Extension of the Center Manifold Approach, Using Rational Fractional Approximants, Applied to Non-Linear Stability Analysis

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EXTENSION OF THE CENTER MANIFOLD APPROACH BY USING THE RATIONAL FRACTIONAL APPROXIMANTS, APPLIED TO NON-LINEAR STABILITY ANALYSIS.

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1 ABSTRACT

A methodology is presented which extends the domain of validity of non-linear systems reduced by using the center manifold approach. This methodology applies the rational fractional approximants in order to enhance the convergence of the series expansions of the center manifold theory. Effectively, a sequence of rational fractional approximants may converge even if the associated series does not; one can than extended our domain of convergence. In this study, the domain of validity of the solution is successfully enhanced by employing rational fractional approximants.
In this paper the basic ideas are outlined, an example is presented and some natural extensions and possible applications of this methodology are briefly described in the conclusions.

2 INTRODUCTION

In recent year non-linear vibration phenomena have been receiving increasing attention. Non-linear techniques have been developed in order to reduce the n-dimensional original system to m-dimensional system (with $m < n$). The most common way to study the behaviour of a non-linear system is to introduce a reduced system that can capture the main features of the original system. One of the most popular method is the center manifold method which has been employed to solve a large variety of bifurcation problems (Nayfeh and Mook [1], Nayfeh and Balachandran [2], Guckenheimer and Holmes [3], Marsden and McCracken [4], Jézéquel and Lamarque [5], Hsu [6-7], Yu [8], Thompson and Stewart [9]). The center manifold approach is a method used in order to reduce the dimension of a system of ordinary differential equation. Generally, this method uses power series expansions in the neighbourhood of an equilibrium point. It can be noted, that the formal center manifold approximation is not difficult to determine. But, obtaining the coefficients associated which each term of the stable variables may pose especially serious difficulties. The only use of the center manifold approach is not very convenient to apply, requiring a great deal of labour, especially for the calculation of the coefficients defined previously.
The fundamental goal of this study is to construct an approximation to the solution of the center manifold approach. This goal will be achieved by implementing the rational fractional approximants after the center manifold theory. Padé approximants and rational fractional approximants (Baker and Graves-Morris [10], Hughes Jones [11], Brezinski [12]) have received very many applications in various branches of science because of their quite

interesting approximation properties and, in particular, their possible convergence outside the domain of convergence of the series they approximate. One will consider this last property of the rational fractional approximants in this paper in order to augment the domain of validity of the series by using the center manifold approach. Moreover, Padé approximants and rational fractional approximants permit to approximate functions given by a formal series expansion.
This new approach is also useful for calculating periodic solutions of non-linear systems. The advantages of the rational fractional approximants is that the results are obtained even if the power series expansions of the center manifold in the neighbourhood of an equilibrium point is not sufficient.
In the following section, a general theory is presented: one will consider the general order case with n center variables and the n-multivariables approximants associated. One will show the general technique to compute and to obtain the coefficients of the center manifold and the multivariables approximants. Firstly, one describes the center manifold theory and interesting features of the combinaison between the center manifold approach, and the rational fractional approximant is introduced. Following the general theory, this non-linear technique will be tested in the case of a system with two degree-of-freedom possessing quadratic and cubic non-linearities. Comparison with the results obtained by considering the complete non-linear system is made and the advantages of this present non-linear technique, by considering the use of the rational approximant after the center manifold approach, is given. One will show that the interest of these rational approximants is that they need less terms that the associated Taylor series in order to obtain an accurate approximation of the behaviour of the complete non-linear system. In any case, the rational approximation has a greater range of validity that the polynomial one. One will demonstrate that the rational approximants permit to obtain an approximation of the solution even if the associated center manifold approximation diverge or is not sufficient in order to approximate the non-linear solution

3 NON-LINEAR ANALYSIS

One begins by presenting the methodology and more particularly the definition of the center manifold approach and the rational fractional approximants.

3.1 THE CENTER MANIFOLD APPROACH

In this section, one briefly describes the reduction of a non-linear system to a lower dimensional form problem by the consideration of the center manifold theory. This method is used in the neighbourhood of a bifurcation point. So, on considers an autonomous m-dimensional ($2 \le m < \infty$) dynamical system defined as follow:

$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mu\right) \tag{1}$$

where μ is a control parameter and $\mathbf{F}$ is a polynomial non-linear function. One assumes that this system has an equilibrium point $\mathbf{x_0}\left(\mu\right)$ such that $\mathbf{F}\left(\mathbf{x_0}, \mu\right) = \mathbf{0}$.

One projects the equations on the basis of its eigenvectors and one considers the augmented system as

$$\begin{cases} \dot{\mathbf{v}}_\mathbf{c} = \mathbf{J_c}\left(\hat{\mu}\right).\mathbf{v_c} + \mathbf{F_c}\left(\mathbf{v_c}, \mathbf{v_s}, \hat{\mu}\right) \quad ; \qquad \dot{\mathbf{v}}_\mathbf{s} = \mathbf{J_s}\left(\hat{\mu}\right).\mathbf{v_s} + \mathbf{F_s}\left(\mathbf{v_c}, \mathbf{v_s}, \hat{\mu}\right) \\ \dot{\hat{\mu}} = 0 \end{cases} \tag{2}$$

where $\mathbf{v_c} \in \mathbb{R}^{\mathrm{n}}$ and $\mathbf{v_s} \in \mathbb{R}^{\mathrm{m-n}}$. By considering the most physically interesting case of the stable equilibrium loosing stability, the unstable manifold is empty and it may assume that $\mathbf{v_s}$ contains the variables associated to the eigenvalues λ with negative real part. $\mathbf{F_c}$ and $\mathbf{F_s}$ are

polynomial non-linear functions. The center manifold theory allows the expression of the variables $\mathbf{v_s}$ as a function of $\mathbf{v_c}$ such that $\mathbf{v_s} = \mathbf{h}(\mathbf{v_c}, \hat{\mu})$. Due to the fact that the expression of $\mathbf{h}$ cannot be solved exactly in most cases, one approximates $\mathbf{v_s} = \mathbf{h}(\mathbf{v_c}, \hat{\mu})$ as a power series in $(\mathbf{v_c}, \hat{\mu})$ of degree q (Carr [13]), without constant and linear terms ($q \geq 2$)

$$\mathbf{v_s} = \mathbf{h}(\mathbf{v_c}, \hat{\mu}) = \sum_{p=i+j+l=2}^{q} \sum_{j=0}^{p} \sum_{l=0}^{p} \mathbf{a_{ijl}} . v_{c1}^{i} . v_{c2}^{j} . \hat{\mu}^{l} \tag{3}$$

where $\mathbf{a_{ijl}}$ are vectors of constant coefficients. This $(m-n)$-dimensional function $\mathbf{h}$ is substituted into the second equation of (2) and then these results are combined with the first equation of (2). By considering the tangency conditions at the fixed point $(\mathbf{0}, \mathbf{0}, 0)$ to the center eigenspace, one obtains

