Nonuniform hyperbolicity for C^1-generic diffeomorphisms

Flavio Abdenur*, Christian Bonatti, and Sylvain Crovisier

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Abstract

We study the ergodic theory of non-conservative C^1-generic diffeomorphisms. First, we show that homoclinic classes of arbitrary diffeomorphisms exhibit ergodic measures whose supports coincide with the homoclinic class. Second, we show that generic (for the weak topology) ergodic measures of C^1-generic diffeomorphisms are nonuniformly hyperbolic: they exhibit no zero Lyapunov exponents. Third, we extend a theorem by Sigmund on hyperbolic basic sets: every isolated transitive set Λ of any C^1-generic diffeomorphism f exhibits many ergodic hyperbolic measures whose supports coincide with the whole set Λ.

In addition, confirming a claim made by R. Mañé in 1982, we show that hyperbolic measures whose Oseledets splittings are dominated satisfy Pesin's Stable Manifold Theorem, even if the diffeomorphism is only C^1.

Keywords: dominated splitting, nonuniform hyperbolicity, generic dynamics, Pesin theory.

MSC 2000: 37C05, 37C20, 37C25, 37C29, 37D30.

1 Introduction

In his address to the 1982 ICM, R. Mañé [M_2] speculated on the ergodic properties of C^1-generic diffeomorphisms. He divided his discussion into two parts, the first dealing with non-conservative (i.e. "dissipative") diffeomorphisms, the second with conservative diffeomorphisms.

In the first part, drawing inspiration from the work of K. Sigmund [Si] on generic measures supported on basic sets of Axiom A diffeomorphisms, Mañé first used his Ergodic Closing Lemma [M_1] to show that ergodic measures of generic diffeomorphisms are approached in the weak topology by measures associated to periodic orbits (this is item (i) of Theorem 3.8 of this paper; we include a detailed proof, since Mañé did not). He then went on to prove that the Oseledets splittings of generic ergodic measures[1] of generic diffeomorphisms are in fact uniformly dominated, and to claim that such conditions – uniformly dominated Oseledets splittings – together with nonuniform hyperbolicity are sufficient to guarantee the existence of smooth local stable manifolds at μ-a.e. point, as in Pesin's Stable Manifold Theorem [Pe].

In the second part, discussing the case of conservative diffeomorphisms, he stated a C^1-generic dichotomy between (some form of) hyperbolicity and an abundance of orbits with zero Lyapunov exponents. In the two-dimensional setting this reduced to a dichotomy between Anosov diffeomorphisms and those having zero exponents at almost every orbit. Mañé never published a proof of this dichotomy.

For conservative diffeomorphisms much progress has been made. The generic dichotomy between hyperbolicity and zero Lyapunov exponents for surface diffeomorphisms, in particular,

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[1]The space $\mathcal{M}_f^{erg}(M)$ of ergodic measures of a diffeomorphism f is a Baire space when endowed with the weak topology, so that its residual subsets are dense; see Subsection 5.1.

was proven by Bochi [Boc1] in 2000, later extended to higher dimensions by Bochi and Viana [BocV], and finally settled in the original (symplectic, in arbitrary dimension) statement of Mañé by Bochi [Boc2] in 2007. Many other important results have been obtained for C^1-generic conservative diffeomorphisms, see for instance [ABC, DW, HT].

By contrast, there was for a long time after Mañé's address little progress towards the development of the ergodic theory for C^1-generic dissipative diffeomorphisms. This is in our view due to the two following obstacles:

- **Obstacle 1: The Absence of Natural Invariant Measures.** Conservative diffeomorphisms are endowed with a natural invariant measure, namely the volume that is preserved. In the dissipative context, hyperbolic basic sets are endowed with some very interesting invariant measures, such as the measure of maximal entropy (see [Bow]), or, in the case of hyperbolic attractors, the Sinai-Ruelle-Bowen measure (see for instance [R]). In the case of C^1-generic dissipative diffeomorphisms, however, it is difficult to guarantee the existence of measures describing most of the underlying dynamics. For instance, Avila-Bochi [AB] have recently shown that C^1-generic maps do not admit absolutely continuous invariant measures.

- **Obstacle 2: The C^1-Generic Lack of C^2-Regularity.** For much of differentiable ergodic theory the hypothesis of C^1 differentiability is insufficient; higher regularity, usually C^2 but at least C^1+Hölder, is required. This is the case for instance of Pesin's Stable Manifold Theorem [Pe] for nonuniformly hyperbolic dynamics [2].

The aim of this paper is to realize *some of Mañé's vision of an ergodic theory for non-conservative C^1-generic diffeomorphisms.* Some of our results confirm claims made without proof by Mañé; others extend Sigmund's work to the nonhyperbolic C^1-generic setting; and still others go beyond the scope of both of these previous works. In any case, our results begin to tackle both of the aforementioned obstacles to a generic ergodic theory. We hope that this work will help the development of a rich ergodic theory for C^1-generic dissipative diffeomorphisms.

Our starting point is the generic *geometric* theory for dissipative diffeomorphisms, that is, the study from the C^1-generic viewpoint of non-statistical properties: transitivity, existence of dominated splittings, Newhouse phenomenon (coexistence of an infinite number of periodic sinks or sources)... There has been, especially since the mid-90's, an explosion of important generic geometric results, thanks largely to Hayashi's Connecting Lemma [H]. It turns out, however, that many of these tools – especially from [ABCDW], [BDP], and [BDPR] – are also useful for the study of generic ergodic problems. Our results on generic ergodic theory follow largely from the combined use of these geometric tools with techniques by Sigmund and Mañé.

Some of our results hold for every diffeomorphism, some require a C^1-generic assumption. We can group them into three types:

a) **Approximations by Periodic Measures.** A classical consequence of Mañé ergodic closing lemma [M_1] is that, for C^1-generic diffeomorphisms, every invariant measure is the weak limit of a convex sum of dirac measures along periodic orbits. We propose some variation on this statement, for instance:

   If f is a C^1-generic diffeomorphism then

   - *any ergodic measure μ is the weak and Hausdorff limit of periodic measures whose Lyapunov exponents converge to those of μ* (Theorem 3.8);

[2] Obstacle 2, unlike Obstacle 1, is of course also a problem in the conservative setting.

– *any (non necessarily ergodic) measure supported on an isolated transitive set Λ is the weak limit of periodic measures supported on Λ* (Theorem 3.5 part (a)).

The idea is to show that, analogously with what occurs from the "geometric" viewpoint with Pugh's General Density Theorem [Pu_1], generically hyperbolic periodic measures are abundant (e.g., dense) among ergodic measures, and so provide a robust skeleton for studying the space of invariant measures.

b) **Geometric Properties of Invariant Measures.** Some of our results deal with the geometric and topological aspects of the invariant measures, such as the sizes of their supports, their Lyapunov exponents and corresponding Lyapunov spaces, and the structure of their stable and unstable sets. For instance:

– *Let Λ be an isolated transitive set of a C^1-generic diffeomorphism f. Then every generic measure with support contained in Λ is ergodic, has no zero Lyapunov exponents (i.e. is nonuniformly hyperbolic) and its support is equal to Λ* (Theorem 3.5 part (b)).

– *Let μ be an ergodic measure without zero Lyapunov exponent, and whose support admits a dominated splitting corresponding to the stable/unstable spaces of μ. Then there exists stable and unstable manifolds a μ-almost every point* (Theorem 3.11).

c) **Ergodic Properties of Invariant Measures.** Finally, many of our results deal with "statistical" properties such as ergodicity and entropy of the invariant measures. For instance:

– *Any homoclinic class coincides with the support of an ergodic measure with zero entropy* (Theorem 3.1).

Some of our results may admit extensions to or analogues in the conservative setting, but we have not explored this direction.

2 Preliminaries

2.1 General definitions

Given a compact boundaryless d-dimensional manifold M, denote by $\mathrm{Diff}^1(M)$ the space of C^1 diffeomorphisms of M endowed with the usual C^1 topology.

Given a diffeomorphism $f \in \mathrm{Diff}^1(M)$, a point $x \in M$, and a constant $\varepsilon > 0$, then the *stable set* of x is

$$W^s(x) := \{y \in M : d(f^k(x), f^k(y)) \to 0 \text{ as } k \to +\infty\}$$

and the *ε-local stable set* of x is

$$W^s_\varepsilon(x) := \{y \in W^s(x) : d(f^k(x), f^k(y)) \leq \varepsilon \text{ for every } k \in \mathbb{N}\}.$$

The *unstable set* $W^u(x)$ and the *ε-local unstable set* $W^s_\varepsilon(x)$ are defined analogously.

Given $f \in \mathrm{Diff}^1(M)$, a compact f-invariant set Λ is *isolated* if there is some neighborhood U of Λ in M such that

$$\Lambda = \bigcap_{k\in\mathbb{Z}} f^k(U).$$

A compact f-invariant set Λ is *transitive* if there is some $x \in \Lambda$ whose forward orbit is dense in Λ. A transitive set Λ is *trivial* if it consists of a periodic orbit.

We denote by $\mathcal{O}(p)$ the orbit of a periodic point p and by $\Pi(p)$ its period.

For $A \in GL(\mathbb{R}, d)$ we denote by $m(A) = \|A^{-1}\|^{-1}$ its minimal dilatation.

2.2 Homoclinic classes

The Spectral Decomposition Theorem splits the nonwandering set of any Axiom A diffeomorphism into *basic sets* which are pairwise disjoint isolated transitive sets. They are the homoclinic classes of periodic orbits. This notion of homoclinic class can be defined in a more general setting:

Definition 2.1. *Let $\mathcal{O}(p)$ be a hyperbolic periodic orbit of $f \in \mathrm{Diff}^1(M)$. Then*

- *the* homoclinic class *of $\mathcal{O}(p)$ is the set*

$$H(\mathcal{O}(p)) := \overline{W^s(\mathcal{O}(p)) \pitchfork W^u(\mathcal{O}(p))};$$

- *given an open set V containing $\mathcal{O}(p)$, the* homoclinic class of $\mathcal{O}(p)$ relative to V *is the set*

$$H_V(\mathcal{O}(p)) := \overline{\{x \in W^s(\mathcal{O}(p)) \pitchfork W^u(\mathcal{O}(p)) : \mathcal{O}(x) \subset V\}}.$$

Although the homoclinic class is associated to the periodic orbit $\mathcal{O}(p)$ of p, we write sometimes $H(p)$ instead of $H(\mathcal{O}(p))$.

Relative homoclinic classes like full homoclinic classes are compact transitive sets with dense subsets of periodic orbits. There is another characterization of homoclinic classes:

Definition 2.2. *Two hyperbolic periodic points p and q having the same stable dimension are* homoclinically related *if*

$$W^s(\mathcal{O}(p)) \pitchfork W^u(\mathcal{O}(q)) \neq \emptyset \text{ and } W^u(\mathcal{O}(p)) \pitchfork W^s(\mathcal{O}(q)) \neq \emptyset$$

If we define Σ_p as the set of hyperbolic periodic points that are homocliically related to p, then Σ_p is f-invariant and its closure coincides with $H(p)$.

In the relative case in an open set V we denote by $\Sigma_{V,p}$ the set of hyperbolic periodic points whose orbit is contained in V and which are homoclinically related with p by orbits contained in V. Once more $H_V(p)$ is the closure of $\Sigma_{V,p}$.

2.3 Invariant measures and nonuniform hyperbolicity

The statements of many of our results involve two different types of weak hyperbolicity: *nonuniform hyperbolicity* and *dominated splittings*. We now recall the first of these two notions.

– The support of a measure μ is denoted by $\mathrm{Supp}(\mu)$. Given Λ a compact f-invariant set of some $f \in \mathrm{Diff}^1(M)$, set

$$\mathcal{M}_f(\Lambda) := \{\mu : \mu \text{ is an } f\text{-invariant Borel probability on } M \text{ such that } \mathrm{Supp}(\mu) \subset \Lambda\},$$

endowed with the weak topology. Then, $\mathcal{M}_f(\Lambda)$ is a compact metric space hence a Baire space.

– We denote by $\mathcal{M}_f^{erg}(\Lambda)$ the set of ergodic measures $\mu \in \mathcal{M}_f(\Lambda)$. This set is a G_δ subset of $\mathcal{M}_f(\Lambda)$ (see Proposition 5.1), and hence is a Baire space.

– Given γ a periodic orbit of $f \in \mathrm{Diff}^1(M)$, its associated *periodic measure* μ_γ is defined by

$$\mu_\gamma := \frac{1}{\#\gamma} \sum_{p \in \gamma} \delta_p.$$

Given Λ a compact f-invariant set of some $f \in \mathrm{Diff}^1(M)$, set

$$\mathcal{P}_f(\Lambda) := \{\mu_\gamma : \gamma \text{ is a periodic orbit in } \Lambda\} \subset \mathcal{M}_f(M),$$

$$Per_f(\Lambda) := \{p : p \text{ is a periodic point in } \Lambda\} \subset M.$$

– Given any ergodic invariant probability μ of a diffeomorphism of a compact manifold of dimension d the *Lyapunov vector of* μ denoted by $L(\mu) \in \mathbb{R}^d$ is the d-uple of the Lyapunov exponents of μ, with multiplicity, endowed with an increasing order.

An ergodic measure $\mu \in \mathcal{M}_f^{erg}(M)$ is *nonuniformly hyperbolic* if the Lyapunov exponents of μ-a.e. $x \in M$ are all non-zero. The *index* of a nonuniformly hyperbolic measure μ is the sum of the dimensions of Lyapunov spaces corresponding to its negative exponents.

A measure $\mu \in \mathcal{M}_f(M)$ is *uniformly hyperbolic* if $\mathrm{Supp}(\mu)$ is a hyperbolic set.

– Given a nonuniformly hyperbolic measure μ then its *hyperbolic Oseledets splitting*, defined at μ-a.e. x, is the Df-invariant splitting given by

$$\tilde{E}^s(x) := \bigoplus_{\lambda_x<0} \tilde{E}(\lambda_x) \text{ and } \tilde{E}^u(x) := \bigoplus_{\lambda_x>0} \tilde{E}(\lambda_x),$$

where $\tilde{E}(\lambda_x)$ is the Lyapunov space corresponding to the Lyapunov exponent λ_x at x.

– A point $x \in M$ is called *irregular for positive iterations* (or shortly *irregular*$^+$) if there is a continuous function $\psi\colon M \to \mathbb{R}$ such that the sequence $\frac{1}{n}\sum_{t=0}^{n-1} \psi(f^t(x))$ is not convergent. A point x is *Lyapunov irregular*$^+$ if the Lyapunov exponents of x are not well-defined for positive iteration. *Irregular*$^-$ and *Lyapunov irregular*$^-$ points are defined analogously, considering negative iterates instead.

A point is *regular* if it is regular$^+$ and regular$^-$ and if furthermore the positive and negative average of any given continuous function converge to the same limit.

2.4 Dominated splitting

We recall the definition and some properties of dominated splittings (see [BDV, Appendix B]).

A Df-invariant splitting $T_\Lambda M = E \oplus F$ of the tangent bundle over an f-invariant set Λ is *dominated* if there exists $N \geq 1$ such that given any $x \in \Lambda$, any unitary vectors $v \in E(x)$ and $w \in F(x)$, then

$$\|D_x f^N(v)\| \leq \frac{1}{2}\|D_x f^N(w)\|.$$

This will be denoted by $E \oplus_< F$.

More generally, a Df-invariant splitting $E_1 \oplus_< \ldots \oplus_< E_t$ of the tangent bundle $T_\Lambda M$ is a *dominated splitting* if given any $\ell \in \{1, \ldots, t-1\}$ then the splitting

$$(E_1 \oplus \ldots \oplus E_\ell) \oplus (E_{\ell+1} \oplus \ldots \oplus E_t)$$

is dominated. A dominated splitting is *non-trivial* if contains at least two non-empty bundles.

