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CRITICAL POINTS BETWEEN VARIETIES GENERATED BY SUBSPACE LATTICES OF VECTOR SPACES

PIERRE GILLIBERT

ABSTRACT. We denote by $\text{Con}_c A$ the semilattice of all compact congruences of an algebra A . Given a variety \mathcal{V} of algebras, we denote by $\text{Con}_c \mathcal{V}$ the class of all semilattices isomorphic to $\text{Con}_c A$ for some $A \in \mathcal{V}$. Given \mathcal{V}_1 and \mathcal{V}_2 varieties of algebras, the *critical point* of \mathcal{V}_1 under \mathcal{V}_2 is defined as $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) = \min\{\text{card } D \mid D \in \text{Con}_c \mathcal{V}_1 - \text{Con}_c \mathcal{V}_2\}$. Given a finitely generated variety \mathcal{V} of modular lattices, we obtain an integer ℓ , depending of \mathcal{V} , such that $\text{crit}(\mathcal{V}; \mathbf{Var}(\text{Sub } F^n)) \geq \aleph_2$ for any $n \geq \ell$ and any field F .

In a second part, using tools introduced in [4], we prove that:

$$\text{crit}(\mathcal{M}_n; \mathbf{Var}(\text{Sub } F^3)) = \aleph_2,$$

for any finite field F and any integer n such that $2 + \text{card } F \leq n \leq \omega$. Similarly $\text{crit}(\mathbf{Var}(\text{Sub } F^3); \mathbf{Var}(\text{Sub } K^3)) = \aleph_2$, for all finite fields F and K such that $\text{card } F > \text{card } K$.

1. INTRODUCTION

We denote by $\text{Con } A$ (resp., $\text{Con}_c A$) the lattice (resp., $(\vee, 0)$ -semilattice) of all congruences (resp., compact congruences) of an algebra A . For a homomorphism $f: A \rightarrow B$ of algebras, we denote by $\text{Con } f$ the map from $\text{Con } A$ to $\text{Con } B$ defined by the rule

$$(\text{Con } f)(\alpha) = \text{congruence of } B \text{ generated by } \{(f(x), f(y)) \mid (x, y) \in \alpha\},$$

for every $\alpha \in \text{Con } A$, and we also denote by $\text{Con}_c f$ the restriction of $\text{Con } f$ from $\text{Con}_c A$ to $\text{Con}_c B$.

A *congruence-lifting* of a $(\vee, 0)$ -semilattice S is an algebra A such that $\text{Con}_c A \cong S$. Given a variety \mathcal{V} of algebras, *the compact congruence class of \mathcal{V}* , denoted by $\text{Con}_c \mathcal{V}$, is the class of all $(\vee, 0)$ -semilattices isomorphic to $\text{Con}_c A$ for some $A \in \mathcal{V}$. As illustrated by [10], even the compact congruence classes of small varieties of lattices are complicated objects. For example, in case \mathcal{V} is the variety of all lattices, $\text{Con}_c \mathcal{V}$ contains all distributive $(\vee, 0)$ -semilattices of cardinality at most \aleph_1 , but not all distributive $(\vee, 0)$ -semilattices (cf. [13]).

Given varieties \mathcal{V}_1 and \mathcal{V}_2 of algebras, the *critical point* of \mathcal{V}_1 and \mathcal{V}_2 , denoted by $\text{crit}(\mathcal{V}_1; \mathcal{V}_2)$, is the smallest cardinality of a $(\vee, 0)$ -semilattice in $\text{Con}_c(\mathcal{V}_1) - \text{Con}_c(\mathcal{V}_2)$ if it exists, or ∞ , otherwise (i.e., if $\text{Con}_c \mathcal{V}_1 \subseteq \text{Con}_c \mathcal{V}_2$).

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Let I be a poset. A *direct system indexed by I* is a family $(A_i, f_{i,j})_{i \leq j \text{ in } I}$ such that A_i is an algebra, $f_{i,j}: A_i \rightarrow A_j$ is a morphism of algebras, $f_{i,i} = \text{id}_{A_i}$, and $f_{i,k} = f_{j,k} \circ f_{i,j}$, for all $i \leq j \leq k$ in I .

Denote by $\text{Sub } V$ the subspace lattice of a vector space V , and by \mathcal{M}_n the variety of lattices generated by the lattice M_n of length two with n atoms, for $3 \leq n \leq \omega$. Using the theory of the *dimension monoid* of a lattice, introduced by F. Wehrung in [11], together with some Von Neumann regular ring theory, we prove in Section 3 that if \mathcal{W} is a finitely generated variety of modular lattices with all subdirectly irreducible members of length less or equal to n , then $\text{crit}(\mathcal{W}; \mathbf{Var}(\text{Sub}(F^n))) \geq \aleph_2$ for any field F . As an immediate application, $\text{crit}(\mathcal{M}_n; \mathcal{M}_3) \geq \aleph_2$ for every n with $3 \leq n \leq \omega$ (cf. Corollary 3.12). Thus, by using the result of M. Ploščica in [8], we obtain the inequality $\text{crit}(\mathcal{M}_m; \mathcal{M}_n) = \aleph_2$ for all m, n with $3 \leq n < m \leq \omega$. Our proof does not rely on the approach used by Ploščica in [9] to prove the inequality $\text{crit}(\mathcal{M}_m^{0,1}; \mathcal{M}_n^{0,1}) \geq \aleph_2$, and it extends that result to the unbounded case. We also obtain a new proof of that result in Section 4, that does not even rely on the approach used by Ploščica in [8] to prove the inequality $\text{crit}(\mathcal{M}_m; \mathcal{M}_n) \leq \aleph_2$.

Let \mathcal{V} be a variety of lattices, let \vec{D} be a diagram of $(\vee, 0)$ -semilattices and $(\vee, 0)$ -homomorphisms. A *congruence-lifting* of \vec{D} in \mathcal{V} is a diagram \vec{L} of \mathcal{V} such that the composite $\text{Con}_c \circ \vec{L}$ is naturally equivalent to \vec{D} .

In Section 4, we give a diagram of finite $(\vee, 0)$ -semilattices that is congruence-liftable in \mathcal{M}_n , but not congruence-liftable in $\mathbf{Var}(\text{Sub } F^3)$, for any finite field F and any n such that $2 + \text{card } F \leq n \leq \omega$. As the diagram of $(\vee, 0)$ -semilattices is indexed by some “good” lattice, we obtain, using results of [4], that $\text{crit}(\mathcal{M}_n; \mathbf{Var}(\text{Sub } F^3)) = \aleph_2$. This implies immediately that $\text{crit}(\mathcal{M}_4; \mathcal{M}_{3,3}) = \aleph_2$. Let F and K be finite fields such that $\text{card } F > \text{card } K$, we also obtain $\text{crit}(\mathbf{Var}(\text{Sub } F^3); \mathbf{Var}(\text{Sub } K^3)) = \aleph_2$.

In a similar way, we prove that $\text{crit}(\mathcal{M}_\omega; \mathcal{V}) = \aleph_2$, for every finitely generated variety of lattices \mathcal{V} such that $M_3 \in \mathcal{V}$.

2. BASIC CONCEPTS

We denote by $\text{dom } f$ the domain of any function f . A *poset* is a partially ordered set. We put

$$Q \downarrow X = \{p \in Q \mid (\exists x \in X)(p \leq x)\}, \quad Q \uparrow X = \{p \in Q \mid (\exists x \in X)(p \geq x)\},$$

for any $X, Q \subseteq P$, and we will write $\downarrow X$ (resp., $\uparrow X$) instead of $P \downarrow X$ (resp., $P \uparrow X$) in case P is understood. We shall also write $\downarrow p$ instead of $\downarrow \{p\}$, and so on, for $p \in P$. A poset P is *lower finite* if $P \downarrow p$ is finite for all $p \in P$. For $p, q \in P$ let $p \prec q$ hold, if $p < q$ and there is no $r \in P$ with $p < r < q$, in this case p is called a *lower cover* of q . We denote by P^\neq the set of all non-maximal elements in a poset P . We denote by $M(L)$ the set of all completely meet-irreducible elements of a lattice L .

A *2-ladder* is a lower finite lattice in which every element has at most two lower covers. S. Z. Ditor constructs in [1] a 2-ladder of cardinality \aleph_1 .

For a set X and a cardinal κ , we denote by:

$$\begin{aligned} [X]^\kappa &= \{Y \subseteq X \mid \text{card } Y = \kappa\}, \\ [X]^{\leq \kappa} &= \{Y \subseteq X \mid \text{card } Y \leq \kappa\}, \\ [X]^{< \kappa} &= \{Y \subseteq X \mid \text{card } Y < \kappa\}. \end{aligned}$$

Denote by \mathcal{P} the category with objects the ordered pairs (G, u) where G is a pre-ordered abelian group and u is an order-unit of G (i.e., for each $x \in G$, there

exists an integer n with $-nx \leq x \leq nx$, and morphisms $f: (G, u) \rightarrow (H, v)$ where $f: G \rightarrow H$ is an order-preserving group homomorphism and $f(u) = v$. Let $\Lambda^{0,1}$ be the category of all bounded lattices with morphisms of lattices preserving bounds.

We denote by Dim the functor that maps a lattice to its *dimension monoid*, introduced by F. Wehrung in [11], we also denote by $\Delta(a, b)$ for $a \leq b$ in L the canonical generators of $\text{Dim } L$. We denote by K_0^ℓ the functor that maps a lattice to the pre-ordered abelian universal group (also called Grothendieck group) of its dimension monoid. If L is a bounded lattice then (the canonical image in $K_0^\ell(L)$ of) $\Delta(0_L, 1_L)$ is an order-unit of $K_0^\ell(L)$. If $f: L \rightarrow L'$ is a 0, 1-preserving homomorphism of bounded lattices, then $K_0^\ell(f): (K_0^\ell(L), \Delta(0_L, 1_L)) \rightarrow (K_0^\ell(L'), \Delta(0_{L'}, 1_{L'}))$ preserves the order-unit.

All our rings are associative but not necessarily unital.