$$\begin{cases} D_{\mathbf{v_c}, \hat{\mu}}(\mathbf{h}(\mathbf{v_c}, \hat{\mu})) . (\mathbf{J_c} . \mathbf{v_c} + \mathbf{F_c}(\mathbf{v_c}, \mathbf{h}(\mathbf{v_c}, \hat{\mu}), \hat{\mu})) = \mathbf{J_s} . \mathbf{h}(\mathbf{v_c}, \hat{\mu}) - \mathbf{F_s}(\mathbf{v_c}, \mathbf{h}(\mathbf{v_c}, \hat{\mu}), \hat{\mu}) \\ \mathbf{h} = \mathbf{0} \qquad D_{\mathbf{v_c}} h_i(\mathbf{0}) = \mathbf{0} \quad 1 \leq i \leq n-2 \qquad \dfrac{\partial \mathbf{h}}{\partial \hat{\mu}} = 0 \end{cases} \tag{4}$$

where h_i $(1 \leq i \leq m-n)$ are the scalar components of $\mathbf{h}$.

To solve (4), one equates the coefficients of the different terms in the polynomials on both sides; one obtains a system of algebraic equations for the coefficients $\mathbf{a_{ijl}}$ of the polynomials. Solving these equations, one obtains an approximation to the center manifold $\mathbf{v_s} = \mathbf{h}(\mathbf{v_c}, \hat{\mu})$. After $\mathbf{h}$ is identified, the reduced order structural dynamic model, which is only a function of $\mathbf{v_c}$, is obtained:

$$\begin{cases} \dot{\mathbf{v}}_\mathbf{c} = \mathbf{J_c}(\hat{\mu}) . \mathbf{v_c} + \mathbf{F_c}(\mathbf{v_c}, \mathbf{h}(\mathbf{v_c}, \hat{\mu}), \hat{\mu}) \\ \dot{\hat{\mu}} = 0 \end{cases} \tag{5}$$

If n of the m eigenvalues have zero real parts, then one reduces the number of equations of the original system from m to n in order to obtain a simplified system.

3.2 RATIONAL FRACTIONAL APPROXIMANTS

The center manifold equations can have complicated non-linear terms, which can be simplified using further non-linear methods. The interest of the rational fractional approximants is that they need less terms than the associated Taylor series in order to obtain an accurate approximation of the limit cycle amplitudes (Baker and Grave-Morris [10]): they allow the computation of an accurate approximation of the non-linear function $f(\mathbf{x})$ even at values of f for which the Taylor series of $f(\mathbf{x})$ diverge. One will consider this last property of the rational fractional approximants in this paper in order to augment the domain of validity of the series previously obtained by using the center manifold approach.

Moreover, the objective is to approximate the non-linear terms by using rational polynomial approximants (Baker and Grave-Morris [10], Hughes Jones [11], Brezinski [12]). The use of the rational approximants allows to simplify the non-linear system and to obtain limit cycles more easily and rapidly.

Let $f(x_1, x_2, ..., x_n)$ be a function of n-variables defined by a formal power series expansion

$$f(x_1, x_2, ..., x_n) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} ... \sum_{i_n}^{\infty} c_{i_1 i_2 ... i_n} x^{i_1} x^{i_2} ... x^{i_n} = \sum_{\mathbf{i} \in S} c_\mathbf{i} \mathbf{x}^\mathbf{i} \tag{6}$$

where

$$\mathbf{x} = (x_1, x_2, ..., x_n); \qquad \mathbf{i} = (i_1, i_2, ..., i_n); \qquad \mathbf{x}^{\mathbf{i}} \equiv x_1^{i_1} x_2^{i_2} ... x_n^{i_n} \tag{7}$$

$$S = \{\mathbf{i} | i_a \text{ a non negative integer, } a \in I_n \} \tag{8}$$

$$I_n = \{1, 2, ..., n\} \tag{9}$$

In this paper, on will consider symmetric-off-diagonal (SOD) rational approximants (Baker and Graves-Morris [10], Hughes [11]) to $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ of the form

$$[M/N]_f(\mathbf{x}) = \frac{\sum_{\mathbf{a} \in S_M} \alpha_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}}{\sum_{\mathbf{b} \in S_N} \beta_{\mathbf{b}} \mathbf{x}^{\mathbf{b}}} \tag{10}$$

where

$$S_M = \{\boldsymbol{\gamma} | 0 \le \gamma_j \le M, j \in I_n\} \qquad \text{and} \qquad S_N = \{\boldsymbol{\gamma} | 0 \le \gamma_j \le N, j \in I_n\} \tag{11}$$

There are $\prod_{i \in I_n} (M_i + 1) + \prod_{i \in I_n} (N_i + 1)$ unknown coefficients in equation (10). By considering this equation, one notes that the coefficients $\alpha_{\mathbf{a}}$ and $\beta_{\mathbf{b}}$ will be determined at most up to a common multiplicative factor. So, one can assume that $\beta_{0,0,...,0} = \beta_0 = 1$. By multiplying the difference between $f(\mathbf{x})$ and $[M/N]_f(\mathbf{x})$ by the denominator of $[M/N]_f(\mathbf{x})$, one obtains

$$\left(\sum_{\mathbf{b} \in S_N} \beta_{\mathbf{b}} \mathbf{x}^{\mathbf{b}} \right) \times \left(\sum_{\mathbf{i} \in S} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \right) - \sum_{\mathbf{a} \in S_M} \alpha_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} = \sum_{\mathbf{j} \in S} e_{\mathbf{j}} \mathbf{x}^{\mathbf{j}} \tag{12}$$

where

$$e_{\mathbf{j}} = 0 \qquad \mathbf{j} \in S_M \bigcup S_N \tag{13}$$

$$e_{\mathbf{j}} = 0 \qquad \mathbf{j} \in A \tag{14}$$

$$\sum_{\mathbf{j} \in A_{2;\mathbf{p}}} e_{\mathbf{j}} = 0 \qquad \mathbf{p} \in P \tag{15}$$

with

$$A = \bigcup_{\mathbf{p} \in P} A_{1;\mathbf{p}} \tag{16}$$

$$A_{1;\mathbf{p}} = \bigcup_{i \in I_{\mathbf{p}}} \{\boldsymbol{\alpha} | n_i' < \alpha_i \le m_i' + n_i' - p; \alpha_j = p_j, j \ne i\} \tag{17}$$

$$A_{2;\mathbf{p}} = \bigcup_{i \in I_{\mathbf{p}}} \{\boldsymbol{\alpha} | \alpha_i = m_i' + n_i' - p + 1; \alpha_j = p_j, j \ne i\} \tag{18}$$

$$P = \left\{ \mathbf{p} \middle| \mathbf{p} \in S_M \cap S_N; I_{\mathbf{p}} = \left\{ j \middle| p_j = \max_{i \in I_n} (p_i) \right\} \text{ has at least two elements} \right\} \tag{19}$$

with $m_i' = \min(m_i, n_i)$ and $n_i' = \max(m_i, n_i)$.