If an invariant set Λ admits a dominated splitting $E_1 \oplus_< \ldots \oplus_< E_t$, then:

a) the splitting $E_1(x) \oplus_< \ldots \oplus_< E_t(x)$ varies continuously with the point $x \in \Lambda$;

b) the splitting $E_1 \oplus \ldots \oplus E_t$ extends to a dominated splitting (also denoted by $E_1 \oplus_< \ldots \oplus_< E_t$) over the closure $\overline{\Lambda}$ of Λ;

c) there is a neighborhood V of $\overline{\Lambda}$ such that every f-invariant subset Υ of V admits a dominated splitting $E'_1 \oplus_< \ldots \oplus_< E'_t$ with $dim(E'_i) = dim(E_i)$ for each $i \in \{1, \ldots, t\}$.

There always exists a (unique) *finest dominated splitting* $F_1 \oplus_< \ldots \oplus_< F_k$ over $T_\Lambda M$, characterized by the following property: given any dominated splitting $E' \oplus_< F'$ over Λ then there is some $\ell \in \{1, \ldots, k-1\}$ such that

$$E' = F_1 \oplus_< \ldots \oplus_< F_\ell \text{ and } F' = F_{\ell+1} \oplus_< \ldots \oplus_< F_k.$$

That is, the finest dominated splitting $F_1 \oplus_< \ldots \oplus_< F_k$ is *minimal* in the sense that every dominated splitting over Λ can be obtained by bunching together bundles of the finest dominated splitting. Equivalently, each of the bundles F_i of the finest dominated splitting is *indecomposable*, in the sense that there exist no subbundles F_i^1 and F_i^2 such that $F_i = F_i^1 \oplus F_i^2$ and

$$F_1 \oplus \ldots \oplus F_{i-1} \oplus F_i^1 \oplus F_i^2 \oplus F_{i+1} \oplus \ldots \oplus F_k$$

is a dominated splitting. Roughly speaking: "there is no domination within each F_i".

The finest dominated splitting "separates Lyapunov exponents". That is, given $\mu \in \mathcal{M}_f(\Gamma)$ an ergodic measure with Oseledets splitting $\tilde{E}_1 \oplus \ldots \oplus \tilde{E}_s$ and corresponding Lyapunov exponents $\lambda_1 < \lambda_2 < \ldots < \lambda_s$ defined at μ-a.e. x, then there are numbers $0 = j_0 < j_1 < j_2 < \ldots j_k = s$ such that for each $i \in \{1, \ldots, k\}$

$$\bigoplus_{j_{i-1} < m \leq j_i} \tilde{E}_m(x) = F_i(x)$$

at μ-a.e. x, where the F_i are the bundles of the finest dominated splitting. In other words, the bundles of the finest dominated splitting can be written as sums of the Lyapunov spaces of the increasing Lyapunov exponents of μ. So we speak of the Lyapunov spaces and of the Lyapunov exponents "inside" each bundle F_i. We denote by $L|_F(\mu)$ the set of Lyapunov exponents of μ inside the bundle F; likewise, given a Lyapunov-regular point $x \in \Lambda$, we denote by $L|_F(x)$ the set of Lyapunov exponents of x inside F.

2.5 Semicontinuity and genericity

Given Y a compact metric space, we denote by $\mathcal{K}(Y)$ the space of compact subsets of Y endowed with the *Hausdorff distance*: given two non-empty sets $K_1, K_2 \in \mathcal{K}(Y)$, set

$$d_H(K_1, K_2) := \inf\{\varepsilon > 0 : B_\varepsilon(K_1) \supset K_2 \text{ and } B_\varepsilon(K_2) \supset K_1\},$$

where $B_\varepsilon(K)$ denotes the ε-ball centered on the set K. (The distance from the empty set to any non-empty set is by convention equal to $\mathrm{Diam}(Y)$.)

Then the space $(\mathcal{K}(Y), d_H)$ is itself a compact (and hence a Baire) metric space.

Definition 2.3. *Given a topological space X and a compact metric space Y, a map $\Phi : X \to \mathcal{K}(Y)$ is*

- lower-semicontinuous *at $x \in X$ if for any open $V \subset Y$ with $V \cap \Phi(x) \neq \emptyset$, there is a neighborhood U of x in X such that $V \cap \Phi(x') \neq \emptyset$ for every $x' \in U$;*
- upper-semicontinuous *at $x \in X$ if for any open $V \subset Y$ containing $\Phi(x)$, there is a neighborhood U of x in X such that V contains $\Phi(x')$ for every $x' \in U$;*
- lower-semicontinuous *(resp,* upper-semicontinuous*) if it is lower-semicontinuous (resp, upper-semicontinuous) at every $x \in X$.*

Now, we can state a result from general topology (see for instance [K]) which is one of the keys to most of the genericity arguments in this paper:

Semicontinuity Lemma. *Given X a Baire space, Y a compact metric space, and $\Phi : X \to \mathcal{K}(Y)$ a lower-semicontinuous (resp, upper-semicontinuous) map, then there is a residual subset $\mathcal{R}$ of X which consists of continuity points of Φ.*

Remark 2.4. In this paper X is usually either $\mathrm{Diff}^1(M)$ (with the C^1 topology) or else $\mathcal{M}_f(M)$ (with the weak topology), while Y is usually M or else $\mathcal{M}(M)$.

In a Baire space, a set is *residual* if it contains a countable intersection of dense open sets. We establish a convention: the phrases "generic diffeomorphisms f (resp., measures μ) satisfy..." and "every generic diffeomorphism f (resp., measure μ) satisfies..." should be read as "there exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ (resp., of $\mathcal{M}_f(\Lambda)$) such that every $f \in \mathcal{R}$ (resp., every $\mu \in \mathcal{R}$) satisfies..."

3 The Main Results

3.1 Homoclinic classes admit ergodic measures with full support

Theorem 3.1. *Let $H(p)$ be a relative homoclinic class of a diffeomorphism $f \in \mathrm{Diff}^1(M)$. Then there is a measure $\mu \in \mathcal{M}_f(H(p))$ which*

i) is ergodic;

ii) has "full support": $\mathrm{Supp}(\mu) = H(p)$*;*

iii) has zero entropy: $h_\mu(f) = 0$.

So any homoclinic class of *any* diffeomorphism exhibits at least one ergodic measure with full support. Theorem 3.1 is in fact a corollary of Theorem 3.1' stated in Section 5.4.

Remark 3.2.
- One intriguing consequence of Theorem 3.1 is this: given f a C^1-generic *conservative* diffeomorphism, then f admits at least one ergodic measure μ whose support coincides with all of M. This follows from Theorem 3.1 and the fact that for C^1-generic conservative diffeomorphisms the manifold M is a homoclinic class (see [BC]).
- We think furthermore that the (f-invariant) volume m is approached in the weak topology by ergodic measures with full support μ; we have not checked this completely, the missing ingredient is a conservative version of the Transition Property Lemma in Subsection 4.2.

3.2 Generic ergodic measure of C^1-generic diffeomorphisms

Methods similar to those used in the proof of Theorem 3.1' yield an analogous result in the wider space of ergodic measures:

Theorem 3.3. *Given a C^1-generic diffeomorphism f then every generic measure μ in $\mathcal{M}_f^{erg}(M)$*

i) has zero entropy: $h_\mu(f) = 0$*;*

ii) is nonuniformly hyperbolic and its Oseledets splitting $\tilde{E}_1 \oplus \ldots \oplus \tilde{E}_k$ is dominated.

In Theorem 3.3 the domination of the Oseledets splitting is due to Mañé [M_2].

Remark 3.4. The splitting above is trivial when μ is supported on a periodic sink or source.

3.3 Isolated transitive sets of C^1-generic diffeomorphisms

Isolated transitive sets are natural generalizations of hyperbolic basic sets. Bonatti-Diaz [BD] used Hayashi's Connecting Lemma [H] to show that every isolated transitive set of a C^1-generic diffeomorphisms is a relative homoclinic class (see also [Ab]). Though at this point it is not known whether every generic diffeomorphism exhibits some isolated transitive set, there are several examples of locally generic diffeomorphisms having some non-hyperbolic isolated transitive sets, for instance nonhyperbolic robustly transitive sets and diffeomorphisms.

Theorem 3.5 below presents a overview of C^1-generic properties satisfied by measures contained in an isolated transitive set.

Theorem 3.5. *Let Λ be an isolated non-trivial transitive set of a C^1-generic diffeomorphism $f \in \mathrm{Diff}^1(M)$ and let $F_1 \oplus \ldots \oplus F_k$ be the finest dominated splitting over Λ. Then*

a) The set $\mathcal{P}_f(\Lambda)$ of periodic measures supported in Λ is a dense subset of the set $\mathcal{M}_f(\Lambda)$ of invariant measures supported in Λ.

b) For every generic measure $\mu \in \mathcal{M}_f(\Lambda)$,

b.i) μ is ergodic;

b.ii) μ has full support: $\mathrm{Supp}(\mu) = \Lambda$*;*

b.iii) μ has zero entropy: $h_\mu(f) = 0$;

b.iv) for μ-a.e. point x the Oseledets splitting coincides with $F_1(x) \oplus \ldots \oplus F_k(x)$;

b.v) μ is nonuniformly *hyperbolic.*

c) There exists a dense subset $\mathcal{D}$ of $\mathcal{M}_f(\Lambda)$ such that every $\nu \in \mathcal{D}$,

c.i) is ergodic;

c.ii) has positive entropy: $h_\nu(f) > 0$;

c.iii) is uniformly *hyperbolic.*

Remark 3.6. 1. The conclusion of Theorem 3.5 does not apply to isolated transitive sets of arbitrary diffeomorphisms: consider for example a normally hyperbolic irrational rotation of the circle inside a two-dimensional manifold.

2. Recently Díaz and Gorodetski [DG] have shown that non-hyperbolic homoclinic classes of C^1-generic diffeomorphisms always support at least one ergodic measure which is not nonuniformly hyperbolic.

Theorem 3.5 parts (a) and (b) is a nonhyperbolic, C^1-generic version of the following theorem by Sigmund on hyperbolic basic sets:

Theorem (Sigmund, 1970). *Given Λ a* hyperbolic *isolated transitive set of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, then the set $\mathcal{P}_f(\Lambda)$ of periodic measures in Λ is a dense subset of the set $\mathcal{M}_f(\Lambda)$ of invariant measures in Λ. Moreover every generic measure $\mu \in \mathcal{M}_f(\Lambda)$ is ergodic,* $\mathrm{Supp}(\mu) = \Lambda$*, and $h_\mu(f) = 0$.*

Remark 3.7. Although this was not stated by Sigmund, the statement of Theorem 3.5 part (c) applies also to the space of measures over any non-trivial hyperbolic basic set.

3.4 Approximation by periodic measures

Many of our results rely in a fundamental way on the approximation of invariant measures by periodic measures. The following theorem is at the heart of the proofs of both Theorem 3.3 and Theorem 3.5.

Theorem 3.8. *Given an ergodic measure μ of a C^1-generic diffeomorphism f, there is a sequence γ_n of periodic orbits such that*

i) the measures μ_{γ_n} converge to μ in the weak topology;

ii) the periodic orbits γ_n converge to $\mathrm{Supp}(\mu)$ *in the Hausdorff topology;*

iii) the Lyapunov vectors $L(\mu_{\gamma_n})$ converge to the Lyapunov vector $L(\mu)$.

As already said, the main novelty here is that, at the same time, the Lyapunov exponents of the periodic measures converge to those of the measure μ. Theorem 3.8 is a generic consequence of the perturbative result Proposition 6.1 which refines Mañé's Ergodic Closing Lemma.

Consider now the finest dominated splitting supported by the ergodic measure μ. Then [BGV] produces perturbations of the derivative of Df along the orbits of the periodic orbits γ_n which make all of the exponents inside a given subbundle coincide. One deduces:

Corollary 3.9. *Given an ergodic measure μ of a C^1-generic diffeomorphism f, let $F_1 \oplus \ldots \oplus F_k$ be the finest dominated splitting over* $\mathrm{Supp}(\mu)$*. Then there is a sequence of periodic orbits γ_k which converges to μ in the weak topology, to* $\mathrm{Supp}(\mu)$ *in the Hausdorff topology, and such that for each $i \in \{1, \ldots, k\}$ the Lyapunov exponents of γ_k inside F_i converge to the mean value λ_{E_i} of the Lyapunov exponents of μ inside the F_i.*

We state another result which allows to approximate measures by periodic measures contained in a homoclinic class.

Theorem 3.10. *For any C^1-generic diffeomorphism $f \in \mathrm{Diff}^1(M)$, any open set $V \subset M$ and any relative homoclinic class $\Lambda = H_V(\mathcal{O})$ of f in V, the closure (for the weak topology) of the set $\mathcal{P}_f(\Lambda)$ of periodic measures supported in $\Lambda \cap V$ is convex.*

In other words, every convex sum of periodic measures in $\Lambda \cap V$ is the weak limit of periodic orbits in $\Lambda \cap V$.

3.5 C^1-Pesin theory for dominated splittings

Theorems 3.1, 3.3, and 3.5 constitute as a group an assault on Obstacle 1. Our next result deals with Obstacle 2. Pugh has built a C^1-diffeomorphism which is a counter-example [Pu_2] to Pesin's Stable Manifold Theorem. It turns out however that Pesin's Stable Manifold Theorem *does* hold for maps which are *only* C^1, as long as the C^1+Hölder hypothesis is replaced by a uniform domination hypothesis on the measure's Oseledets splitting. This has been already done by Pliss [Pl] in the case when all the exponents are strictly negative. The difficulty for applying Pliss argument when the measure has positive and negative exponents is that we have no control on the geometry of iterated disks tangent to the stable/unstable directions. The dominated splitting provides us this control solving this difficulty.

Theorem 3.11 below is a simpler statement of our complete result stated in Section 8, where we show that Pesin's Stable Manifold Theorem applies to ergodic nonuniformly hyperbolic measures with dominated hyperbolic Oseledets splitting .

Theorem 3.11. *Let $\mu \in \mathcal{M}_f(M)$ be an ergodic nonuniformly hyperbolic measure of a diffeomorphism $f \in \mathrm{Diff}^1(M)$. Assume that its hyperbolic Oseledets splitting $\tilde{E}^s \oplus \tilde{E}^u$ is dominated.*

Then, for μ-a.e. x, there is $\varepsilon(x) > 0$ such that the local stable set $W^s_{\varepsilon(x)}(x)$ is an embedded C^1 disc, tangent at x to $\tilde{E}^s(x)$ and contained in the stable set $W^s(x)$. Furthermore, one can choose $\varepsilon(x)$ in such a way that $x \mapsto \varepsilon(x)$ is a measurable map and such that the family $W^s_{\varepsilon(x)}(x)$ is a measurable family of discs.

In other words:

(Nonuniform hyperbolicity) + (Uniform domination) $\Rightarrow$ (Pesin Stable Manifold Theorem).
We note that its statement includes no genericity assumption on f or on μ. It has already been used, in a C^1-generic context, to obtain results on ergodic measures of diffeomorphisms far from homoclinic tangencies [Y_1].

Theorem 3.11 seems to be a folklore result. Indeed, R. Mañé [M_2] announced this result without proof[3] in his ICM address. Although no one seems to have written a full proof under our very general hypotheses, some authors have used the conclusion implicitly in their work. Gan [G], for instance, uses this kind of idea to extend Katok's celebrated result on entropy and horseshoes of $C^{1+\alpha}$ surface diffeomorphisms to a C^1-diffeomorphisms.

Theorems 3.3 and 3.5 show that dominated hyperbolic Oseledets splittings occur quite naturally in the C^1-generic context. We thus obtain:

Corollary 3.12. *Let f be a C^1-generic diffeomorphism. Then for any generic ergodic measure μ, μ-a.e. x exhibits a C^1 stable local manifold $W^s_{loc}(x)$ tangent to $E(x)$ at x as in Theorem 3.11.*

Corollary 3.13. *Let Λ be an isolated transitive set of a C^1-generic diffeomorphism f. Then for any generic ergodic measure μ, μ-a.e. x exhibits a C^1 stable local manifold $W^s_{loc}(x)$ tangent to $E(x)$ at x as in Theorem 3.11.*

3.6 Genericity of irregular points

Our final two results make precise some informal statements of Mañé[4] regarding the irregularity of generic points of C^1-generic diffeomorphisms.

