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Gauss and Poisson Approximation: Applications to CDOs Tranche Pricing

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Abstract

This article describes a new numerical method, based on Stein's method and zero bias transformation, to compute CDO tranche prices. We propose first order correction terms for both Gauss and Poisson approximations and the approximation errors are discussed. We then combine the two approximations to price CDOs tranches in the conditionally independent framework using a realistic local correlation structure. Numerical tests show that the method provides robust results with a very low computational burden.

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1 Introduction

In the growing credit derivatives market, correlation products like CDO play a major role. On one hand, CDO enables financial institutions to transfer efficiently the credit risk of a pool of names in a global way. On the other hand, the investors can choose to invest on different tranches according to their risk aversion.

The key term to value a CDO tranche is the law of the cumulative loss of the underlying portfolio $l_t = \sum_{i=1}^{n} \frac{N_i}{N}(1 - R_i) I_{\{\tau_i \leq t\}}$, where N_i, R_i, τ_i are respectively the notional value, the recovery rate and the default time of name i and N the total portfolio notional defined by $N = \sum_{i=1}^{n} N_i$. The critical issue for CDO pricing is to compute the value $\mathbb{E}[(l_t - k)_+]$, which is a call function on the cumulative loss.

The standard market model for the default correlation is the factor model ([1],[14]), where the default events are supposed to be conditionally independent given a common factor U. In the literature, there exist approximation methods for the conditional distribution and then integrating (see [11], [18]). In fact, conditional on U, the cumulative loss l_t can be written as a sum of independent random variables. Using the central limit theorem, it is then natural to apply Gauss or Poisson approximation to compute the conditional cumulative losses distribution.

The binomial-normal approximation has been studied in various financial problems. It is well known that the price of an European option calculated in the binomial tree model converges to its Black-Scholes price when the discretization number tends towards infinity. In particular, Diener and Diener [9] have proved that in this symmetric binomial case, the convergence speed is of order $O(1/n)$. In the credit analysis, Vasicek [24] has introduced the normal approximation to a homogeneous portfolio of loans. As the default probabilities are in general small and not equal to 1/2, the convergence speed is of order $O(1/\sqrt{n})$ in the general case.

Other numerical methods such as the saddle-point method ([16], [17], [2]) have been proposed. The saddle-point method consists in expanding around the saddle point to approximate a function of the cumulant generating function of conditional losses. It coincides with the normal approximation when choosing some particular point. In the inhomogeneous case, it is rather costly to find the saddle-point numerically. In addition, although proven efficient by empirical tests, there is no discussion of the error estimations in aforementioned papers.

The Poisson approximation, less discussed in the financial context, is known to be robust for small probabilities in the approximation of binomial laws. One usually asserts that the normal approximation remains robust when $np \geq 10$. If np is small, the binomial law approaches a Poisson law. In our case, the size of the portfolio is fixed for a standard synthetic CDO tranche and $n \approx 125$. On the other hand, the conditional default probability

$p(U)$ varies in the interval $(0,1)$ according to its explicit form with respect to the factor U. Hence we may encounter both cases and it is mandatory to study the convergence speed since n is finite.

Stein's method is an efficient tool to estimate the approximation errors in the limit theorem problems. In this paper, we provide, by combining Stein's method and the zero bias transformation, first-order correction terms for both Gauss and Poisson approximations. Error estimations of corrected approximations are obtained. These first order approximations can be applied to conditional distributions in the general factor framework and the CDOs tranches prices can then be obtained by integration across the common factors.

Thanks to the simple form of the formulas, we reduce largely the computational burden for CDOs prices. In addition, the summand variables are not required to be identically distributed, which corresponds to inhomogeneous CDO tranches. We present in Section 2 the theoretical results and Section 3 and 4 are devoted to numerical tests on CDOs. Section 5 contains the conclusion and perspective remarks. We gather at last some technical results and proofs in Appendix.

2 First-Order Correction of Conditional Losses

2.1 First-order Gaussian correction

In the classical binomial-normal approximation, the expectation of functions of conditional losses can be calculated using a Gaussian expectation. More precisely, the expectation $\mathbb{E}[h(W)]$ where W is the sum of conditionally independent individual loss variables can be approximated by $\Phi_{\sigma_W}(h)$ defined by

$$\Phi_{\sigma_W}(h) = \frac{1}{\sqrt{2\pi}\sigma_W} \int_{-\infty}^{\infty} h(u) \exp\Big(- \frac{u^2}{2\sigma_W^2} \Big) \mathrm{d}u \tag{1}$$

where σ_W is the standard deviation of W. The error of this zero-order approximation is of order $O(1/\sqrt{n})$ by the well-known Berry-Esseen inequality using the Wasserstein distance ([19], [8]) except in the symmetric case.

We shall improve the approximation quality by finding a correction term such that the corrected error is of order $O(1/n)$ even in the asymmetric case. Some regularity conditions are required on the considered function h. Notably, the call function, not possessing second order derivative, is difficult to analyze. In the following theorem, we give the corrector term for regular enough functions. The explicit error bound and the proof can be found in Appendix 6.2.1.

Theorem 2.1 *Let $X_1, \ldots, X_n$ be independent mean zero random variables (r.v.) such that $\mathbb{E}[X_i^4]$ $(i = 1, \ldots, n)$ exists. Let $W = X_1 + \cdots + X_n$*

and $\sigma_W^2 = \mathrm{Var}[W]$. *For any function* h *such that* $\|h''\|$ *exists, the normal approximation* $\Phi_{\sigma_W}(h)$ *of* $\mathbb{E}[h(W)]$ *has corrector*

$$C_h = \frac{\mu_{(3)}}{2\sigma_W^4}\Phi_{\sigma_W}\left(\left(\frac{x^2}{3\sigma_W^2} - 1\right)xh(x)\right) \tag{2}$$

where $\mu_{(3)} = \sum_{i=1}^n \mathbb{E}[X_i^3]$. *The corrected approximation error is bounded by*

$$\left|\mathbb{E}[h(W)] - \Phi_{\sigma_W}(h) - C_h\right| \leq \alpha\big(h, X_1, \ldots, X_n\big)$$

where $\alpha\big(h, X_1, \ldots, X_n\big)$, *precised later in* (22), *depends on* h'' *and on the moments of* X_i *up to the fourth order.*

The corrector is written as the product of two terms: the first one depends on the moments of X_i up to the third order and the second one is a normal expectation of some polynomial function multiplying h. Both terms are simple to calculate, even in the inhomogeneous case.

To adapt to the definition of the zero bias transformation, which will be introduced in Section 6.1.1, and also to obtain a simple representation of the corrector, the variables X_i's are set to be of zero expectation in Theorem 2.1. This condition requires a normalization step when applying the theorem to conditional losses. A useful example concerns the centered Bernoulli random variables which take two real values and have zero expectation.

Note that the moments of X_i play an important role here. In the symmetric case we have $\mu_{(3)} = 0$ and as a consequence $C_h = 0$ for any function h. Therefore, C_h can be viewed as an asymmetric corrector in the sense that, after correction, the approximation realizes the same error order as in the symmetric case.

To specify the convergence order of the corrector, let us consider the normalization of an homogeneous case where X_i are i.i.d. random variables whose moments may depend on n. Notice that

$$\Phi_{\sigma_W}\left(\left(\frac{x^2}{3\sigma_W^2} - 1\right)xh(x)\right) = \sigma_W \Phi_1\left(\left(\frac{x^2}{3} - 1\right)xh(\sigma_W x)\right).$$

To ensure that the above expectation term is of constant order, we often suppose that the variance of W is finite and does not depend on n. In this case, we have $\mu_{(3)} \sim O(1/\sqrt{n})$ and the corrector C_h is also of order $O(1/\sqrt{n})$. Consider now the percentage default indicator variable $I_{\{\tau_i \leq t\}}/n$, whose conditional variance given the common factor is equal to $p(1-p)/n^2$ where p is the conditional default probability of the ith credit, identical for all in the homogeneous case and depends on the common factor. Hence, we shall fix p to be zero order and let $X_i = (I_{\{\tau_i \leq t\}} - p)/\sqrt{n}$. Then σ_W is of constant order as stated above. Finally, for the percentage conditional loss, the corrector is of order $O(1/n)$ because of the remaining coefficient $1/\sqrt{n}$.

