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# Model of chatter vibrations and stability analysis of a non-linear wiper system

S. Berger<sup>1</sup>, P. Ragot<sup>1</sup>, JJ. Sinou<sup>2</sup>, E. Aubry<sup>1</sup>

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**Abstract** – *This paper presents a two dimensional phenomenological non-linear model of a windscreen wiper system. This non-linear model describes the instabilities arising from the frictional contact between the glass and the rubber. The objective of this paper is to illustrate the fact that a very simple non-linear phenomenological system may be used to model and simulate chatter vibrations for wiper systems stability analysis.*

*First, the complete non-linear phenomenological system which represents the non-linear wiper system is described. Then, the stability analysis of the system is performed by considering the eigenvalues of the linearized system around the non-linear equilibrium points. Finally, the non-linear behaviour of the mechanical system which corresponds to the “limit cycles” is investigated using a Runge-Kutta method.*

**Keywords:** *friction, limit cycles, mechanical system, stability analysis, vibration*

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## I. Introduction

A great number of studies have been carried out on the dynamic behaviour of dry friction systems, such as braking systems or windscreen-wipers from [1]-[13].

It is classically admitted that one of the most important phases in studying windscreen-wipers systems is the determination of the friction mechanism. According to Ibrahim [7], Oden and Martins [8], and Sinou et al. [10], [11], there are four general mechanisms for friction-induced system instability: stick-slip, variable dynamic friction coefficient, sprag-slip [13] and geometric coupling of degrees of freedom. The first two approaches rely on changes in the friction coefficient with the relative sliding speed to affect the system stability [9]. The latter two approaches utilize kinematic constraints and modal coupling to develop instability when the friction coefficient is constant [5], [11]. The sprag-slip action was described by Spurr [13] and does not depend on a friction coefficient varying with the relative rotation speed of the brake disc. Next, a number of investigations have been carried out by considering kinematic constraint or geometric instability which can be seen as an extension of the sprag-slip model. This mechanism involves the coupling of the different degrees of freedom and has been studied by numerous researchers [2], [5], [10], [11]. Their investigations demonstrated that instability may occur even if the friction coefficient is constant. In their studies, the authors consider the latter two approaches which use a modal coupling mechanism involving two system modes which couple together due to the friction interface, so as to develop the instability of the wiper

system with a constant friction coefficient. As previously explained, this instability is defined as a geometric coupling [11] where two system modes move closer in frequency as the friction coefficient varies.

It is well known that the dynamic behaviour of the wiper system is highly sensitive to the design parameters and the physical parameters such as the friction coefficient, the value of the applied load. So, due to the presence of flutter instability, the performance of such mechanical wiper systems may be seriously decreased. Thus, as friction-induced vibration is the main characteristic of the non-linear dynamics of wiper systems, the objective is now to study the stability analysis and the non-linear dynamics of a non-linear wiper system when it is subjected to self-induced oscillations.

This paper first presents the development of a non-linear phenomenological system which represents the non-linear dynamic of wiper systems. Secondly, the stability analysis of the non-linear system is performed and the effects of various physical parameters on the stability are studied. Finally, the last section illustrates the possibility to obtain the limit cycles of the non-linear wiper system when the equilibrium points of the non-linear system are unstable.

## II. Nomenclature

$\theta$  : angle between stem  $T_1$  and vector  $\vec{x}$  (rad)

$\phi$  : angle between stem  $T_2$  and vector  $\vec{x}$  (rad)

$v$  : displacement according to the vector  $\vec{y}$  of the solid S (m)

$u$  : displacement according to the vector  $\vec{x}$  of the

- solid S (m)
- $\alpha$ : leading angle (rad)
- $\mu$ : friction coefficient for the contact at point C between stem  $T_2$  and the surface ( $\pi$ )
- $I_1$ : Axial mass inertia moment of stem  $T_1$  for the axis  $\vec{Z}$  through point  $G_1$  ( $\text{kg.m}^2$ )
- $I_2$ : Axial mass inertia moment of stem  $T_2$  for the axis  $\vec{Z}$  through point  $G_2$  ( $\text{kg.m}^2$ )
- $k_1$ : torsion stiffness between the solid S and stem  $T_1$  ( $\text{N.m / rad}$ )
- $K_1$ : linear stiffness between the solid S and the rigid framework ( $\text{N / m}$ )
- $k_2$ : torsion stiffness between the solid S and stem  $T_2$  ( $\text{N.m / rad}$ )
- $F_0$ : load (N)
- $L_1$ : length of stem  $T_1$  (m)
- $L_2$ : length of stem  $T_2$  (m)
- $L_{G1}$ : distance  $AG_1$  with  $G_1$  centre of inertia  $T_1$  (m)
- $L_{G2}$ : distance  $BG_2$  avec  $G_2$  centre of inertia  $T_2$  (m)
- $m$ : mass of the solid S (kg)
- $m_1$ : mass of stem  $T_1$  (kg)
- $m_2$ : mass of stem  $T_2$  (kg)