Theorem 3.14. *Given any C^1-generic diffeomorphism $f \in \mathrm{Diff}^1(M)$ there is a residual subset $R \subset M$ such that every $x \in R$ is irregular.*

This result does not hold if we replace regular points by regular$^+$ points: every point in the basin of a (periodic) sink is regular$^+$. We conjecture that if one excludes the basins of sinks, generic points of C^1-generic diffeomorphisms are irregular$^+$. Our next result is that this conjecture is true in the setting of *tame* diffeomorphisms[5].

Recall that a diffeomorphism is called *tame* if all its chain recurrence classes are robustly isolated (see [BC]). The set of tame diffeomorphisms is a C^1-open set which strictly contains the set of Axiom A+no cycle diffeomorphisms. The chain recurrent set of C^1-generic tame diffeomorphisms consist of finitely many pairwise disjoint homoclinic classes. Our result is :

Theorem 3.15. *If f is a C^1-generic tame diffeomorphism then there is a residual subset $R \subset M$ such that if $x \in M$ and $\omega(x)$ is not a sink, then x is both irregular$^+$ and Lyapunov- irregular$^+$.*

[3] He did provide the following one-line proof: "This follows from the results of Hirsch, Pugh, and Shub." Since the ingredients for the proof we provide in Section 8 are all classical and were available in 1982, we believe that Mañé did indeed know how to prove it, but never wrote the proof (possibly because at the time there was little motivation for obtaining a Pesin theory for maps which are C^1 but *not* C^1+Hölder).

[4] "In general, regular points are few from the topological point of view – they form a set of first category". [M_3, Page 264]

[5] Indeed a recent result by J. Yang [Y_2] allows us to extend Theorem 3.15 to C^1-generic diffeomorphisms far from tangencies: Yang announced that, in this setting, generic points belongs to the stable set of homoclinic classes.

3.7 Layout of the Paper

The remainder of this paper is organized as follows:

- In Section 4 we prove an ergodic analogue of Pugh's General Density Theorem which we call *Mañé's Ergodic General Density Theorem*. It implies items (i) and (ii) of Theorem 3.8. We also prove a "generalized specification property" satisfied by C^1-generic diffeomorphisms inside homoclinic classes: this gives Theorem 3.10. One deduces from these results the parts (a) and (c) of Theorem 3.5.
- In Section 5 we state and prove some abstract results on ergodicity, support, and entropy of generic measures. We show then how these abstract results yield Theorem 3.1, item (i) of Theorem 3.3 and items (b.i), (b.ii), and (b.iii) of Theorem 3.5.
- In Section 6 we control the Lyapunov exponents of the periodic measures provided by Mañé's ergodic closing lemma. This implies the item (iii) of Theorem 3.8.
- In Section 7 we prove Corollary 3.9 and we combine most of the previous machinery with some new ingredients in order to obtain our results on nonuniform hyperbolicity of generic measures: item (ii) of Theorem 3.3 and items (b.iv) and (b.v) of Theorem 3.5.
- In Section 8 we construct an adapted metric for the Oseledets splittings and then use it to prove Theorem 3.11.
- Finally, in Section 9 we prove Theorems 3.14 and 3.15.

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4 Approximation of invariant measures by periodic orbits

4.1 Mañé's Ergodic General Density Theorem

In [M$_3$] Mañé states without proof the following fact (called Mañé's Ergodic General Density Theorem):

Theorem 4.1. *For any C^1-generic diffeomorphism f, the convex hull of periodic measures is dense in $\mathcal{M}_f(M)$.*

More precisely, every measure $\mu \in \mathcal{M}_f(M)$ is approached in the weak topology by a measure ν which is the convex sum of finitely many periodic measures and whose support $\mathrm{Supp}(\nu)$ *is arbitrarily close to* $\mathrm{Supp}(\mu)$.

We now prove a more precise result which corresponds to items (i) and (ii) of Theorem 3.8: the ergodic measures are approached by periodic measures in the weak and Hausdorff senses. In Section 6 we shall modify the proof in order to include also the approximation of the mean Lyapunov exponents in each bundle of the finest dominated splitting (item (iii)).

Theorem 4.2. *Given $\mu \in \mathcal{M}_f(M)$ an ergodic measure of a C^1-generic diffeomorphism f, then for every neighborhood $\mathcal{V}$ of μ in $\mathcal{M}_f(M)$ and every neighborhood $\mathcal{W}$ of* $\mathrm{Supp}(\mu)$ *in $\mathcal{K}(M)$ there is some periodic measure μ_γ of f such that $\mu_\gamma \in \mathcal{V}$ and* $\mathrm{Supp}(\mu_\gamma) \in \mathcal{W}$.

Just as the C^1-generic density of $Per_f(M)$ in $\Omega(f)$ follows from Pugh's Closing Lemma [Pu$_1$], Theorem 4.2 follows from Mañé's Ergodic Closing Lemma [M$_1$] (discussed below).

Definition 4.3. *A (recurrent) point x of $f \in \mathrm{Diff}^1(M)$ is* well-closable *if given any $\varepsilon > 0$ and any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ there is some $g \in \mathcal{U}$ such that $x \in Per_g(M)$ and moreover*

$$d(f^k(x), g^k(x)) < \varepsilon$$

for all $k \geq 0$ smaller than the period of x by g.

That is, a point x is well-closable if its orbit can be closed via a small C^1-perturbation in such a way that the resulting periodic point "shadows" the original orbit along the periodic point's entire orbit. Mañé proved that almost every point of any invariant measure is well-closable:

Ergodic Closing Lemma. *Given $f \in \mathrm{Diff}^1(M)$ and $\mu \in \mathcal{M}_f(M)$, μ-a.e. x is well-closable.*

Birkhoff's ergodic theorem and Mañé's ergodic closing lemma implies:

Corollary 4.4. *Given $f \in \mathrm{Diff}^1(M)$ and an ergodic measure $\mu \in \mathcal{M}_f(M)$, then for any neighborhoods $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ and $\mathcal{W}$ of μ in $\mathcal{M}(M)$ and any $\varepsilon > 0$ there is $g \in \mathcal{U}$ having a periodic orbit γ such that $\mu_\gamma \in \mathcal{W}$ and the Hausdorff distance between $\mathrm{Supp}(\mu)$ and γ is less than ε.*

Proof of Theorem 4.2. We consider $X := \mathcal{M}(M) \times \mathcal{K}(M)$ endowed with the product metric. The space $\mathcal{K}(\mathcal{M}(M) \times \mathcal{K}(M))$ of compact subsets of $\mathcal{M}(M) \times \mathcal{K}(M)$ is a compact metric space when endowed with the Hausdorff distance. Consider the map $\Phi : \mathrm{Diff}^1(M) \longrightarrow \mathcal{K}(\mathcal{M}(M) \times \mathcal{K}(M))$, which associates to each diffeomorphism f the closure of the set of pairs (μ_γ, γ) where γ is a periodic orbit of f.

Kupka-Smale Theorem asserts that there is a residual set $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that every periodic orbit of $g \in \mathcal{R}$ is hyperbolic. Then the robustness of hyperbolic periodic orbits implies that the map Φ is lower-semicontinuous at $g \in \mathcal{R}$. Applying the Semicontinuity Lemma (see Section 2.5) to $\Phi|_{\mathcal{R}}$, we obtain a residual subset $\mathcal{S}$ of $\mathcal{R}$ (and hence of $\mathrm{Diff}^1(M)$) such that every $g \in \mathcal{S}$ is a continuity point of $\Phi|_{\mathcal{R}}$. We shall now show that each such continuity point satisfies the conclusion of Lemma 4.2:

Consider $g \in \mathcal{S}$ and μ an ergodic measure of g. Fix an open neighborhood $\mathcal{Z}_0$ of $(\mu, \mathrm{Supp}(\mu))$ in $\mathcal{K}(\mathcal{M}(M) \times \mathcal{K}(M))$. We need to prove that there exists a pair (μ_γ, γ) in $\mathcal{Z}_0$, where γ is a periodic orbit of g. Fix now a compact neighborhood $\mathcal{Z} \subset \mathcal{Z}_0$ of $(\mu, \mathrm{Supp}(\mu))$; it is enough to prove that $\Phi(g) \cap \mathcal{Z} \neq \emptyset$.

Applying the Corollary 4.4 to g, we obtain an arbitrarily small C^1-perturbation g' of g having a periodic orbit γ such that simultaneously μ_γ is weak-close to μ and γ is Hausdorff-close to $\mathrm{Supp}(\mu)$. With another arbitrarily small C^1-perturbation g'' we make γ hyperbolic and hence robust, while keeping μ_γ close to μ and γ close to $\mathrm{Supp}(\mu)$. With yet another small C^1-perturbation g''', using the robustness of γ, we guarantee that $g''' \in \mathcal{R}$ and $(\mu_\gamma, \gamma) \in \mathcal{Z} \cap \Phi(g''')$.

By letting g''' tends to g, using the continuity of $\Phi|_{\mathcal{R}}$ at g and the compactnes of $\mathcal{Z}$ one gets that $\mathcal{Z} \cap \Phi(g) \neq \emptyset$ as announced. □

Theorem 4.1 now follows by combining Theorem 4.2 with the the following "approximative" version of the Ergodic Decomposition Theorem, which is easily deduced from the standard statement:

Ergodic Decomposition Theorem. *Given a homeomorphisms f of a compact metric space M and $\mu \in \mathcal{M}_f(M)$, then for any neighborhood $\mathcal{V}$ of μ in $\mathcal{M}_f(M)$ there are a finite set of ergodic measures $\mu_1, \ldots, \mu_k \in \mathcal{M}_f(M)$ and positive numbers $\lambda_1, \ldots, \lambda_k$ with $\lambda_1 + \ldots + \lambda_k = 1$ such that*

$$\lambda_1\mu_1 + \ldots + \lambda_k\mu_k \in \mathcal{V}.$$

That is, any invariant measure may be approached by finite combinations of its ergodic components.

Proof of Theorem 4.1. Let μ be an invariant measure of a C^1-generic diffeomorphism f as in the statement of Lemma 4.2. Fix a neighborhood $\mathcal{V}$ of μ in $\mathcal{M}_f(M)$ and a number $\varepsilon > 0$.

Let $\tilde{\mathcal{V}}$ denote $\mathcal{V} \cap \mathcal{M}_f(\mathrm{Supp}(\mu))$: it is a neighborhood of μ in $\mathcal{M}_f(\mathrm{Supp}(\mu))$. By the Ergodic Decomposition Theorem applied to $f|_{\mathrm{Supp}(\mu)}$ and $\tilde{\mathcal{V}}$ there is a convex combination

$$\lambda_1\mu_1 + \ldots + \lambda_k\mu_k$$

of ergodic measures which belongs to $\mathcal{V}$ and supported in $\mathrm{Supp}(\mu)$. Now, by Theorem 4.2, each ergodic component μ_i is weak-approached by periodic measures μ_{γ_i} of f whose support γ_i is contained in the ε-neighborhood of $\mathrm{Supp}(\mu_i)$ and hence of $\mathrm{Supp}(\mu)$.

Now the convex sum $\nu = \lambda_1\mu_{\gamma_1} + \ldots + \lambda_k\mu_{\gamma_k}$ is close to μ for the weak topology and its support is contained in the ε-neigborhood of $\mathrm{Supp}(\mu)$. As the support of a measure varies lower-semicontinuously with the measure in the weak topology, we get that $\mathrm{Supp}(\mu)$ and $\mathrm{Supp}(\nu)$ are close in the Hausdorff distance.

□

4.2 Periodic measures in homoclinic classes of C^1-generic diffeomorphisms

Through the use of Markov partitions, Bowen [Bow] showed that every hyperbolic basic set Λ contains periodic orbits with an arbitrarily prescribed itinerary (this is known as the *specification property*). So the invariant probabilities supported in Λ are approached in the weak topology by periodic orbits in Λ. An intermediary step for this result consists in proving that every convex sum of periodic measures in Λ is approached by periodic orbits in Λ. One thus defines:

Definition 4.5. *A set of periodic points $\Sigma \subset Per_f(M)$ has the* barycenter property *if, for any two points $p, q \in \Sigma$, any $\lambda \in [0,1]$, and $\varepsilon > 0$, there exists $x \in \Sigma$ and pairwise disjoint sets $I, J \subset \mathbb{N} \cap [0, \Pi(x))$ such that*

1. $\lambda - \varepsilon < \frac{Card(I)}{\Pi(p)} < \lambda + \varepsilon$ *and* $(1-\lambda) - \varepsilon < \frac{Card(J)}{\Pi(q)} < (1-\lambda) + \varepsilon$,
2. $d(f^m(x), f^m(p)) < \varepsilon$ *for every* $m \in I$ *and* $d(f^m(x), f^m(q)) < \varepsilon$ *for every* $m \in J$.

The barycenter property implies that for any two periodic points p, q in Σ and $\lambda \in (0,1)$ there is some periodic point $x \in \Sigma$, of very high period, which spends a portion approximately equal to λ of its period shadowing the orbit of p and a portion equal to $1-\lambda$ shadowing the orbit of q. As a consequence we get:

Remark 4.6. If a set $\Sigma \subset \mathcal{P}_f(M)$ has the barycenter property then the closure of $\{m_{\mathcal{O}(p)}, p \in \Sigma\} \subset \mathcal{M}_f(M)$ is convex.

Consider now the set Σ_p of periodic points homoclinically related to a hyperbolic periodic point p of an arbitrary diffeomorphism f. Then Σ_p is contained in an increasing sequence of basic sets contained in the homoclinic class $H(p)$. For this reason, it remains true that every convex sum of periodic measure μ_{γ_i} with $\gamma_i \in \Sigma(p)$ is approached by a periodic orbit in the basic set. From the transition property in [BDP], we thus have:

Proposition 4.7. *For any open set $V \subset M$ and any hyperbolic periodic point p whose orbit is contained in V, the set $\Sigma_{V,p}$ of periodic points related to p in V satisfies the barycenter property.*

Proposition 4.7 does not hold a priori for the set of periodic orbits in an homoclinic class $H(\mathcal{O})$ in particular in the case where $H(\mathcal{O})$ contains periodic points of different indices (which thus are not homoclinically related). However when two hyperbolic periodic orbits γ_1, γ_2 of different

indices are related by a heterodimensional cycle, [ABCDW] shows that one can produce, by arbitrarily small C^1-perturbations, periodic orbits which spend a prescribed proportion of time shadowing the orbit of γ_i, $i = 1, 2$. Furthermore, if γ_1 and γ_2 are robustly in the same chain recurrence class, then the new orbits also belongs to the same class. This allows one to prove that the barycenter property holds generically:

Proposition 4.8. *Let f be a C^1-generic diffeomorphism and $\mathcal{O}$ a hyperbolic periodic orbit and $V \subset M$ be an open set. Then the set of periodic orbits contained in $V \cap H_V(\mathcal{O})$ satisfies the barycenter property.*

Notice that this proposition together with Remark 4.6 implies Theorem 3.10.

Proof. We first give the proof for whole homoclinic classes (i.e. when $V = M$). According to [BD], for every C^1-generic diffeomorphism f and every periodic point p, q of f the homoclinic classes are either equal or disjoint; furthermore, if $H(p) = H(q)$ then there is an open neighborhood $\mathcal{U}$ of f such that for every generic $g \in \mathcal{U}$ the homoclinic classes of the continuations of p and q for g are equal; moreover if $H(p) = H(q)$ and if p and q have the same index, then they are homoclinically related. Hence the barycenter property is satisfied for pairs of point of the same index in an homoclinic class.

Hence we now assume that $H(\mathcal{O})$ contains periodic points p and q with different indices, and we fix some number $\lambda \in (0, 1)$. We want to prove the barycenter property for p, q and λ. Notice that the homoclinic classes of p and q are not trivial and from [BC], one may assume that they coincide with $H(\mathcal{O})$. The next lemma will allow us to assume that p and q have all their eigenvalues real, of different modulus, and of multiplicity equal to 1.