- We denote by \mathbb{L} the functor from the category of regular rings to the category of lattices, that maps a ring to the lattice of its principal right ideals and any homomorphism $f: R \rightarrow S$ of regular rings to the 0-lattice homomorphism $\mathbb{L}(f): \mathbb{L}(R) \rightarrow \mathbb{L}(S)$, $I \mapsto f(I)S$.
- We denote by V the functor from the category of unital ring with morphisms preserving units to the category of commutative monoids, that maps a unital ring R to the commutative monoid of all isomorphism classes of finitely generated projective right R -modules and any homomorphism $f: R \rightarrow S$ of unital rings to the monoid homomorphism $V(f): V(R) \rightarrow V(S)$, $\sum_i e_i R \mapsto \sum_i f(e_i)S$.

We denote by $\text{Id } R$ (resp., $\text{Id}_c R$) the lattice of all two-sided ideals (resp., finitely generated two-sided ideals) of any ring R . We denote by $\text{Sub } E$ the subspace lattice of a vector space E . We denote by $M_n(F)$ the F -algebra of $n \times n$ matrices with entries from F , for every field F and every positive integer n . A *matricial F -algebra* is an F -algebra of the form $M_{k_1}(F) \times \cdots \times M_{k_n}(F)$, for positive integers k_1, \dots, k_n .

For a finitely generated projective right module P over a unital ring R , we denote by $[P]$ the corresponding element in $K_0(R)$, that is, the stable isomorphism class of P . We refer to [6, Section 15] for the required notions about the K_0 functor.

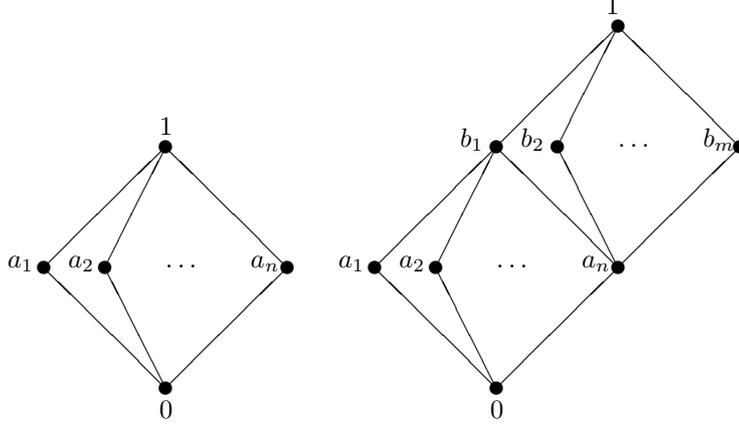
A K_0 -*lifting* of a pre-ordered abelian group with order-unit (G, u) is a regular ring R such that $(K_0(R), [R]) \cong (G, u)$. A K_0 -lifting of a diagram $\vec{G}: I \rightarrow \mathcal{P}$ is a diagram $\vec{R}: I \rightarrow \mathcal{P}$ such that $(K_0(-), [-]) \circ \vec{R} \cong \vec{G}$.

We denote by ∇ the functor that sends a monoid to its maximal semilattice quotient, that is, $\nabla(M) = M/\approx$ where \approx is the smallest congruence of M such that M/\approx is a semilattice. We denote by $\overline{\nabla}$ the functor that maps a partially pre-ordered abelian group G to $\nabla(G^+)$ where G^+ is the monoid of all positive elements of G .

We denote by $\mathbf{Var}(L)$ (resp., $\mathbf{Var}_0(L)$, resp., $\mathbf{Var}_{0,1}(L)$) the variety of lattices (resp., lattices with 0, resp., bounded lattices) generated by a lattice L .

A lattice K is a *congruence-preserving extension* of a lattice L , if L is a sublattice of K and $\text{Con}_c i: \text{Con } L \rightarrow \text{Con } K$ is an isomorphism, where $i: L \rightarrow K$ is the inclusion map.

We denote by M_n and $M_{n,m}$ the lattices represented in Figure 1, for $3 \leq m, n \leq \omega$, and by \mathcal{M}_n and $\mathcal{M}_{n,m}$, respectively, the lattice varieties that they generate. We also denote by \mathcal{M}_n^0 the variety of lattices with 0 generated by M_n , and so on.

FIGURE 1. The lattices M_n and $M_{n,m}$.

A lattice L satisfies *Whitman's condition* if for all a, b, c , and d in L :
 $a \wedge b \leq c \vee d$ implies either $a \leq c \vee d$ or $b \leq c \vee d$ or $a \wedge b \leq c$ or $a \wedge b \leq d$.
 The lattice M_n satisfies Whitman's condition for all $n \geq 3$.

3. LOWER BOUNDS FOR SOME CRITICAL POINTS

The following proposition is proved in [11, Proposition 5.5].

Proposition 3.1. *Let L be a modular lattice without infinite bounded chains. Let P be the set of all projectivity classes of prime intervals of L . Denote by $|a, b|_\xi$ the number of occurrences of an interval in ξ in any maximal chain of the interval $[a, b]$. Then there exists an isomorphism $\pi: \text{Dim } L \rightarrow (\mathbb{Z}^+)^{(P)}$ such that $\pi(\Delta(a, b)) = (|a, b|_\xi \mid \xi \in P)$ for all $a \leq b$ in L .*

This makes it possible to prove the following lemma, which gives an explicit description of $K_0^\ell(L)$ for every modular lattice L of finite length (in such a case the set P is finite).

Lemma 3.2. *Let L be a modular lattice of finite length, set $X = \text{M}(\text{Con } L)$. Then there exists an isomorphism $\pi': K_0^\ell(L) \rightarrow \mathbb{Z}^X$ such that*

$$\pi'(\Delta(a, b)) = (\text{lh}([a/\theta, b/\theta]) \mid \theta \in X), \quad \text{for all } a \leq b \text{ in } L.$$

In particular $(K_0^\ell(L), \Delta(0, 1))$ is isomorphic to $(\mathbb{Z}^X, (\text{lh}(L/\theta))_{\theta \in X})$.

Proof. For any $\xi \in P$ denote by θ_ξ the largest congruence of L that does not collapse any prime intervals in ξ . It follows from the modularity of L that the assignment $\xi \mapsto \theta_\xi$ defines a bijection from P onto X . Moreover any prime interval not in ξ is collapsed by θ_ξ , for any $\xi \in P$. Let $a \leq b$ in L , let $\xi \in P$. Let $a_0 \prec a_1 \prec \dots \prec a_n$ in L such that $a_0 = a$ and $a_n = b$. Let $0 \leq r_1 < r_2 < \dots < r_s < n$ be all the integers such that $[a_{r_k}, a_{r_{k+1}}] \in \xi$ for all $1 \leq k \leq s$. Thus $|a, b|_\xi = s$. Set $r_{s+1} = n$. As $[a_{r_k}, a_{r_{k+1}}] \in \xi$ and $[a_{r_k+t}, a_{r_{k+t+1}}] \notin \xi$ for all $1 \leq t \leq r_{k+1} - r_k - 1$, we obtain that

$$a_{r_k}/\theta_\xi \prec a_{r_{k+1}}/\theta_\xi = a_{r_{k+2}}/\theta_\xi = \dots = a_{r_{k+1}}/\theta_\xi, \quad \text{for all } 1 \leq k \leq s.$$

Thus the following covering relations hold:

$$a/\theta_\xi = a_{r_1}/\theta_\xi \prec a_{r_2}/\theta_\xi \prec \cdots \prec a_{r_s}/\theta_\xi \prec a_{r_{s+1}}/\theta_\xi = b/\theta_\xi.$$

So $\text{lh}([a/\theta_\xi, b/\theta_\xi]) = s = |a, b|_\xi$. We conclude the proof by using Proposition 3.1. \square

Proposition 3.3. *The following natural equivalences hold*

$$\begin{array}{lll} (i) & \nabla \circ \text{Dim} \cong \text{Con}_c & \text{on lattices} \\ (ii) & \nabla \circ V \cong \text{Con}_c \circ \mathbb{L} & \text{on regular rings} \end{array}$$

Proof. (i) follows from [11, Corollary 2.3], while (ii) is contained in [6, Corollary 2.23]; see also the proof of [12, Proposition 4.6]. \square

We shall always apply this result to unital regular rings R such that $V(R)$ is cancellative (i.e., R is unit-regular), so $K_0(R)^+ = V(R)$, and to lattices L such that $\text{Dim } L$ is cancellative, so $K_0^\ell(L)^+ \cong \text{Dim } L$. Here G^+ denotes the positive cone of G , for any partially pre-ordered abelian group G .

The following theorem is proved in [6, Theorem 15.23].

Theorem 3.4. *Let F be a field, let R be a matricial F -algebra, and let S be a unit-regular F -algebra.*

- (1) *Given any morphism $f: (K_0(R), [R]) \rightarrow (K_0(S), [S])$ in the category \mathcal{P} , there exists a unital F -algebra map $\phi: R \rightarrow S$ such that $K_0(\phi) = f$.*
- (2) *if $\phi, \psi: R \rightarrow S$ are F -algebra maps, then $K_0(\phi) = K_0(\psi)$ if and only if there exists an inner automorphism θ of S such that $\phi = \theta \circ \psi$.*

The following lemma is folklore.

Lemma 3.5. *Let F be a field, let $\mathbf{u} = (u_k)_{1 \leq k \leq n}$ be a family of positive integers, let $R = \prod_{k=1}^n M_{u_k}(F)$. Then $(K_0(R), [R]) \cong (\mathbb{Z}^n, \mathbf{u})$.*

Lemma 3.6. *Let F be a field. Let I be a 2-ladder, let $G_i = (\mathbb{Z}^{n_i}, \mathbf{u}^i = (u_k^i)_{1 \leq k \leq n_i})$ such that \mathbf{u}^i is an order-unit, let $R_i = \prod_{k=1}^{n_i} M_{u_k^i}(F)$ for all $i \in I$. Let $f_{i,j}: G_i \rightarrow G_j$ for all $i \leq j$ in I such that $\vec{G} = (G_i, f_{i,j})_{i \leq j \text{ in } I}$ is a direct system in \mathcal{P} . Then there exists a direct system $(R_i, \phi_{i,j})_{i \leq j \text{ in } I}$ of F -matricial rings which is a K_0 -lifting of $(G_i, f_{i,j})_{i \leq j \text{ in } I}$ such that $\phi_{i,j}$ is a unital morphism for all $i \leq j$ in I .*

Proof. By Lemma 3.5 there exists an isomorphism $\tau_i: (K_0(R_i), [R_i]) \rightarrow G_i = (\mathbb{Z}^{n_i}, \mathbf{u}^i)$ in \mathcal{P} , for all $i \in I$. Let $g_{i,j} = \tau_j^{-1} \circ f_{i,j} \circ \tau_i$, for all $i \leq j$ in I .