### III. Description of the non linear system

The non-linear mechanical system considered here has been defined starting from studies [6], [15] on a windscreen wiper blade. Fig 1(a) shows the physical system under study. Based on previous work considering a finite element model and experimental tests [6], a two-dimensional phenomenological system is developed to model instability at the contact frictional surface between the glass and the rubber. The observation of chatter vibrations with a high-speed camera shows that the rubber blade vibrates in a bending mode mainly [6].

The non-linear model is illustrated in Fig. 1(b). The rigid solid S represents the blade holder. The two rigid stems  $T_1$  and  $T_2$  whose respective lengths are  $L_1$  and  $L_2$  and masses  $m_1$  and  $m_2$  describe the rubber blade. The top of the rubber blade is clamped in the blade holder whereas the end of the rubber blade is in contact with the glass. It may be noted that the two rigid stems  $T_1$  and  $T_2$  allow the considerations of the first two bending modes of the rubber blade.

A uniform load  $F_0$  which represents the uniform pressure under the rubber blade for a classic wiper blade is applied on the top of the rigid solid S. As indicated in Fig. 1(a) and (b), the previous load and the wiping motion introduce a leading angle between the rubber blade and the glass surface. The resulting contact (see point C in Fig. 1(b)) generates a frictional blade-glass interaction. As explained by a lot of researchers [2], [3], [7], [11], [13], [15], the friction contact between a pad (i.e. the rubber blade in this study) and a moving

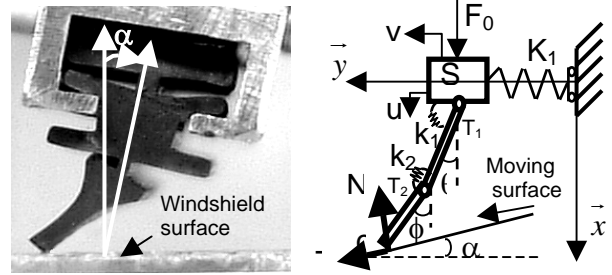
surface (i.e. the glass) may generate unstable motion (flutter instabilities) due to mode coupling depending on the contact angle between the two elements.

The three elements of the rubber blade (i.e. the rigid solid S and the two rigid stems  $T_1$  and  $T_2$ ) are connected to each other by perfect pivot links coupled with a torsion stiffness  $k_1$  (between the solid S and stem  $T_1$ ), and with a torsion stiffness  $k_2$  (between stem  $T_1$  and stem  $T_2$ ). The solid S is linked to the rigid framework by a return spring of stiffness  $K_1$ , as shown in Fig. 1. As previously explained, stem  $T_2$  rubs on the moving surface of the glass. It is assumed that the contact at point C is permanent and it is modelled using Coulomb's law :

$$T = \mu N \quad (1)$$

where  $T$  is the tangential component and  $N$  the normal component of the friction force.

The non-linear wiper system has three degrees of freedom: the generalized coordinates chosen are the displacement  $u$  according to the vector  $\vec{X}$  of the solid S, the angle  $\theta$  between stem  $T_1$  and vector  $\vec{X}$ , and the angle  $\phi$  between stem  $T_2$  and vector  $\vec{X}$ , respectively.



(a) Physical system

(b) Non linear model

Fig. 1: Windscreen wiper blade

The principle of virtual work helps to obtain the equations of motion for the non-linear system:

$$\begin{aligned}
 & - (m + m_1 + m_2) \ddot{u} + \left( \frac{m_1}{2} + m_2 \right) (L_1 \sin\theta) \ddot{\theta} \\
 & + \left( \frac{m_1}{2} + m_2 \right) (L_1 \cos\theta) \dot{\theta}^2 + \left( \frac{m_2 L_2}{2} \sin\phi \right) \ddot{\phi} \\
 & + \left( \frac{m_2 L_2}{2} \cos\phi \right) \dot{\phi}^2 - N' + F_0 = 0
 \end{aligned} \quad (2)$$

$$\begin{aligned}
& - (m + m_1 + m_2) \ddot{v} - \left(\frac{m_1}{2} + m_2\right) (L_1 \cos\theta) \ddot{\theta} \\
& + \left(\frac{m_1}{2} + m_2\right) (L_1 \sin\theta) \dot{\theta}^2 - \left(\frac{m_2 L_2}{2} \cos\phi\right) \ddot{\phi} \quad (3) \\
& + \left(\frac{m_2 L_2}{2} \sin\phi\right) \dot{\phi}^2 + f N' - K_1 v = 0
\end{aligned}$$