Next, the equations obtained by matching coefficients in (12) are (Hughes Jones [11])

$$\sum_{\boldsymbol{\psi} \in S_N} \beta_{\boldsymbol{\psi}} c_{\mathbf{a}-\boldsymbol{\psi}} = \alpha_{\mathbf{a}} \qquad \mathbf{a} \in S_M \tag{20}$$

$$\sum_{\boldsymbol{\psi} \in S_N} \beta_{\boldsymbol{\psi}} c_{\mathbf{a}-\boldsymbol{\psi}} = 0 \qquad \mathbf{a} \in (S_N \setminus S_M) \bigcup A \tag{21}$$

$$\sum_{\mathbf{a}\in A_{2;\mathbf{p}}} \sum_{\boldsymbol{\psi}\in S_N} \beta_{\boldsymbol{\psi}} c_{\mathbf{a}-\boldsymbol{\psi}} = 0 \qquad \mathbf{p} \in P \tag{22}$$

where $c_{\boldsymbol{\alpha}} \equiv 0$ if $\alpha_i < 0$ for at least one $i \in I_n$. Then, after normalising $\beta_{0,0,\ldots,0}$,..., $\beta_{0,0}$ and β_0 to unity, the computation of the coefficients $\beta_{\boldsymbol{\psi}}$ can be achieved by solving the linear equations which arise from (21) and (22). Next, the linear equations given by (20) enable the coefficients $\alpha_{\mathbf{a}}$ to be determined, with the coefficients $\beta_{\boldsymbol{\psi}}$ found previously.

4 EXAMPLE

In this section, one will illustrate the extension of the domain of validity by using the rational fractional approximants: this non-linear methodology will be tested in the case of a system with two degree-of-freedom possessing quadratic and cubic non-linearities. This model illustrates a brake system. One will present more particularly the extension of the domain of validity by using the rational fractional approximants after the center manifold approach. On will show that a sequence of rational fractional approximants may converge even if the associated series, defined by using the center manifold approach, does not.

This example deals with the study of instability phenomena in non-linear model with a constant brake friction. It outlines stability analysis and one develops the non-linear strategy, based on the center manifold and the rational approximants in order to study the non-linear dynamical behaviour of a system in the neighbourhood of a critical steady-state equilibrium point.

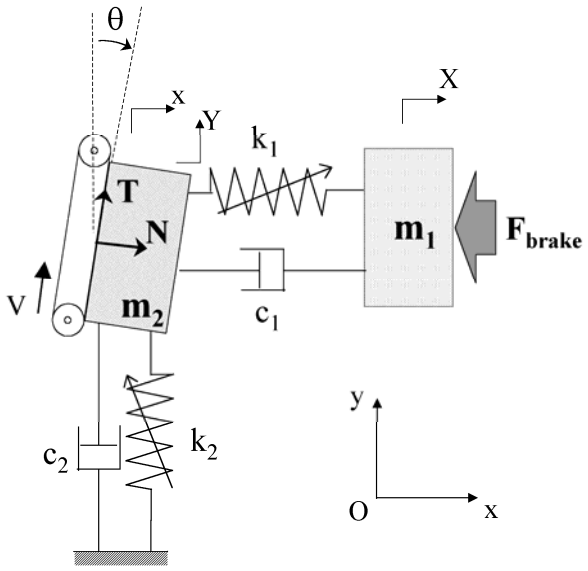

Figure 1: Non-linear system

4.1 NON-LINEAR SYSTEM

The non-linear system to be considered is shown in Figure 1. It illustrates braking system (Sinou [14]). The instability observed by considering this system is more particularly grabbing vibration: it was observed on brake control and front axle assembly. According to experimental investigations, the frequency spectrum is in the 50–100 Hz range. For this study, one assumes that the variation of the brake friction coefficient can be negligible, although this is not always the case for modeling brake systems (Ibrahim [16-17], Oden and Martins [18]). As a result, one considers the sprag-slip theory (Spurr [15]) based on dynamic coupling due to

buttressing with a constant brake friction coefficient. This vibration results from coupling between the torsional mode of the front axle (k_2, m_2) and the normal mode of the brake control (k_1, m_1). In order to simulate braking system placed crosswise due to overhanging caused by static force effect, one considers the moving belt slopes with an angle θ. This slope couples the normal and tangential degree-of-freedom induced only by the brake friction coefficient. Therefore, one considers the non-linear behaviour dynamic of the brake command of the system (k_1, m_1), and the non-linear behaviour dynamic of the front axle assembly and the suspension (k_2, m_2) are concerned, respectively. One expresses this non-linear stiffness as a quadratic and cubic polynomial in the relative displacement:

$$k_1 = \mathrm{k}_{11} + \mathrm{k}_{12}.\Delta + \mathrm{k}_{13}.\Delta^2 \qquad k_2 = \mathrm{k}_{21} + \mathrm{k}_{22}.\delta + \mathrm{k}_{23}.\delta^2 \tag{23}$$

where Δ is the relative displacement between the normal displacement in the y-direction of the mass m_1 and the mass m_2 (one has $\Delta = x - X$), and δ the translational displacement defined by the frictional x-direction of the mass m_2 (one has $\delta = Y$).

One assumes that the tangential force T is generated by the brake friction coefficient μ, considering the Coulomb's friction law $T = \mu.N$. The three equations of motion can be expressed as

$$\begin{cases} m_1\ddot{X} + c_1\left(\dot{X} - \dot{x}\right) + k_{11}\left(X - x\right) + k_{12}\left(X - x\right)^2 + k_{13}\left(X - x\right)^3 = -F_{brake} \\ m_2\ddot{Y} + c_2\dot{Y} + k_{21}Y + k_{22}Y^2 + k_{23}Y^3 = -N\sin\theta + T\cos\theta \\ m_2\ddot{x} + c_1\left(\dot{x} - \dot{X}\right) + k_{11}\left(x - X\right) + k_{12}\left(x - X\right)^2 + k_{13}\left(x - X\right)^3 = N\cos\theta + T\sin\theta \end{cases} \tag{24}$$

Using the transformations $x = Y\tan\theta$ and $\mathbf{x} = \{X \;\; Y\}^{\mathrm{T}}$, and considering the Coulomb's friction law $T = \mu.N$, the non-linear *2*-degrees-of-freedom system has the form

$$\mathbf{M}.\ddot{\mathbf{x}} + \mathbf{C}.\dot{\mathbf{x}} + \mathbf{K}.\mathbf{x} = \mathbf{F} + \mathbf{F_{NL}}\left(\mathbf{x}\right) = \mathbf{F} + \mathbf{f_{(2)}}.\mathbf{x}\otimes\mathbf{x} + \mathbf{f_{(3)}}.\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x} \tag{25}$$

where $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$ and $\mathbf{x}$ are the acceleration, velocity, and displacement response 2-dimensional vectors of the degrees-of-freedom, respectively. $\otimes$ defines the Kronecker product (Stewart [19]). $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix and $\mathbf{K}$ is the stiffness matrix. $\mathbf{F}$ is the vector force due to brake command and $\mathbf{F_{NL}}$ contains moreover the quadratic $\mathbf{f_{(2)}}$ and cubic $\mathbf{f_{(3)}}$ non-linear terms. These expressions are given in Annexe A.