For $i = j = 0$ (the smallest element of I), we put $\phi_{0,0} = \text{id}_{R_0}$. Let $i \in I$ with a lower cover i' . It follows from Theorem 3.4(1) that there exists $\psi_{i',i}: R_{i'} \rightarrow R_i$ such that $K_0(\psi_{i',i}) = g_{i',i}$.

If i has only i' as lower cover, assume that we have a direct system $(R_j, \phi_{j,k})_{j \leq k \leq i'}$ lifting $(G_j, f_{j,k})_{j \leq k \leq i'}$. Set $\phi_{j,i} = \psi_{i',i} \circ \phi_{j,i'}$ for all $j < i$, and $\phi_{i,i} = \text{id}_{R_i}$. It is easy to see that $(R_i, \phi_{j,k})_{j \leq k \leq i}$ is a direct system lifting $(G_j, f_{j,k})_{j \leq k \leq i}$.

Let i have two distinct lower covers i' and i'' , and set $\ell = i' \wedge i''$. Assume that we have direct system $(R_j, \phi_{j,k})_{j \leq k \leq i'}$ and $(R_j, \phi_{j,k})_{j \leq k \leq i''}$ lifting $(G_j, f_{j,k})_{j \leq k \leq i'}$ and $(G_j, f_{j,k})_{j \leq k \leq i''}$ respectively. The following equalities hold

$$K_0(\psi_{i',i} \circ \phi_{\ell,i'}) = K_0(\psi_{i',i}) \circ K_0(\phi_{\ell,i'}) = g_{i',i} \circ g_{\ell,i'} = g_{\ell,i}$$

Similarly $K_0(\psi_{i'',i} \circ \phi_{\ell,i''}) = g_{\ell,i} = K_0(\psi_{i',i} \circ \phi_{\ell,i'})$, thus, by Theorem 3.4(2), there exists an inner automorphism θ of R_i such that $\theta \circ \psi_{i'',i} \circ \phi_{\ell,i''} = \psi_{i',i} \circ \phi_{\ell,i'}$.

Put $\phi_{i',i} = \psi_{i',i}$ and $\phi_{i'',i} = \theta \circ \psi_{i'',i}$. Thus $\phi_{i',i} \circ \phi_{i' \wedge i'',i'} = \phi_{i'',i} \circ \phi_{i' \wedge i'',i'}$, so we can construct a direct system $(R_j, \phi_{j,k})_{j \leq k \leq i}$.

Hence, by induction, we obtain a direct system $(R_i, \phi_{i,j})_{i \leq j \text{ in } I}$ such that $K_0(\phi_{i,j}) = g_{i,j}$ for all $i \leq j$ in I as required. \square

Lemma 3.7. *Let F be a field. Let L be a bounded modular lattice such that all finitely generated sublattices of L are of finite length. Assume that $\text{card } L \leq \aleph_1$. Then there exists a locally matricial ring R such that $\text{Con } L \cong \text{Con } \mathbb{L}(R)$ and $\mathbb{L}(R) \in \mathbf{Var}_{0,1}(\text{Sub } F^n \mid n < \omega)$.*

Moreover if there exists $n < \omega$ such that $n \geq \text{lh}(K)$ for each simple lattice $K \in \mathbf{Var}(L)$ of finite length, then there exists a locally matricial ring R such that $\text{Con } L \cong \text{Con } \mathbb{L}(R)$ and $\mathbb{L}(R) \in \mathbf{Var}_{0,1}(\text{Sub } F^n)$.

Proof. Let I be a 2-ladder of cardinality \aleph_1 . Pick a surjection $\rho: I \rightarrow L$ and denote by L_i the sublattice of L generated by $\rho(I \downarrow i) \cup \{0, 1\}$, for each $i \in I$. Furthermore, denote by $f_{i,j}: L_i \rightarrow L_j$ the inclusion map, for all $i \leq j$ in I . Then $\vec{L} = (L_i, f_{i,j})_{i \leq j \text{ in } I}$ is a direct system of modular lattices of finite length and 0, 1-lattice embeddings.

Assume that there exists $n < \omega$ such that $n \geq \text{lh}(K)$ for each simple lattice $K \in \mathbf{Var}(L)$ of finite length. Let $\vec{G} = K_0^\ell \circ \vec{L}$, set $X_i = \text{M}(\text{Con } L_i)$ for all $i \in I$, and set $r_x^i = \text{lh}(L_i/x)$ for each $x \in X_i$. As L_i/x is a subdirectly irreducible modular lattice of finite length, it is simple, thus $r_x^i \leq n$, for all $i \in I$ and all $x \in X_i$. By Lemma 3.2, $G_i \cong (\mathbb{Z}^{X_i}, (r_x^i)_{x \in X_i})$ for all $i \in I$.

Set $R_i = \prod_{x \in X_i} M_{r_x^i}(F)$. By Lemma 3.5, $(K_0(R_i), [R_i]) \cong (\mathbb{Z}^{X_i}, (r_x^i)_{x \in X_i}) \cong G_i$. By Lemma 3.6, there exists a direct system $\vec{R} = (R_i, \phi_{i,j})_{i \leq j \text{ in } I}$ with morphisms preserving units, such that:

$$K_0 \circ \vec{R} \cong \vec{G} = K_0^\ell \circ \vec{L}. \quad (3.1)$$

Moreover:

$$\mathbb{L}(R_i) \cong \mathbb{L} \left(\prod_{x \in X_i} M_{r_x^i}(F) \right) \cong \prod_{x \in X_i} \mathbb{L}(M_{r_x^i}(F)) \cong \prod_{x \in X_i} \text{Sub}(F^{r_x^i}) \in \mathbf{Var}_{0,1}(\text{Sub } F^n).$$

Let $R = \varinjlim \vec{R}$. As \mathbb{L} preserves direct limits, $\mathbb{L}(R) \cong \varinjlim (\mathbb{L} \circ \vec{R})$, but $\mathbb{L} \circ \vec{R}$ is a diagram of $\mathbf{Var}_{0,1}(\text{Sub } F^n)$, so $\mathbb{L}(R) \in \mathbf{Var}_{0,1}(\text{Sub } F^n)$. Moreover the following isomorphisms hold:

$$\begin{aligned} \text{Con}_c(\mathbb{L}(R)) &\cong \overline{\text{V}}(K_0(R)) && \text{by Proposition 3.3} \\ &\cong \overline{\text{V}}(K_0(\varinjlim \vec{R})) \\ &\cong \overline{\text{V}}(\varinjlim (K_0 \circ \vec{R})) && \text{as } K_0 \text{ preserves direct limits} \\ &\cong \overline{\text{V}}(\varinjlim (K_0^\ell \circ \vec{L})) && \text{by (3.1)} \\ &\cong \overline{\text{V}}(K_0^\ell(\varinjlim \vec{L})) && \text{as } K_0^\ell \text{ preserves direct limits} \\ &\cong \overline{\text{V}}(K_0^\ell(L)) \\ &\cong \text{Con}_c L && \text{by Proposition 3.3.} \end{aligned}$$

The other case, without restriction on finite lengths of simple lattices, is similar. \square

Lemma 3.7 works for bounded lattices, however any lattice can be embedded into a bounded lattice. In the rest of this section, using this result, we extend Lemma 3.7 to unbounded lattices.

Lemma 3.8. *Let L be a lattice, let $L' = L \sqcup \{0, 1\}$ such that 0 is the smallest element of L' and 1 is the largest. Let $f: L \hookrightarrow L'$ be the inclusion map. Then $\text{Con}_c f$ is a $(\vee, 0)$ -homomorphism-embeddings and $(\text{Con}_c f)(\text{Con}_c L)$ is an ideal of $\text{Con}_c L'$.*

Proof. Let $\theta \in \text{Con}_c L$, let $L'_\theta = (L/\theta) \sqcup \{0, 1\}$ such that 0 is the smallest element of L'_θ and 1 is its largest element. The following map

$$g: L' \rightarrow L'_\theta$$

$$x \mapsto \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ x/\theta & \text{if } x \in L \end{cases}$$

is a lattice homomorphism, and $\ker g = \theta \cup \{(0, 0), (1, 1)\}$, so the latter is a congruence of L' . It follows that $(\text{Con}_c f)(\theta) = \theta \cup \{(0, 0), (1, 1)\}$. Thus $\text{Con}_c f$ is an embedding. Let $\beta = \bigvee_{i=1}^n \Theta_{L'}(x_i, y_i) \in \text{Con}_c L'$, such that $\beta \subseteq (\text{Con}_c f)(\theta)$. We can assume that $x_i \neq y_i$ for all $1 \leq i \leq n$. Thus, as $(x_i, y_i) \in \theta \cup \{(0, 0), (1, 1)\}$, $(x_i, y_i) \in \theta$ for all $1 \leq i \leq n$. Let $\alpha = \bigvee_{i=1}^n \Theta_L(x_i, y_i)$, then $(\text{Con}_c f)(\alpha) = \beta$. Thus $(\text{Con}_c f)(\text{Con}_c L)$ is an ideal of $\text{Con}_c L'$. \square

F. Wehrung proves the following proposition in [12, Corollary 4.4]; the result also applies to the non-unital case, with a similar proof.

Proposition 3.9. *For any regular ring R , $\text{Con}_c \mathbb{L}(R)$ is isomorphic to $\text{Id}_c R$.*

Lemma 3.10. *Let R be a regular ring, and let I be a two-sided ideal of R . Then the following assertions hold*

- (1) *The set I is a regular subring of R .*
- (2) *Any right (resp., left) ideal of I is a right (resp., left) ideal of R .*
- (3) *In particular $\text{Id}(I) = \text{Id}(R) \downarrow I$, and $\mathbb{L}(I) = \mathbb{L}(R) \downarrow I$.*

Proof. The assertion (1) follows from [6, Lemma 1.3].