The X_i's are not required to have the same distribution: we can handle easily different recovery rates (as long as they are independent r.v.) by computing the moments of the product variables $(1-R_i)I_{\{\tau_i \le t\}}$. The corrector depends only on the moments of R_i up to the third order. Note however that the dispersion of the recovery rates alongside the dispersion of the notional values can have an impact on the order of the corrector.

We now concentrate on the call function $h(x) = (x-k)_+$. The Gauss approximation corrector is given in this case by

$$C_h = \frac{\mu_{(3)}}{6\sigma_W^2} k \phi_{\sigma_W}(k) \tag{3}$$

where ϕ_σ is the density function of the distribution $N(0, \sigma^2)$. When the strike $k = 0$, the corrector $C_h = 0$. On the other hand, the function $k \exp\left(-\frac{k^2}{2\sigma_W^2}\right)$ reaches its maximum and minimum values when $k = \sigma_W$ and $k = -\sigma_W$ respectively, and then tends to zero quickly.

The numerical computation of this corrector is extremely simple since there is no need to take expectation. Observe however that the call function is a Lipschitz function with $h'(x) = I_{\{x>k\}}$ and h'' exists only in the distribution sense. Therefore, we can not apply directly Theorem 2.1 and the error estimation deserves a more subtle analysis. The main tool we used to establish the error estimation for the call function is a concentration inequality of Chen and Shao [7]. For detailed proof, interested reader may refer to El Karoui and Jiao [10].

We shall point out that the regularity of the function h is essential in the above result. For more regular functions, we can establish correction terms of corresponding order. However, for the call function, the second order correction can not bring further improvement to the approximation results in general.

2.2 First-order Poisson correction

Following the same idea, if V is a random variable taking non-negative integers, then we may approximate $\mathbb{E}[h(V)]$ by a Poisson function

$$\mathcal{P}_{\lambda_V}(h) = \sum_{l=0}^{n} \frac{\lambda_V^l}{l!} e^{-\lambda_V} h(l).$$

The Poisson approximation is robust under some conditions, for example, when $V \sim B(n,p)$ and $np < 10$. We shall improve the Poisson approximation by presenting a corrector term as above. We remark that due to the property that a Poisson distributed random variable takes non-negative integer values, the variables Y_i's in Theorem 2.2 are discrete integer random variables. Similar as in the Gaussian case, the proof of the following theorem is postponed to Appendix 6.2.2.

Theorem 2.2 *Let $Y_1, \ldots, Y_n$ be independent random variables taking non-negative integer values such that $\mathbb{E}[Y_i^3]$ $(i = 1, \ldots, n)$ exist. Let $V = Y_1 + \cdots + Y_n$ with expectation and variance $\lambda_V = \mathbb{E}[V]$ and $\sigma_V^2 = \mathrm{Var}[V]$. Then, for any bounded function h defined on $\mathbb{N}_+$, the Poisson approximation $\mathcal{P}_{\lambda_V}(h)$ of $\mathbb{E}[h(V)]$ has corrector*

$$C_h^{\mathcal{P}} = \frac{\sigma_V^2 - \lambda_V}{2} \mathcal{P}_{\lambda_V}(\Delta^2 h) \tag{4}$$

where $\mathcal{P}_\lambda(h) = \mathbb{E}[h(\Lambda)]$ with $\Lambda \sim \mathcal{P}(\lambda)$ and $\Delta h(x) = h(x+1) - h(x)$. The corrected approximation error is bounded by

$$\left| \mathbb{E}[h(V)] - \mathcal{P}_{\lambda_V}(h) - C_h^{\mathcal{P}} \right| \leq \beta(h, Y_1, \ldots, Y_n)$$

where $\beta(h, Y_1, \ldots, Y_n)$, precised later in (30)*, depends on h and on the moments of Y_i up to the third order.*

The Poisson corrector $C_h^{\mathcal{P}}$ is of similar form with the Gaussian one and contains two terms as well: one term depends on the moments of Y_i and the other is a Poisson expectation.

Since Y_i's are $\mathbb{N}_+$-valued random variables, they can represent directly the default indicators $I_{\{\tau_i \leq t\}}$. This fact limits however the recovery rates to be identical or proportional for all credits. We now consider the order of the corrector. Suppose that λ_V does not depend on n to ensure that $\mathcal{P}_{\lambda_V}(\Delta^2 h)$ is of constant order. Then in the homogeneous case, the conditional default probability $p \sim O(1/n)$. For the percentage conditional losses, as in the Gaussian case, the corrector is of order $O(1/n)$ with the coefficient $1/n$.

Consider the call function $h(x) = (x-k)_+$ where k is a positive integer. Since $\Delta^2 h(x) = I_{\{x=k-1\}}$, its Poisson approximation corrector is given by

$$C_h^{\mathcal{P}} = \frac{\sigma_V^2 - \lambda_V}{2(k-1)!} e^{-\lambda_V} \lambda_V^{k-1}. \tag{5}$$

The corrector vanishes when the expectation and the variance of the sum variable V are equal. The difficulty here is that the call function is not bounded. However, we can prove that Theorem 2.2 holds for any function of linear increasing speed (see [10]).

2.3 Other approximation methods

There exist other approximation methods, notably the Gram-Charlier expansions and the saddle-point method, to calculate the portfolio products prices. Both methods can be applied to conditional losses. We use the notations as in Section 2.1.

The Gram-Charlier expansion has been used to approximate the bond prices [22] through expansion of the density function of a given random variable by using the Gaussian density and cumulants. We shall note that in the

expectation term (2), the polynomial function multiplied by h corresponds to the first order Gram-Charlier expansion. However, the higher order terms obtained by our method are different. For comparison of Gram-Charlier and other expansions (Edgeworth expansion for example), one can consult [22].

The saddle-point method consists of writing the considered expectation function as an integral which contains the cumulant generating function $\mathcal{K}(\xi) = \ln \mathbb{E}[\exp(\xi W)]$. For example,

$$\mathbb{E}[(W-k)_+] = \lim_{A\to+\infty} \frac{1}{2\pi i} \int_{c-iA}^{c+iA} \frac{\exp(\mathcal{K}(\xi) - \xi k)}{\xi^2} d\xi \tag{6}$$

where c is a real number. In general, the expansion of the integrand function is made around its critical point — the saddle point, where the integrand function decreases rapidly and hence is most dense. By choosing some particular point to make the expansion, the saddle point method may coincide with the classical Gauss approximation. In [2], the saddle point is chosen such that $\mathcal{K}'(\xi_0) = k$ and the authors propose to approximate $\mathbb{E}[(W-k)_+]$ by expansion formulas of increasing precision orders.

It is important to note that in the saddle-point method, the first step is to obtain the value of the saddle point ξ_0, which is equivalent to find the solution of the equation $\mathcal{K}'(\xi) = k$. In the homogeneous case where X_i's are i.i.d. random variables given by $X_i = \gamma(I_{\{\tau_i \le t\}} - p)$ with $\gamma = 1/\sqrt{np(1-p)}$, we have the explicit solution

$$\xi_0 = \sqrt{np(1-p)} \ln\Big(\frac{\sqrt{np(1-p)} + k(1-p)}{\sqrt{np(1-p)} - kp}\Big).$$

However, in the inhomogeneous case, it consists in solving the equation numerically, which can be rather tedious. In addition, although empirically proved to be efficient, there is no discussions on convergence speed for the above two methods.

3 Numerical Tests on Conditional Losses

Before going further on applications to the CDOs pricing, we would like in this section to perform some basic testings of the preceding formulas. In the sequel, we consider the call value $\mathbb{E}[(l-k)_+]$ where $l = n^{-1}\sum_{i=1}^n (1 - R_i)\xi_i$ and the ξ_i's are independent Bernoulli random variable with success probability equal to p_i.