Lemma 4.9. *Let f be a C^1-generic diffeomorphism and p be a periodic point of f whose homoclinic class is non-trivial. Then for every $t \in (0, 1)$ and $\varepsilon > 0$ there is a periodic point p_ε homoclinically related with p, and a segment $I = \{i, i + 1, \dots, i + j\} \subset [0, \Pi(p_\varepsilon) - 1] \cap \mathbb{N}$ with $1 - \frac{j}{\Pi(p)} < \varepsilon$ such that:*

- *for every $k \in \{0, \dots, j\}$ one has $d(f^{i+k}(p_\varepsilon), f^k(p)) < \varepsilon$,*
- *the eigenvalues of $Df^{\Pi(p_\varepsilon)}(p_\varepsilon)$ are real; have different modulus, and multiplicity equal to 1;*
- *p and p_ε have the same index of p.*

Proof. The proof consists in considering periodic orbits of very large period shadowing the orbit of an homoclinic intersection associated to p. An arbitrarily small perturbation of the derivative of such orbits produces eigenvalues that are real, have different modulus and multiplicity 1. As this property is an open property, the genericity assumption implies that f already exhibits the announced periodic orbits, without needing perturbations. □

Notice that if, for every $\varepsilon > 0$, the barycenter property is satisfied for p_ε, q and λ, then it also holds for p, q and λ. Hence we may assume that the points p and q have different indices and have all their eigenvalues real, of different modulus, and of multiplicity equal to 1. For fixing the idea one assume $\dim W^s(p) < \dim W^s(q)$. Furthermore $H(p) = H(q)$ from [BD] and this property persists for any C^1-generic diffeomorphism close to f.

The end of the proof now follows from [ABCDW]; however there is no precise statement in this paper of the result we need. For this reason we recall here the steps of the proof. First by using Hayashi connecting lemma, one creates an heterodimensional cycle associated to the points p and q: one has $W^u(\mathcal{O}(p)) \pitchfork W^s(\mathcal{O}(q)) \neq \emptyset$ and $W^s(\mathcal{O}(p)) \cap W^u(\mathcal{O}(q)) \neq \emptyset$. Then [ABCDW, Lemma 3.4] linearizes the heterodimensional cycle producing an *affine heterodimensional cycle*. This heterodimensional cycle [ABCDW, Section 3.2] produces, for every large ℓ, m, a periodic

point $r_{\ell,m}$ whose orbit spends exactly $\ell.\Pi(p)$ times shadowing the orbit of p and $m.\Pi(q)$ times shadowing the orbit of q and an bounded time outside a small neighborhood of these two orbits. So, we can choose ℓ and m such that the orbit of $r_{\ell,m}$ spends a proportion of time close to the orbit of p which is almost λ and a proportion of time close to the orbit of q which is almost $1-\lambda$. Furthermore, one has $W^u(\mathcal{O}(p)) \pitchfork W^s(r_{\ell,m}) \neq \emptyset$ and- $W^u(r_{\ell,m}) \pitchfork W^s(\mathcal{O}(q)) \neq \emptyset$. One deduces that, for any C^1-generic diffeomorphism in an open set close to f, $H(r_{\ell,m}) = H(p) = H(q) = H(\mathcal{O})$. Since f is generic, the class $H(\mathcal{O})$ for f already contained periodic orbits that satisfy the barycenter property.

In the proof for relative homoclinic classes, there are several new difficulties: the relative homoclinic class of $\mathcal{O}$ in an open set V is the closure of periodic orbits in V related to $\mathcal{O}$ by orbits in V, but some periodic orbit may also be contained in the closure of V. Furthermore the set of open sets is not countable: hence the set of relative homoclinic classes is not countable, leading to some difficulty for performing an argument of genericity. We solved these difficulties by considering the set $Per_f(H_V(\mathcal{O}) \cap V)$ of periodic orbits of $H_V(\mathcal{O})$ which are contained in V. Then, if V is an increasing union of open subsets $\cdots \subset V_n \subset V_{n+1} \subset \cdots$ then

$$Per_f(H_V(\mathcal{O}) \cap V) = \bigcup_n Per_f(H_{V_n}(\mathcal{O}) \cap V_n).$$

This argument allows us to deal with a countable family of open sets V_i, $i \in \mathbb{N}$. One now argues in a very similar way as before (just taking care that all the orbits we use are contained in the open set V). □

4.3 Approximation of measures in isolated transitive sets

One of the main remaining open question for C^1-generic diffeomorphisms is

Question 1. *Given a C^1-generic diffeomorphism $f \in \mathrm{Diff}^1(M)$ and a homoclinic class $H(p)$ of f, is $\mathcal{P}_f(H(p))$ dense in $\mathcal{M}_f(H(p))$? In other words, is every measure supported on $H(p)$ approached by periodic orbits inside the class?*

The fact that we are not able to answer to this question is the main reason for which we will restrict the study to isolated transitive set classes, in this section.

An argument by Bonatti-Diaz [BD], based on Hayashi Connecting Lemma, shows that isolated transitive sets Λ of C^1-generic diffeomorphisms are relative homoclinic classes:

Theorem 4.10. *[BD] Given Λ an isolated transitive set of a C^1-generic diffeomorphism f and let V be an isolating open neighborhood of Λ, then*

$$\Lambda = H_V(\mathcal{O})$$

for some periodic orbit $\mathcal{O} \subset \Lambda$ of f.

Proof of Theorem 3.5 part (a). Let Λ be an isolated transitive set of a C^1-generic diffeomorphism f and μ be an invariant measure supported on Λ. According to Theorem 4.1, the measure $\mu \in \mathcal{M}_f(M)$ is approached in the weak topology by a measure ν which is the convex sum of finitely many periodic measures and whose support $\mathrm{Supp}(\nu)$ is arbitrarily close to $\mathrm{Supp}(\mu)$.

On the other hand Λ is the relative homoclinic class $H_V(p)$ of some periodic point $p \in \Lambda$ in some isolating open neighborhood V; as the support of ν is close to the support of μ one gets that $\mathrm{Supp}(\nu)$ is contained in V. As V is an isolated neighborhood of Λ the measure ν is in fact supported in Λ: hence it is the convex sum of finitely many periodic measures in $\mathcal{P}_f(\Lambda)$.

As Λ is compact and contained in V it does not contain periodic orbits on the boundary of V. Hence Theorem 3.10 implies that the closure of the set $\mathcal{P}_f(\Lambda)$ is convex; this implies that ν belongs to the closure of $\mathcal{P}_f(\Lambda)$, ending the proof. □

Proof of Theorem 3.5 part (c). Let Λ be a (non-trivial) isolated transitive set of a C^1-generic diffeomorphism f. By Theorem 4.10, every periodic point p in Λ has homoclinic class equal to Λ, and hence exhibits some transverse homoclinic orbit. This implies that there are hyperbolic horseshoes Γ arbitrarily close to this homoclinic orbit. The points in Γ spend arbitrarily large fractions of their orbits shadowing the orbit $\mathcal{O}(p)$ of p as closely as we want.

Every horseshoe Γ supports ergodic measures ν which have positive entropy. Since each such ν is supported in a hyperbolic horseshoe, it follows that ν is also uniformly hyperbolic. Now, because the periodic horseshoe Γ shadows $\mathcal{O}(p)$ along most of its orbit, it follows that ν is close in the weak topology to the periodic measure $\mu_{\mathcal{O}(p)}$ associated to the orbit of p.

Since by the Theorem 3.5 part (a) the set of periodic measures $\mathcal{P}_f(\Lambda)$ is dense in $\mathcal{M}_f(\Lambda)$, then it follows that the set of ergodic, positive-entropy, and uniformly hyperbolic measures ν as above is also dense in $\mathcal{M}_f(\Lambda)$. □

5 Ergodicity, Support, Entropy

In this section we prove three "abstract" results on generic measures, dealing respectively with their ergodicity, support, and entropy. These results, together with the Theorem 3.5 part (a), respectively imply items (b.i), (b.ii), and (b.iii) of Theorem 3.5. We also use these general results to obtain Theorem 3.1 and item (i) of Theorem 3.3.

5.1 Ergodicity

Let Λ be an isolated transitive set of a C^1-generic diffeomorphism f. By Theorem 3.5 (a), ergodicity is a dense property in $\mathcal{M}_f(\Lambda)$, since periodic measures are ergodic.

Since dense G_δ sets are residual, we need only prove that ergodicity is G_δ in the weak topology in order to conclude that ergodicity is generic in $\mathcal{M}_f(\Lambda)$. And indeed we have the following general result (which implies in particular item (b.i) of Theorem 3.5):

Proposition 5.1. *Let X be a compact metric space, $A : X \to X$ be a continuous map, and $\mathcal{M}_A(X)$ denote the space of A-invariant Borel probabilities on X, endowed with the weak topology. Then ergodicity is a G_δ property in $\mathcal{M}_A(X)$. In particular, if there exists a dense subset $\mathcal{D}$ of $\mathcal{M}_A(X)$ which consists of ergodic measures, then every generic measure in $\mathcal{M}_A(X)$ is ergodic.*

Proof. Let $\psi \in C^0(X)$ be a continuous real-valued function on X. The set

$$\mathcal{M}^{erg}_{A,\psi}(X) := \left\{ \mu \in \mathcal{M}_A(X) : \int \psi \, d\mu = \lim_{k\to+\infty} \frac{1}{k} \sum_{j=1}^{k} \psi(A^j(x)) \text{ for } \mu\text{-a.e. } x \right\}$$

of measures which are "ergodic with respect to ψ" is given by

$$\mathcal{M}^{erg}_{A,\psi}(X) = \bigcap_{\ell\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \left\{ \mu \in \mathcal{M}_A(X) : \int \| \frac{1}{n} \sum_{j=1}^{n} \psi(fA^j(x)) - \int \psi \, d\mu \| \, d\mu(x) < \frac{1}{\ell} \right\}.$$

In particular $\mathcal{M}^{erg}_{A,\psi}(X)$ is a G_δ set: the integral in the right-hand side of the bracket varies continuously with the measure μ, and so the set defined within the brackets is open in $\mathcal{M}_A(X)$; this shows that $\mathcal{M}^{erg}_{A,\psi}(X)$ is a countable intersection of open sets.

Now let $\{\psi_k\}_{k\in\mathbb{N}}$ be a countable dense subset of $C^0(X)$. By the argument above, for each $k \in \mathbb{N}$ there is some G_δ subset $\mathcal{S}_k$ of $\mathcal{M}_A(X)$ consisting of measures which are ergodic with respect to ψ_k. The measures μ which belong to the residual subset $\mathcal{S}$ of $\mathcal{M}_A(X)$ obtained by

intersecting the $\mathcal{S}_k$'s are precisely the measures which are simultaneously ergodic with respect to every ψ_k. Using standard approximation arguments one can show that such μ are ergodic with respect to *any* $\psi \in C^0(X)$, and hence are ergodic. □

Remark 5.2. Proposition 5.1 implies in particular that the space $\mathcal{M}_f^{erg}(\Lambda)$ of ergodic measures of a diffeomorphism f is a Baire space when endowed with the weak topology. Indeed, any G_δ subset $\mathcal{A}$ of a compact metric space is Baire, since $\mathcal{A}$ is residual in $\overline{\mathcal{A}}$.

5.2 Full Support

Given Λ an isolated transitive set of a C^1-generic diffeomorphism f, then Λ is a homoclinic class, and hence has a dense subset $Per_f(\Lambda)$ of periodic points. This last fact suffices to prove that generic measures on Λ have full support (item (b.ii) of Theorem 3.5), as the following general result shows:

Proposition 5.3. *Let X be a compact metric space, $A : X \to X$ be a continuous map, and $\mathcal{M}_A(X)$ denote the space of A-invariant Borel probabilities on X, endowed with the weak topology. Then every generic measure μ in $\mathcal{M}_A(X)$ satisfies*

$$\mathrm{Supp}(\mu) = \bigcup_{\nu \in \mathcal{M}_A(X)} \mathrm{Supp}(\nu).$$

In particular, if the set of periodic points of A is dense in X, then every generic μ satisfies $\mathrm{Supp}(\mu) = X$.

Proof. Consider the map

$$\begin{aligned} \Phi : \mathcal{M}_A(X) &\to \mathcal{K}(X) \\ \mu &\mapsto \mathrm{Supp}(\mu). \end{aligned}$$

It is easy to see that Φ is lower-semicontinuous. By the Semicontinuity Lemma, there is a residual subset $\mathcal{S}$ of $\mathcal{M}_A(X)$ which consists of continuity points of Φ. The following claim then concludes the proof:

Claim. *Given $\mu \in \mathcal{M}_A(X)$ a continuity point of Φ, then* $\mathrm{Supp}(\mu) = \bigcup_{\nu \in \mathcal{M}_A(X)} \mathrm{Supp}(\nu)$.

Let us now prove the claim. One considers any measure $\nu \in \mathcal{M}_A(X)$. The measures $(1-\lambda)\mu + \lambda\,\nu$ converge to μ as λ goes to zero, and hence their supports, which equal $\mathrm{Supp}(\mu) \cup \mathrm{Supp}(\nu)$, converge to $\mathrm{Supp}(\mu)$. This implies that $\mathrm{Supp}(\nu)$ is contained in $\mathrm{Supp}(\mu)$ and concludes the proof of the claim. □

5.3 Zero Entropy

The next abstract result shall allow us to prove item (b.iii) of Theorem 3.5, that is, that generic measures of an isolated transitive set have zero entropy:

Proposition 5.4. *Let X be a compact metric space, $A : X \to X$ be a continuous map, and $\mathcal{M}_A(X)$ denote the space of A-invariant Borel probabilities on X, endowed with the weak topology. Assume that there exists a sequence of measurable finite partitions $\{\mathcal{P}_k\}_{k\in\mathbb{N}}$ of X such that*

1) the partition $\mathcal{P}_{k+1}$ is finer than $\mathcal{P}_k$ for every $k \in \mathbb{N}$;

2) the product $\bigvee_{k\in\mathbb{N}} \mathcal{P}_k$ is the Borel σ-algebra of X.

Assume also that there is a dense subset $\mathcal{D}$ of $\mathcal{M}_A(X)$ such that every $\mu \in \mathcal{D}$ satisfies $\mu(\partial\mathcal{P}_k) = 0$ and $h(\mu, \mathcal{P}_k) = 0$, for every $k \in \mathbb{N}$. Then there is a residual subset $\mathcal{S}$ of $\mathcal{M}_A(X)$ such that every $\mu \in \mathcal{S}$ satisfies $h(\mu) = 0$.

Proof. By the Kolmogorov-Sinai theorem given any $\mu \in \mathcal{M}_A(X)$ then the entropy $h(\mu)$ of μ is equal to $sup_{\,k\in\mathbb{N}}\,\{h(\mu,\mathcal{P}'_k)\}$.

By assumption, given $k \in \mathbb{N}$ and $\nu \in \mathcal{D}$ then $\nu(\partial\mathcal{P}_k) = 0$. Thus ν is a point of upper-semicontinuity for the map

$$\Theta_k : \mathcal{M}_A(X) \to \mathbb{R}$$

$$\mu \mapsto h(\mu, \mathcal{P}_k).$$

Since $\Theta_k(\nu) = 0$ at every $\nu \in \mathcal{D}$, it follows that every ν is in fact a continuity point of Θ_k. Since $\mathcal{D}$ is dense in $\mathcal{M}_A(X)$, these conditions imply that there exists a residual subset $\mathcal{S}_k$ of $\mathcal{M}_A(X)$ consisting of measures μ such that $h(\mu, \mathcal{P}_k) = 0$.

Setting $\mathcal{S} := \bigcap_{k\in\mathbb{N}} \mathcal{S}_k$ we obtain a residual subset of $\mathcal{M}_A(X)$ which consists of measures μ such that $h(\mu, \mathcal{P}_k) = 0$ for every $k \in \mathbb{N}$, and hence which by the Kolmogorov-Sinai theorem satisfy $h(\mu) = 0$. □

We may now prove item (b.iii) of Theorem 3.5:

Corollary 5.5. *Given Λ an isolated transitive set of a C^1-generic diffeomorphism f, then there is a residual subset $\mathcal{S}$ of $\mathcal{M}_f(\Lambda)$ such that every $\mu \in \mathcal{S}$ has zero entropy.*

Proof. From Theorem 3.5, part (a), there is a dense subset $\mathcal{D}$ of $\mathcal{M}_f(\Lambda)$ which consists of periodic measures.