Let J be a right ideal of I , let $a \in J$, let $x \in R$. As I is regular there exists $y \in I$ such that $a = aya$, so $ax = ayax$, but $a \in I$, so $yax \in I$, moreover J is a right ideal of I , so $ax = ayax \in J$. Thus J is a right ideal of R . Similarly any left ideal of I is a left ideal of R . Thus $\text{Id}(I) = \text{Id}(R) \downarrow I$.

Let $a \in R$ idempotent. If $aR \subseteq I$, then $a \in I$, so $aI \subseteq aR = aaR \subseteq aI$, and so $aI = aR$, thus $aR \in \mathbb{L}(I)$. So $\mathbb{L}(I) = \mathbb{L}(R) \downarrow I$. \square

Theorem 3.11. *Let F be a field. Let \mathcal{V} be a variety of modular lattices (resp., a variety of bounded modular lattices). Assume that all finitely generated lattices of \mathcal{V} are of finite length. Then*

$$\text{crit}(\mathcal{V}; \mathbf{Var}_0(\text{Sub } F^n \mid n \in \omega)) \geq \aleph_2 \quad (\text{resp., } \text{crit}(\mathcal{V}; \mathbf{Var}_{0,1}(\text{Sub } F^n \mid n \in \omega)) \geq \aleph_2).$$

Moreover for $L \in \mathcal{V}$ of cardinality at most \aleph_1 , there exists a regular ring A such that $\text{Con } L \cong \text{Con}(\mathbb{L}A)$ and $\mathbb{L}(A) \in \mathbf{Var}_0(\text{Sub } F^n)$ (resp., $\mathbb{L}(A) \in \mathbf{Var}_{0,1}(\text{Sub } F^n)$).

If there exists $n < \omega$ such that $\text{lh}(K) \leq n$ for each simple lattice $K \in \mathcal{V}_1$ of finite length, then:

$$\text{crit}(\mathcal{V}; \mathbf{Var}_0(\text{Sub } F^n)) \geq \aleph_2 \quad (\text{resp., } \text{crit}(\mathcal{V}; \mathbf{Var}_{0,1}(\text{Sub } F^n)) \geq \aleph_2).$$

Moreover for $L \in \mathcal{V}$ of cardinality at most \aleph_1 , there exists a regular ring A such that $\text{Con } L \cong \text{Con}(\mathbb{L}A)$ and $\mathbb{L}(A) \in \mathbf{Var}_0(\text{Sub } F^n)$ (resp., $\mathbb{L}(A) \in \mathbf{Var}_{0,1}(\text{Sub } F^n)$).

Observe that $\mathbb{L}(A)$ is, in addition, relatively complemented; in particular, it is congruence-permutable.

Proof. The bounded case is an immediate application of Lemma 3.7.

Let \mathcal{V} be a variety of modular lattices in which finitely generated lattices are of finite length. Let $L \in \mathcal{V}$ such that $\text{card } L \leq \aleph_1$, let $L' = L \sqcup \{0, 1\}$ as in Lemma 3.8 and let D be the ideal of $\text{Con}_c L'$ corresponding to $\text{Con}_c L$. By Chapter I, Section 4, Exercise 14 in [5] we have $L' \in \mathcal{V}$, thus, by Lemma 3.7, there exists a regular ring R such that $\mathbb{L}(R) \in \mathbf{Var}_0(\text{Sub } F^n)$, and $\text{Con}_c(\mathbb{L}R) \cong \text{Con}_c L'$. By Proposition 3.9, $\text{Con}_c(\mathbb{L}R) \cong \text{Id}_c R$. Let I be the ideal of R corresponding to D . Then $\text{Con } L \cong \text{Id } D \cong \text{Id } R \downarrow I \cong \text{Id } I \cong \text{Con}(\mathbb{L}I)$. Moreover $\mathbb{L}(I) = \mathbb{L}(R) \downarrow I$ belongs to \mathcal{V}_2 . \square

We obtain the following generalization of M. Ploščica results in [9].

Corollary 3.12. *Let m, n be ordinals such that $3 \leq n < m \leq \omega$. Then the equality $\text{crit}(\mathcal{M}_m; \mathcal{M}_n) = \aleph_2$ holds.*

Proof. Every simple lattice of \mathcal{M}_n has length at most two. Moreover, $\text{Sub } \mathbb{F}_2^2 \cong M_3 \in \mathcal{M}_n$, where \mathbb{F}_2 is the two-element field. Thus, by Theorem 3.11, $\text{crit}(\mathcal{M}_m; \mathcal{M}_n) \geq \aleph_2$.

Conversely, M. Ploščica proves in [8] that there exists a $(\vee, 0)$ -semilattice of cardinality \aleph_2 , congruence-liftable in \mathcal{M}_m , but not congruence-liftable in \mathcal{M}_n . So $\text{crit}(\mathcal{M}_m; \mathcal{M}_n) \leq \aleph_2$. \square

In Section 4 we shall give another $(\vee, 0)$ -semilattice of cardinality \aleph_2 , congruence-liftable in \mathcal{M}_n , but not congruence-liftable in \mathcal{M}_m .

4. AN UPPER BOUND OF SOME CRITICAL POINTS

Using the results of [4], we first prove that if a simple lattice of a variety of lattices \mathcal{V}_1 has larger length than all simple lattices of a finitely generated variety of lattice \mathcal{V}_2 , then $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) \leq \aleph_0$.

Remark 4.1. Let $x \prec y$ in a lattice L . Let $(\alpha_i)_{i \in I}$ be a family of congruences of L , if $(x, y) \in \bigvee_{i \in I} \alpha_i$, then $(x, y) \in \alpha_i$ for some $i \in I$. In particular there exists a largest congruence separating x and y . Such a congruence is completely meet-irreducible, and in a lattice of finite height all completely meet-irreducible congruences are of this form.

Lemma 4.2. *Let L be a lattice and $n \geq 0$. If $\text{Con}_c L \cong 2^n$ then $\text{lh}(L) \geq n$. Moreover, if C is a maximal chain of L , then $\text{Con}_c f$ is surjective, where $f: C \rightarrow L$ is the inclusion map.*

Proof. The result that $\text{lh}(L) \geq n$ is obvious if L has no maximal chain or if L is unbounded. So we can choose a maximal chain $0 = x_0 \prec x_1 \prec \dots \prec x_m = 1$ of L . Put $C = \{x_0, x_1, \dots, x_m\}$. Let $f: C \rightarrow L$ be the inclusion map. The complement of $\Theta_L(x_k, x_{k+1})$ is the largest congruence separating x_k and x_{k+1} , thus is completely meet-irreducible, so $\Theta_L(x_k, x_{k+1})$ is an atom of $\text{Con } L$, for all $0 \leq k < m$. Moreover:

$$\bigvee_{k=0}^{m-1} \Theta_L(x_k, x_{k+1}) = \Theta_L(0, 1)$$

Thus $\text{Con}_c f$ is surjective, so $m \geq n$, hence $\text{lh}(L) \geq n$. \square

Theorem 4.3. *Let \mathcal{V}_1 be a variety of lattices (resp., a variety of bounded lattices), let \mathcal{V}_2 be a finitely generated variety of lattices, let D be a finite $(\vee, 0)$ -semilattice. If there exists a lifting $K \in \mathcal{V}_1$ of D of length greater than every lifting of D in \mathcal{V}_2 , then $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) \leq \aleph_0$. Moreover if \mathcal{V}_1 is a finitely generated variety of modular lattices and \mathcal{V}_2 is not trivial, then $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) = \aleph_0$.*

Proof. As D is finite, taking a sublattice, we can assume that $\text{card } K \leq \aleph_0$. Let n be the greatest length of a lifting of D in \mathcal{V}_2 . As $\text{lh}(K) > n$, there exists a chain C of K of length $n+1$ (resp., we can assume that C has 0 and 1). Let $f: C \rightarrow K$ be the inclusion map. Assume that there exists a lifting $g: C' \rightarrow K'$ of $\text{Con}_c f$ in \mathcal{V}_2 . As f is an embedding, g is also an embedding. As $\text{Con}_c K' \cong \text{Con}_c K \cong D$, $\text{lh}(K') \leq n$. Moreover $\text{Con}_c C' \cong \text{Con}_c C \cong 2^{n+1}$, thus, by Lemma 4.2, $\text{lh}(C') = n+1$. So $n \geq \text{lh}(K') \geq \text{lh}(C') = n+1$; a contradiction.

Therefore $\text{Con}_c f$ has no lifting in \mathcal{V}_2 . So, as $\text{card } K \leq \aleph_0$ and by [4, Corollary 7.6], $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) \leq \aleph_0$ (in the bounded case f preserves bounds, thus the result of [4] also applies).

Moreover if \mathcal{V}_1 is a finitely generated variety of modular lattices, then the finite $(\vee, 0)$ -semilattices with congruence-lifting in \mathcal{V}_1 are the finite Boolean lattices. Finite Boolean lattices are also liftable in \mathcal{V}_2 . Hence $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) = \aleph_0$. \square

The following corollary is an immediate application of Theorem 4.3 and Theorem 3.11. It shows that the critical point between a finitely generated variety of modular lattices and a variety generated by a lattice of subspaces of a finite vector space, cannot be \aleph_1 .

Corollary 4.4. *Let \mathcal{V} be a finitely generated variety of modular lattices, let F be a finite field, let $n \geq 1$ be an integer. If there exists a simple lattice in $K \in \mathcal{V}$ such that $\text{lh}(K) > n$, then $\text{crit}(\mathcal{V}; \mathbf{Var}(\text{Sub } F^n)) = \aleph_0$, else $\text{crit}(\mathcal{V}; \mathbf{Var}(\text{Sub } F^n)) \geq \aleph_2$.*

We shall now give a diagram of $(\vee, 0)$ -semilattices \vec{S} , congruence-liftable in \mathcal{M}_n , such that for every finitely generated variety \mathcal{V} , generated by lattices of length at most three, the diagram \vec{S} is congruence-liftable in \mathcal{V} if and only if $M_n \in \mathcal{V}$.

Let $n \geq 3$ be an integer. Set $\underline{n} = \{0, 1, \dots, n-1\}$, and set:

$$I_n = \{P \in \mathfrak{P}(\underline{n}) \mid \text{either } \text{card}(P) \leq 2 \text{ or } P = \underline{n}\}.$$

Denote by a_0, \dots, a_{n-1} the atoms of M_n . Set $A_P = \{a_x \mid x \in P\} \cup \{0, 1\}$, for all $P \in I_n$. Let $f_{P,Q}: A_P \rightarrow A_Q$ be the inclusion map for all $P \subseteq Q$ in I_n . Then $\vec{A} = (A_P, f_{P,Q})_{P \subseteq Q \text{ in } I_n}$ is a direct system in $\mathcal{M}_n^{0,1}$. The diagram \vec{S} is defined as $\text{Con}_c \circ \vec{A}$.