$$\begin{aligned}
& - \left(I_1 + m_1 L_{g_1}^2 + m_2 L_1^2\right) \ddot{\theta} + \left(\frac{m_1}{2} + m_2\right) (L_1 \sin\theta) \ddot{u} \\
& - \left(\frac{m_1}{2} + m_2\right) (L_1 \cos\theta) \ddot{v} - \left(\frac{m_2 L_1 L_2}{2}\right) \cos(\theta - \phi) \ddot{\phi} \quad (4) \\
& - \left(\frac{m_2 L_1 L_2}{2}\right) \sin(\theta - \phi) \dot{\phi}^2 + N' L_1 \sin\theta = 0
\end{aligned}$$

$$\begin{aligned}
& - \left(I_2 + m_2 L_{g_2}^2\right) \ddot{\phi} + \left(\frac{m_2 L_2}{2}\right) (\sin\phi) \ddot{u} \\
& - \left(\frac{m_2 L_2}{2}\right) (\cos\phi) \ddot{v} - \left(\frac{m_2 L_1 L_2}{2}\right) \cos(\theta - \phi) \ddot{\theta} \quad (5) \\
& + \left(\frac{m_2 L_1 L_2}{2}\right) \sin(\theta - \phi) \dot{\theta}^2 + N' L_2 \sin\phi \\
& + f N' L_2 \cos\phi + k_2 (\theta - \phi) = 0
\end{aligned}$$

Coefficient  $f$  is the apparent coefficient friction [3]. It depends both on the friction coefficient  $\mu$  at point C and on the leading angle  $\alpha$ , as shown in the following formula:

$$f = \frac{T'}{N'} = \frac{\mu \cos \alpha - \sin \alpha}{\cos \alpha + \mu \sin \alpha} \quad (6)$$

$T'$  and  $N'$  are the projections of the friction force onto  $\bar{x}$  and  $\bar{y}$  respectively (Fig. 1).

As it is assumed that the contact at point C is permanent, the linked equation is written:

$$\begin{aligned}
& \tan\alpha (v + L_1 \sin\theta + L_2 \sin\phi) \\
& = -u - L_1 \cos\theta - L_2 \cos\phi + L_1 + L_2 \quad (7)
\end{aligned}$$

Considering the linked equation, the equations of motion of the non-linear system may be reduced to the form:

$$\begin{aligned}
& \mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} + \mathbf{B}(\mathbf{X}) \dot{\mathbf{X}} + \mathbf{FNL}(\mathbf{X}, \dot{\mathbf{X}}) = \mathbf{0} \quad (8) \\
& \text{with } \mathbf{X} = [v \quad \theta \quad \phi]^T
\end{aligned}$$

$\ddot{\mathbf{X}}$ ,  $\dot{\mathbf{X}}$  and  $\mathbf{X}$  are the acceleration, velocity, and displacement response 3-dimensional vectors of the degrees of freedom, respectively.  $\mathbf{M}$  and  $\mathbf{K}$  are constant matrices. Matrix  $\mathbf{B}(\mathbf{X})$  and vector  $\mathbf{FNL}(\mathbf{X}, \dot{\mathbf{X}})$  contain the non-linear expressions (cf. appendix).

## IV. Stability analysis

The first step in the solution procedure is to obtain the steady state operating point for the full set of the non-linear system by the determination of the equilibrium solutions. These equilibrium solutions  $\mathbf{X}_0 = (v_0, \theta_0, \phi_0)^T$  (i.e.  $\ddot{\mathbf{X}} = \dot{\mathbf{X}} = \mathbf{0}$ ) are obtained by solving the non-linear static equations for a given load  $F_0$ . These equilibrium points satisfy the following conditions [14]:

$$\mathbf{K}\mathbf{X}_0 + \mathbf{FNL}(\mathbf{X}_0, \mathbf{0}) = \mathbf{0} \quad (9)$$

This system of equations is solved using an algorithm of the Newton-Raphson type. The difficulty is not to find the solutions, but to be sure to find all the solutions. This study not only proposes to find the first equilibrium point of the non-linear wiper system, but also all the solutions of the non-linear static equation (9). This helps to perform a complete stability analysis of the non-linear wiper system by considering the stability of each equilibrium point of the system. For this, the field of study is discretized in a very fine way and the initialization value of the Newton-Raphson series describes all the discretization points. The resolution of the system allows the determination of the stationary equilibrium positions  $\mathbf{X}_0$  of the non-linear system.