3.1 Validity Domain of the Approximations

We begin by testing the accuracy of the corrected Gauss and Poisson approximations for different values of $np = \sum_{i=1}^n p_i$ in the case $R_i = 0$, $n = 100$ and for different values of k. The benchmark value is obtained through the

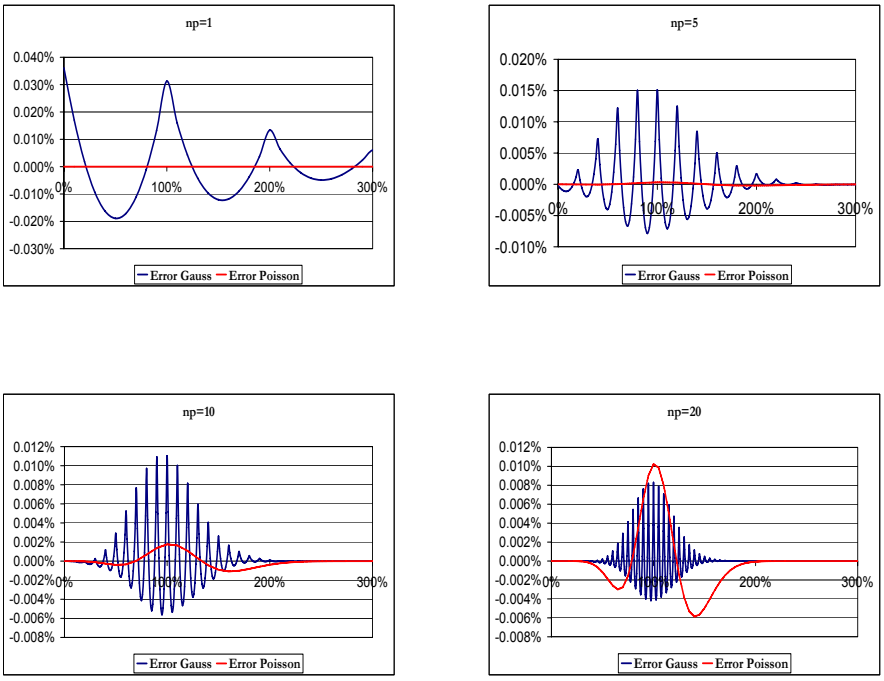

Figure 1: Gauss and Poisson approximation errors for various values of np as a function of the strike over the expected loss.

recursive methodology well known by the practitioners ([15], [5]) which computes the loss distribution by reducing the portfolio size by one name at each recursive step.

In Figure 1 we plot the differences between the corrected Gauss approximation and the benchmark (Error Gauss) and the corrected Poisson approximation and the benchmark (Error Poisson) for different values of np as a function of the call strike over the expected loss. Note that when the tranche strike equals the expected loss, the normalized strike value in the Gaussian case equals zero due to the centered random variables, which means that the correction vanishes. We also observe that the Gaussian error is maximal around this point.

We observe on these graphs that the Poisson approximation outperforms the gaussian one for approximately $np < 15$. On the contrary, for large value of np, the Gauss approximation is the best one. Because of the correction, the threshold between the Gauss-Poisson approximation is higher than the classical one $np \approx 10$. In addition, the threshold may be chosen rather flexibly around 15. Combining the two approximations, the minimal error

of the two approximations is relatively larger in the overlapping area when np is around 15. However, we obtain satisfactory results even in this case. In all the graphs presented, the error of the mixed approximation is inferior than 1 bp.

Our tests are made with inhomogeneous p_i's obtained as

$$p_i = p \exp(\sigma W_i - 0.5\sigma^2)$$

(log-normal random variable with expectation p and volatility σ) where W_i is a family of independent standard normal random variables and values of σ ranging from 0% to 100%. Qualitatively, the results were not affected by the heterogeneity of the p_i's.

Observe that there is oscillation in the Gauss approximation error, while the Poisson error is relatively smooth. This phenomenon is related to the discretisation impact of discrete distributions.

As far as a unitary computation is concerned (one call price), the Gauss and Poisson approximation perform much better than the recursive methodology: we estimate that these methodologies are 200 times faster. To be fair with the recursive methodology one has to recall that by using it we obtain not only a given call price but the whole loss distribution with which we can obtain several strikes values at the same time. In that case, our approximations outperform the recursive methodology by a factor ≥ 30 with six strike values (3%, 6%, 9%, 12%, 15%, 22%).

3.2 Saddle-point method and Gauss approximation

We now compare numerically the saddle-point method in [2] and the Gauss approximation for different values of np. In Figure 2 are presented the errors of the first order Gauss approximation, and of the first and the second saddle point approximations, as a function of the call strike over the expected loss. The errors of the second order saddle-point method are comparable with the Gauss approximation in all tests. Note that the saddle-point method has also been discussed for non-normal distributions (see [25]) and deserves further studies for CDOs computations.

The tests are applied to the homogeneous case for constant values of p and the calculation times for the saddle point method have outperformed the first order Gauss approximation. However this is no longer true in the inhomogeneous case.

3.3 Stochastic Recovery Rate - Gaussian case

We then consider the case of stochastic recovery rate and check the validity of the Gauss approximation in this case. Following the standard in the industry (Moody's assumption), we will model the R_i's as independent beta

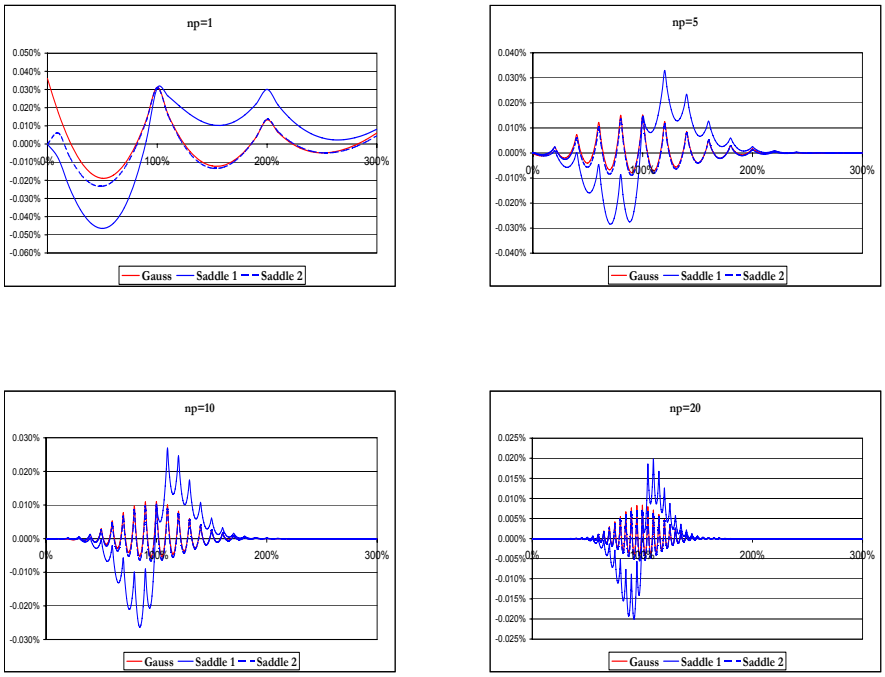

Figure 2: Gauss and Saddle-point methods approximation errors for various values of np as a function of the strike over the expected loss.

random variables with expectation 50% and standard deviation 26%. In addition, R_i is independent of ξ_i.

An application of Theorem 2.1 is used so that the first order corrector term takes into account the first three moments of the random variables R_i. To describe the obtained result, let us first introduce some notation. Let $\mu_{R_i}, \sigma^2_{R_i}$ and $\gamma^3_{R_i}$ be the first three centered moments of the random variable R_i, namely

$$\mu_{R_i} = \mathbb{E}R_i, \quad \sigma^2_{R_i} = \mathbb{E}(R_i - \mu_{R_i})^2, \quad \gamma^3_{R_i} = \mathbb{E}(R_i - \mu_{R_i})^3.$$

We also define $X_i = n^{-1}(1 - R_i)\xi_i - \mu_i$ where $\mu_i = n^{-1}(1 - \mu_{R_i})p_i$ and $p_i = \mathbb{E}\xi_i$. Let W be $\sum_{i=1}^n X_i$. We have

$$\sigma^2_W := \mathrm{Var}(W) = \sum_{i=1}^n \sigma^2_{X_i} \quad \text{where} \quad \sigma^2_{X_i} = \frac{p_i}{n^2}\Big[\sigma^2_{R_i} + (1 - p_i)(1 - \mu_{R_i})^2\Big].$$

Finally, if $\tilde{k} = k - \sum_{i=1}^n \mu_i$, we have the following approximation

$$\mathbb{E}(l - \tilde{k})_+ \simeq \Phi_{\sigma_W}(\,\cdot\, - k)_+ + \frac{1}{6}\frac{1}{\sigma^2_W}\sum_{i=1}^n \mathbb{E}\big[X_i^3\big]\tilde{k}\phi_{\sigma_W}(\tilde{k})$$

where

$$\mathbb{E}\big[X_i^3\big] = \frac{p_i}{n^3}\Big[(1 - \mu_{R_i})^3\big(1 - p_i)(1 - 2p_i) + 3(1 - p_i)(1 - \mu_{R_i})\sigma^2_{R_i} - \gamma^3_{R_i}\Big].$$

The benchmark is obtained using standard Monte Carlo integration with 10^6 simulations. In Figure 3, we display the difference between the approximated call price and the benchmark as a function of the strike normalized by the expected loss. We also consider the lower and upper 95% confidence interval for the Monte Carlo results. As in the standard case, one observes that the greater the value of np the better the approximation. Furthermore, the stochastic recovery brings a smoothing effect as the conditional loss no longer follows a binomial law.