4.2 HOPF BIFURCATION POINT

The first step is the static problem, the determination of the Hopf bifurcation point and the stability analysis associated. The equilibrium point $\mathbf{x_0}$ is obtained by solving the non-linear static equations for a given net brake hydraulic pressure:

$$\mathbf{K}.\mathbf{x_0} = \mathbf{F} + \mathbf{F_{NL}}\left(\mathbf{x_0}\right) \tag{26}$$

The stability is investigated by calculating the Jacobian of the system at the equilibrium points. A representation of the evolution of the eigenvalues in the complex plane against brake friction coefficient is given in Figure 2. As long as the real part of all the eigenvalues remains negative, the system is stable. When at least one of the eigenvalues has a positive real part, the dynamical system is unstable. The imaginary part of this eigenvalue represents the frequency of the unstable mode. The Hopf bifurcation point occurs at $\mu_0 = 0,28$. The frequency ω_0 of the unstable mode, obtained for $\mu = \mu_0$ is near 50 Hz.

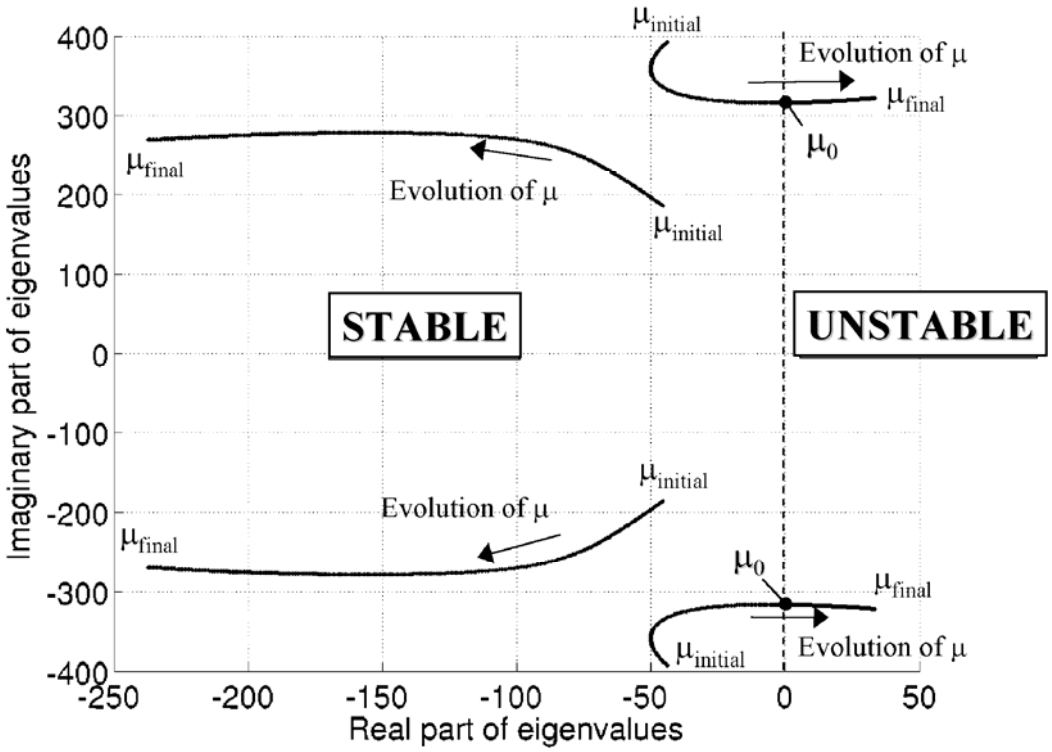

Figure 2: Determination of the Hopf bifurcation point in the complex plane

4.3 NON-LINEAR ANALYSIS

The complete non-linear expressions are expressed at the Hopf bifurcation point and by considering the equilibrium point $\mathbf{x}_0$ for small perturbations $\bar{\mathbf{x}}$, in order to conduct a complex non-linear analysis. The complete non-linear equations can be written as follow:

$$\mathbf{M}.\ddot{\bar{\mathbf{x}}} + \mathbf{C}.\dot{\bar{\mathbf{x}}} + \mathbf{K}.\bar{\mathbf{x}} = \mathbf{F}_{\mathbf{NL}}\left(\bar{\mathbf{x}}\right) \tag{27}$$

where $\ddot{\bar{\mathbf{x}}}$, $\dot{\bar{\mathbf{x}}}$ and $\bar{\mathbf{x}}$ are the acceleration, velocity, and displacement response of the degrees-of-freedom, respectively. $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix and $\mathbf{K}$ is the stiffness matrix. $\mathbf{F}_{\mathbf{NL}}\left(\bar{\mathbf{x}}\right)$ contains the non-linear terms near the Hopf bifurcation point for a given equilibrium point.

The evolutions of the displacements, velocities and the limit cycle amplitudes associated can be calculated by using classical Runge-Kutta numerical methods. As illustrated in Figure 3 and in Figure 4, one may obtain the displacement of $X(t)$ and $\dot{X}(t)$ for example. One observes that the displacement and velocity growth until one obtain the periodic oscillations. Figure 5 and Figure 6 show the evolution of the limit cycle amplitudes $\left(X(t),\dot{X}(t)\right)$ and $\left(Y(t),\dot{Y}(t)\right)$, respectively. By integrating the full system (Runge-Kutta 4), it is possible to obtain the limit cycle amplitudes of the non-linear system, but this procedure is rather expensive and consumes considerable resources both in terms of the computation time and in terms of the data storage requirements. So, the understanding of the behaviour of this non-

linear system requires a simplification and a reduction of the equations. This is why the center manifold approach and the rational approximants will be used in order to reduce and in order to simplify this non-linear system.