Let now $\{\mathcal{P}_k\}_{k\in\mathbb{N}}$ be a sequence of finite partitions of M into zero-codimension submanifolds of M and their boundaries such that:

1) $\partial\mathcal{P}_k \cap Per_f(\Lambda) = \emptyset$ for all $k \in \mathbb{N}$;

2) the partition $\mathcal{P}_{k+1}$ is finer than $\mathcal{P}_k$ for every $k \in \mathbb{N}$;

3) the product $\bigvee_{k\in\mathbb{N}} \mathcal{P}_k$ is the Borel σ-algebra of M.

Then the intersection of each of the partitions $\mathcal{P}_k$ with Λ yields a sequence of partitions $\{\mathcal{P}'_k\}_{k\in\mathbb{N}}$ of Λ which satisfy conditions (1)-(3) above (replacing M by Λ in condition (3)).

Clearly this sequence of partitions $\{\mathcal{P}'_k\}_{k\in\mathbb{N}}$ satisfies the hypotheses of Proposition 5.4 above, with $X = \Lambda$ and $\mathcal{D}$ the set of periodic measures supported in Λ. So there is a residual subset of $\mathcal{M}_f(\Lambda)$ consisting of measures with zero entropy. □

We can also use Proposition 5.4 to prove that generic measures of C^1-generic diffeomorphisms have zero entropy. Indeed, by Theorem 4.1, given a C^1-generic diffeomorphism f then the set of finite convex combinations of periodic measures of f is dense in $\mathcal{M}_f(\Lambda)$. Moreover, given a sequence of partition $\{\mathcal{P}_k\}_{k\in\mathbb{N}}$ of M as in the proof of Proposition 5.4, then each such combination satisfies the hypotheses of Proposition 5.4 above. So we obtain the item (i) of Theorem 3.3.

Corollary 5.6. *For any C^1-generic diffeomorphism f, every generic measure μ of f satisfies $h(\mu) = 0$.*

5.4 Ergodic measures whose support fills a homoclinic class

Given an open set $V \subset M$ and a hyperbolic periodic point p with orbit contained in V, recall that $\Sigma_{V,p}$ denotes the set of periodic orbits heteroclinically related to (the orbit of) p by orbits contained in V. Let $\mathcal{M}(\Sigma_{V,p})$ denote the set of periodic measures associated to orbits in $\Sigma_{V,p}$ and let $\mathcal{CM}(\Sigma_{V,p})$ denote the closure (in the weak topology) of the convex hull of $\mathcal{M}(\Sigma_{V,p})$. We may now state:

Theorem 3.1'. *Let $H_V(p)$ be a homoclinic class of some $f \in \mathrm{Diff}^1(M)$. Then there is a residual subset $\mathcal{S}$ of $\mathcal{CM}(\Sigma_{V,p})$ such that every $\mu \in \mathcal{S}$ is ergodic, satisfies $\mathrm{Supp}(\mu) = H_V(p)$, and has zero entropy.*

In particular, at least one such measure exists, implying Theorem 3.1. So it turns out that the proof of Theorem 3.1 – whose statement includes no genericity conditions at all – ultimately relies on generic arguments on the space of measures supported in $\Sigma_{V,p}$; this is a good illustration of the capacity of genericity arguments to yield non-generic results.

Proof of Theorem 3.1'. By Proposition 4.7, the set $\mathcal{M}(\Sigma_{V,p})$ of periodic measures associated to orbits in $\Sigma_{V,p}$ constitutes a dense subset of $\mathcal{CM}(\Sigma_p)$. That is, we have that $\mathcal{CM}(\Sigma_p) = \overline{\mathcal{M}(\Sigma_{V,p})}$.

Now, each element of $\mathcal{M}(\Sigma_{V,p})$ is ergodic and so it follows by Lemma 5.1 that there is some residual subset $\mathcal{S}_1$ of $\mathcal{CM}(\Sigma_p)$ such that every $\mu \in \mathcal{S}_1$ is ergodic. Lemma 5.3 implies that there is some residual subset $\mathcal{S}_2$ of $\mathcal{CM}(\Sigma_p)$ such that the support of every $\mu \in \mathcal{S}_2$ coincides with $\overline{\Sigma_{V,p}} = H_V(p)$. And by Lemma 5.4 there is some residual subset $\mathcal{S}_3$ of $\mathcal{CM}(\Sigma_p)$ such that every $\mu \in \mathcal{S}_3$ has zero entropy. Set $\mathcal{S} := \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3$ and we are done. □

6 Approximation of Lyapunov Exponents by Periodic Orbits

One deduces Theorem 3.8 from the following perturbative result:

Proposition 6.1. *Let μ be an ergodic invariant probability measure of a diffeomorphism f of a compact manifold M. Fix a C^1-neighborhood $\mathcal{U}$ of f, a neighborhood $\mathcal{V}$ of μ in the space of probability measures with the weak topology, a Hausdorff-neighborhood $\mathcal{K}$ of the support of μ, and a neighborhood O of $L(\mu)$ in $\mathbb{R}^d$. Then there is $g \in \mathcal{U}$ and a periodic orbit γ of g such that the Dirac measure μ_γ associated to γ belongs to $\mathcal{V}$, its support belongs to $\mathcal{K}$, and its Lyapunov vector $L(\mu_\gamma)$ belongs to O.*

Proof of Theorem 3.8. The proof is similar to the proof of Theorem 4.2. Note first that it is enough to prove the Theorem restricted to a small C^1-neighborhood $\mathcal{U}$ of an arbitrary diffeomorphism $f_0 \in \mathrm{Diff}^1(M)$. In particular, one may assume that $\log \|Df\|$ and $\log \|Df^{-1}\|$ are bounded by some constant $S > 0$ for any $f \in \mathcal{U}$.

Let X be the space of triples (μ, K, L) where μ is a probability measure on M, $K \subset M$ is a compact set, and $L \in [-S, S]^d$, endowed with the product topology of the weak topology on the probability measures, the Hausdorff topology on the compact subspaces of M, and the usual topology on $\mathbb{R}^d$.

To any periodic orbit γ of a diffeomorphism f we associate a triple $x_\gamma = (\mu_\gamma, \gamma, L(\mu_\gamma))$. We denote by X_f the closure of the set $\{x_\gamma, \gamma \in Per(f)\}$. This is a compact subset of X, and hence an element of the space $\mathcal{K}(X)$ of compact subsets of X endowed with the Hausdorff topology.

One easily verifies that the map $f \mapsto X_f$ is lower semi-continuous on the set of Kupka-Smale diffeomorphisms, which is residual in $\mathcal{U}$. As a consequence, this map is continuous on a residual subset $\mathcal{R} \cap \mathcal{U}$ of the set of Kupka-Smale diffeomorphisms, hence of $\mathcal{U}$.

Consider $f \in \mathcal{R} \cap \mathcal{U}$ and μ an ergodic probability measure of f. Proposition 6.1 allows us to create a periodic orbit γ such that x_γ is arbitrarily close to $(\mu, \mathrm{Supp}(\mu), L(\mu))$; a small perturbation makes this periodic orbit hyperbolic, and hence persistent by perturbations; a new small perturbation yields a Kupka-Smale diffeomorphism. Since f is a continuity point of $g \mapsto X_g$ in the set of Kupka-Smale diffeomorphisms, one has shown that $(\mu, \mathrm{Supp}(\mu), L(\mu))$ belongs to X_f, which implies the theorem. □

6.1 Approximation by perturbation: proof of Proposition 6.1

We fix an ergodic measure μ of a diffeomorphism f. Let $\lambda_1 < \cdots < \lambda_k$ be the Lyapunov exponents of μ and for every i let d_i be the multiplicity of the Lyapunov exponent λ_i.

We consider a regular point x for μ in the following sense:

- The probability measures $\frac{1}{n}\sum_0^{n-1}\delta_{f^i(x)}$ and $\frac{1}{n}\sum_0^{n-1}\delta_{f^{-i}(x)}$ converge to μ as $n \to +\infty$.
- x has well-defined Lyapunov exponents and its exponents are those of μ. Moreover there is a splitting $T_xM = E_1 \oplus \cdots \oplus E_k$, such that:
  - $dim(E_i) = d_i$;
  - the number $\frac{1}{n}\log(\|Df^n(u)\|)$ converges uniformly to λ_i on the set of unit vectors u of E_i as n tends to $+\infty$;
  - the angle between between the Lyapunov spaces $Df^n(E_i)$ and $Df^n(E_j)$ decreases at most subexponentially:
$$\lim_{n\to+\infty}\frac{1}{n}\log\sin(\angle E_iE_j) = 0.$$
- x is well closable: for any C^1-neighborhood $\mathcal{U}$ of f, any $\varepsilon > 0$ any $N > 0$ there is $n > N$ and $g \in \mathcal{U}$ such that x is periodic of period n for g and $d(g^i(x), f^i(x)) < \varepsilon$ for $i \in \{1, \dots, n\}$.

The set of regular points for μ has full measure for μ, according to the Birkhoff ergodic theorem, the Oseledets subadditive theorem, and Mañé's ergodic closing lemma. In particular, such a point x exists. Fix a local chart at x such that E_i coincides with the space

$$E_i = \{0\}^{\sum_1^{i-1} d_\ell} \times \mathbb{R}^{d_i} \times \{0\}^{\sum_{i+1}^{k} d_\ell}$$

and a Riemannian metric on M which coincides with the Euclidian metrics on this local chart. For $i \le j$ we denote $E_{i,j} = E_i \oplus \cdots \oplus E_j$.

Given a number $C > 0$ and two linear subspaces $E, F \subset T_xM$ having the same dimension, we will say that *the inclination of F with respect to E is less than C* if F is transverse to the orthogonal space $E^\perp$ and if F is the graph of a linear map $\varphi\colon E \to E^\perp$ of norm bounded by C.

We divide the proof of Proposition 6.1 into two main steps stated now. In the first step we build the perturbation, and in the second step we verify the announced properties.

Lemma 6.2. *For every C^1-neighborhood $\mathcal{U}$ of f, and any $\varepsilon > 0$ there are:*

- *a number $C > 0$,*
- *a sequence (ε_n) of positive numbers with $\lim_{n\to\infty}\varepsilon_n = 0$,*
- *a sequence of integers $t_n \to +\infty$,*
- *a sequence of linear isometries $P_n \in O(\mathbb{R}, d)$ such that $\|P_n - Id\| < \varepsilon$,*
- *a sequence of diffeomorphisms $f_n \in \mathcal{U}$,*

with the following properties:

a) The point x is periodic of period t_n for f_n.

b) The distance $d(f^t(x), f_n^t(x))$ remains bounded by ε_n for $t \in \{0, \dots, t_n\}$. In particular the point $f^{t_n}(x)$ belongs to the local chart we fixed at $x = f_n^{t_n}(x)$. This allows us to consider the derivative $Df^{t_n}(x)$ as an element of $GL(\mathbb{R}, d)$.

c) The expression of $Df_n^{t_n}(x)$ in the local coordinates at x coincides with $P_n \circ Df^{t_n}(x)$.

d) For every $i \leq j \in \{1, \dots, k\}$ the inclination of $Df_n^{t_n}(x).E_{i,j}$ with respect to $E_{i,j}$ is less than C.

As x is a regular point of μ one gets that the Dirac measures along the (periodic) orbits of x for f_n converge weakly to μ, and that the orbits themselves converge to the support of μ in the Hausdorff topology as $n \to \infty$. Then we conclude the proof of Proposition 6.1 by proving:

Lemma 6.3. *The Lyapunov vectors of the orbits of x by f_n converge to the Lyapunov vector of μ for f when $n \to +\infty$.*

6.2 Building the perturbations: Proof of Lemma 6.2

We cover M by finitely many local charts $\varphi_i \colon V_i \to \mathbb{R}^d$ and we choose open subsets $W_i \subset V_i$, relatively compact in V_i, such that the W_i cover M. For every $t \in \mathbb{Z}$ we fix $i(t)$ such that $f^t(x) \in W_{i(t)}$.

Shrinking ε if necessary we may assume that:

- every 10ε perturbation of f in $\mathrm{Diff}^1(M)$ is contained in $\mathcal{U}$,
- ε is smaller than the infima of the distances between W_i and the complement of V_i.

We fix $\varepsilon_0 \in (0, \varepsilon)$ such that for any $y \in W_i$ and any point z such that $d(y, z) < \varepsilon_0$ then:

- The distance $d(f(y), f(z))$ is smaller than ε. In particular given any j with $f(y) \in W_j$ one has $z \in V_i$ and $f(z) \in V_j$.
- There is a linear map $A \colon T_zM \to T_{f(z)}M$ such that $\|A - Df(z)\| < \varepsilon$ and such that the expressions of A and of $Df(y)$ in the charts V_i and V_j coincide.
- If g is a ε_0-C^1-perturbation of f then $d(g(y), f(y)) < \varepsilon$, so that $g(y) \in V_j$. Furthermore, there is a linear map $B \colon T_yM \to T_{g(y)}M$ such that $\|B - Dg(y)\| < \varepsilon$ and such that the expressions of B and of $Df(y)$ in the charts V_i and V_j coincide.

We fix now a sequence $0 < \varepsilon_i < \varepsilon_0$ decreasing to 0. As the point x is well closable, there is a sequence of ε_n-perturbations h_n of f and integers $t_n \in \mathbb{N}$ with $t_n \to +\infty$ such that:

- the point x is periodic of period t_n for h_n;
- the distance $d(f^t(x), h_n^t(x))$ remains bounded by ε_n for $t \in \{0, \dots, t_n\}$.

The diffeomorphism f_n built below will preserve the orbit of x by h_n, and hence items a) and b) of the lemma will be satisfied.

As h_n is ε_n-close to f, for every $t \in \{0, t_n - 1\}$ the map $B_t \colon T_{h_n^t(x)}M \to T_{h_n^{t+1}(x)}M$ whose expression in the coordinates $V_{i(t)}, V_{i(t+1)}$ is $Df(h_n^t(x))$ satisfies $\|B_t - Dh_n(h_n^t(x))\| < \varepsilon$. As $d(f^t(x), h_n^t(x)) < \varepsilon_n$, the linear map $A_t \colon T_{h_n^t(x)}M \to T_{f(h_n^t(x))}M$ whose expression in the coordinates $V_{i(t)}, V_{i(t+1)}$ is $Df(f^t(x))$ satisfies $\|A_t - Df(h_n^t(x))\| < \varepsilon$. One deduces

$$\|Dh_n(h_n^t(x)) - Df^(f^t(x))\| < 2\varepsilon.$$

By Franks Lemma, there is a diffeomorphism g_n such that $g_n = h_n$ on the periodic orbit of x, g_n is 3ε-C^1-close to h_n, $g_n = h_n$ out of an arbitrarily small neighborhood of the orbit of x and such that the expression of Dg at the point $g_n^t(x) = h_n^t(x)$ in the coordinates $V_{i(t)}, V_{i(t+1)}$ is A_t, i.e. the same as $Df(f^t(x))$.

As a consequence g_n is a 4ε-perturbation of f satisfying:

- g_n preserves the orbit of x by h_n,
- the expression of $Dg_n^{t_n}(x)$ in the local coordinates $V_{i(0)}$ is the same as $Df^{t_n}(x)$.

In order to conclude the proof of Lemma 6.2 we need only to control the inclinations. For that we will prove

Claim 6.4. *Given any $\eta > 0$ and given an integer $l > 0$ there is $C > 0$ such that, given any pair of l-uples $(F_1, \dots, F_l), (G_1, \dots, G_l)$ of vector subspaces of $\mathbb{R}^d$ such that dim F_j = dim G_j for all $j \in \{1, \dots, l\}$, there is an orthogonal matrix P such that $\|P - id\| < \eta$, and the inclination of $P(G_j)$ with respect to F_i is less that C.*

Proof. The proof is done by induction on l. Assuming the result obtained for $l-1$ and $\eta/2$, we perform a very small perturbation of the matrix P to control the inclination of $P(G_l)$ with respect to F_l while keeping the other inclinations smaller than $2C$. □

Let $K > 0$ be a bound on $\|Dg\|$ for any $g \in \mathcal{U}$ and fix $\eta \in (0, \varepsilon K^{-1})$. There exists $C > 0$ such that the claim is satisfied for any $\ell = d^2$. Hence, there exists $P_n \in O(\mathbb{R}, d)$ with $\|P_n - id\| < \eta \leq \varepsilon$ such that $P_n Dg_n(g_n^{t_n - 1}(x))$ is an ε-perturbation of $Dg_n(g_n^{t_n-1}(x))$. Now, applying once more Franks Lemma, we obtain a sequence f_n satisfying:

- f_n is a 6ε-perturbation of f, and hence belongs to $\mathcal{U}$,
- f_n preserves the orbit of x by h_n, and hence satisfies items a) and b) of the lemma,
- the expression of $Df_n^{t_n}(x)$ in the local coordinates $V_{i(0)}$ is $P_n \circ Df^{t_n}(x)$,
- for $i \leq j \in \{1, \dots, k\}$ the inclination of $Df_n^{t_n}(x).E_{i,j}$ with respect to $E_{i,j}$ is less than C.