The Poisson approximation, due to constraint of integer valued random variables, can not be used directly in the stochastic recovery rates case. We try however to take the mean value of R_i's as the uniform recovery rate without improving the results except for very low strike (equal to a few bp).

4 Application to CDOs portfolios

In this section, we want to test on real life examples the approaches developed in the preceding sections. We will work in the conditionally independent framework. In other words, we will assume that conditionally on a risk factor U the default indicators of the names in the considered pool are independent.

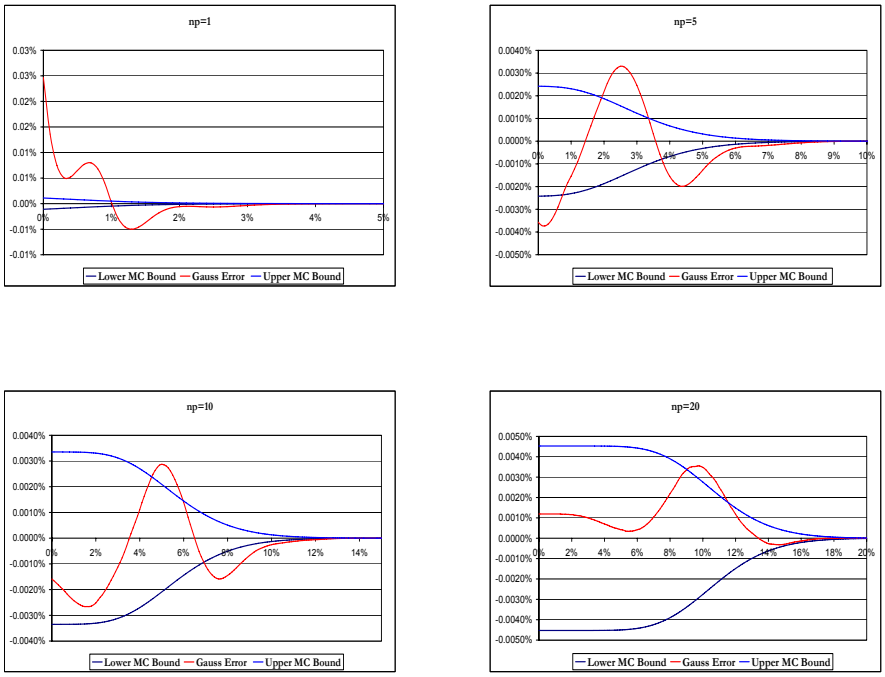

Figure 3: Gauss approximation error in the stochastic recovery case for various values of np as a function of the strike. Comparison with Monte Carlo 1,000,000 simulations. 95% confidence interval.

Let us introduce the notation. We consider a portfolio of n issuers and we will assume for the sake of simplicity that the weight of each firm in the portfolio is the same is equal to $1/n$ and that the recovery rate for each of these credits is constant and equal to a fixed value R fixed at 40%. The percentage loss on the pool for a given time horizon t is defined as

$$l_t = \frac{1-R}{n} \sum_{i=1}^{n} I_{\{\tau_i \leq t\}} \, .$$

The first step to value properly a CDO is to define the correlation between the default events.

4.1 Modelling the Correlation

Practically, one defines a correlation structure using the conditionally independent framework. In a nutshell, this tantamounts to postulate the existence of a random variable U (that we may assume uniformly distributed on (0, 1) without loss of generality) such that, conditionally on U, the events $E_i = \{\tau_i \leq t\}$ are independent. To completely specify a correlation model, one has to choose a function F such that

$$\int_0^1 F(p,u)\mathrm{d}u = p, \quad 0 \leq F \leq 1.$$

If $p_i = \mathbb{P}[E_i]$, one simply interprets $F(p_i, u)$ as $\mathbb{P}[E_i|U = u]$.

The standard Gaussian copula case with correlation ρ corresponds to the function F defined by

$$F(p,u) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(p) - \sqrt{\rho}\mathcal{N}^{-1}(u)}{\sqrt{1-\rho}}\right)$$

where $\mathcal{N}(x)$ is the distribution function of $N(0,1)$. The main drawback of the Gaussian correlation approach is the fact that one cannot find a unique model parameter ρ to price all the observed market tranches on a given basket. This phenomenon is referred to as *correlation skew* by the market practitioners.

In our tests, we will apply a more general approach ([23], [4]) in which the function F is defined in a non parametric way in order to retrieve the observed market prices of tranches. This function has been calibrated on market prices of the five year tranches 0%-3%, 3%-6%, 6%-9%, 9%-12%, 12%-15% and 15%-22% on a bespoke basket whose prices are observed on a monthly basis.

4.2 CDOs Payoff

Let us describe the payoff of a CDO. Let a and b be the attachment and detachment point expressed in percentage and let

$$l_t^{a,b} = (l_t - a)_+ - (l_t - b)_+$$

be the loss on the tranche $[a, b]$. The outstanding notional on the tranche is defined as

$$c_t^{a,b} = 1 - \frac{l_t^{a,b}}{b-a}$$

whereas the tranche survival probability is given by $q(a, b, t) = \mathbb{E}[c_t^{a,b}]$ computed under any risk-neutral probability.

The value of the default leg and the premium leg of a continuously compounded CDO of maturity T are given respectively by the following formulas:

$$\text{Default Leg} = -(b-a)\int_0^T B(0,t)q(a,b,\mathrm{d}t),$$
$$\text{Premium Leg} = \text{Spread} \times (b-a)\int_0^T B(0,t)q(a,b,t)\mathrm{d}t$$

where $B(0, t)$ is the value at time 0 of a zero coupon maturing at time t. We here assume that the interest rates are deterministic. Thanks to the integration by part formula, the fair spread is then computed as follow

$$\text{Fair Spread} = \frac{1 - B(0,T)q(a,b,T) + \int_0^T q(a,b,t)B(0,\mathrm{d}t)}{\int_0^T B(0,t)q(a,b,t)\mathrm{d}t}. \tag{7}$$

To compute the value of the preceding integrals, we begin by approximating the logarithm of the functions q and B by linear splines with monthly pillars for q (adding the one week point for short term precision) and weekly pillars for B. The integrals are then computed using time step of length one week. Performing these operations boils down to compute call prices value of the form $C(t, k) = \mathbb{E}[(l_t - k)_+]$.

4.3 Gauss Approximation

We describe in this subsection the Gauss approximation that can be used to compute in an efficient way the call prices in any conditionally independent model.

Let μ_i and σ_i be respectively the expectation and standard deviation of the random variable $\chi_i = n^{-1}(1-R)I_{\{\tau_i \le t\}}$. Let $X_i = \chi_i - \mu_i$ and $W = \sum_{i=1}^n X_i$, so that the expectation and standard deviation of the random variable W are 0 and $\sigma_W := \sqrt{\sum_{i=1}^n \sigma_i^2}$ respectively. Let also p_i be the default probability of issuer i. We want to calculate

$$C(t,k) = \mathbb{E}[(l_t - k)_+] = \mathbb{E}[(W - \tilde{k})_+]$$

where $\tilde{k} = k - \sum_{i=1}^{n} \mu_i$. Remark that p_i and k are in fact all functions of the common factor.

Assuming that the random variable X_i's are mutually independent, the result of Theorem 2.1 may be stated in the following way

$$C(t,k) \simeq \int_{-\infty}^{+\infty} \mathrm{d}x\, \phi_{\sigma_W}(x)(x-\tilde{k})_+ + \frac{1}{6}\frac{1}{\sigma_W^2}\sum_{i=1}^{n} \mathbb{E}[X_i^3]\tilde{k}\phi_{\sigma_W}(\tilde{k}) \tag{8}$$

where $\mathbb{E}[X_i^3] = \frac{(1-R)^3}{n^3} p_i(1-p_i)(1-2p_i)$. The first term on the right-hand side of (8) is the Gauss approximation that can be computed in closed form thanks to Bachelier formula whereas the second term is a correction term that account for the non-normality of the loss distribution.