In order to use the non-linear methods (the center manifold approach and the rational approximants), one writes the non-linear equation in state variables $\mathbf{y} = \left\{ \overline{\mathbf{x}} \quad \dot{\overline{\mathbf{x}}} \right\}^T$

$$\dot{\mathbf{y}} = \mathbf{A}.\mathbf{y} + \sum_{i=1}^{4}\sum_{j=1}^{4} \mathbf{p}_{(2)}^{\mathbf{ij}}.y_i.y_j + \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} \mathbf{p}_{(3)}^{\mathbf{ijk}}.y_i.y_j.y_k \tag{28}$$

where y_i defines the i^{th}-term of $\mathbf{y}$. $\mathbf{A}$, $\mathbf{p}_{(2)}^{\mathbf{ij}}$ and $\mathbf{p}_{(3)}^{\mathbf{ijk}}$ are the 30×30 matrix, quadratic and cubic non-linear terms, respectively. One has $\mathbf{A} = -\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{M}^{-1}.\mathbf{K} & -\mathbf{M}^{-1}.\mathbf{C} \end{bmatrix}$, $\mathbf{p}_{(2)} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}.\mathbf{q}_{(2)} \end{Bmatrix}$ and $\mathbf{p}_{(3)} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}.\mathbf{q}_{(3)} \end{Bmatrix}$.

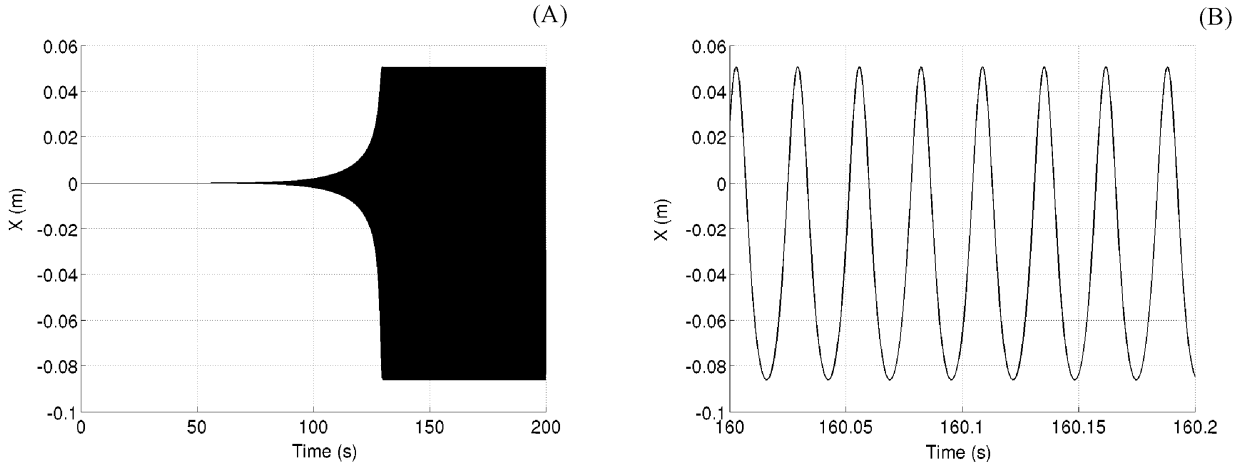

Figure 3: (A) Evolution of the displacement X(t) by using Runge-Kutta 4 ($\mu = 1.01\mu_0$)
(B) Zoom

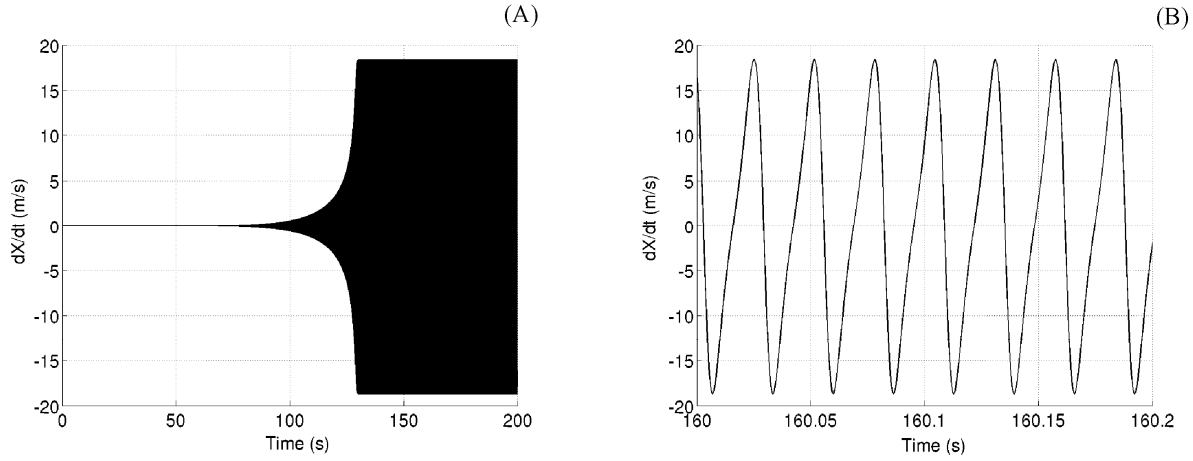

Figure 4: (A) Evolution of the velocity $\dot{X}(t)$ by using Runge-Kutta 4 ($\mu = 1.01\mu_0$)
(B) Zoom

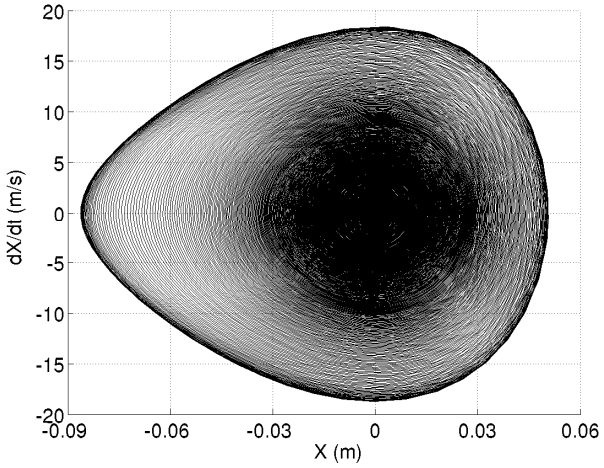

Figure 5: Evolution of the limit cycle amplitude $\left(X, \dot{X}\right)$ by using Runge-Kutta 4 ($\mu = 1.01\mu_0$)

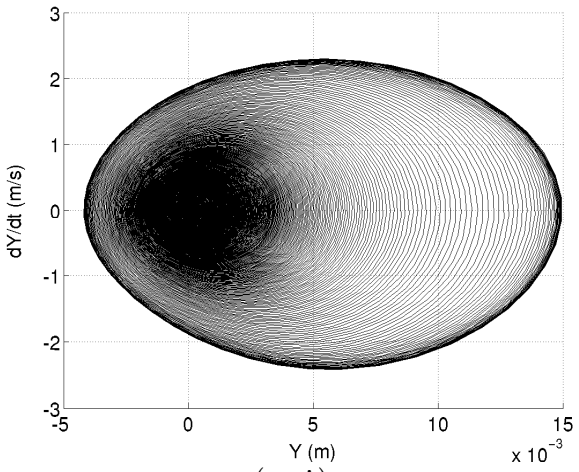

Figure 6: Evolution of the limit cycle amplitude $\left(Y, \dot{Y}\right)$ by using Runge-Kutta 4 **(** $\mu = 1.01\mu_0$ **)**