This ends the proof of Lemma 6.2

6.3 Lyapunov exponents: Proof of Lemma 6.3

We consider the Lyapunov spaces $E_1, \dots, E_k$ of x and for every $j \in \{1, \dots, k-1\}$ we denote $F_j = E_1 \oplus \cdots \oplus E_j = E_{1,j}$ and $G_j = E_{j+1} \oplus \cdots \oplus E_k = E_{j+1,k}$. Recall that $m(A)$ denotes the minimal expansion of a linear automorphism $A \in GL(\mathbb{R}, d)$.

Lemma 6.5. *For any $\nu > 0$ there is $n_\nu \geq 1$ such that for any $n \geq n_\nu$ and $j \in \{1, \dots, k-1\}$ one has:*

$$\tfrac{1}{n}\log(\|Df^n|_{F_i}\|) \;\leq\; \lambda_j + \tfrac{1}{2}\nu \quad \textit{and} \quad \tfrac{1}{n}\log(m(Df^n|_{G_i})) \;\geq\; \lambda_{j+1} - \nu, \tag{1}$$

$$\tfrac{1}{n}\log(m(Df^{-n}|_{F_i})) \;\geq\; -\lambda_j - \tfrac{1}{2}\nu \quad \textit{and} \quad \tfrac{1}{n}\log(\|(Df^{-n}|_{G_i}\|) \;\leq\; -\lambda_{j+1} + \nu. \tag{2}$$

Proof. This is an easy consequence of Oseledets theorem: the rate of expansion on the Lyapunov space E_i converges uniformly to the Lyapunov exponents by positive and negative iterations, together with the fact that the angles between the images of the Lyapunov spaces decrease subexponentially with the number of iterations. □

For $K > 0$, let $\mathcal{C}^u_{j,K}$ be the cone of vectors whose inclination with respect to G_j is smaller than K:

$$\mathcal{C}^u_{j,K} = \{v = v^s + v^u \in T_xM : v^s \in F^j, v^u \in G_j, \|v^s\| \leq K\|v^u\|\}.$$

We denote by $\mathcal{C}^s_{j,K}$ the closure of $T_xM \setminus \mathcal{C}^u_{j,K}$. Note that, one has $Df_n^{t_n}(G_j) \subset \mathcal{C}^u_{j,C} \subset \mathcal{C}^u_{j,2C} \subset \mathcal{C}^u_{j,4C}$.

Lemma 6.6. *For every $\nu > 0$ there is $n'_\nu > 0$ such that for any $n \geq n'_\nu$ and $j \in \{1, \dots, k\}$ one has:*

- *The cone $\mathcal{C}^u_{j,4C}$ is strictly invariant; more precisely:*

$$Df_n^{t_n}(\mathcal{C}^u_{j,4C}) \subset \mathcal{C}^u_{j,2C} \subset \mathcal{C}^u_{j,4C}.$$

  As a consequence, the cone $\mathcal{C}^s_{j,4C}$ is strictly invariant by $Df_n^{-t_n}$.

- *For every unit vector $v \in \mathcal{C}^u_{j,4C}$ one has*

$$\frac{1}{t_n} \log \|Df_n^{t_n}(v)\| \geq \lambda_{j+1} - \nu.$$

- *For every unit vector $w \in \mathcal{C}^s_{j,4C}$ one has*

$$\frac{-1}{t_n} \log \|Df_n^{-t_n}(w)\| \leq \lambda_j + \nu.$$

Proof. Consider $v = v^s + v^u \in \mathcal{C}^u_{j,4C}$, $v^s \in F_j$ and $v^u \in G_j$. By definition we have $\|v^s\| \leq 4C\|v^u\|$. For $n \geq n_\nu$ we get from equation (1) that:

$$\begin{aligned} \|Df^n(v^s)\| &\leq e^{n(\lambda_j + \frac{1}{2}\nu)}\|v^s\| \\ &\leq 4Ce^{n(\lambda_j + \frac{1}{2}\nu)}\|v^u\| \\ &\leq 4Ce^{n(\lambda_j - \lambda_{j+1} + \nu)}\|Df^n(v^u)\|. \end{aligned}$$

Hence,

$$\frac{\|Df^n(v)\|}{\|v\|} \geq \frac{\|v^u\|}{\|v\|} \frac{\|Df^n(v^u)\| - \|Df^n(v^s)\|}{\|v^u\|} \geq \frac{1}{\sqrt{1+16C^2}}(1 - 4Ce^{n(\lambda_j - \lambda_{j+1} + \nu)})e^{n(\lambda_{j+1} - \frac{\nu}{2})}.$$

Notice that $4Ce^{n(\lambda_j - \lambda_{j+1} + \nu)}$ tends to 0 when $n \to +\infty$. In particular for n large one has:

$$\inf \left\{ \frac{1}{n} \log \frac{\|Df^n(v)\|}{\|v\|}, v \in \mathcal{C}^u_{j,4C} \right\} \geq \lambda_{j+1} - \frac{3}{4}\nu.$$

Recall that the expression in the chart at x of $Df_n^{t_n}$ is the same as $P_n \circ Df^{t_n}$, where P_n is an isometry. It follows that for n large enough one has

$$\inf \left\{ \frac{1}{n} \log \frac{\|Df_n^{t_n}(v)\|}{\|v\|}, v \in \mathcal{C}^u_{j,4C} \right\} \geq \lambda_{j+1} - \nu.$$

Furthermore, P_n has been chosen in such a way that $Df_n^{t_n}(v^u)$ belongs to the cone $\mathcal{C}^u_{j,C}$. As a consequence, for $4Ce^{t_n(\lambda_j - \lambda_{j+1} + \nu)}$ small enough the vectors $Df_n^{t_n}(v) = Df_n^{t_n}(v^u) + Df_n^{t_n}(v^s)$ belong to $\mathcal{C}^u_{j,2C}$, for all $v \in \mathcal{C}^u_{j,4C}$. This proves the two first items of the Lemma 6.6.

Consider now $w = w^s + w^u \in \mathcal{C}^s_{j,4C}$ with $w^s \in F_j$ and $w^u \in G_j$. By hypothesis one has $\|w^u\| \leq \frac{1}{4C}\|w^s\|$. Let us decompose $\tilde{w} := P^{-1}(w)$ as $\tilde{w} = \tilde{w}^s + \tilde{w}^u$ with $\tilde{w}^s \in F_j$,$\tilde{w}^u \in G_i$. Since $\|P_n^{-1} - id\| = \|P_n - id\| < \varepsilon$, one deduces that

$$\|\tilde{w}^u\| \leq \frac{\frac{1}{4C} + \varepsilon}{1 - \varepsilon}\|w^s\|.$$

We denote by $\bar{w}^s$ and $\bar{w}^u$ the vectors of $T_{f^{t_n}(x)}M$ whose expressions in the local coordinates at x are equal to those of $\tilde{w}^s$ and $\tilde{w}^u$, respectively. Note that, by construction, $Df_n^{-t_n}(w) = Df^{-t_n}(\bar{w}^u) + Df^{-t_n}(\bar{w}^s)$. The proof of the third item consists now in estimating and comparing the norms $\|Df^{t_n}(\bar{w}^u)\|$ and $\|Df^{t_n}(\bar{w}^s)\|$ using equation (2) instead of equation (1), in a similar way as above. □

Let us now end the proof of Lemma 6.3.

Proof of Lemma 6.3. Fix ν smaller than $\frac{1}{10}\inf_{i\neq j}\{|\lambda_i - \lambda_j|\}$ and consider $n > n'_\nu$. Then Lemma 6.6 implies:

- $Df_n^{t_n}(x)$ admits a (unique) invariant vector space G_i^n of dimension $\dim(G_i)$ in $\mathcal{C}^u_{i,4C}$.
- The restriction of $Df_n^{t_n}(x)$ to G_i^n has a minimal dilatation larger than $\lambda_{i+1} - \nu$.
- $Df_n^{t_n}(x)$ admits a (unique) invariant vector space F_i^n of dimension $\dim(F_i)$ in $\mathcal{C}^s_{i,4C}$.
- The restriction of $Df_n^{t_n}(x)$ to F_i^n has norm smaller than $\lambda_i + \nu$.

Set $E_i^n := F_i^n \cap G_{i-1}^n$. It is a vector space of dimension at least $\dim(F_i^n) + \dim(G_{i-1}^n) - d = \dim(E_i)$. Furthermore, one has

$$\lambda_i - \nu \leq m(Df_n^{t_n}(x)|_{E_i^n}) \leq \|Df_n^{t_n}(x)|_{E_i^n}\| \leq \lambda_i + \nu.$$

As $\lambda_i + \nu < \lambda_{i+1} - \nu$, one deduces that the sum $E_1^n + \cdots + E_k^n$ is a direct sum. It follows that $\dim(E_i^n) \leq \dim(E_i)$, and hence $\dim(E_i^n) = \dim(E_i)$. Hence x has $\dim(E_i)$ Lyapunov exponents contained in $[\lambda_i - \nu, \lambda_i + \nu]$. This proves that for n large the Lyapunov vector of the measure associated to the f_n-orbit of x is ν-close to the Lyapunov vector of μ, ending the proof of Lemma 6.3. □

7 Generic Nonuniform Hyperbolicity

In this section we obtain the nonuniform hyperbolicity of the generic measures over an isolated transitive set (items (b.iv) and (b.v) of Theorem 3.5), and also of the generic ergodic measures of C^1-generic diffeomorphisms (item (ii) of Theorem 3.3). We also give the proof of Corollary 3.9 which approximates an ergodic measure by period measures whose Lyapunov exponents are almost constant on the bundles of the finest dominated splitting.

7.1 Approximation by periodic orbits with mean Lyapunov exponents

Since it is very similar to the proofs of Lemma 4.2 and Theorem 3.8, we now only sketch out the proof of Corollary 3.9. This uses [BGV], which constructs perturbations on sets of periodic orbits which exhibit a lack of domination. In our context we may state this tool in the following way:

Theorem 7.1 ([BGV]). *Let $\{\gamma_k\}$ be a family of hyperbolic periodic orbits of $f \in \mathrm{Diff}^1(M)$ and $F_1 \oplus_< \cdots \oplus_< F_k$ be the finest dominated splitting over $\overline{\cup_{k\in\mathbb{N}}\gamma_k}$. Assume that there is no infinite subset Γ of $\cup_{k\in\mathbb{N}}\gamma_k$ such that the finest dominated splitting over $\overline{\Gamma}$ is strictly finer than $F_1 \oplus_< \cdots \oplus_< F_k$. Then given any $\varepsilon > 0$ there is an ε-perturbation g of f such that g exhibits a periodic orbit, coinciding with one of the original orbits, and whose Lyapunov exponents inside each bundle F_i all coincide.*

Proof of Corollary 3.9. Consider X the space of triples (μ, K, L) where μ is a probability measure on M, $K \subset M$ is a compact set, and L is a vector in $\mathbb{R}^d$, endowed with the usual product topology. If γ is a periodic orbit, we denote by x_γ the triple $(\mu_\gamma, \gamma, L(\mu_\gamma))$ (the measure associated to γ, its support γ, and its Lyapunov vector $L(\mu_\gamma)$). As in the proof of Theorem 3.8, the map $f \mapsto X_f$, which to each diffeomorphism f associates the closure X_f of the set $\{x_\gamma, \gamma \in Per(f)\}$, is continuous on a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$.

Consider now, for such a C^1-generic f, a triple of the form $(\mu, supp(\mu), v)$, where μ is a generic (and hence ergodic) measure supported in $\mathrm{Supp}(\mu)$ and v is the vector given by

$$v = \left\{\frac{\int \log \|det\, Df|_{F_1}\|\, d\mu}{\dim(F_1)}\right\}^{\dim(F_1)} \times \left\{\frac{\int \log \|det\, Df|_{F_2}\|\, d\mu}{\dim(F_2)}\right\}^{\dim(F_2)} \times \ldots \times \ldots \times \left\{\frac{\int \log \|det\, Df|_{F_k}\|\, d\mu}{\dim(F_k)}\right\}^{\dim(F_k)},$$

where the F_i are the bundles of the finest dominated splitting on $\mathrm{Supp}(\mu)$.

We claim that $(\mu, \mathrm{Supp}(\mu), v) \in X_f$, which proves Proposition 3.9. By Theorem 3.8, there is a sequence of periodic orbits γ_k such that $(\mu_{\gamma_k}, \gamma_k, L(\gamma_k))$ accumulate on $(\mu, \mathrm{Supp}(\mu), L(\mu))$. Since these orbits Hausdorff-accumulate on $\mathrm{Supp}(\mu)$, it follows that for large enough K the set $\{\gamma_k\}_{k\geq K}$ admits as its finest dominated splitting a continuation of the dominated splitting $F_1 \oplus_< \ldots \oplus_< F_k$ over $\mathrm{Supp}(\mu)$, so that no subsequence of $\{\gamma_k\}_{k\geq K}$ admits a finer dominated splitting. Now an application of Theorem 7.1 yields after a small perturbation a periodic orbit γ' whose Lyapunov exponents inside each F_i all coincide. Up to performing a new perturbation we obtain a triple $(\mu_{\gamma''}, \gamma'', L(\mu_{\gamma''}))$ close to $(\mu, \mathrm{Supp}(\mu), v)$ for some C^1-generic $g \in \mathcal{G}$ arbitrarily close to f. Since f is a continuity point of $f \mapsto X_f$, one gets that $(\mu, \mathrm{Supp}(\mu), v) \in X_f$, ending the proof. □

7.2 Proof of Theorem 3.5, items (b.iv) and (b.v)

In [BocV] arguments involving flags are used to obtain semicontinuity properties of the Lyapunov exponents and Lyapunov spaces *when the diffeomorphism f varies* and keeping constant a volume measure μ on M. An application of the Semicontinuity Lemma then shows that C^1-generic (conservative) diffeomorphisms are continuity points for the set of Lyapunov exponents and their corresponding Lyapunov spaces.

In our dissipative setting, identical arguments yield semicontinuous variation of the exponents *when the measure μ varies* and keeping the diffeomorphism f fixed. The Semicontinuity Lemma then yields that generic measures are continuity points for the Lyapunov exponents. That is, we have:

Proposition 7.2. *Given Λ a compact invariant set of a diffeomorphism f, then there is a residual subset $\mathcal{S}^*$ of $\mathcal{M}_f^{erg}(\Lambda)$ which consists of ergodic measures μ which are continuity points for the map*

$$\Phi : \mathcal{M}_f^{erg}(\Lambda) \to \mathbb{R}^d$$
$$\mu \mapsto L(\mu),$$

where $L(\mu) = (\lambda_1^\mu, \ldots, \lambda_d^\mu)$ denotes the Lyapunov vector of μ.

Remark 7.3. Here we state the continuity restricted to the ergodic measures simply because that makes it easier to state the continuity; furthermore, we shall only use the continuity on the set of ergodic measures.