In the sequel, we will compute the value of the call option on a loss distribution by making use of the approximation formula (8). In the conditionally independent case, one can indeed write

$$\mathbb{E}[(l_t - k)_+] = \int \mathbb{P}_U(\mathrm{d}u)\mathbb{E}[(l_t - k)_+ | U = u]$$

where U is the latent variable describing the general state of the economy. As the default time are conditionally independent upon the variable U, the integrand may be computed in closed form using (8).

4.4 Poisson Approximation

We describe in this subsection the poisson approximation that can also be used to compute in an efficient way the call prices in the conditionally independent model.

Recall that $\mathcal{P}_\lambda$ is the Poisson measure of intensity λ. Let $\lambda_i = p_i$ and $\lambda_V = \sum_{i=1}^{n} \lambda_i$ where now $V = \sum_{i=1}^{n} Y_i$ with $Y_i = I_{\{\tau_i \le t\}}$. We want to calculate

$$C(t,k) = \mathbb{E}[(l_t - k)_+] = \mathbb{E}[(n^{-1}(1-R)V - k)_+].$$

Recall that the operator Δ is such that $(\Delta f)(x) = f(x+1) - f(x)$. We also let the function h be defined by $h(x) = (n^{-1}(1-R)x - k)_+$.

Assuming that the random variables Y_i's are mutually independent, we may write according to the results of Theorem 2.2 that

$$C(t,k) \simeq \mathcal{P}_{\lambda_V}(h) - \frac{1}{2}\Big(\sum_{i=1}^{n} \lambda_i^2\Big)\mathcal{P}_{\lambda_V}(\Delta^2 h) \tag{9}$$

where

$$\mathcal{P}_{\lambda_V}(\Delta^2 h) = n^{-1}(1-R)\mathrm{e}^{-\lambda_V}\frac{\lambda_V^{m-1}}{(m-1)!}$$

and $m = kn/(1-R)$ is suppose to beinteger. Recall that here k represents the percentage attachment or detachment points. The formula (9) may be used to compute the conditional call price in the same way as in the preceding subsection. The true prices can be then obtained by taking integration with respect to the common factor.

4.5 Real Life CDO Pricing

In this subsection, we finally use both first order approximations to compute CDO leg values and break even as described in formula (7). As this formula involves conditioning on the latent variable U, we are either in the validity domain of the Poisson approximation or in the validity domain of the Gauss approximation. Taking into account the empirical facts underlined in Section 3, we choose to apply the Gauss approximation for the call value as soon as $\sum_i F(p_i, u) > 15$ and the Poisson approximation otherwise. All the subsequent results are benchmarked using the recursive methodology.

Our results for the quoted tranches are gathered in the following table. Level represents the premium leg for a spread of 1 bp and break even is the spread of CDO as described in (7).

Attach	Detach	Output	REC	Approx.
0%	3%	Default Leg	2.1744%	2.1752%
		Level	323.2118%	323.2634%
		Break Even	22.4251%	22.4295%
3%	6%	Default Leg	0.6069%	0.6084%
		Level	443.7654%	443.7495%
		Break Even	4.5586%	4.5702%
6%	9%	Default Leg	0.1405%	0.1404%
		Level	459.3171%	459.3270%
		Break Even	1.0197%	1.0189%
9%	12%	Default Leg	0.0659%	0.0660%
		Level	462.1545%	462.1613%
		Break Even	0.4754%	0.4758%
12%	15%	Default Leg	0.0405%	0.0403%
		Level	463.3631%	463.3706%
		Break Even	0.2910%	0.2902%
15%	22%	Default Leg	0.0503%	0.0504%
		Level	464.1557%	464.1606%
		Break Even	0.1549%	0.1552%
0%	100%	Default Leg	3.1388%	3.1410%
		Level	456.3206%	456.3293%
		Break Even	1.1464%	1.1472%

In the following table and Figure 4 are presented the error on the break even expressed in bp. One should note that in all cases the error is less than 1.15 bp way below the market bid-ask uncertainty that prevail on the bespoke CDO business.

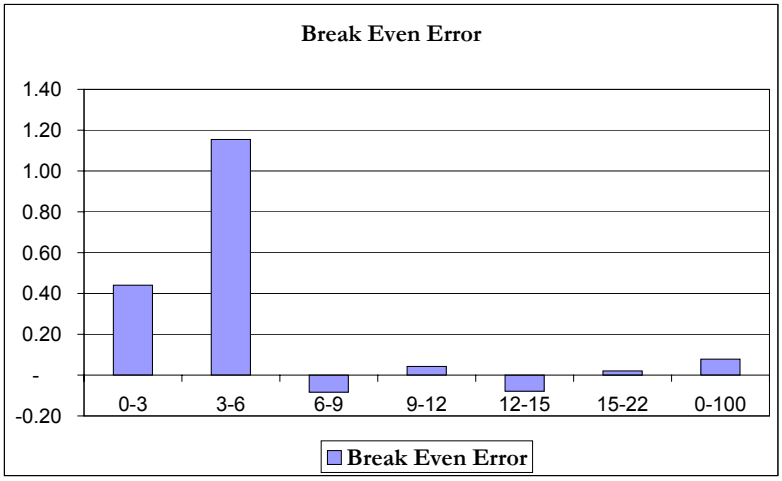

Figure 4: Break even error for the quoted tranches expressed in bp.

	Error
0-3	0.44
3-6	1.15
6-9	- 0.08
9-12	0.04
12-15	- 0.08
15-22	0.02
0-100	0.08

Trying to understand better these results, we display now in the following tables and Figure 5 the same results but for equity tranches. We observe on this graph that the error is maximum for the tranche 0%-6% which correspond to our empirical finding (see Figure 1) that the approximation error is maximum near the expected loss of the portfolio (equal here to 4.3%).

Attach	Detach	Output	REC	Mixte
0%	3%	Default Leg	2.1744%	2.1752%
		Level	323.2118%	323.2634%
		Break Even	22.4251%	22.4295%
0%	6%	Default Leg	2.7813%	2.7836%
		Level	383.4886%	383.5114%
		Break Even	12.0878%	12.0969%
0%	9%	Default Leg	2.9218%	2.9240%
		Level	408.7648%	408.7853%
		Break Even	7.9422%	7.9476%
0%	12%	Default Leg	2.9877%	2.9900%
		Level	422.1122%	422.1302%
		Break Even	5.8984%	5.9025%
0%	15%	Default Leg	3.0282%	3.0303%
		Level	430.3624%	430.3788%
		Break Even	4.6909%	4.6940%
0%	22%	Default Leg	3.0785%	3.0807%
		Level	441.1148%	441.1280%
		Break Even	3.1723%	3.1744%
0%	100%	Default Leg	3.1388%	3.1410%
		Level	456.3206%	456.3293%
		Break Even	1.1464%	1.1472%

	Error
0-3	0.44
0-6	0.92
0-9	0.55
0-12	0.41
0-15	0.31
0-22	0.21
0-100	0.08

4.6 Sensitivity analysis

We are finally interested in calculating the sensitivity with respect to p_j. As for the Greek of the classical option theory, direct approximations using the finite difference method implies large errors. We hence propose the following procedure.