Lemma 4.5. *Let $\vec{B} = (B_P, g_{P,Q})_{P \subseteq Q \text{ in } I_n}$ be a congruence-lifting of $\text{Con}_c \circ \vec{A}$ by lattices, with all the maps $g_{P,Q}$ inclusion maps, for all $P \subseteq Q$ in I . Let $u < v$ in B_0 . Let $P \in I_n$ then:*

$$\Theta_{B_P}(u, v) = B_P \times B_P, \quad \text{the largest congruence of } B_P.$$

Let $\vec{\xi} = (\xi_P)_{P \in I}: \text{Con}_c \circ \vec{A} \rightarrow \text{Con}_c \circ \vec{B}$ be a natural equivalence. Let $x, y \in \underline{n}$ distinct. Let $b_x \in [u, v]_{B_{\{x\}}}$ and $b_y \in [u, v]_{B_{\{y\}}}$. Set $P = \{x, y\}$. Let $c \in \{0, 1\}$. Then the following assertions hold:

- (1) If $\Theta_{B_{\{x\}}}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))$, then $\Theta_{B_P}(u, b_x) = \xi_P(\Theta_{A_P}(c, a_x))$.

- (2) If $\Theta_{B_{\{z\}}}(u, b_z) = \xi_{\{z\}}(\Theta_{A_{\{z\}}}(c, a_z))$ for all $z \in \{x, y\}$, then $b_x \wedge b_y = u$.
(3) If $\Theta_{B_{\{x\}}}(b_x, v) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))$, then $\Theta_{B_P}(b_x, v) = \xi_P(\Theta_{A_P}(c, a_x))$.
(4) If $\Theta_{B_{\{z\}}}(b_z, v) = \xi_{\{z\}}(\Theta_{A_{\{z\}}}(c, a_z))$ for all $z \in \{x, y\}$, then $b_x \vee b_y = v$.
(5) If $\Theta_{B_{\{x\}}}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))$ and $\Theta_{B_{\{y\}}}(b_y, v) = \xi_{\{y\}}(\Theta_{A_{\{y\}}}(c, a_y))$, then $b_x \leq b_y$.

Proof. As $f_{P,Q}$ preserves bounds, $\text{Con}_c f_{P,Q}$ preserves bounds, thus $\text{Con}_c g_{P,Q}$ preserves bounds, for all $P \leq Q$ in I_n . Let $u < v$ in B_\emptyset . As B_\emptyset is simple, $\Theta_{B_\emptyset}(u, v)$ is the largest congruence of B_\emptyset . Moreover, $\text{Con}_c g_{\emptyset, P}$ preserves bounds, for all $P \in I_n$. Hence:

$$\Theta_{B_P}(u, v) = B_P \times B_P, \quad \text{the largest congruence of } B_P.$$

- (1) The following equalities hold:

$$\begin{aligned} \Theta_{B_P}(u, b_x) &= (\text{Con}_c g_{\{x\}, P})(\Theta_{B_{\{x\}}}(u, b_x)) \\ &= (\text{Con}_c g_{\{x\}, P})(\xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))) && \text{by assumption} \\ &= \xi_P \circ (\text{Con}_c f_{\{x\}, P})(\Theta_{A_{\{x\}}}(c, a_x)) \\ &= \xi_P(\Theta_{A_P}(c, a_x)). \end{aligned}$$

- (2) The following containments hold:

$$\begin{aligned} \Theta_{B_P}(u, b_x \wedge b_y) &\subseteq \Theta_{B_P}(u, b_x) \cap \Theta_{B_P}(u, b_y) \\ &= \xi_P(\Theta_{A_P}(c, a_x)) \cap \xi_P(\Theta_{A_P}(c, a_y)) && \text{by (1)} \\ &= \xi_P(\Theta_{A_P}(c, a_x) \cap \Theta_{A_P}(c, a_y)) \\ &= \xi_P(\text{id}_{A_P}) = \text{id}_{B_P}. \end{aligned}$$

so $u = b_x \wedge b_y$.

- (3) Similar to (1).

- (4) The following containments hold:

$$\begin{aligned} \Theta_{B_P}(b_x \vee b_y, v) &\subseteq \Theta_{B_P}(b_x, v) \cap \Theta_{B_P}(b_y, v) \\ &= \xi_P(\Theta_{A_P}(c, a_x)) \cap \xi_P(\Theta_{A_P}(c, a_y)) && \text{by (3)} \\ &= \xi_P(\Theta_{A_P}(c, a_x) \cap \Theta_{A_P}(c, a_y)) \\ &= \xi_P(\text{id}_{A_P}) = \text{id}_{B_P}. \end{aligned}$$

so $v = b_x \vee b_y$.

- (5) The following containments hold:

$$\begin{aligned} \Theta_{B_P}(b_y, b_x \vee b_y) &\subseteq \Theta_{B_P}(u, b_x) \cap \Theta_{B_P}(b_y, v) \\ &= \xi_P(\Theta_{A_P}(c, a_x)) \cap \xi_P(\Theta_{A_P}(c, a_y)) && \text{by (1) and (3)} \\ &= \xi_P(\Theta_{A_P}(c, a_x) \cap \Theta_{A_P}(c, a_y)) \\ &= \xi_P(\text{id}_{A_P}) = \text{id}_{B_P}. \end{aligned}$$

so $b_y = b_x \vee b_y$, thus $b_x \leq b_y$. \square

The following lemma shows that if we have some ‘‘small’’ enough congruence-lifting of $\text{Con}_c \circ \vec{A}$ in a variety, then M_n belongs to this variety.

Lemma 4.6. *Let $\vec{B} = (B_P, g_{P,Q})_{P \leq Q}$ in I_n be a congruence-lifting of $\text{Con}_c \circ \vec{A}$ by lattices. Assume that $B_{\{x\}}$ is a chain of length two for all $x \in \underline{n}$. Then M_n can be embedded into $B_{\underline{n}}$.*

Proof. Let $\vec{\xi} = (\xi_P)_{P \in I_n} : \text{Con}_c \circ \vec{A} \rightarrow \text{Con}_c \circ \vec{B}$ a natural equivalence. As $f_{P,Q}$ is an embedding, $\text{Con}_c f_{P,Q}$ separates 0, so $\text{Con}_c g_{P,Q}$ separates 0, and so we can assume that $g_{P,Q}$ is the inclusion map from B_P into B_Q , for all $P \leq Q$ in I_n .

Let $u < v$ in B_\emptyset . By Lemma 4.5, $\Theta_{B_{\{x\}}}(u, v)$ is the largest congruence of $B_{\{x\}}$. Moreover $B_{\{x\}}$ is the 3-element chain, so u is the smallest element of $B_{\{x\}}$ while v is its largest element. Denote by b_x the middle element of $B_{\{x\}}$.

The congruence $\xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x))$ is join-irreducible, thus it is either equal to $\Theta_{B_{\{x\}}}(u, b_x)$ or to $\Theta_{B_{\{x\}}}(b_x, v)$. Set:

$$X' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) = \Theta_{B_{\{x\}}}(u, b_x)\},$$

$$X'' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) = \Theta_{B_{\{x\}}}(b_x, v)\}.$$

As $\Theta_{A_{\{x\}}}(0, a_x)$ is the complement of $\Theta_{A_{\{x\}}}(a_x, 1)$ and $\Theta_{B_{\{x\}}}(u, b_x)$ is the complement of $\Theta_{B_{\{x\}}}(b_x, v)$, we also get that:

$$X' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1)) = \Theta_{B_{\{x\}}}(b_x, v)\}$$

$$X'' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1)) = \Theta_{B_{\{x\}}}(u, b_x)\}.$$

Moreover $\underline{n} = X' \cup X''$. As $\text{card } \underline{n} \geq 3$, either $\text{card } X' \geq 2$ or $\text{card } X'' \geq 2$.

Assume that $\text{card } X' \geq 2$. Let x, y in X' distinct. By Lemma 4.5(2), $b_x \wedge b_y = u$. By Lemma 4.5(4), $b_x \vee b_y = v$.

Let x, y in X' distinct. As $\xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) = \Theta_{B_{\{x\}}}(u, b_x)$ and $\xi_{\{z\}}(\Theta_{A_{\{z\}}}(0, a_z)) = \Theta_{B_{\{z\}}}(b_z, v)$ for each $z \in X''$, it follows from Lemma 4.5(5) that $b_x \leq b_z$. Similarly, as $\xi_{\{z\}}(\Theta_{A_{\{z\}}}(a_z, 1)) = \Theta_{B_{\{z\}}}(u, b_z)$ and $\xi_{\{y\}}(\Theta_{A_{\{y\}}}(a_y, 1)) = \Theta_{B_{\{y\}}}(b_y, v)$, it follows from Lemma 4.5(5) that $b_z \leq b_y$. Thus $b_x \leq b_y$. So $u = b_x \wedge b_y = b_x > u$, a contradiction.

Thus $X'' = \emptyset$, so $X' = \underline{n}$, and so $\{u, b_0, b_1, \dots, b_n, v\}$ is a sublattice of $B_{\underline{n}}$ isomorphic to M_n . The case $\text{card } X'' \geq 2$ is similar. \square

We shall now use a tool introduced in [4] to prove that having a congruence-lifting of $\text{Con}_c \circ \vec{A}$ is equivalent to having a congruence-lifting of some $(\vee, 0)$ -semilattice of cardinality \aleph_2 . This requires the following infinite combinatorial property, proved by A. Hajnal and A. Máté in [7], see also [3, Theorem 46.2]. This property is also used by M. Ploščica in [8].

Proposition 4.7. *Let $n \geq 0$ be an integer, let α be an ordinal, let $\kappa \geq \aleph_{\alpha+2}$, let $f: [\kappa]^3 \rightarrow [\kappa]^{<\aleph_\alpha}$. Then there exists $Y \in [\kappa]^n$ such that $a \notin f(\{b, c\})$ for all distinct $a, b, c \in Y$.*

Now recall the definition of supported poset and norm-covering introduced in [4, Section 4].