Then, the stability of the non-linear system is investigated in the linearized equations, assuming small perturbations  $\bar{\mathbf{X}}$  about the equilibrium point  $\mathbf{X}_0$  of the non-linear system:

$$\mathbf{X} = \mathbf{X}_0 + \bar{\mathbf{X}} \quad (10)$$

Replacing Equation (10) into Equation (11) leads to:

$$\mathbf{M}\ddot{\bar{\mathbf{X}}} + \mathbf{K}(\bar{\mathbf{X}} + \mathbf{X}_0) \dot{\bar{\mathbf{X}}} + \mathbf{B}(\bar{\mathbf{X}} + \mathbf{X}_0) \bar{\mathbf{X}} + \mathbf{FNL}(\bar{\mathbf{X}} + \mathbf{X}_0, \dot{\bar{\mathbf{X}}}) = \mathbf{0}$$

Supposing that  $\mathbf{B}$  and  $\mathbf{FNL}$  belong to the C1 class, developing the non-linear expressions as Taylor series and keeping the first order leads to:

$$\mathbf{B}(\bar{\mathbf{X}} + \mathbf{X}_0) \bar{\mathbf{X}} \approx \mathbf{M}_{\text{NL}}|_{\mathbf{X}_0} \ddot{\bar{\mathbf{X}}} \quad (11)$$

$$\mathbf{FNL}(\bar{\mathbf{X}} + \mathbf{X}_0) \approx \mathbf{FNL}(\bar{\mathbf{X}} + \mathbf{X}_0) + \mathbf{K}_{\text{NL}}|_{\mathbf{X}_0} \bar{\mathbf{X}} \quad (12)$$

Substituting the two linearized expression of the non-linear terms  $\mathbf{B}$  and  $\mathbf{FNL}$  in the complete non-linear expression (6) and by using the non-linear static equation (9), the linearized system is given by

$$(\mathbf{M} + \mathbf{M}_{\text{NL}}) \ddot{\bar{\mathbf{X}}} + (\mathbf{K} + \mathbf{K}_{\text{NL}}) \bar{\mathbf{X}} = \mathbf{0} \quad (13)$$

where  $\mathbf{M}_{\text{NL}}$  and  $\mathbf{K}_{\text{NL}}$  define the linearized expressions of  $\mathbf{B}(\mathbf{X})$  and  $\mathbf{FNL}(\mathbf{X}, \dot{\mathbf{X}})$  at the stationary state configuration, respectively.

The study of the poles  $\lambda$  of this system of equations (13) helps to determine the stability or instability of the system, as well as the type of instability (flutter, divergence). So, if at least one of the real parts of the poles is positive, the system is unstable; otherwise, it is stable. The imaginary parts of the poles give the

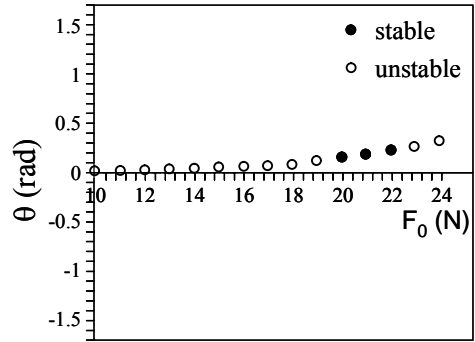
pulsation of the modes considered. The poles  $\lambda$  of the system are the six roots of the polynomial:

$$P(\lambda) = \det(\lambda^2(\mathbf{M} + \mathbf{M}_{NL}) + (\mathbf{K} + \mathbf{K}_{NL})) \quad (14)$$

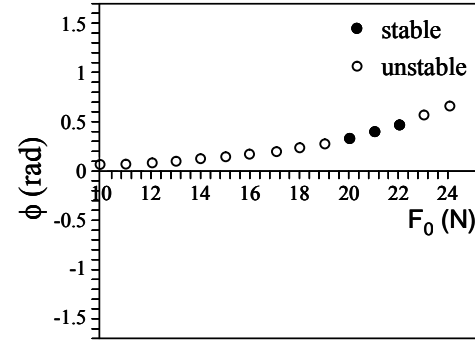
It is important to note that initially damping is neglected in the model for the sake of simplicity. However, it is known that damping can have a significant influence on the stability of mechanical systems: if damping is equally distributed on the two modes involved in the mode coupling phenomenon, added damping stabilizes the mechanical system; this phenomenon is called the ‘lowering’ effect. If damping is spread non-equally over the two modes, added damping may act in an unintuitive way by destabilizing the system; this phenomenon is referred as the ‘smoothing effect’ and destabilizing paradox.. For more details, we refer the interested reader to the following paper [12].

To illustrate the previous developments, two parametric stability studies are carried out on the present non-linear wiper system.