4.4 EXTENSION OF THE DOMAIN OF VALIDITY

Next the first objective is to reduce the non-linear system of 4-degree-of-freedom by using the center manifold approach near the Hopf bifurcation point.
At the Hopf bifurcation point, this previous system can be rewritten as illustrated in equation (2). In this example, the center variables are two ($\mathbf{v_c} \in \mathbb{R}^2$ and $\mathbf{v_c} = \{v_{c1}\, v_{c2}\}^T$) and the stable variables are two ($\mathbf{v_s} \in \mathbb{R}^2$ and $\mathbf{v_s} = \{v_{s1}\, v_{s2}\}^T$), as illustrated in Figure 2. As explained previously, using existence theorem of the center manifold theory (Carr [13]), there exists an center manifold for the system (28) such that the dynamics of (28), for a given control parameter $\hat{\mu}$, is determined by

$$\begin{cases} \dot{\mathbf{v}}_\mathbf{c} = \mathbf{J_c}.\mathbf{v_c} + \mathbf{F_c}\left(\mathbf{v_c}, \mathbf{v_s}, \hat{\mu}\right) \\ \dot{\hat{\mu}} = 0 \end{cases} \tag{29}$$

with $\mathbf{v_s}$ is approximated as indicated in equations (3) and (4). $\mathbf{F_c}$ defines the quadratic and cubic non-linear terms of equation (28). This non-linear system can be rewritten as follow

$$\dot{\mathbf{v}}_{\mathbf{c}} = \mathbf{f}\left(\mathbf{v}_{\mathbf{c}}\right) = \mathbf{f}\left(v_{c1}, v_{c2}\right) = \sum_{p=1}^{3q}\sum_{i=0}^{p} \mathbf{c}_{\mathbf{i,p-i}}.v_{c1}^{i}.v_{c2}^{p-i} \tag{30}$$

where q defines the degree of the power series $\mathbf{h}$ in equation (3).The right side of this equation is a function of 2-variables defined by a formal power series expansion. In this case, on may consider symmetric-off-diagonal (SOD) rational approximants (Baker and Graves-Morris [10], Hughes Jones and Makinson [20]) to $f\left(v_{c1}, v_{c2}\right)$ of the form

$$\left[M/N\right]_f\left(v_{c1}, v_{c2}\right) = \frac{\sum_{(i,j)\in S_M} \alpha_{ij}.v_{c1}^{i}v_{c2}^{j}}{\sum_{(i,j)\in S_N} \beta_{ij}.v_{c1}^{i}v_{c2}^{j}} \tag{31}$$

where $S_M = \left\{(i,j)\middle| 0 \le i \le M, 0 \le j \le M\right\}$ and $S_N = \left\{(i,j)\middle| 0 \le i \le N, 0 \le j \le N\right\}$. There are $(M+1)^2 + (N+1)^2$ unknown coefficients. As explained previously, d_{00} can be normalised to unity and the other coefficients α_{ij} and β_{ij} can be determined; equations (20), (21) and (22) restricted to two variables becomes.

$$d_{00} = 1 \tag{32}$$

$$\sum_{i=0}^{a}\sum_{j=0}^{b} \beta_{ij} c_{a-i;b-j} = \alpha_{ab} \qquad 0 \le a \le m, 0 \le b \le m \tag{33}$$

$$\sum_{i=0}^{a}\sum_{j=0}^{n} \beta_{ij} c_{a-i;b-j} = 0 \qquad 0 \le a \le m, m \le b \le m+n-a \tag{34}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{b} \beta_{ij} c_{a-i;b-j} = 0 \qquad m \le a \le m+n-b, 0 \le b \le m \tag{35}$$

$$\sum_{i=0}^{\sigma}\sum_{j=0}^{n} \left(\beta_{ij} c_{\sigma-i;m+n+1-\sigma-j} + \beta_{ij} c_{m+n+1-\sigma-i;\sigma-j}\right) = 0 \qquad 1 \le \sigma \le n \tag{36}$$

Next, the previous system can be written as follow

$$\dot{\mathbf{v}}_{\mathbf{c}} = \mathbf{f}^{\mathbf{NL}}(\mathbf{v}_{\mathbf{c}}) = \left[M/N\right]_f\left(v_{c1}, v_{c2}\right) = \left\{\frac{\sum_{i=0}^{m}\sum_{j=0}^{m} \alpha_{1.ij}.v_{c1}^{i}.v_{c2}^{j}}{\sum_{i=0}^{n}\sum_{j=0}^{n} \beta_{1.ij}.v_{c1}^{i}.v_{c2}^{j}} \quad \frac{\sum_{i=0}^{m}\sum_{j=0}^{m} \alpha_{2.ij}.v_{c1}^{i}.v_{c2}^{j}}{\sum_{i=0}^{n}\sum_{j=0}^{n} \beta_{2.ij}.v_{c1}^{i}.v_{c2}^{j}}\right\}^{\mathrm{T}} \tag{37}$$

where T defines the transpose matrix. The advantages of these rational fractional approximants are that the results are obtained even if the power series expansions of the center manifold in the neighbourhood of an equilibrium point is not sufficient. Effectively, one observes that the limit cycles for the center manifold of order 2,3,4 or 5, diverge, as illustrated in Figure 7 and Figure 8. However, the $\left[8/7\right]_{\mathbf{f}}\left(v_{c1}, v_{c2}\right)$ symmetric-off-diagonal rational approximants are applied in order to simplify the non-linear expression of the non-linear equation (29). These rational approximants are determined by using the center manifold approach of order 5 (one knows that the associated limit cycle diverge). By using the $\left[8/7\right]_{\mathbf{f}}\left(v_{c1}, v_{c2}\right)$ approximants, one observes that the limit cycles are acceptable, as illustrated in Figure 9 and Figure 10. So, in this case, the rational fractional approximants permit to enhance the convergence of the series expansions of the center manifold theory. Moreover, the sequence of rational fractional approximants converges even if the associated series does not; one can than extended our domain of convergence. In this case, the domain of validity of

the solution is successfully enhanced by employing rational fractional approximants. Good agreements are found between the original and reduced system. However, the methods require few computer resources: effectively, the use of the rational approximants permit to consider lower order approximation for the center manifold approach. Moreover, the obtaining the coefficients associated which each term of the stable variables may pose especially serious difficulties. This is why the only use of the center manifold approach is not very convenient to apply, requiring a great deal of labour, especially for the calculation of the coefficients defined previously.

