded below*, Amer. J. Math. 122, 745-755 (2000).
- [18] C. Y. XIA, *Large volume growth and the topology of open manifolds*, Math. Z. 239, 515-526 (2002).

BAZANFARÉ, DÉPARTEMENT DE MATHÉMATIQUES ET INFORMATIQUE
UNIVERSITÉ ABDOU MOUMOUNI
B.P. 10662, NIAMEY, NIGER
E-mail address: bmahaman@yahoo.fr