Definition 4.8. A finite subset V of a poset U is a *kernel*, if for every $u \in U$, there exists a largest element $v \in V$ such that $v \leq u$. We denote this element by $V \cdot u$.

We say that U is *supported*, if every finite subset of U is contained in a kernel of U .

We denote by $V \cdot \mathbf{u}$ the largest element of $V \cap \mathbf{u}$, for every kernel V of U and every ideal \mathbf{u} of U . As an immediate application of the finiteness of kernels, we obtain that any intersection of a nonempty set of kernels of a poset U is a kernel of U .

Definition 4.9. A *norm-covering* of a poset I is a pair $(U, |\cdot|)$, where U is a supported poset and $|\cdot|: U \rightarrow I$, $u \mapsto |u|$ is an order-preserving map.

A *sharp ideal* of $(U, |\cdot|)$ is an ideal \mathbf{u} of U such that $\{|v| \mid v \in \mathbf{u}\}$ has a largest element. For example, for every $u \in U$, the principal ideal $U \downarrow u$ is sharp; we shall often identify u and $U \downarrow u$. We denote this element by $|\mathbf{u}|$. We denote by $\text{Id}_s(U, |\cdot|)$ the set of all sharp ideals of $(U, |\cdot|)$, partially ordered by inclusion.

A sharp ideal \mathbf{u} of $(U, |\cdot|)$ is *extreme*, if there is no sharp ideal \mathbf{v} with $\mathbf{v} > \mathbf{u}$ and $|\mathbf{v}| = |\mathbf{u}|$. We denote by $\text{Id}_e(U, |\cdot|)$ the set of all extreme ideals of $(U, |\cdot|)$.

Let κ be a cardinal number. We say that $(U, |\cdot|)$ is κ -*compatible*, if for every order-preserving map $F: \text{Id}_e(U, |\cdot|) \rightarrow \mathfrak{P}(U)$ such that $\text{card } F(\mathbf{u}) < \kappa$ for all $\mathbf{u} \in \text{Id}_e(U, |\cdot|)^=$, there exists an order-preserving map $\sigma: I \rightarrow \text{Id}_e(U, |\cdot|)$ such that:

- (1) The equality $|\sigma(i)| = i$ holds for all $i \in I$.
- (2) The containment $F(\sigma(i)) \cap \sigma(j) \subseteq \sigma(i)$ holds for all $i \leq j$ in I .

Lemma 4.10. *Let X be a set, let $(A_x)_{x \in X}$ be a family of sets, let:*

$$U = \bigsqcup_{P \in [X]^{< \omega}} \prod_{x \in P} A_x.$$

We view the elements of U as (partial) functions and “to be greater” means “to extend”. Then U is a supported poset.

Proof. Let V be a finite subset of U . Let $Y_x = \{u_x \mid u \in V \text{ and } x \in \text{dom } u\}$ for all $x \in X$. Let $D = \bigcup_{u \in V} \text{dom } u$. Let:

$$W = \{u \in U \mid \text{dom } u \subseteq D \text{ and } (\forall x \in \text{dom } u)(u_x \in Y_x)\}$$

the set D , and the sets Y_x for $x \in X$ are all finite, so W is finite.

Let $u \in U$, let $P = \{x \in \text{dom } u \mid x \in D \text{ and } u_x \in Y_x\}$. Then $u \upharpoonright P \in W$. Moreover let $w \in W$ such that $w \leq u$. Let $x \in \text{dom } w$, then $x \in D$, and $u_x = w_x \in Y_x$, thus $\text{dom } w \subseteq P$, so $w \leq u \upharpoonright P$. Therefore $u \upharpoonright P$ is the largest element of $W \downarrow u$. \square

Using Lemma 4.10 and Proposition 4.7 we can construct a \aleph_0 -compatible lower finite norm-covering of I_n , the poset constructed earlier.

Lemma 4.11. *Let α be an ordinal. Let $U = \bigsqcup_{P \in \mathfrak{P}(\underline{n})} \aleph_{\alpha+2}^P$, partially ordered by inclusion. Let*

$$|\cdot|: U \rightarrow I_n$$

$$u \mapsto |u| = \begin{cases} \text{dom } u & \text{if } \text{card}(\text{dom } u) \leq 2 \\ \underline{n} & \text{otherwise.} \end{cases}$$

Then $(U, |\cdot|)$ is a \aleph_α -compatible lower finite norm-covering of I_n . Moreover $\text{card } U = \aleph_{\alpha+2}$.

Proof. By Lemma 4.10, the set U is supported. Moreover $|\cdot|$ preserves order, so $(U, |\cdot|)$ is a norm-covering of I_n . The poset U is lower finite.

Extreme ideals are of the form $\downarrow u$, where $u \in U$ and $\text{dom } u \in I_n$, so we identify the corresponding extreme ideal with u . Thus $\text{Id}_e(U, |\cdot|) = \{u \in U \mid \text{dom } u \in I_n\}$.

Let $F: \text{Id}_e(U, |\cdot|) \rightarrow \mathfrak{P}(U)$ be an order-preserving map such that $\text{card } F(\mathbf{u}) < \aleph_\alpha$ for all $\mathbf{u} \in \text{Id}_e(U, |\cdot|)^\omega$, let

$$G: [\aleph_{\alpha+2}]^2 \rightarrow [\aleph_{\alpha+2}]^{<\aleph_\alpha}$$

$$s \mapsto \bigcup \left\{ \text{im } v \mid u \in \bigcup_{P \in I_n - \{\underline{n}\}} s^P \text{ and } v \in F(u) \right\}.$$

By Proposition 4.7, there exists $A \subset \aleph_{\alpha+2}$ such that $\text{card } A = n$ and $a \notin G(\{b, c\})$ for all distinct $a, b, c \in A$. Let $u: \underline{n} \rightarrow A$ be a one-to-one map. Let $\phi: I_n \rightarrow \text{Id}_e(U, |\cdot|)$, $P \mapsto u \upharpoonright P$. Then $|\phi(P)| = P$. Let $P < Q$ in I_n , let $v \in F(u \upharpoonright P) \downarrow (u \upharpoonright Q)$. Let $x \in \text{dom } v - P$. As $P \in I_n$, and $P \neq \underline{n}$, $\text{card } P \leq 2$. Let $P' = \{y, z\} \subseteq \underline{n}$, such that y, z are distinct, $P \subseteq P'$, and $x \notin P'$. Let $s = \{u_x, u_y\}$, then $u \upharpoonright P' \in s^{P'}$, as $v \in F(u \upharpoonright P) \subseteq F(u \upharpoonright P')$, $v_x \in G(s)$. Moreover $u_x, u_y, u_z \in A$ are distinct, thus $u_x \notin G(\{u_y, u_z\}) = G(s)$, so $v_x \neq u_x$ in contradiction with $v \leq u$, so $\text{dom } v \subseteq P$, and so $v \leq u \upharpoonright P$. \square

Using the results of [4] together with Lemma 4.11, we obtain the following result.

Lemma 4.12. *Let \mathcal{V}_1 be a variety of algebras with a countable similarity type, let \mathcal{V}_2 be a finitely generated congruence-distributive variety such that $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) > \aleph_2$. Let $\vec{D}: I_n \rightarrow \mathcal{S}$ be a diagram of finite $(\vee, 0)$ -semilattices. If \vec{D} is congruence-liftable in \mathcal{V}_1 , then \vec{D} is congruence-liftable in \mathcal{V}_2 .*

Proof. In this proof we use, but do not give, many definitions of [4]. By Lemma 4.11 there exists $(U, |\cdot|)$ a \aleph_0 -compatible lower finite norm-covering of I_n such that $\text{card } U = \aleph_2$. Let J be a one-element ordered set. By [4, Lemma 3.9], \mathcal{V}_2 is $(\text{Id}_e(U, |\cdot|)^\omega, J, \aleph_0)$ -Löwenheim-Skolem.

Let $\vec{A} = (A_P, f_{P,Q})_{P \leq Q \text{ in } I_n}$ be a congruence-lifting of \vec{D} in \mathcal{V}_1 . As $\text{Conc } A_P$ is finite, using [4, Lemma 3.6], taking sublattices we can assume that A_P is countable for all $P \in I_n$. By [4, Lemma 6.7], there exists an U -quasi-lifting $(\tau, \text{Cond}(\vec{A}, U))$ of \vec{D} in \mathcal{V}_1 . Moreover:

$$\text{card } \text{Cond}(\vec{A}, U) \leq \sum_{V \in [U]^{<\omega}} \text{card} \left(\prod_{u \in V} A_{|u|} \right) \leq \sum_{V \in [U]^{<\omega}} \aleph_0 \leq \aleph_2$$

As $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) > \aleph_2$, there are $B \in \mathcal{V}_2$ and an isomorphism $\xi: \text{Conc } \text{Cond}(\vec{A}, U) \rightarrow \text{Conc } B$. So $(\tau \circ \xi, B)$ is an U -quasi-lifting of \vec{D} . Moreover \mathcal{V}_2 is $(\text{Id}_e(U, |\cdot|)^\omega, J, \aleph_0)$ -Löwenheim-Skolem, hence, by [4, Theorem 6.9], with $I = I_n$, there exists a congruence-lifting of \vec{D} in \mathcal{V}_2 . \square

A similar proof, using Lemma 3.6, Lemma 3.7, Lemma 6.7, and Theorem 6.9 in [4] together with Lemma 4.11, yields the following generalization of Lemma 4.12.

Lemma 4.13. *Let $\alpha \geq 1$ be an ordinal. Let \mathcal{V}_1 and \mathcal{V}_2 be varieties of algebras, with similarity types of cardinality $< \aleph_\alpha$. Let $\vec{D} = (D_P, \varphi_{P,Q})_{P \leq Q \text{ in } I_n}$ be a direct system of $(\vee, 0)$ -semilattices. Assume that the following conditions hold:*

- (1) $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) > \aleph_{\alpha+2}$.
- (2) $\text{card}(D_P) < \aleph_\alpha$, for all $P \in I_n - \{\underline{n}\}$.
- (3) $\text{card}(D_{\underline{n}}) \leq \aleph_{\alpha+2}$.
- (4) \vec{D} is congruence-liftable in \mathcal{V}_1 .