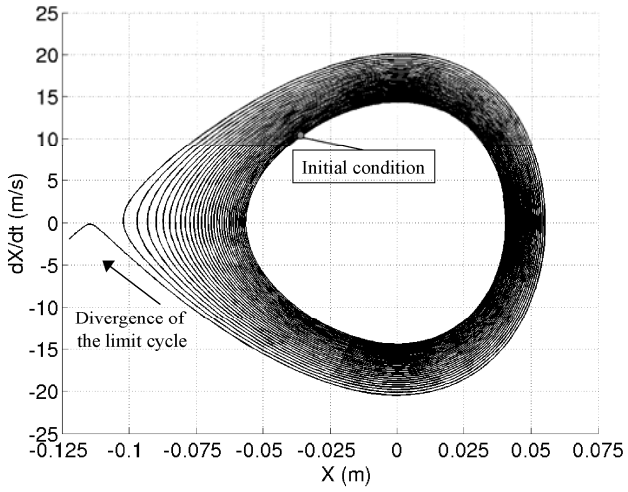

Figure 7: Evolution of the limit cycle amplitude $\left(X,\dot{X}\right)$ by using the center manifold approach of order 5 ($\mu = 1.01\mu_0$)

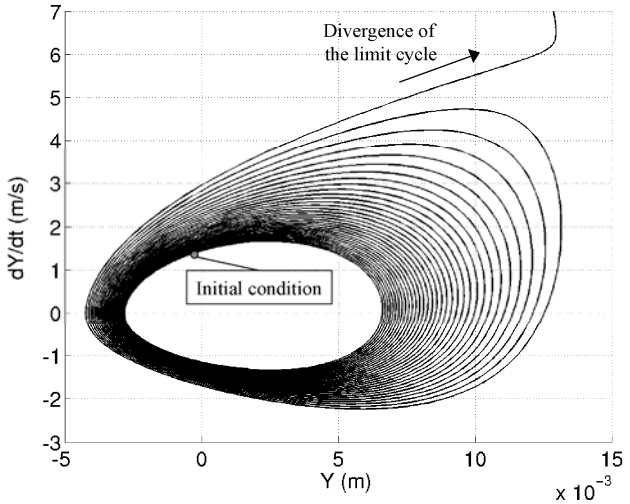

Figure 8: Evolution of the limit cycle amplitude $\left(Y,\dot{Y}\right)$ by using the center manifold approach of order 5 ($\mu = 1.01\mu_0$)

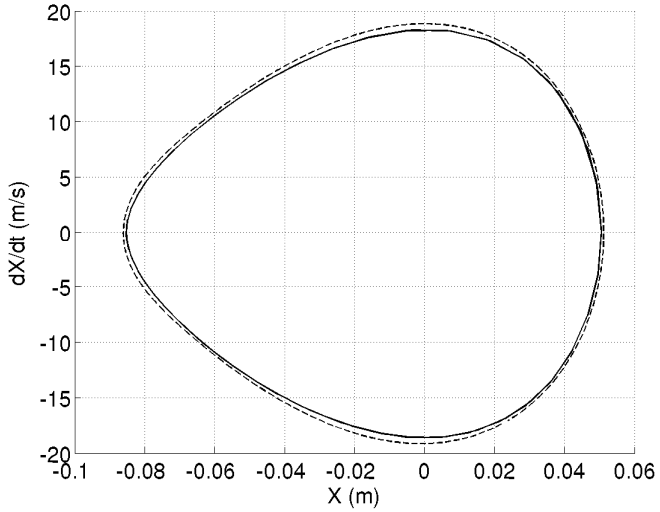

Figure 9: Evolution of the limit cycle amplitude $\left(X, \dot{X}\right)$ by using the rational fractional approximants ($\mu = 1.01\mu_0$)

——— Original system - - - - - Reduced system

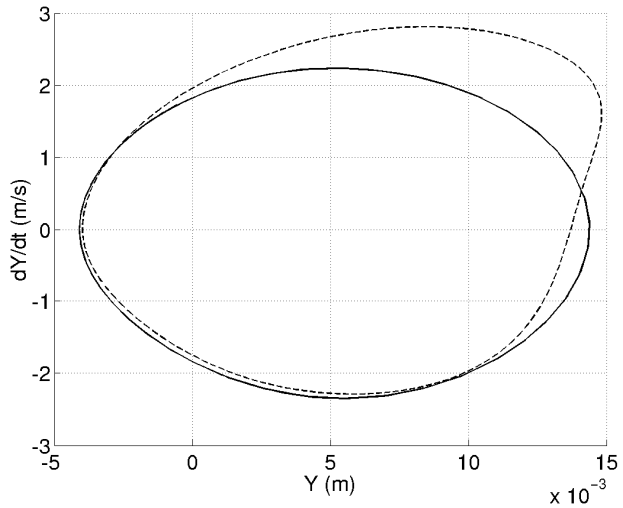

Figure 10: Evolution of the limit cycle amplitude $\left(Y, \dot{Y}\right)$ by using the rational fractional approximants ($\mu = 1.01\mu_0$)

——— Original system - - - - - Reduced system

Moreover, they are two important points to make here. First, this procedure used 316 non-linear terms in order to obtain an estimation of the limit cycle amplitude, as indicated in Table 1; in the case of the center manifold approach, 512 non-linear terms are not sufficient in order to obtain the limit cycle amplitudes. So, one extends the domain of validity of the problem and one simplifies the non-linear terms. Secondly, one obtains a good agreement with the complete non-linear system. This procedure permits to reduce the number of degree-of-freedom of the original non-linear system and to simplify the non-linear terms.

This scheme can be applied for a complex system with n-degree-of-freedom and with m eigenvalues with zero real parts at the Hopf bifurcation point. By comparison with the use of the center manifold theory alone, this proposed methodology appears to be particularly interesting for cases of large non-linear systems with strongly nonlinearities.

	CPU time	Number of degree-of-freedom	Number of non-linear terms
Original System in state variables (Runge Kutta 4)	30 minutes	4	96
Reduced system (center manifold of order 5)	**divergence**	2	512
Reduced and simplify system (center manifold of order 5 + [8/7] approximants)	5 minutes	2	316

Table 1: Comparisons of the CPU time,the number of degree-of-freedom and the number of non-linear terms

5 CONCLUSION

This procedure consisting of applying the multivariable approximants next the center manifold approach appears very interesting in regard to computational time and it necessitate fewer computer ressources due to the number of stables coefficients used to obtained the limit cycle amplitude. Effectively, a sequence of rational fractional approximants may converge even if the associated series does not; one can than extended our domain of convergence. In this work, the domain of validity of the solution is successfully enhanced by employing rational fractional approximants. The rational fractional approximants show superior performance over series approximations. Finally, The center manifold theory and the rational approximants allow to reduce the number of equations of the original system and to simplify the non-linear terms in order to obtain a simplified system, without loosing the dynamics of the original system, as well as the contributions of the non-linear terms.