The first parametric study shows the influence of force  $F_0$ . Fig. 2 and 3 show respectively the evolution of the angles  $\theta$  and  $\phi$  and the values of the system’s poles according to  $F_0$  for  $\mu=0.528$ . There are six two-by-two conjugate roots; so, for the sake of simplicity, only three are presented. For a slight variation of the load  $F_0$ , 3 zones appear, corresponding each to a state: steady equilibrium, unstable equilibrium with flutter and unstable equilibrium with divergence. For  $23 \leq F_0 \leq 24$ , the equilibrium position is unstable because of the presence of at least one positive real part. Moreover, as the imaginary parts of two poles are equal (they are also combined), there are only two modes instead of three. It is a coalescence of two modes. This instability is of the flutter type and corresponds to an instability caused by the coupling of two modes. These results are in perfect agreement with experimental tests carried out on the wiper system [6], [15]. They clearly show that unstable motions and chatter instabilities may be described by considering a highly non-linear phenomenological system. For  $20 \leq F_0 \leq 22$ , the equilibrium positions are stable because the real parts resulting from the three modes are zero. The two modes which were confused, separate. For  $10 \leq F_0 \leq 19$ , the system is in the position of divergent unstable equilibrium because one of the poles is a strictly positive reality.

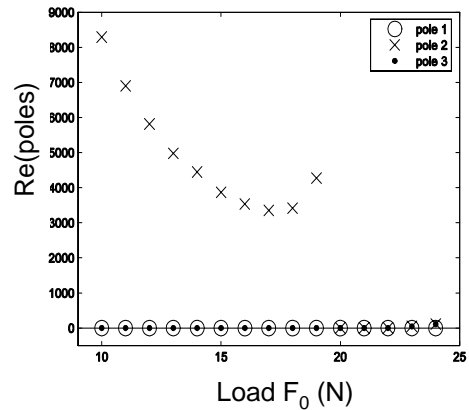


(a) Angle  $\theta$

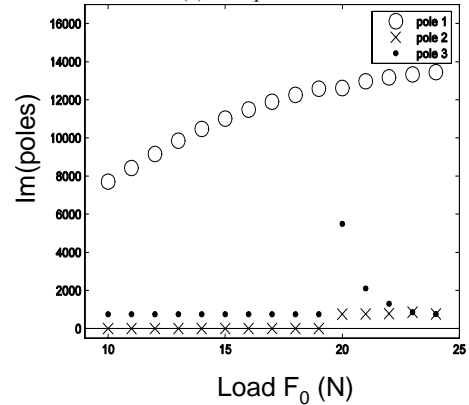


(b) Angle  $\phi$

Fig. 2 : Equilibrium and stability of the system according to  $F_0$  for  $\mu=0.528$  (unstable=white dot, stable=black dot)



(a) Real parts



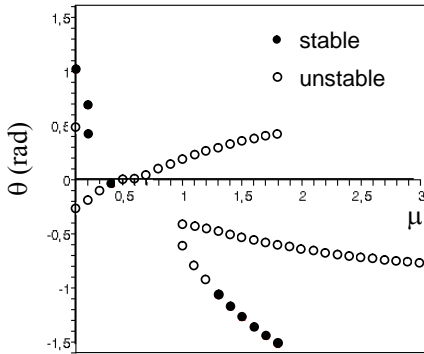
(b) Imaginary parts

Fig. 3 : Real and imaginary parts of the poles according to  $F_0$  for  $\mu=0.528$

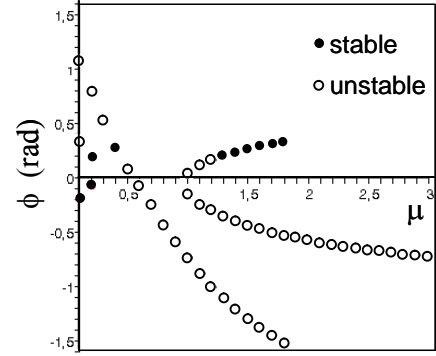
The second study shows the influence of friction coefficient  $\mu$  for various values of the angular stiffness  $k_2$ . Fig. 4 to 6 describe the evolution of the equilibrium positions  $\phi$  and  $\theta$  with their stability, according to the friction coefficient for various angular stiffnesses  $k_2$ .

These various graphs highlight the system's highly non-linear behaviour and the high sensitivity to the parameters. For the same set of parameters, the system may be in several different geometrical configurations (one to three positions of equilibrium). The system's behaviour may present discontinuity zones, even with a continuous evolution of the various parameters. The direction of variation of  $\mu$  may also influence the system's behaviour (Fig. 4). Moreover, the stability of these points of equilibrium can change rather rapidly and several times with a slight variation of one parameter (Fig. 5).