Then \vec{D} is congruence-liftable in \mathcal{V}_2 .

The following lemma implies, in particular, that a modular lattice of length three is a congruence-preserving extension of one of its subchains.

Lemma 4.14. *Let L be a lattice of length at most three, let u, v in L such that $\Theta_L(u, v) = L \times L$. If $\text{Con}_c L \cong 2^2$, then there exists $x \in L$ with $u < x < v$ such that L is a congruence-preserving extension of the chain $C = \{u, x, v\}$.*

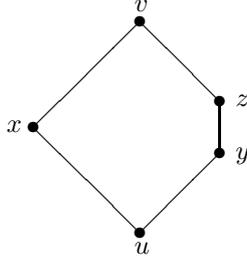


FIGURE 2. The lattice N_5 .

Proof. As $\text{Con}_c L \cong 2^2$, $\text{lh}([u, v]) \geq 2$. If $\text{lh}([u, v]) = 2$, then let $C = \{u, x, v\}$, where x is any element such that $u < x < v$. Let $i: C \rightarrow L$ the inclusion map. The morphism $\text{Con}_c i: \text{Con}_c C \rightarrow \text{Con}_c L$ is onto, moreover $\text{Con}_c C \cong 2^2 \cong \text{Con}_c L$, so $\text{Con}_c i$ is an isomorphism.

Now assume that $[u, v]$ has length three. As $\text{lh}(L) \leq 3$, $\text{lh}(L) = 3$, u is the smallest element of L , and v is the largest element.

Assume that L has a sublattice isomorphic to N_5 , as labelled in Figure 2. Then $C = \{u, y, z, v\}$ is a maximal chain of L . Let $i: C \rightarrow L$ be the inclusion map. By Lemma 4.2, $\text{Con}_c i$ is surjective. Thus, as $\text{Con} L \cong 2^2$, and $\Theta_L(u, y)$, $\Theta_L(y, z)$, and $\Theta_L(z, v)$ are all the atoms of $\text{Con} L$,

$$\Theta_L(y, z) \subseteq \Theta_L(u, y) \cap \Theta_L(y, z) \cap \Theta_L(z, v) = \text{id}_L,$$

a contradiction. Thus L does not contain any lattice isomorphic to N_5 , that is, L is modular.

As $\text{Con} L \cong 2^2$ and $\text{lh}(L) = 3$, L is not distributive. Hence there exists a sublattice of L isomorphic to M_3 , let $a < x_1, x_2, x_3 < b$ be its elements. As L is modular, $[a, x_1]_L \cong [x_1, b]_L$, thus $\text{lh}([a, b]_L)$ is even. But $2 \leq \text{lh}([a, b]_L) \leq 3$, so $\text{lh}([a, b]_L) = 2$, thus $a < x_1 < b$. This chain can be completed into a maximal chain $c < a < x_1 < b$ or $a < x_1 < b < c$. By symmetry, we may assume that $b < c$. Observe that $a = u$ and $c = v$. Set $C = \{u, b, v\}$ and $C_1 = \{u, x_1, b, v\}$. Let $i: C \rightarrow L$ and $i_1: C_1 \rightarrow L$ be the inclusion maps. As C_1 is a maximal chain, $\text{Con}_c i_1$ is onto. As $\Theta_L(u, x_1) = \Theta_L(x_1, b) = \Theta_L(u, b)$, $\text{Con}_c i_1$ and $\text{Con}_c i$ have the same image, thus $\text{Con}_c i$ is onto, so $\text{Con}_c i$ is an isomorphism. \square

The result of Lemma 4.14 does not extend to length four or more. The lattice of Figure 3 is not a congruence-preserving extension of any chain with extremities u and v .

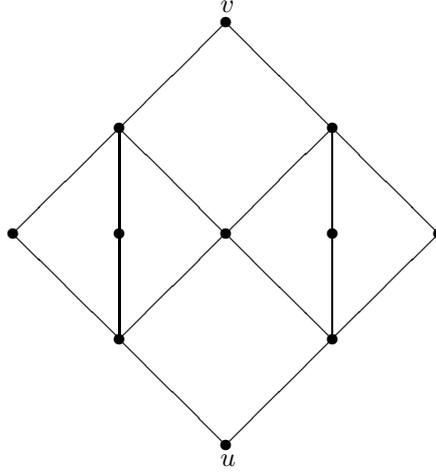


FIGURE 3. Lemma 4.14 does not extend to lattices of greater length.

Lemma 4.15. *Let $n \geq 4$ be an integer, let \mathcal{V} be a finitely generated variety of lattices such that $M_n \notin \mathcal{V}$. If $\text{lh}(K) \leq 3$ for each simple lattice K of \mathcal{V} , then $\text{crit}(\mathcal{M}_n^{0,1}; \mathcal{V}) \leq \aleph_2$.*

Proof. Assume that $\text{crit}(\mathcal{M}_n^{0,1}; \mathcal{V}) > \aleph_2$. As $M_n \in \mathcal{M}_n^{0,1}$, \vec{A} is a diagram of $\mathcal{M}_n^{0,1}$ indexed by I_n . By Lemma 4.12, the diagram $\text{Con}_c \circ \vec{A}$ has a congruence-lifting $\vec{B} = (B_P, g_{P,Q})_{P \leq Q \text{ in } I_n}$ in \mathcal{V} . As $\text{Con } B_{\underline{n}} \cong 2$, the lattice $B_{\underline{n}}$ is simple, thus, by assumption on \mathcal{V} , $\text{lh}(B_{\underline{n}}) \leq 3$, and so $\text{lh}(B_{\{x\}}) \leq 3$, for all $x \in \underline{n}$. The lattice B_{\emptyset} is simple, so, taking a sublattice, we can assume that $B_{\emptyset} = \{u, v\}$, with $u < v$. By Lemma 4.14, we can assume that $B_{\{x\}}$ is a chain of length two, for each $x \in \underline{n}$. So by Lemma 4.6, M_n is a sublattice of $B_{\underline{n}}$, and so $M_n \in \mathcal{V}$, a contradiction. \square

Theorem 4.16. *Let \mathcal{V}_1 be a finitely generated variety of modular lattices and \mathcal{V}_2 be finitely generated variety of lattices. Let $n \geq 3$ be an integer such that $M_n \in \mathcal{V}_1 - \mathcal{V}_2$. If $\text{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}_1$, then $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) \leq \aleph_2$. Moreover if either $\text{lh}(K) \leq 2$ for each simple $K \in \mathcal{V}_1$ and $M_3 \in \mathcal{V}_2$ or $\text{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}_1$ and $\text{Sub } F^3 \in \mathcal{V}_2$ for some field F , then $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) = \aleph_2$.*

Proof. By Lemma 4.15, $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) \leq \aleph_2$.

Assume that $\text{lh}(K) \leq 2$ for each simple $K \in \mathcal{V}_1$ and $M_3 \in \mathcal{V}_2$. As $\text{Sub } \mathbb{F}_2^2 \cong M_3 \in \mathcal{V}_2$, it follows from Theorem 3.11 that $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) \geq \aleph_2$.

Assume that $\text{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}_1$ and $\text{Sub } F^3 \in \mathcal{V}_2$ for some field F , it follows from Theorem 3.11 that $\text{crit}(\mathcal{V}_1; \mathcal{V}_2) \geq \aleph_2$. \square

Similarly we obtain the following critical points.

Corollary 4.17. *The following equalities hold*

$$\begin{aligned} \text{crit}(\mathcal{M}_n; \mathcal{M}_{m,m}) &= \aleph_2; \\ \text{crit}(\mathcal{M}_n^{0,1}; \mathcal{M}_{m,m}) &= \aleph_2; \\ \text{crit}(\mathcal{M}_n^{0,1}; \mathcal{M}_{m,m}^{0,1}) &= \aleph_2; \\ \text{crit}(\mathcal{M}_n; \mathcal{M}_{m,m}^0) &= \aleph_2; \\ \text{crit}(\mathcal{M}_n; \mathcal{M}_m^0) &= \aleph_2, \end{aligned} \quad \text{for all } n, m \text{ with } 3 \leq m < n \leq \omega.$$

Proof. Let $n' \leq n$ be an integer such that $m < n' < \omega$. As $M_{n'} \notin \mathcal{M}_{m,m}$, it follows from Lemma 4.15 that $\text{crit}(\mathcal{M}_{n'}^{0,1}; \mathcal{M}_{m,m}) \leq \aleph_2$, thus:

$$\text{crit}(\mathcal{M}_n^{0,1}; \mathcal{M}_{m,m}) \leq \aleph_2. \quad (4.1)$$

Moreover $M_3 \in \mathcal{M}_{m,m}$, simple lattices of $\mathcal{M}_{m,m}$ are of length at most 3, and finitely generated lattices of \mathcal{M}_n are of finite length (and even finite). Thus, by Theorem 3.11

$$\text{crit}(\mathcal{M}_n; \mathcal{M}_{m,m}^0) \geq \aleph_2. \quad (4.2)$$

Similarly:

$$\text{crit}(\mathcal{M}_n^{0,1}; \mathcal{M}_{m,m}^{0,1}) \geq \aleph_2. \quad (4.3)$$

All the desired equalities are immediate consequences of (4.1), (4.2), and (4.3). \square

As an immediate consequence we obtain:

Corollary 4.18. $\text{crit}(\mathcal{M}_{4,3}; \mathcal{M}_{3,3}) \leq \aleph_2$.

This question was suggested by M. Ploščica.

Lemma 4.19. *Let F be field. Then $M_n \in \mathbf{Var}(\text{Sub } F^3)$ if and only if $n \leq 1 + \text{card } F$.*

Proof. If F is infinite then the result is obvious. So we can assume that F is finite.

If $n \leq 1 + \text{card } F$, then M_n is a sublattice of $M_{1+\text{card } F} \cong \text{Sub } F^2 \in \mathbf{Var}(\text{Sub } F^3)$, thus $M_n \in \mathbf{Var}(\text{Sub } F^3)$.