REFERENCES

1. Nayfeh, A.H., and Mook, D.T.., *Nonlinear Oscillations*, John Wiley & Sons, 1979.
2. Nayfeh, A.H., and Balachandran, B., *Applied Nonlinear Dynamics : Analytical, Comptational and Experimental Methods*, John Wiley & Sons, 1995.
3. Guckenheimer, J., and Holmes, P., Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, 1986.
4. J.E. Marsden, J.E., and McCracken, M., *Applied Mathematical Sciences. The Hopf Bifurcation and its Applications*, Spring-Verlag, 1976.
5. Jezequel, L. and Lamarque, C.H., 'Analysis of Non-linear Dynamical Systems by the Normal Form Theory', *in Journal of Sound and Vibration*, 1991, 149, pp. 429-459.

6. Hsu, L., 'Analysis of critical and post-critical behaviour of non-linear dynamical systems by the normal form method, part I : Normalisation formulae', *in Journal of Sound and Vibration,* 1983, 89, pp. 169-181.
7. Hsu., L., 'Analysis of critical and post-critical behaviour of non-linear dynamical systems by the normal form method, part II : Divergence and flutter', *in Journal of Sound and Vibration,* 1983, 89, pp. 183-194.
8. Yu, P., 'Computation of normal forms via a perturbation tehcnique', *in Journal of Sound and Vibration,*1998, 211, pp. 19-38.
9. Thompson, J.M.T., and Stewart, H.B., *Nonlinear Dynamics and Chaos*, Wiley, Chichester, 1986.
10. Baker, G.A. and Graves-Morris, P., *Padé Approximants*, Cambridge University Press, 1996.
11. Hughes Jones, R., 'General rational approximants in N-variables', *in Journal of Approximation Theory* , 1976, 16, pp. 201-233.
12. Brezinski, C., 'Extrapolation Algorithms and Padé Approximations: a historical survey', *in Applied Numerical Mathematics,* 1983, 20, pp. 299-318.
13. Carr, J., *Application of Center Manifold*, Springer-Verlag, 1981.
14. Sinou, J-J., *Synthèse non-linéaire des Systèmes vibrants - Application aux systèmes de freinage,* Student Press, Ecole Centrale de Lyon, 2002.

15. R.T. Spurr,R.T., 'A theory of brake squeal', *in Proc. Auto. Div., Instn. Mech. Engrs*, 1961, n°1, pp. 33-40.
16. Ibrahim, R.A., 'Friction-Induced Vibration, Chatter, Squeal and Chaos : Part I - Mechanics of Contact and Friction' , *in ASME Applied Mechanics Review*, 1994, 47, n°7, pp. 209-226..
17. Ibrahim, R.A., 'Friction-Induced Vibration, Chatter, Squeal and Chaos : Part II – Dynamics and Modeling', *in ASME Applied Mechanics Review*, 1994, 47, n°7, pp. 227-253.
18. Oden, J.T., and Martins, J.A.C, 'Models and Computational Methods for Dynamic friction Phenomena', *in Computer Methods in Apllied Mechanics and Engineering*, 1985, 52, pp. 527-634.
19. Stewart, G.W. and Sun, J.G, *Computer Science and Scientific Computing. Matrix Perturbation Theory,* Academic Press, 1990.
20. Hughes Jones, R., and Makinson, G.J., 'The generation of Chisholm rational approximants to power series in two variables', *in Journal. Inst. math. Appl.*, 1974, 13, pp. 299-310.

APPENDIX A

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2\left(\tan^2\theta + 1\right) \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_1 & -c_1\tan\theta \\ c_1\left(-\tan\theta + \mu\right) & c_1\left(\tan^2\theta - \mu\tan\theta\right) + c_2\left(1 + \mu\tan\theta\right) \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_{11} & -k_{11}\tan\theta \\ k_{11}\left(-\tan\theta + \mu\right) & k_{21}\left(1 + \mu\tan\theta\right) + k_{11}\left(\tan^2\theta - \mu\tan\theta\right) \end{bmatrix}$$

$$\mathbf{F_{NL}} = \begin{Bmatrix} -k_{12}\left(Y - X\tan\theta\right)^2 - k_{13}\left(Y - X\tan\theta\right)^3 \\ \left(-\tan\theta + \mu\right)\left(k_{12}\left(X\tan\theta - Y\right)^2 + k_{13}\left(X\tan\theta - Y\right)^3\right) + k_{22}\left(1 + \mu\tan\theta\right)X^2 + k_{23}\left(1 + \mu\tan\theta\right)X^3 \end{Bmatrix}$$

$$\mathbf{F} = \begin{Bmatrix} -F_{brake} \\ 0 \end{Bmatrix}$$

APPENDIX B : PARAMETER VALUES

$F_{brake} = 1N$	brake force
$m_1 = 1kg$	equivalent mass of first mode
$m_2 = 1kg$	equivalent mass of second mode
$c_1 = 5N/m.\sec$	equivalent damping of first mode
$c_2 = 300N/m.\sec$	equivalent damping of second mode
$k_{11} = 1.10^5 N/m$	coefficient of linear term of stiffness k_1
$k_{12} = 1.10^6 N/m^2$	coefficient of quadratic term of stiffness k_1
$k_{13} = 1.10^6 N/m^3$	coefficient of cubic term of stiffness k_1
$k_{21} = 1.10^5 N/m$	coefficient of linear term of stiffness k_2
$k_{22} = 1.10^5 N/m^2$	coefficient of quadratic term of stiffness k_2
$k_{23} = 1.10^5 N/m^3$	coefficient of cubic term of stiffness k_2
$\theta = 0,2rad$	sprag-slip angle

APPENDIX C: NOMENCLATURE

$\mathbf{x}$	vector
$\dot{\mathbf{x}}$	vector of velocity
$\ddot{\mathbf{x}}$	vector of acceleration
$\mathbf{x_0}$	equilibrium point
$\bar{\mathbf{x}}$	small pertubation
$\mathbf{C}$	damping matrix
$\mathbf{K}$	stiffness matrix
$\mathbf{M}$	mass matrix
$\mathbf{F}$	vector force due to the net hydraulic pressure
$\mathbf{a_{ijl}}$	vector of the coefficients of the center manifold
$\mathbf{v_c}$	vector of center variables
$\mathbf{v_s}$	vector of stable variables
$\mathbf{h}$	vector of the polynomial approximation of stable variables in center variables
$\mathbf{J_s}$	Jacobian matrix of stable variables
$\mathbf{J_c}$	Jacobian matrix of center variables
$\mathbf{F_c}$	vector function of quadratic and cubic terms for the center variables
$\mathbf{F_s}$	vector function of quadratic and cubic terms for the stable variables
α_{ij}	coefficients of the denominator of the rational approximants
β_{ij}	coefficients of the numerator of the rational approximants
μ	brake friction coefficient
μ_0	brake friction coefficient at the Hopf bifurcation point