Fig. 7 shows the evolution of the values of the system's poles according to friction coefficient  $\mu$  for  $k_2=0.6$  N.m / rad. Only two poles are plotted, as the values of pole 3 are too high to be represented on the same graph. For a slight variation of the friction coefficient [0.8 , 1.1], 3 zones appear, corresponding each again to a state. For  $\mu=0.8$ , the equilibrium position is unstable. It is a coalescence of two modes which corresponds to a flutter instability. For  $0.85 < \mu < 0.95$ , the equilibrium positions are stable, the two modes which merged now separate. For  $\mu > 1$ , the positions are unstable again, the system is divergent.

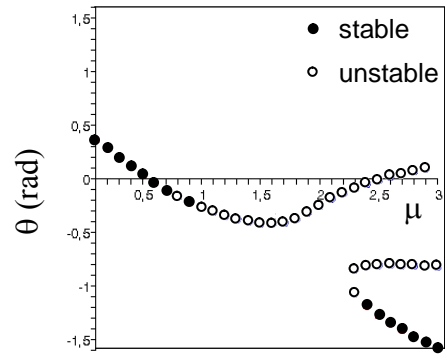


(a) Angle  $\theta$

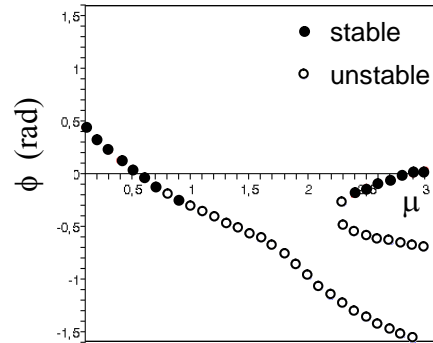


(b) Angle  $\phi$

Fig. 4 : Equilibrium and stability of the system according to  $\mu$  for  $k_2=0.5$  N.m / rad (unstable= white dot, stable= black dot)



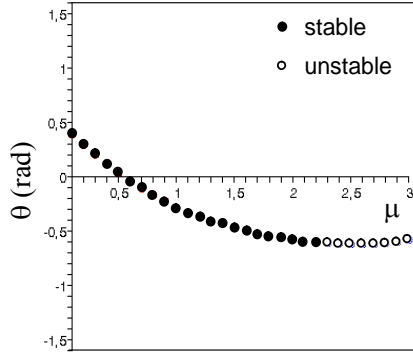
(a) Angle  $\theta$



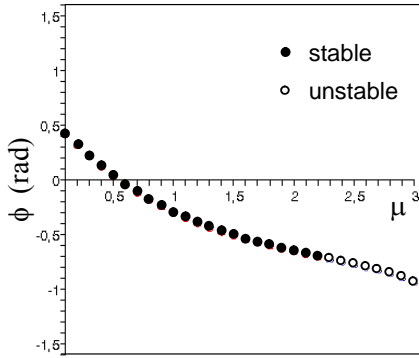
(b) Angle  $\phi$

Fig. 5 : Equilibrium and stability of the system according to  $\mu$  for  $k_2=0.6$  N.m / rad

(unstable= white dot, stable= black dot)

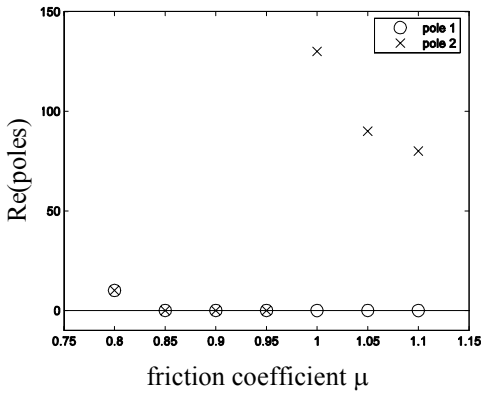


(a) Angle  $\theta$

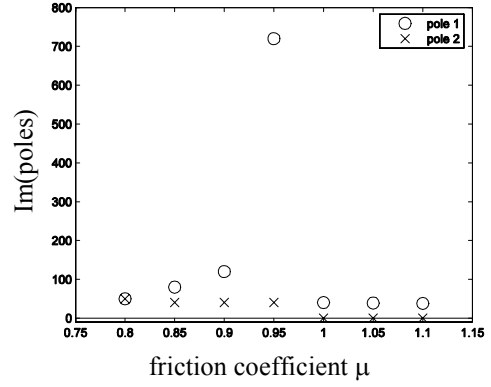


(b) Angle  $\phi$

Fig. 6 : Equilibrium and stability of the system according to  $\mu$  for  $k_2=0.7 \text{ N.m / rad}$  (unstable= white dot, stable= black dot)



(a) Real parts



(b) Imaginary parts

Fig. 7 : Real and imaginary parts of the poles according to  $\mu$  for  $k_2=0.6 \text{ N.m / rad}$

## V. Non linear behaviour –Limit cycles

If the static point is unstable (i.e. flutter unstable), the system oscillates around its position of equilibrium with amplitudes which depend on the value of the parameters, in particular of the friction coefficient. In fact, these oscillations are self-sustained oscillations which decrease the performance of the system and can degrade it through fatigue. Thus, it is primordial to quantify the level of the system's vibratory amplitude. To do so, the amplitudes of the limit cycles of the non linear system are studied.