Let $l_t^j = \omega_j(1-R_j)I_{\{\tau_j \le t\}}$. Then for all $j = 1, \cdots, n$,

$$(l_t - k)_+ = I_{\{\tau_j \le t\}}\Big(\sum_{i\,:\,i\neq j} l_t^i + \omega_j(1-R_j) - k\Big)_+ + I_{\{\tau_j > t\}}\Big(\sum_{i\,:\,i\neq j} l_t^i - k\Big)_+ .$$

As a consequence, we may write

$$\begin{aligned}\mathbb{E}[(l_t-k)_+|U] = F(p_j,U)\mathbb{E}\Big[\Big(\sum_{i\,:\,i\neq j} l_t^i + \omega_j(1-R_j) - k\Big)_+\Big|U\Big] \\ + \big(1-F(p_j,U)\big)\mathbb{E}\Big[\Big(\sum_{i\,:\,i\neq j} l_t^i - k\Big)_+\Big|U\Big].\end{aligned}$$

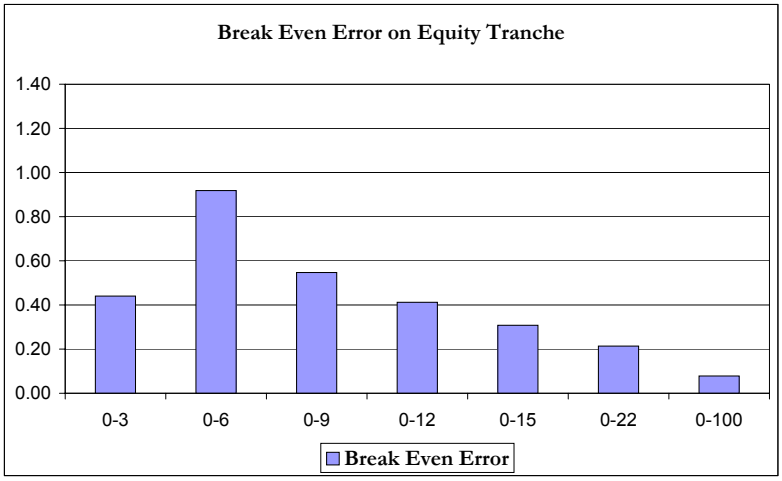

Figure 5: Break even error for the equity tranches expressed in bp.

Since the only term which depends on p_j is the function $F(p_j, U)$, we obtain

$$\frac{\partial C(t,k)}{\partial p_j} = \int_0^1 \mathrm{d}u \partial_1 F(p_j, u) \mathbb{E}\Big[\Big(\sum_{i\,:\,i\neq j} l_t^i + \omega_j(1-R_j) - k\Big)_+ - \Big(\sum_{i\,:\,i\neq j} l_t^i - k\Big)_+ \Big| U = u\Big] \tag{10}$$

where we compute the call spread using the mixed approximation for the partial total loss.

We test this approach in the case where $R = 0$ on a portfolio of 100 names such that one fifth of the names have a default probability of 25 bp, 50 bp, 75 bp, 100 bp and 200 bp respectively for an average default probability of 90 bp. We compute the derivatives of call prices with respect to each individual name probability according to the formula (10) and we benchmark this result by the sensitivities given by the recursive methodology.

We find out that in all tested cases (strike ranging from 3% to 20%) the relative error on these derivatives is less than 1% except for strike higher than 15% for which the relative error is around 2%. Note however that in this case the absolute error is less than 0.1 bp for derivatives whose value is ranging from 2 bp to 20 bp. In Figure 6 we plot these derivatives for a strike value of 3% computed using the recursive and approximated methodology. We may remark that the approximated methodology always overvalue the derivatives value. However in the case of a mezzanine tranche (call spread) this effect will be offset. We consider these results as very satisfying.

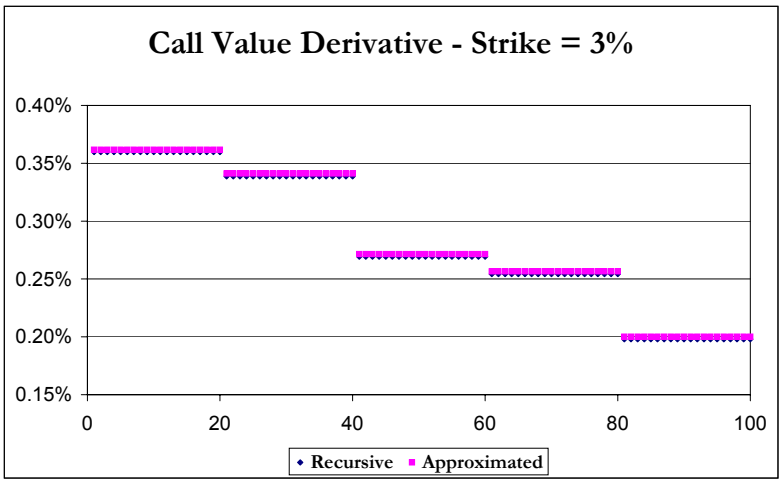

Figure 6: Derivatives with respect to individual name default probability using the recursive and approximated methodology.

5 Conclusion

We propose in this paper a combination of first order Gauss and Poisson approximations. Various numerical tests have been effectuated. Notably, we have provided an empirical threshold for choosing between the two approximations. Comparisons between other numerical methods (saddle-point, Monte Carlo and recursive) show that our method provides very satisfactory results when computing prices and sensitivities for CDOs tranches . Furthermore, it outperforms in terms of computation time thanks to explicit formulas of correctors.

Further research work consists in some extensions where we hope to treat random recovery rates when using the Poisson approximation. In the framework of Stein's method and zero bias transformation, this may involve certain alternative distribution other than the Poisson one.

6 Appendix

Theorem 2.1 and Theorem 2.2 are obtained through Stein's method and zero bias transformation. We now present the theoretical framework and proofs for both theorems.

6.1 Zero bias transformation and Stein's method

Stein's method is an efficient tool to study the approximation problems. In his pioneer paper, Stein [20] first proposed this method to study the Gauss approximation in the central limit theorem. The method has been extended to the Poisson case by Chen [6]. In this framework, the zero bias transformation has been introduced by Goldstein and Reinert [12] for the Gaussian distribution, which provides practical and concise notation for the estimations.

Generally speaking, the zero bias transformation is characterized by some functional relationship implied by the reference distributions, Gauss or Poisson, such that the "distance" between one distribution and the reference distribution can be measured by the "distance" between the distribution and its zero biased one.

6.1.1 The Gaussian case

In the Gaussian case, the zero bias transformation is motivated by the following observation of Stein: a random variable (r.v.) Z has the central normal distribution $N(0, \sigma^2)$ if and only if $\mathbb{E}[Zf(Z)] = \sigma^2\mathbb{E}[f'(Z)]$ for all regular enough functions f. In a more general context, for any mean zero r.v. X of variance $\sigma^2 > 0$, a r.v. X^* is said to have its zero biased distribution if the following relationship (11) holds for any function f such that the expectation terms are well-defined

$$\mathbb{E}[Xf(X)] = \sigma^2\mathbb{E}[f'(X^*)]. \tag{11}$$

The distribution of X^* is unique with density function given by $p_{X^*}(x) = \sigma^{-2}\mathbb{E}[XI_{\{X>x\}}]$.

The central normal distribution is invariant by the zero bias transformation. In fact, X^* and X have the same distribution if and only if X is a normal variable of mean zero.

For any given function h, the error of the Gaussian approximation of $\mathbb{E}[h(X)]$ can be estimated by combining the *Stein's equation*

$$xf(x) - \sigma^2 f'(x) = h(x) - \Phi_\sigma(h), \tag{12}$$

where $\Phi_\sigma(h)$ is given by (1). By (11) and (12), we have

$$\begin{aligned}\mathbb{E}[h(X)] - \Phi_\sigma(h) &= \mathbb{E}[Xf_h(X) - \sigma^2 f_h'(X)] \\ &= \sigma^2\mathbb{E}[f_h'(X^*) - f_h'(X)] \le \sigma^2\|f_h''\|\,\mathbb{E}[|X^* - X|].\end{aligned} \tag{13}$$

where f_h is the solution of (12). Here the property of the function f_h and the difference between X and X^* are important for the estimations.

The Stein's equation can be solved explicitly. If $h(t)\exp(-\frac{t^2}{2\sigma^2})$ is integrable on $\mathbb{R}$, then one solution of (12) is given by

$$f_h(x) = \frac{1}{\sigma^2\phi_\sigma(x)}\int_x^\infty (h(t) - \Phi_\sigma(h))\phi_\sigma(t)dt \tag{14}$$

where $\phi_\sigma(x)$ is the density function of $N(0,\sigma^2)$. The function f_h is one order more differentiable than h. Stein has established that $\|f_h''\| \le 2\|h'\|/\sigma^2$ if h is absolutely continuous.

For the term $X - X^*$, the estimations are easy when X and X^* are independent by using a symmetrical term $X^s = X - \tilde{X}$ where $\widetilde{X}$ is an independent duplicate of X:

$$\mathbb{E}[|X^* - X|^k] = \frac{1}{2(k+1)\sigma^2}\mathbb{E}\big[|X^s|^{k+2}\big], \quad \forall k \in \mathbb{N}_+. \tag{15}$$

For the sum of independent random variables, Goldstein and Reinert [12] have introduced a construction of zero bias transformation using a random index design to choose the weight of each summand variable.