Now assume that $M_n \in \mathbf{Var}(\text{Sub } F^3)$. By Jónsson's Lemma, M_n is a homomorphic image of a sublattice of $\text{Sub } F^3$. As M_n satisfies Whitman's condition, it follows from the Davey-Sands Theorem [2, Theorem 1] that M_n is projective in the class of all finite lattices. Therefore, as $\text{Sub } F^3$ is finite, M_n is a sublattice of $\text{Sub } F^3$. Thus there exist distinct subspaces $A, B, V_1, V_2, \dots, V_n$ of F^3 such that $V_i \cap V_j = A$ and $V_i + V_j = B$, for all $1 \leq i < j \leq n$. Let i, j, k distinct. Then:

$$\dim V_i + \dim V_j = \dim B + \dim A = \dim V_i + \dim V_k.$$

Thus $\dim V_j = \dim V_k$. But $\dim A < \dim V_1 < \dim B \leq \dim F^3 = 3$. If $\dim A = 1$, then M_n is isomorphic to $\{A/A, V_1/A, \dots, V_n/A, B/A\}$ which is a sublattice of $\text{Sub}(B/A)$, with $\dim B/A = 2$. If $\dim A = 0$, then:

$$\dim B = \dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2 = 2 \cdot \dim V_1.$$

Thus $\dim B$ is even, moreover $\dim B \leq 3$, hence $\dim B = 2$.

In both cases M_n is a sublattice of $\text{Sub } E$ for some F -vector space E of dimension two. But $\text{Sub } E \cong M_{1+\text{card } F}$, thus $n \leq 1 + \text{card } F$. \square

Corollary 4.20. *Let F be a finite field and let $n > 1 + \text{card } F$. Then:*

$$\begin{aligned} \text{crit}(\mathcal{M}_n; \mathbf{Var}(\text{Sub } F^3)) &= \aleph_2; \\ \text{crit}(\mathcal{M}_n; \mathbf{Var}_0(\text{Sub } F^3)) &= \aleph_2; \\ \text{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}(\text{Sub } F^3)) &= \aleph_2; \\ \text{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}_{0,1}(\text{Sub } F^3)) &= \aleph_2. \end{aligned}$$

Proof. By Lemma 4.19, $M_n \notin \mathbf{Var}(\text{Sub } F^3)$, moreover simple lattices of $\mathbf{Var}(\text{Sub } F^3)$ are of length at most three. Thus, by Lemma 4.15:

$$\text{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}(\text{Sub } F^3)) \leq \aleph_2. \quad (4.4)$$

As each simple lattice of \mathcal{M}_n is of length at most two, it follows from Theorem 3.11 that

$$\text{crit}(\mathcal{M}_n; \mathbf{Var}_0(\text{Sub } F^n)) \geq \aleph_2, \quad \text{and } \text{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}_{0,1}(\text{Sub } F^n)) \geq \aleph_2. \quad (4.5)$$

All the other desired equalities are consequences of (4.4), (4.5). \square

Corollary 4.21. *Let F and K be finite fields. If $\text{card } F > \text{card } K$ then:*

$$\begin{aligned} \text{crit}(\mathbf{Var}(\text{Sub } F^3); \mathbf{Var}(\text{Sub } K^3)) &= \aleph_2; \\ \text{crit}(\mathbf{Var}(\text{Sub } F^3); \mathbf{Var}_0(\text{Sub } K^3)) &= \aleph_2; \\ \text{crit}(\mathbf{Var}_{0,1}(\text{Sub } F^3); \mathbf{Var}(\text{Sub } K^3)) &= \aleph_2; \\ \text{crit}(\mathbf{Var}_{0,1}(\text{Sub } F^3); \mathbf{Var}_{0,1}(\text{Sub } K^3)) &= \aleph_2. \end{aligned}$$

Proof. By Lemma 4.19, $M_{1+\text{card } F} \notin \mathbf{Var}(\text{Sub } K^3)$, moreover simple lattices of $\mathbf{Var}(\text{Sub } K^3)$ are of length at most three. Thus, by Lemma 4.15:

$$\text{crit}(\mathbf{Var}_{0,1}(\text{Sub } F^3); \mathbf{Var}(\text{Sub } K^3)) \leq \aleph_2. \quad (4.6)$$

As each simple lattice of $\mathbf{Var}(\text{Sub } F^3)$ is of length at most three, it follows from Theorem 3.11 that:

$$\text{crit}(\mathbf{Var}(\text{Sub } F^3); \mathbf{Var}_0(\text{Sub } K^n)) \geq \aleph_2, \quad (4.7)$$

$$\text{crit}(\mathbf{Var}_{0,1}(\text{Sub } F^3); \mathbf{Var}_{0,1}(\text{Sub } K^n)) \geq \aleph_2. \quad (4.8)$$

All the other desired equalities are consequences of (4.6), (4.7), (4.8). \square

Lemma 4.22. *Let \mathcal{V} be a finitely generated variety of lattices (resp., a finitely generated variety of lattices with 0), let $m \geq 2$ an integer. Assume that for each simple lattice K of \mathcal{V} , there do not exist $b_0, b_1, \dots, b_{m-1} > u$ in K such that $b_i \wedge b_j = u$ (resp., $b_i \wedge b_j = 0$), for all $0 \leq i < j \leq m-1$. Then $\text{crit}(\mathcal{M}_{2m-1}^{0,1}; \mathcal{V}) \leq \aleph_2$.*

Proof. Set $n = 2m - 1 \geq 3$. Let $\vec{A} = (A_P, f_{P,Q})_{P \leq Q \text{ in } I_n}$ be the direct system of $\mathcal{M}_n^{0,1}$ defined in the beginning of this section. Assume that $\text{crit}(\mathcal{M}_n^{0,1}; \mathcal{V}) > \aleph_2$. By Lemma 4.12, there exists a congruence-lifting $\vec{B} = (B_P, g_{P,Q})_{P \leq Q \text{ in } I_n}$ of $\text{Con}_c \circ \vec{A}$ in \mathcal{V} . Let $\vec{\xi} = (\xi_P)_{P \in I_n}: \text{Con}_c \circ \vec{A} \rightarrow \text{Con}_c \circ \vec{B}$ be a natural equivalence. Taking a sublattice of B_\emptyset , we can assume that B_\emptyset is a chain $u < v$. Moreover, as the map $f_{P,Q}$ is an inclusion map, we can assume that $g_{P,Q}$ is an inclusion map, for all $P \leq Q$ in I_n .

Let $x \in \underline{n}$. By Lemma 4.5, $\Theta_{B_{\{x\}}}(u, v)$ is the largest congruence of $B_{\{x\}}$. Thus:

$$\Theta_{B_{\{x\}}}(u, v) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) \vee \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1))$$

Therefore there exist $t_0^x = u < t_1^x < \dots < t_{r+1}^x = v$ in $B_{\{x\}}$ such that, for all $0 \leq i \leq r$:

$$\text{either } (t_i^x, t_{i+1}^x) \in \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) \text{ or } (t_i^x, t_{i+1}^x) \in \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1))$$

Set $b_x = t_1^x$. Put:

$$X' = \{x \in \underline{n} \mid \Theta_{B_{\{x\}}}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x))\}$$

$$X'' = \{x \in \underline{n} \mid \Theta_{B_{\{x\}}}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1))\}.$$

By symmetry we can assume that $\text{card } X' \geq \text{card } X''$ (we can replace the diagram \vec{A} by its dual if required). As $\underline{n} = X' \cup X''$ and $\text{card } \underline{n} = n = 2m - 1$, $\text{card } X' \geq m$. Let $x, y \in X'$ distinct, it follows from Lemma 4.5(2) that $b_x \wedge b_y = u$. So we obtain a family of elements $(b_x)_{x \in X'}$ greater than u such that $b_x \wedge b_y = u$ (resp., $b_x \wedge b_y = u = 0$) for all $x \neq y$ in X' , a contradiction. \square

With a similar proof using Lemma 4.13 instead of Lemma 4.12 we obtain the following lemma.

Lemma 4.23. *Let \mathcal{V} be a variety of lattices (resp., a variety of lattices with 0), let $m \geq 2$ an integer. Assume that for each simple lattice K of \mathcal{V} , there do not exist $b_0, b_1, \dots, b_{m-1} > u$ in K such that $b_i \wedge b_j = u$ (resp., $b_i \wedge b_j = 0$), for all $0 \leq i < j \leq m - 1$. Then $\text{crit}(\mathcal{M}_{2m-1}^{0,1}; \mathcal{V}) \leq \aleph_3$.*

Theorem 4.24. *Let \mathcal{V} be either a finitely generated variety of lattices or a finitely generated variety of lattices with 0. If $M_3 \in \mathcal{V}$ then:*

$$\text{crit}(\mathcal{M}_\omega; \mathcal{V}) = \aleph_2;$$

$$\text{crit}(\mathcal{M}_\omega^0; \mathcal{V}) = \aleph_2.$$

Let \mathcal{V} be a finitely generated variety of bounded lattices. If $M_3 \in \mathcal{V}$ then:

$$\text{crit}(\mathcal{M}_\omega^{0,1}; \mathcal{V}) = \aleph_2.$$

Proof. Let \mathcal{V} be a finitely generated variety of lattices, let m be the maximal cardinality of a simple lattice of \mathcal{V} . Thus the assumptions of Lemma 4.22 are satisfied, so *a fortiori* $\text{crit}(\mathcal{M}_{2m-1}^{0,1}; \mathcal{V}) \leq \aleph_2$, and so $\text{crit}(\mathcal{M}_\omega^{0,1}; \mathcal{V}) \leq \aleph_2$.

Denote by \mathbb{F}_2 the two-element field. Let \mathcal{V} be a variety of lattices with 0 (resp., with 0 and 1), such that $M_3 \in \mathcal{V}$. The variety \mathcal{M}_ω is locally finite, thus all finitely generated lattices of \mathcal{M}_ω are of finite length. Moreover all simple lattices of \mathcal{M}_ω have length at most two. Thus, by Theorem 3.11:

$$\text{crit}(\mathcal{M}_\omega; \mathbf{Var}_0(\text{Sub } \mathbb{F}_2^2)) \geq \aleph_2 \text{ (resp., } \text{crit}(\mathcal{M}_\omega^{0,1}; \mathbf{Var}_{0,1}(\text{Sub } \mathbb{F}_2^2)) \geq \aleph_2).$$

Moreover $\text{Sub } \mathbb{F}_2^2 \cong M_3$, so $\text{crit}(\mathcal{M}_\omega; \mathcal{V}) \geq \aleph_2$. \square

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