To determine the limit cycles, it is necessary to integrate the system (8) of differential equations of the second order which describe the system's behaviour. It must be turned into the system of the first order below:

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}) \quad (13)$$

where

$$\mathbf{F}(\mathbf{z}) = \mathbf{M}_{\text{CL}}(\mathbf{z})^{-1} \mathbf{A} \mathbf{z} + \mathbf{M}_{\text{CL}}(\mathbf{z})^{-1} \mathbf{T}(\mathbf{z})$$

$$\mathbf{z}(t) = (\dot{\mathbf{X}}(t), \mathbf{X}(t))^T$$

$$\mathbf{M}_{\text{CL}}(\mathbf{z}) = \begin{bmatrix} \mathbf{M} + \mathbf{B}(\mathbf{z}) & \mathbf{0} \\ \mathbf{0} & \text{Id} \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ -\text{Id} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{T}(\mathbf{z}) = \begin{pmatrix} -\mathbf{FNL}(\mathbf{z}) \\ \mathbf{0} \end{pmatrix}$$

The temporal resolution can be carried out starting from a discretization of the equations with formulas of the Runge-Kutta type. It consists in determining  $\mathbf{z}(t)$ , starting from an initial state at a given time interval  $[t_0, t_f]$ .

To consider one case, the static and stability studies showed the existence of a flutter unstable equilibrium. The wiper system was initially positioned near its equilibrium position without velocity. The temporal resolution was carried out from 0s to 20s. The system evolved around its equilibrium position with constant

amplitude. Fig. 8 to 10 present the limit cycles of the variable states. It must be noticed that if the system is initially positioned in an area around its equilibrium, the system oscillates with the same limit cycle. On the contrary, the amplitude of the limit cycles depends on the value of the friction coefficient.

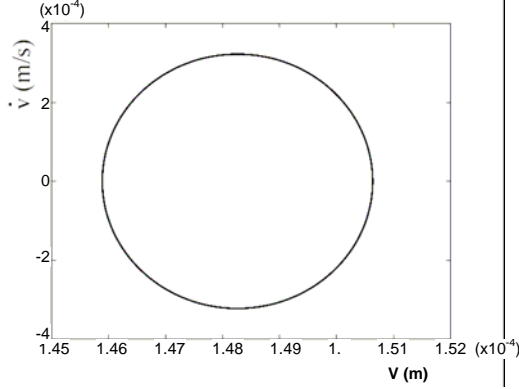


Fig. 8: Limit cycle of  $v$

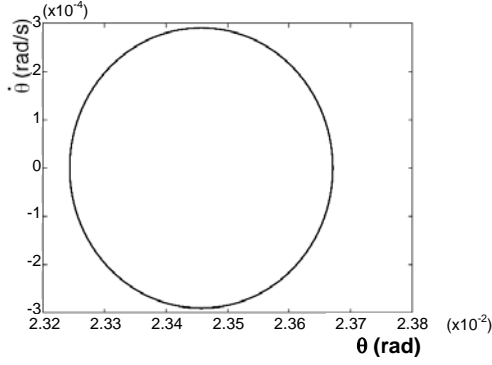


Fig. 9: Limit cycle of  $\theta$

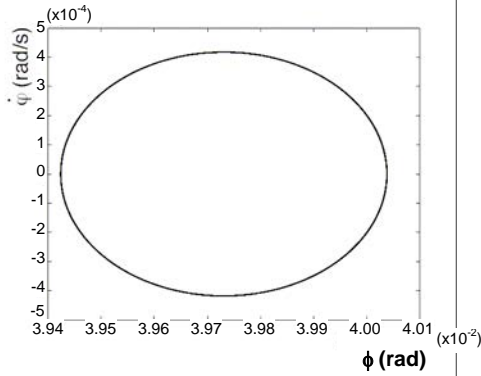


Fig. 10: Limit cycle of  $\phi$

## VI. Conclusion

This paper has presented the stability analysis and dynamics of a non-linear windscreen wiper blade. A

non-linear phenomenological system has been developed in order to analyze chatter vibrations. It has been shown that unstable motions may be described by considering only the mode coupling of the rubber blade, the contact friction and the leading angle between the rubber blade and the surface glass during wiping motion. The stability of all the equilibrium points of the non-linear system has been determined. Parametric studies have shown the system's highly non linear aspect and high sensitivity to the parameters, in terms of equilibrium and stability. Finally, the non-linear behaviour and the limit cycles of this non-linear mechanical system when the equilibrium points are unstable have been obtained using a Runge-Kutta method.