Proposition 6.1 *(Goldstein and Reinert) Let X_i $(i = 1,\dots,n)$ be independent zero-mean r.v. of finite variance $\sigma_i^2 > 0$ and X_i^* having the X_i-zero normal biased distribution. We assume that $(\bar{X},\bar{X}^*) = (X_1,\dots,X_n,X_1^*,\dots,X_n^*)$ are independent r.v. Let $W = X_1 + \cdots + X_n$ and denote its variance by σ_W^2. We also use the notation $W^{(i)} = W - X_i$. Let us introduce a random index I independent of $(\bar{X},\bar{X}^*)$ such that $\mathbb{P}(I = i) = \sigma_i^2/\sigma_W^2$. Then $W^* = W^{(I)} + X_I^*$ has the W-zero biased distribution.*

Although W and W^* are dependent, the above construction based on a random index choice enables us to obtain the estimation of $W - W^*$:

$$\mathbb{E}\big[|W^* - W|^k\big] = \frac{1}{2(k+1)\sigma_W^2}\sum_{i=1}^n \mathbb{E}\big[|X_i^s|^{k+2}\big], \quad \forall k \in \mathbb{N}_+. \tag{16}$$

6.1.2 The Poisson case

The Poisson case is similar to the Gaussian one. Recall that Chen [6] has observed that a non-negative integer-valued random variable Λ of expectation λ follows the Poisson distribution if and only if $\mathbb{E}[\Lambda g(\Lambda)] = \lambda\mathbb{E}[g(\Lambda+1)]$ for any bounded function g. Similarly as in the Gaussian case, let us consider a random variable Y taking non-negative integer values and $\mathbb{E}[Y] = \lambda < \infty$. A r.v. Y^* is said to have the *Y-zero Poisson biased distribution* if for any function g such that $\mathbb{E}[Yg(Y)]$ exists, we have

$$\mathbb{E}[Yg(Y)] = \lambda\mathbb{E}[g(Y^*+1)]. \tag{17}$$

The Stein's Poisson equation is also introduced by Chen [6]:

$$yg(y) - \lambda g(y+1) = h(y) - \mathcal{P}_\lambda(h) \tag{18}$$

where $\mathcal{P}_\lambda(h) = \mathbb{E}[h(\Lambda)]$ with $\Lambda \sim P(\lambda)$. Hence, we obtain the error of the Poisson approximation

$$\mathbb{E}[h(V)] - \mathcal{P}_\lambda(h) = \mathbb{E}\big[Vg_h(V) - \lambda_V g_h(V+1)\big] = \lambda_V \mathbb{E}\big[g_h(V^*+1) - g_h(V+1)\big] \tag{19}$$

where V is a non-negative integer-valued r.v. with expectation λ_V, the function g_h is the solution of (18) and is given by

$$g_h(k) = \frac{(k-1)!}{\lambda^k} \sum_{i=k}^{\infty} \frac{\lambda^i}{i!} \big(h(i) - \mathcal{P}_\lambda(h)\big). \tag{20}$$

It is unique except at $k = 0$. However, the value $g(0)$ does not enter into our calculations afterwards.

We consider now the sum of independent random variables. Let $Y_i (i = 1, \cdots, n)$ be independent non-negative integer-valued r.v. with positive expectations λ_i and Y_i^* having the Y_i-Poisson zero biased distribution. Denote by $V = Y_1 + \cdots + Y_n$ and $\lambda_V = \mathbb{E}[V]$. Let I be a random index independent of $(\bar{Y}, \bar{Y}^*)$ satisfying $P(I = i) = \lambda_i/\lambda_V$. Then $V^{(I)} + Y_I^*$ has the V-Poisson zero biased distribution where $V^{(i)} = V - Y_i$.

For any integer $l \geq 1$, assume that Y and Y_i have to up $(l+1)$-order moments. Then

$$\mathbb{E}[|Y^* - Y|^l] = \frac{1}{\lambda}\mathbb{E}\big[Y|Y^s - 1|^l\big], \qquad \mathbb{E}[|V^* - V|^l] = \frac{1}{\lambda_V}\sum_{i=1}^{n}\mathbb{E}\big[Y_i|Y_i^s - 1|^l\big].$$

Finally, recall that Chen has established $\|\Delta g_h\| \leq 6\|h\| \min\big(\lambda^{-\frac{1}{2}}, 1\big)$ with which we obtain the following zero order estimation

$$|\mathbb{E}[h(V)] - \mathcal{P}_{\lambda_V}(h)| \leq 6\|h\| \min\Big(\frac{1}{\sqrt{\lambda_V}}, 1\Big) \sum_{i=1}^{n} \mathbb{E}\big[Y_i|Y_i^s - 1|\big]. \tag{21}$$

There also exist other estimations of error bound (e.g. Barbour and Eagleson [3]). However we are more interested in the order than the constant of the error.

6.2 Proof of Theorem 2.1 and 2.2

We shall use in the sequel without comment the notation introduced in Section 6.1.

6.2.1 The normal case: Theorem 2.1

We now give the explicit form of the corrected approximation error and the proof to establish it. With the notation of Theorem 2.1, the corrected error bound $\alpha(h, X_1, \cdots, X_n)$ is given by

$$\begin{aligned}&\Big|\mathbb{E}[h(W)] - \Phi_{\sigma_W}(h) - C_h\Big| \\ &\leq \|f_h^{(3)}\| \left(\frac{1}{12}\sum_{i=1}^n \mathbb{E}\big[|X_i^s|^4\big] + \frac{1}{4\sigma_W^2}\Big|\sum_{i=1}^n \mathbb{E}[X_i^3]\Big| \sum_{i=1}^n \mathbb{E}\big[|X_i^s|^3\big] + \frac{1}{\sigma_W}\sqrt{\sum_{i=1}^n \sigma_i^6} \right).\end{aligned} \tag{22}$$

Note that the existence of $f_h^{(3)}$ requires that h is second order derivable.

Proof. By taking first order Taylor expansion, we obtain

$$\begin{aligned}&\mathbb{E}[h(W)] - \Phi_{\sigma_W}(h) = \sigma_W^2 \mathbb{E}[f_h'(W^*) - f_h'(W)] \\ &= \sigma_W^2 \mathbb{E}[f_h''(W)(W^* - W)] + \sigma_W^2 \mathbb{E}\Big[f_h^{(3)}\big(\xi W + (1-\xi)W^*\big)\xi(W^* - W)^2\Big]\end{aligned} \tag{23}$$

where ξ is a random variable on $[0, 1]$ independent of all X_i and X_i^*. Firstly, we notice that the remaining term is bounded by

$$\mathbb{E}\Big[\Big|f_h^{(3)}\big(\xi W + (1-\xi)W^*\big)\xi(W^* - W)^2\Big|\Big] \leq \frac{\|f_h^{(3)}\|}{2}\mathbb{E}[(W^* - W)^2] \leq \frac{\|f_h^{(3)}\|}{12\sigma_W^2}\sum_{i=1}^n \mathbb{E}\big[|X_i^s|^4\big]. \tag{24}$$

Secondly, we consider the first term of equation (23). Since X_I^* is independent of W, we have

$$\mathbb{E}[f_h''(W)(W^* - W)] = \mathbb{E}[f_h''(W)(X_I^* - X_I)] = \mathbb{E}[X_I^*]\mathbb{E}[f_h''(W)] - \mathbb{E}[f_h''(W)X_I]. \tag{25}$$

For the first term $\mathbb{E}[X_I^*]\mathbb{E}[f_h''(W)]$ of (25), we write it as the sum of two parts

$$\mathbb{E}[X_I^*]\mathbb{E}[f_h''(W)] = \mathbb{E}[X_I^*]\Phi_{\sigma_W}(f_h'') + \mathbb{E}[X_I^*]\mathbb{E}[f_h''(W) - \Phi_{\sigma_W}(f_h'')].$$

The first term $\mathbb{E}[X_I^*]\Phi_{\sigma_W}(f_h'')$ of the right-hand side is the candidate of the corrector. For the second term, we apply the zero order estimation and get

$$\Big|\mathbb{E}[X_I^*]\Big(\mathbb{E}[f_h''(W)] - \Phi_{\sigma_W}(f_h'')\Big)\Big| \leq \frac{\|f_h^{(3)}\|}{4\sigma_W^4}\Big|\sum_{i=1}^n \mathbb{E}[X_i^3]\Big| \sum_{i=1}^n \mathbb{E}\big[|X_i^s|^3\big]. \tag{26}$$

For the second term $\mathbb{E}[f_h''(W)X_I]$ of (25), we use a technique of conditional expectation by observing that $\mathbb{E}[f_h''(W)X_I] = \mathbb{E}\big[f_h''(W)\mathbb{E}[X_I|\bar{X}, \bar{X}^*]\big]$.