We are now ready to prove the hyperbolicity of generic measures over isolated transitive sets of generic diffeomorphisms. In fact, we will prove something stronger:

Proposition 7.4. *Let Λ be an isolated transitive set of a C^1-generic diffeomorphism f, with finest dominated splitting $F_1 \oplus_< \ldots \oplus_< F_k$ over $T_\Lambda M$. Then there is a residual subset $\mathcal{S}$ of $\mathcal{M}_f(\Lambda)$ such that for any measure $\mu \in \mathcal{S}$ and any $i \in \{1, \ldots, k\}$ there is only one Lyapunov exponent λ_i of μ in F_i, which furthermore is non-zero.*

Remark 7.5. The Proposition above shows that even if Λ is nonhyperbolic, and thus contains periodic orbits of distinct indices (see [BDPR]), the generic hyperbolic measures it supports may all have the same index. Indeed, by the Proposition the indices of the generic μ's are restricted

by the (dimensions of the) bundles of the finest dominated splitting over Λ. There are examples of nonhyperbolic robustly transitive sets – and hence of isolated transitive sets of C^1-generic diffeomorphisms – whose finest dominated splitting has only two bundles E and F, see [BonV]. Thus in such examples all of the generic measures provided by Proposition 7.4 above must have the same index (namely, the dimension of E), even though the set Λ is nonhyperbolic and thus contains periodic orbits of distinct indices.

Proof of Proposition 7.4. Let Λ be a non-trivial isolated transitive set of a C^1-generic diffeomorphism f. Let us fix any bundle $F = F_i$ of the finest dominated splitting. Given any $\mu \in \mathcal{M}_f(\Lambda)$, we set

$$I(\mu) := \int \log \|det\, Df|_F\|\, d\mu. \tag{3}$$

Note that since F is a continuous bundle, $I(\mu)$ varies continuously with μ in the weak topology.

On the other hand, if $\mu \in \mathcal{M}_f^{erg}(\Lambda)$ then

$$I(\mu) = \sum_{\tilde{F}(\lambda_i)\subset F} \lambda_i \dim(\tilde{F}(\lambda_i)),$$

where λ_i and $F(\lambda_i)$ are respectively the Lyapunov exponents and the Lyapunov spaces inside F.

For any periodic measure μ_γ, Franks lemma allows one to perturb the diffeomorphism in $\mathrm{Diff}^1(M)$ so that each sum $\lambda_i + \cdots + \lambda_j$ for $1 \le i \le j \le d$ is different from zero. An easy genericity argument hence implies that under a C^1-genericity assumption on f, the quantity $I(\mu)$ never vanishes on the periodic measures of f.

Now, by Theorem 4.2, there exists a dense set $\mathcal{D} \subset \mathcal{M}_f(\Lambda)$, consisting of hyperbolic periodic measures such that $I(\nu) \neq 0$ for every $\nu \in \mathcal{D}$. Since the integral $I(\mu)$ varies continuously with μ, we conclude that $I(\mu) \neq 0$ in an open and dense subset of $\mathcal{M}_f(\Lambda)$.

If $\mu \in \mathcal{M}_f(\Lambda)$ is a generic measure, we know that it is ergodic, that $\mathrm{Supp}(\mu) = \Lambda$, that $I(\mu) \neq 0$, and that (by proposition 7.2) it is a continuity point for the map $\nu \to L(\nu)$ defined on $\mathcal{M}_f^{erg}(\Lambda)$. Using Corollary 3.9 there is a sequence of periodic orbits γ_ℓ such that $L_{|F}(\mu_{\gamma_\ell})$ converges to some single value λ_F. Since μ is a continuity point for $\nu \to L_{|F}(\nu)$ it follows that λ_F is the only Lyapunov exponent of μ in F: this proves that the finest dominated splitting on $\mathrm{Supp}(\mu)$ coincides with the Oseledets splitting of μ. Moreover we must have

$$\lambda_F = \frac{I(\mu)}{\dim(F)} \neq 0,$$

implying that μ is nonuniformly hyperbolic. □

7.3 Proof of Theorem 3.3

The argument is very similar to the proof of Proposition 7.4. Since f is C^1-generic, then for each periodic orbit γ, the sum $\lambda_i + \cdots + \lambda_j$ for $1 \le i \le j \le d$ is different from zero, where $\lambda_1, \ldots, \lambda_d$ denote the Lyapunov exponents of γ with multiplicities.

Any generic ergodic measure μ is a continuity point of the map $\mu \mapsto \mathrm{Supp}(\mu)$ on $\mathcal{M}_f^{erg}(\Lambda)$. As a consequence the finest dominated splitting $F_1 \oplus_< \cdots \oplus_< F_k$ on $\mathrm{Supp}(\mu)$ extends to the support of any ergodic measure ν close to μ in the weak topology. In particular any bundle $F = F_i$ of the finest splitting extends to $\mathrm{Supp}(\nu)$ and the map $\nu \mapsto I(\nu)$, giving the sum of the Lyapunov exponents of ν inside F, varies continuously with ν on a neighborhood of μ. By Theorem 4.2, there exists a sequence of periodic measures μ_γ which converges to μ and such that $I(\mu_\gamma) \neq 0$. Since μ is generic, one thus gets $I(\mu) \neq 0$.

By Proposition 7.2, μ it a continuity point for the map $\nu \to L(\nu)$ defined on $\mathcal{M}_f^{erg}(\Lambda)$. Using Corollary 3.9 the periodic measures may be chosen so that the Lyapunov exponents in $L_{|F}(\mu_{\gamma_\ell})$ converge to some single value λ_F. It follows that λ_F is the only Lyapunov exponent of μ in F: this proves that the finest dominated splitting on $\mathrm{Supp}(\mu)$ coincides with the Oseledets splitting of μ. The argument proves that λ_F is non-zero, and hence that μ is nonuniformly hyperbolic.

8 Invariant Manifolds for Dominated Hyperbolic Measures

In this section we will prove a stronger version of Theorem 3.11 stated in Proposition 8.9. Fix a C^1-diffeomorphism f of the manifold M and an ergodic measure μ whose support admits a dominated splitting $E \oplus_< F$. One assumes that E is non-uniformly contracted for μ (i.e. the Lyapunov exponents of μ in E are all negative); notice that we do not assume that vectors in F are (non-uniformly) expanded. We will prove the existence of stable manifolds tangent to E for μ-almost every point, and control the rate of approximation of the points in these stable manifolds.

8.1 Adapted metrics

We first build an Euclidian metric on the tangent space at μ-almost every point, depending in a measurable way on the point, and which is adapted to the tangent dynamics.

Definition 8.1. *We say that a sequence (A_n) of positive numbers* varies sub-exponentially *if for any $\eta > 0$, there exists a constant $C > 0$ such that*

$$C^{-1}.e^{-\eta.n} < A_n < C.e^{\eta.n}$$

for every $n \in \mathbb{N}$.

Proposition 8.2. *Let f be a C^1-diffeomorphism and μ be an ergodic invariant probability measure. Assume that there is a $Df-$invariant continuous subbundle $E \subset T_{\mathrm{Supp}(\mu)}M$ defined over the support of μ. Let λ_E^+ be the maximal Lyapunov exponent of the measure μ in E.*

Then for any $\varepsilon > 0$ there exists an integer $N \geq 1$ and a measurable function A from M to $(0, +\infty)$ such that :

- *the sequences $(A(f^n(x)))_{n\in\mathbb{N}}$ and $(A(f^{-n}(x)))_{n\in\mathbb{N}}$ vary sub-exponentially for each $x \in M$;*
- *if $\|.\|'_x$ denotes the metric on E_x defined by*

$$\|v\|'_x = \sum_{0\leq k<N} e^{-k.(\lambda_E^+ + \varepsilon)}.A(f^k(x)).\|D_x f^k.v\|,$$

then for μ-almost every point x, for every $v \in E_x$ one has

$$\|D_x f.v\|'_{f(x)} \leq e^{\lambda_E^+ + \varepsilon}.\|v\|'_x.$$

Remark 8.3. Since the integer N is uniformly bounded, the (measurable) metric $\|.\|'$ is quasi-conformally equivalent to the initial metric $\|.\|$.

Before proving Proposition 8.2, let us first explain how the Lyapunov exponents may be computed as a limit of Birkhoff sums given by the derivative of f.

Lemma 8.4. *Let f be a C^1-diffeomorphism, μ be an ergodic invariant probability measure, and $E \subset T_{\mathrm{Supp}(\mu)}M$ be a Df-invariant continuous subbundle defined over* $\mathrm{Supp}(\mu)$*. Let λ_E^+ be the upper Lyapunov exponent in E of the measure μ.*

Then, for any $\varepsilon > 0$, there exists an integer N_ε such that, for μ-almost every point $x \in M$ and any $N \geq N_\varepsilon$, the Birkhoff averages

$$\frac{1}{k.N}\sum_{\ell=0}^{k-1}\log\|Df^N_{|E}(f^{\ell.N}(x))\|$$

converge towards a number contained in $[\lambda^+_E, \lambda^+_E + \varepsilon)$, when k goes to $+\infty$.

Proof. The exponent λ^+_E is given by:

$$\lambda^+_E = \lim_{n\to+\infty}\frac{1}{n}\int\log\|Df^n_{|E}\|d\mu.$$

One fixes an integer $n_0 \geq 1$ large enough so that for any $n \geq n_0$ we have:

$$\left|\frac{1}{n}\int\log\|Df^n_{|E}\|d\mu - \lambda^+_E\right| \leq \frac{\varepsilon}{2}. \tag{4}$$

The measure μ is ergodic for the dynamics of f, but it may happen that μ is not ergodic for f^{n_0}. Hence, it decomposes as

$$\mu = \frac{1}{m}\left(\mu_1 + \cdots + \mu_m\right),$$

where $m \in \mathbb{N}\setminus\{0\}$ divides n_0 and each μ_i is an ergodic f^{n_0}-invariant measure such that $\mu_{i+1} = f_*\mu_i$ for each $i \pmod m$. Let $A_1 \cup \cdots \cup A_m$ be a measurable partition of (M,μ) such that $f(A_i) = A_{i+1}$ for each $i \pmod m$ and $\mu_i(A_i) = 1$.

Note that by (4), there exists $i_0 \in \{1,\ldots,m\}$, such that

$$\frac{1}{n_0}\int\log\|Df^{n_0}_{|E}\|d\mu_{i_0} \leq \lambda^+_E + \frac{\varepsilon}{2}. \tag{5}$$

For $N \geq 1$, and μ-a.e. point x, one decomposes the segment of f-orbit of length N of x as $(x, f(x), \ldots, f^{j-1}(x))$, $(f^j(x), \ldots, f^{j+(r-1).n_0-1}(x))$ and $(f^{j+(r-1).n_0}(x), \ldots, f^{N-1}(x))$ such that $j < n_0$, $j + r.n_0 \geq N$ and all the points $f^j(x), f^{j+n_0}(x), \ldots, f^{j+r.n_0}$ belong to A_{i_0}. One deduces that

$$\|Df^N_{|E}(x)\| \leq \|Df^j_{|E}(x)\|.\ \Big(\|Df^{n_0}_{|E}(f^j(x))\|\ldots \\ \ldots\|Df^{n_0}_{|E}(f^{j+(r-2).n_0}(x))\|\Big).\|Df^{N-(j+(r-1).n_0)}_{|E}(f^{j+(r-1).n_0}(x))\|.$$

Hence, for μ-almost every point one has:

$$\log\|Df^N_{|E}(x)\| \leq 2n_0.C_f + \sum_{s=0}^{r-2}\log\|Df^{n_0}_{|E}(f^{j+s.n_0}(x))\|,$$

where C_f is an upper bound for both $\log\|Df\|$ and $\log\|Df^{-1}\|$.

The point $f^j(x)$ is regular for the dynamics (μ_{i_0}, f^{n_0}). One deduces that the average $\frac{1}{k.n_0}\sum_{\ell=0}^{k-1}\log\|Df^{n_0}_{|E}(f^{j+\ell.n_0}(x))\|$ converges to $\frac{1}{n_0}\int\log\|Df^{n_0}_{|E}\|d\mu_{i_0}$. Hence

$$\lim_{k\to+\infty}\frac{1}{k.N}\sum_{\ell=0}^{k-1}\log\|Df^N_{|E}(f^{\ell.N}(x))\| \leq \frac{2n_0.C_f}{N} + \lim_{k\to+\infty}\frac{1}{k.n_0}\sum_{\ell=0}^{k-1}\log\|Df^{n_0}(f^{j+\ell.n_0}(x))\|.$$

Hence, choosing $N > \frac{4n_0.C_f}{\varepsilon}$ and using the inequality (5), one gets

$$\lim_{k\to+\infty}\frac{1}{k.N}\sum_{\ell=0}^{k-1}\log\|Df^N_{|E}(f^{\ell.N}(x))\|<\lambda^+_E+\varepsilon.$$

One the other hand, using that the norms are sub-multiplicative, one gets

$$\lim_{k\to+\infty}\frac{1}{k.N}\sum_{\ell=0}^{k-1}\log\|Df^N_{|E}(f^{\ell.N}(x))\|\geq\lim_{n\to+\infty}\frac{1}{n}\log\|Df^n(x)\|=\lambda^+_E.$$

□

One now comes to the proof of Proposition 8.2: one considers a constant $\varepsilon>0$ and an integer $N\geq 0$ given by Lemma 8.4 such that at μ-almost every point, the Birkhoff averages for f^N of the functions $x\mapsto\frac{1}{N}\log\|Df^N_{|E}(x)\|$ converge towards some numbers in $[\lambda^+_E,\lambda^+_E+\varepsilon)$. In particular the sequence $\sum_{\ell=0}^{k-1}\log\|Df^N_{|E}(f^{\ell.N}(x))\|$ is bounded by $k.N.(\lambda^+_E+\varepsilon)$ when k is large.

This allows us to define the quantity

$$A(x)=\max_{k\geq 0}\left(e^{-k.N.(\lambda^+_E+\varepsilon)}.\prod_{\ell=0}^{k-1}\|Df^N_{|E}(f^{\ell.N}(x))\|\right),\tag{6}$$

with the convention $\prod_{\ell=0}^{k-1}\|Df^N_{|E}(f^{\ell.N}(x))\|=1$ for $k=0$. Note that $A(x)\geq 1$, by definition.

The Proposition 8.2 now follows from the next two lemmas.