## Appendix

The governing nonlinear system equation is

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} + \mathbf{B}(\mathbf{X})\dot{\mathbf{X}} + \mathbf{FNL}(\mathbf{X}, \dot{\mathbf{X}}) = \mathbf{0}$$

$$\text{with } \mathbf{X} = [v \quad \theta \quad \phi]^T$$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_3 & 0 \\ 0 & 0 & b_7 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & -(k_1 + k_2) & k_2 \\ 0 & k_2 & -k_2 \end{bmatrix}$$

$$\mathbf{B}(\mathbf{X}) = \begin{bmatrix} B_{11}(\theta, \phi) & B_{12}(\theta, \phi) & B_{13}(\theta, \phi) \\ B_{21}(\theta, \phi) & B_{22}(\theta, \phi) & B_{23}(\theta, \phi) \\ B_{31}(\theta, \phi) & B_{32}(\theta, \phi) & B_{33}(\theta, \phi) \end{bmatrix}$$

with

$$B_{11}(\theta, \phi) = 1, \quad B_{12}(\theta, \phi) = p_2 \sin(\theta + \gamma_1),$$

$$B_{13}(\theta, \phi) = p_3 \sin(\phi + \gamma_2)$$

$$B_{21}(\theta, \phi) = q_1 \cos(\theta - \gamma_3), \quad B_{22}(\theta, \phi) = q_3 \cos(2\theta - \gamma_5) + b_3$$

$$B_{23}(\theta, \phi) = q_7 \sin(\phi - \alpha) \sin(\theta + \gamma_7) - q_8 \cos\theta \cos(\gamma_5 + \gamma_4)$$

$$B_{31}(\theta, \phi) = r_1 \cos(\phi - \gamma_9),$$

$$B_{32}(\theta, \phi) = r_3 \sin(\phi + \gamma_4) \sin(\theta + \gamma_{10}) - r_4 \cos\theta \cos(\phi - \alpha)$$

$$B_{33}(\theta, \phi) = r_5 \cos(2\phi - \gamma_{11}) + b_7$$

$$\mathbf{FNL}(\mathbf{X}, \dot{\mathbf{X}}) = \begin{bmatrix} g_1(v, \theta, \phi, \dot{v}, \dot{\theta}, \dot{\phi}) \\ g_2(v, \theta, \phi, \dot{v}, \dot{\theta}, \dot{\phi}) \\ g_3(v, \theta, \phi, \dot{v}, \dot{\theta}, \dot{\phi}) \end{bmatrix}$$

with

$$g_1(v, \theta, \phi, \dot{v}, \dot{\theta}, \dot{\phi}) = -p_6 F_0 - \dot{\theta}^2 p_2 \cos(\theta + \gamma_1)$$

$$- \dot{\phi}^2 p_3 \cos(\phi + \gamma_2) - p_4 (L_1 \cos(\theta - \alpha)$$

$$+ L_2 \cos(\phi - \alpha)) - p_5$$



$$g_2(v, \theta, \phi, \dot{v}, \dot{\theta}, \dot{\phi}) = -q_6 F_0 \sin(\theta + \gamma_4) - \dot{\theta}^2 (q_4 \cos(2\theta - \gamma_6) + b_4) - \dot{\phi}^2 (q_9 \cos(\phi - \alpha) \sin(\theta + \gamma_8) + q_8 \cos \theta \sin(\phi + \gamma_4)) - q_5 \sin(\theta + \gamma_4)(L_1 \cos(\theta - \alpha) + L_2 \cos(\phi - \alpha) + q_{10})$$

$$g_3(v, \theta, \phi, \dot{v}, \dot{\theta}, \dot{\phi}) = -r_7 F_0 \sin(\phi + \gamma_4) - \dot{\theta}^2 (r_3 \sin(\phi + \gamma_4) \cos(\theta + \gamma_{10}) + r_4 \sin \theta \cos(\phi - \alpha)) - \dot{\phi}^2 (-r_5 \sin(2\phi - \gamma_{11}) + b_8) - r_6 \sin(\phi + \gamma_4)(L_1 \cos(\theta - \alpha) + L_2 \cos(\phi - \alpha) + r_8)$$

$b_i, p_i, q_i, r_i, \gamma_i$  are coefficients which depend on the system's parameters.

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