Then by the Cauchy-Schwartz inequality,

$$\Big|\mathbb{E}[f_h''(W)X_I]\Big| = \Big|\mathrm{cov}\big[f_h''(W), \mathbb{E}[X_I|\bar{X}, \bar{X}^*]\big]\Big| \leq \frac{1}{\sigma_W^2}\sqrt{\mathrm{Var}[f_h''(W)]}\sqrt{\sum_{i=1}^n \sigma_i^6}.$$

Notice that $\mathrm{Var}[f_h''(W)] = \mathrm{Var}[f_h''(W) - f_h''(0)] \leq \mathbb{E}[(f_h''(W) - f_h''(0))^2] \leq \|f_h^{(3)}\|^2\sigma_W^2$. So

$$\Big|\mathbb{E}[f_h''(W)X_I]\Big| \leq \frac{\|f_h^{(3)}\|}{\sigma_W}\sqrt{\sum_{i=1}^n \sigma_i^6}\,. \tag{27}$$

Finally, it suffices to write

$$\begin{aligned}\mathbb{E}[h(w)] - \Phi_{\sigma_W}(h) = \sigma_W^2\Big(&\mathbb{E}[X_I^*]\Phi_{\sigma_W}(f_h'') + \mathbb{E}[X_I^*]\big[\mathbb{E}[f_h''(W)] - \Phi_{\sigma_W}(f_h'')\big] - \mathbb{E}[f_h''(W)X_I]\Big)\\ &+ \sigma_W^2\mathbb{E}\Big[f_h^{(3)}\big(\xi W + (1-\xi)W^*\big)\xi(W^* - W)^2\Big].\end{aligned} \tag{28}$$

Combining (24), (26) and (27), we let the corrector to be $C_h = \sigma_W^2\mathbb{E}[X_I^*]\Phi_{\sigma_W}(f_h'')$ and we deduce the error bound $\alpha(h, X_1, \cdots, X_n)$ as in (22).

The last step is to use the invariant property of the normal distribution under zero bias transformation and the Stein's equation to obtain

$$\sigma_W^2\Phi_{\sigma_W}(f_h'') = \Phi_{\sigma_W}(xf_h') = \frac{1}{\sigma_W^2}\Phi_{\sigma_W}\Big(\big((\frac{x^2}{3\sigma_W^2} - 1)\big)xh(x)\Big).$$

□

6.2.2 The Poisson case: Theorem 2.2

Proof. Let us first recall the discrete Taylor formula. For any integers x and any positive integer $k \geq 1$,

$$g(x+k) = g(x) + k\Delta g(x) + \sum_{j=0}^{k-1}(k-1-j)\Delta^2 g(x+j).$$

Similar as in the Gaussian case, we apply the above formula to right-hand side of $\mathbb{E}[h(V)] - \mathcal{P}_{\lambda_V}(h) = \lambda_V\mathbb{E}[g_h(V^*+1) - g_h(V+1)]$ and we shall make decompositions. Since $V^* - V$ is not necessarily positive, we take expansion

around $V^{(i)}$ for the following three terms respectively and obtain

$$\begin{aligned}
&\mathbb{E}\big[g_h(V^*+1)-g_h(V+1)-\Delta g_h(V+1)(V^*-V)\big]\\
=&\sum_{i=1}^n \frac{\lambda_i}{\lambda_V}\Big(\mathbb{E}\big[g_h(V^{(i)}+1)+Y_i^*\Delta g_h(V^{(i)}+1)+\sum_{j=0}^{Y_i^*-1}(Y_i^*-1-j)\Delta^2 g_h(V^{(i)}+1+j)\big]\\
&\qquad-\mathbb{E}\big[g_h(V^{(i)}+1)+Y_i\Delta g_h(V^{(i)}+1)+\sum_{j=0}^{Y_i-1}(Y_i-1-j)\Delta^2 g_h(V^{(i)}+1+j)\big]\\
&\qquad-\mathbb{E}\big[\Delta g_h(V^{(i)}+1)(Y_i^*-Y_i)+\sum_{j=0}^{Y_i-1}(Y_i^*-Y_i)\Delta^2 g_h(V^{(i)}+1+j)\big]\Big)
\end{aligned}$$

which implies that the remaining term is bounded by

$$\begin{aligned}
&\Big|\mathbb{E}\big[g_h(V^*+1)-g_h(V+1)-\Delta g_h(V+1)(V^*-V)\big]\Big|\\
\le&\|\Delta^2 g_h\|\sum_{i=1}^n\frac{\lambda_i}{\lambda_V}\left(\mathbb{E}\Big[\binom{Y_i^*}{2}\Big]+\binom{Y_i}{2}\Big]+\mathbb{E}\big[|Y_i(Y_i^*-Y_i)|\big]\right).
\end{aligned}$$

We then make decomposition

$$\begin{aligned}
\mathbb{E}\big[\Delta g_h(V+1)(V^*-V)\big]=&\mathcal{P}_{\lambda_V}(\Delta g_h(x+1))\mathbb{E}[Y_I^*-Y_I]+\mathrm{cov}\big(Y_I^*-Y_I,\Delta g_h(V+1)\big)\\
&+\big(\mathbb{E}[\Delta g_h(V+1)]-\mathcal{P}_{\lambda_V}(\Delta g_h(x+1))\big)\,\mathbb{E}[Y_I^*-Y_I].
\end{aligned}\tag{29}$$

Similar as in the Gaussian case, the first term of (29) is the candidate of the corrector. For the second term, we use again the technique of conditional expectation and obtain

$$\mathrm{cov}\big[\Delta g_h(V+1),Y_I^*-Y_I\big]\le\frac{1}{\lambda_V}\mathrm{Var}\big[\Delta g_h(V+1)\big]^{\frac12}\Big(\sum_{i=1}^n\lambda_i^2\mathrm{Var}[Y_i^*-Y_i]\Big)^{\frac12}.$$

For the last term of (29), we have by the zero order estimation

$$\big(\mathbb{E}[\Delta g_h(V+1)]-\mathcal{P}_{\lambda_V}(\Delta g_h(x+1))\big)\mathbb{E}[Y_I^*-Y_I]\le\frac{6\|\Delta g_h\|}{\lambda_V}\Big(\sum_{i=1}^n\mathbb{E}[Y_i|Y_i^s-1|]\Big)^2.$$

It remains to observe that $\mathcal{P}_{\lambda_V}(\Delta g_h(x+1))=\frac12\mathcal{P}_{\lambda_V}(\Delta^2 h)$ and let the corrector to be

$$C_h^{\mathcal{P}}=\frac{\lambda_V}{2}\mathcal{P}_{\lambda_V}(\Delta^2 h)\mathbb{E}[Y_I^*-Y_I].$$

Combining all these terms, we obtain the error bound $\beta(h,Y_1,\cdots Y_n)$ as

below

$$\big|\mathbb{E}[h(V)] - \mathcal{P}_{\lambda_V}(h) - C_h^{\mathcal{P}}\big| \leq \|\Delta^2 g_h\| \sum_{i=1}^{n} \lambda_i \mathbb{E}\Big[|Y_i^* - Y_i|\big(|Y_i^* - Y_i| - 1\big)\Big]$$
$$+ \mathrm{Var}\big[\Delta g_h(V+1)\big]^{\frac{1}{2}} \Big(\sum_{i=1}^{n} \lambda_i^2 \mathrm{Var}[Y_i^* - Y_i]\Big)^{\frac{1}{2}} + 6\|\Delta g_h\| \Big(\sum_{i=1}^{n} \mathbb{E}[Y_i | Y_i^s - 1|]\Big)^2 \tag{30}$$

where $\|\Delta g_h\| \leq 6\|h\|$ and $\|\Delta^2 g_h\| \leq 2\|\Delta g_h\|$.

□

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