Lemma 8.5. *At μ-almost every point x, the metric*

$$\|v\|'_x=\sum_{0\leq j<N}e^{-j.(\lambda^+_E+\varepsilon)}.A(f^j(x)).\|D_xf^j_{|E}.v\|,$$

on E_x satisfies

$$\|D_xf.v\|'_{f(x)}\leq e^{\lambda^+_E+\varepsilon}.\|v\|'_x.$$

Proof. We write :

$$\begin{aligned}\|D_xf.v\|'_{f(x)}&=\sum_{j=0}^{N-2}e^{-j.(\lambda^+_E+\varepsilon)}.A(f^{j+1}(x)).\|D_xf^{j+1}.v\|+e^{-(N-1).(\lambda^+_E+\varepsilon)}.A(f^N(x)).\|Df^N(v)\|\\&\leq e^{\lambda^+_E+\varepsilon}.\sum_{j=1}^{N-1}e^{-j.(\lambda^+_E+\varepsilon)}.A(f^j(x)).\|D_xf^j.v\|+e^{-(N-1).(\lambda^+_E+\varepsilon)}.A(f^N(x)).\|Df^N_{|E}\|.\|v\|.\end{aligned}$$

Hence one obtains the required estimate from the following:

Claim 8.6.

$$A(f^N(x)).\|Df^N_{|E}(x)\|\leq e^{N.(\lambda^+_E+\varepsilon)}.A(x).$$

The proof of the claim is the following computation:

$$\begin{aligned}A(f^N(x))&=\max_{k\geq 0}\left(e^{-k.N.(\lambda^+_E+\varepsilon)}.\textstyle\prod_{\ell=0}^{k-1}\|Df^N_{|E}(f^{(\ell+1).N}(x))\|\right)\\&=e^{N.(\lambda^+_E+\varepsilon)}.\max_{k\geq 0}\left(e^{-(k+1).N.(\lambda^+_E+\varepsilon)}.\textstyle\prod_{\ell=1}^{k}\|Df^N_{|E}(f^{\ell.N}(x))\|\right)\\&=e^{N.(\lambda^+_E+\varepsilon)}.\max_{k\geq 1}\left(e^{-k.N.(\lambda^+_E+\varepsilon)}.\textstyle\prod_{\ell=1}^{k-1}\|Df^N_{|E}(f^{\ell.N}(x))\|\right).\end{aligned}$$

Hence

$$\begin{aligned}A(f^N(x)).\|Df^N_{|E}(x)\|&=e^{N.(\lambda^+_E+\varepsilon)}.\max_{k\geq 1}\left(e^{-k.N.(\lambda^+_E+\varepsilon)}.\textstyle\prod_{\ell=0}^{k-1}\|Df^N_{|E}(f^{\ell.N}(x))\|\right)\\&\leq e^{N.(\lambda^+_E+\varepsilon)}.\max_{k\geq 0}\left(e^{-k.N.(\lambda^+_E+\varepsilon)}.\textstyle\prod_{\ell=0}^{k-1}\|Df^N_{|E}(f^{\ell.N}(x))\|\right)\\&=e^{N.(\lambda^+_E+\varepsilon)}\cdot A(x).\end{aligned}$$

This ends the proofs of the claim and of Lemma 8.5. □

Lemma 8.7. *At μ-almost every point x, the sequences $(A(f^n(x)))_{n\in\mathbb{N}}$ and $(A(f^{-n}(x)))_{n\in\mathbb{N}}$ vary sub-exponentially.*

Proof. For $k \in \mathbb{N}$ we consider the Birkhoff sum S_k of the function $x \mapsto -N(\lambda_E^+ + \varepsilon) + \log \|Df^N_{|E}(x)\|$ relative to the dynamics of f^N. For μ-a.e. point x, the Birkhoff average $\frac{S_k(x)}{k}$ converges when k tends to $+\infty$ towards a number $\lambda < 0$. One deduces that for any small $\eta > 0$, there exists $C > 0$ such that we have for any $k \in \mathbb{N}$:

$$(\lambda - \eta).k - C \leq S_k(x) \leq (\lambda + \eta).k + C.$$

For any integer $n \geq 0$, one has $S_k(f^{n.N}(x)) = S_{k+n}(x) - S_n(x)$ so that:

$$(\lambda - \eta)k - 2\eta n - 2C \leq S_k(f^{n.N}(x)) \leq (\lambda + \eta)k + 2\eta n + 2C.$$

In particular, using that λ is negative and $\eta < |\lambda|$, we get

$$0 \leq \max_{k\geq 0} S_k(f^{n.N}(x)) \leq 2\eta n + 2C. \tag{7}$$

This implies the subexponentiality of the sequence $\left(A(f^{n.N}(x))\right)_{n\in\mathbb{N}}$ since

$$\log A(f^{n.N}(x)) = \max_{k\geq 0} S_k(f^{n.N}(x)).$$

The subexponentiality of the sequence $(A(f^n(x)))_{n\in\mathbb{N}}$ follows from the subexponentiality of the sequence $\left(A(f^{n.N}(x))\right)_{n\in\mathbb{N}}$.

We now show the subexponentiality of the sequence $\left(A(f^{-n.N}(x))\right)_{n\in\mathbb{N}}$ for μ-almost every point. We first notice that, for $k > n$, one can decompose $S_k(f^{-n.N}(x))$ in $S_n(f^{-n.N}(x)) + S_{k-n}(x)$. Hence we have

$$\max_{k\geq 0} S_k(f^{-n.N}(x)) \leq \max_{0\leq k\leq n} S_k(f^{-n.N}(x)) + \max_{k\geq 0} S_k(x)$$

The subexponentiality of the sequence $\left(A(f^{-n.N}(x))\right)_{n\in\mathbb{N}}$ thus follows from the following claim:

Claim 8.8. *For any $\eta > 0$, there is a contant $C > 0$ such that for any $n \geq 0$,*

$$0 \leq \max_{0\leq k\leq n} S_k(f^{-n.N}(x)) \leq 2\eta n + 2C.$$

For proving the claim, we consider the Birkhoff sum $\widetilde{S_k}$ of the function $x \mapsto -N(\lambda_E^+ + \varepsilon) + \log \|Df^N_{|E}(x)\|$ for the dynamics of f^{-N}. For μ-a.e. point x, the Birkhoff averages $\frac{\widetilde{S_k}(x)}{k}$ and $\frac{S_k(x)}{k}$ converges (when k tends to $+\infty$) towards the same number $\lambda < 0$. Applying to f^{-N} the same argument we applied to f^N for proving the inequality (7), this gives that for any $\eta > 0$, there exists $C > 0$ such that for any $k, n \geq 0$ the following inequality holds:

$$\widetilde{S_k}(f^{-n.N}(x)) \leq (\lambda + \eta)k + 2\eta n + 2C. \tag{8}$$

One concludes the claim (and hence the lemma) by noticing that, for every $0 \leq k \leq n$, one has

$$S_k(f^{-n.N}(x)) = \widetilde{S_k}(f^{-(n-k).N}(x)),$$

which implies

$$0 \leq \max_{0\leq k\leq n} S_k(f^{-n.N}(x)) \leq 2\eta n + 2C.$$

□

8.2 Building the invariant manifolds

In this section we build the local stable manifolds at the regular points of an ergodic measure μ, associated to a dominated splitting $E \oplus_< F$ on the support of the measure μ, under the assumption that the largest Lyapunov exponent λ_E^+ of μ in E is negative.

We introduce a cone field on a neighborhood of $\mathrm{Supp}(\mu)$: for any $K > 0$, there exists a continuous splitting $E' \oplus F'$ on a neighborhood of $\mathrm{Supp}(\mu)$ which allows us to define the cones

$$\mathcal{C}_x^E = \{v = v_1 + v_2 \in T_xM = E'_x \oplus F'_x,\ \|v_2\| \leq K\|v_1\|\}.$$

Moreover, at any point $x \in \mathrm{Supp}(\mu)$, we have $E_x \subset \mathcal{C}_x^E$.

Theorem 3.11 is a direct consequence of the next proposition:

Proposition 8.9. *Let f be a C^1-diffeomorphism and μ be an ergodic invariant probability measure whose support admits a dominated splitting $E \oplus_< F$. Let $\lambda_E^+ < \lambda_F^-$ be the maximal Lyapunov exponent in E and the minimal Lyapunov exponent in F of the measure μ.*

If λ_E^+ is strictly negative, then at μ-almost every point $x \in M$, there exists an injectively immersed C^1-manifold $W^E(x)$ with $\dim W^E(x) = \dim E$, tangent to E_x, and which is a stable manifold: for any $\lambda \leq 0$ contained in $(\lambda_E^+, \lambda_F^-)$ and μ-a.e. point x, we have

$$W^E(x) = \left\{ y \in M,\ d(f^n(x), f^n(y)).e^{-\lambda.n} \underset{n\to+\infty}{\longrightarrow} 0 \right\}.$$

Moreover, at μ-a.e. point x there exists a local manifold $W^E_{loc}(x) \subset W^E(x)$ satisfying:

1. *$W^E_{loc}(x)$ is an embedded C^1-disk centered at x, of radius $L^E(x)$ and tangent to $\mathcal{C}^E$;*
2. *the sequence $\left(L^E(f^n(x))\right)_{n\in\mathbb{Z}}$ varies sub-exponentially;*
3. *$W^E(x) = \bigcup_{n\geq 0} f^{-n}(W^E_{loc}(f^n(x)))$.*

The aim of Sections 8.2 and 8.3 is the proof of Proposition 8.9. In this section (Section 8.2) we build the local stable manifolds $W^E_{loc}(x)$ and we prove the exponential decay of $d(f^n(x), f^n(y))$ for $y \in W^E_{loc}(x)$. Section 8.3 ends the proof by showing that this exponential decay characterizes the points in the stable manifold. In fact the proof does not use the adapted metric built in the previous section, but the function A provided by Proposition 8.2.

Our main tool is the plaque family theorem [HPS, theorem 5.5] of Hirsch-Pugh-Shub:

Theorem (Plaque family theorem, Hirsch-Pugh-Shub). *Let f be a C^1-diffeomorphism and K be an f-invariant compact set admiting a dominated splitting $E \oplus_< F$. Then, there exists a continuous family $(\widehat{D_x^E})_{x\in K}$ of embedded C^1-disks such that:*

- *for every $x \in K$, the disk $\widehat{D_x^E}$ is centered at x and tangent to E_x;*
- *the family $(\widehat{D_x^E})$ is locally invariant: there exists $\delta_0 > 0$ such that for each $x \in K$, the disk centered at x of radius δ_0 and contained in $\widehat{D_x^E}$ is mapped by f into $\widehat{D^E_{f(x)}}$.*

In order to prove Proposition 8.9, we fix a small positive constant $\varepsilon < -\frac{\lambda_E^+}{3}$. In the previous section we obtained an integer N and a measurable map $A \geq 1$ associated to the bundle E and to ε, which is well-defined on the set of μ-regular points. Let $C_f > 1$ be a bound on the norm of the derivative Df.

One also chooses a small constant $\delta_1 \in (0, \delta_0)$, so that for any point $x \in \mathrm{Supp}(\mu)$, any point $y \in \widehat{D_x^E}$ with $d(x, y) < \delta_1$ and any vector $v \in T_y\widehat{D_x^E}$, we have

$$\|D_y f^N.v\| \leq e^{N.\varepsilon}\|Df_E^N(x)\|.\|v\|. \tag{9}$$

Consider now $\delta > 0$. For every μ-regular point x we denote by $D_x \subset \widehat{D^E_x}$ the disk centered at x of radius $L^E(x) = \delta/A(x)$. By choosing δ small enough, the disk D_x is tangent to the cone field $\mathcal{C}^E$.

The next lemma shows that, choosing $\delta > 0$ small enough, the disk D_x is contained in the stable manifold at x.

Lemma 8.10. *For δ smaller than $C_f^{-N}.\delta_1$, and μ-a.e. point $x \in M$, each forward iterate $f^n(D_x)$ of the disk D_x is contained in the corresponding disk $\widehat{D^E_{f^n(x)}}$ and has a diameter bounded by δ_1.*

Moreover, the diameter $\mathrm{Diam}(f^n(D_x))$ *tend exponentially fast to* 0 *when $n \to +\infty$; more precisely, the sequence* $\left(\mathrm{Diam}(f^n(D_x)).e^{-n(\lambda_E^+ + 3\varepsilon)}\right)_{n\geq 0}$ *goes to* 0 *when n tends to $+\infty$.*

Proof. One proves the first part of the lemma inductively on n. Let us assume that all the forward iterates $f^m(D_x)$ up to an integer $n-1 \geq 0$ are contained in the corresponding disk $\widehat{D^E_{f^n(x)}}$ and have diameters bounded by δ_1. Since $\delta_1 < \delta_0$ one first concludes that the iterate $f^n(D_x)$ also is contained in the disk $\widehat{D^E_{f^n(x)}}$. We will prove that its diameter also is bounded by δ_1. Let $k \geq 0$ denote the largest integer such that $k.N \leq n$. By the estimate (9), one deduces the following upper bound

$$\mathrm{Diam}\left(f^n(D_x)\right) \leq C_f^N \ . \quad e^{k.N.\varepsilon} \prod_{0\leq \ell < k} \left\| Df^N_{|E}(f^{\ell N}(x)) \right\| \quad . \ \mathrm{Diam}(D_x),$$

Now by definition (6) of A one deduces

$$\mathrm{Diam}\left(f^n(D_x)\right) \leq C_f^N \ . \quad e^{k.N.(\lambda_E^+ + 2\varepsilon)} A(x) \quad . \frac{\delta}{A(x)}. \tag{10}$$

By our choice of ε and δ, this gives as required:

$$\mathrm{Diam}\left(f^n(D_x)\right) < e^{k.N.(\lambda_E^+ + 2\varepsilon)}\delta_1 \leq \delta_1. \tag{11}$$

In order to get the second part of the lemma, one considers again the estimate (11) which has been now established for all the forward iterates of D_x. It implies:

$$\mathrm{Diam}(f^n(D_x)).e^{-n(\lambda_E^+ + 3\varepsilon)} \leq e^{-(n-kN).(\lambda_E^+ + 2\varepsilon)}.e^{-n\varepsilon}\delta_1 \leq e^{-N.(\lambda_E^+ + 2\varepsilon)}.e^{-n\varepsilon}\delta_1,$$

which goes to 0 as $n \to +\infty$. □

Corollary 8.11. *There is $\delta_2 > 0$ such that, for every $\lambda > \lambda_E^+$, for μ-almost every point $x \in M$, for any point $y \in \widehat{D^E_x}$ one has*

$$sup_{n\geq 0}\, d(f^n(x), f^n(y)) \leq \delta_2 \Longrightarrow \begin{cases} f^n(y) \in \widehat{D^E_{f^n(x)}} \text{ for all } n \geq 0, \\ \lim_{n\to+\infty} e^{-\lambda.n} d(f^n(x), f^n(y)) = 0. \end{cases}$$

Proof. We choose $\delta_2 \in (0, \delta_1)$ such that $\mu\{x \in M, L^E(x) > \delta_2\} > 0$ Note that such a δ_2 exists because L^E is a positive measurable map which is strictly positive at μ-almost every point.

By definition of δ_0, if $y \in \widehat{D^E_x}$ satisfies $d(x,y) \leq \delta_0$ then $f(y) \in \widehat{D^E_{f(x)}}$. As $\delta_2 < \delta_0$ a simple inductive argument shows that, for every $x \in \mathrm{Supp}(\mu)$ and every $y \in \widehat{D^E_x}$ one has

$$\sup_{n\geq 0} d(f^n(x), f^n(y)) \leq \delta_2 \Longrightarrow f^n(y) \in \widehat{D^E_{f^n(x)}} \text{ for all } n \geq 0.$$

Now, the ergodicity of μ implies that, for μ-almost every point x, there are infinitely many $n > 0$ for which $L^E(f^n(x)) > \delta_2$, implying that $f^n(y) \in D_{f^n(x)}$. Now Lemma 8.10 implies that

$\left(d(f^n(x), f^n(y)).e^{-n(\lambda_E^+ + 3\varepsilon)}\right)_{n\geq 0}$ goes to 0 when n tends to $+\infty$. In particular $d(f^n(x), f^n(y))$ tends to 0.

For ending the proof, we fix now $\lambda > \lambda_E^+$. We choose $\varepsilon_1 \in (0, \frac{\lambda_E^+}{3})$ such that $\lambda_E^+ + 3\varepsilon_1 < \lambda$. This gives us a new function A_1 and a new function L_1^E and thus a new family of disks $D_{x,1} \subset \widehat{D_x^E}$, and finally a new number $\delta_{2,1}$ such that $\mu\{x \in M, L_1^E(x) > \delta_{2,1}\} > 0$. Notice that one may apply Lemma 8.10 to ε_1. Thus, the same argument as above proves that, for μ-almost every x and every $y \in \widehat{D_x^E}$ one has

$$\sup_{n\geq 0} d(f^n(x), f^n(y)) \leq \delta_2 \Longrightarrow \lim_{n\to+\infty} e^{-\lambda.n} d(f^n(x), f^n(y)) = 0.$$

As a countable intersection of sets with μ-measure equal to 1 has measure equal to 1, by choosing a sequence of λ decreasing to λ_E^+ one gets that for μ-almost every x, every $\lambda > \lambda_E^+$ and every $y \in \widehat{D_x^E}$ one has

$$\sup_{n\geq 0} d(f^n(x), f^n(y)) \leq \delta_2 \Longrightarrow \lim_{n\to+\infty} e^{-\lambda.n} d(f^n(x), f^n(y)) = 0.$$

□

As a direct corollary of Lemma 8.10 and Corollary 8.11 one gets:

Corollary 8.12. *For μ-almost every point $x \in M$, for any point $y \in D_x$, for every $\lambda > \lambda_E^+$ one has*

$$\lim_{n\to+\infty} e^{-\lambda.n} d(f^n(x), f^n(y)) = 0.$$

8.3 Characterization of the invariant manifolds $W^E(x)$ by the speed of approximation

Lemma 8.13 below will end the proof of Proposition 8.9 (and therefore of Theorem 3.11) by showing that the speed of approximation of $y \in D_x$ given by Corollary 8.12 provides a characterization of the points in the stable manifold $W^E(x)$.

Lemma 8.13. *For μ-almost every point x, for every point $y \in M$ such that*

$$\lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0,$$

we have the following dichotomy:

- *either there is $n > 0$ such that $f^n(y) \in D_{f^n(x)}$ (and so we have exponential convergence);*
- *or for every $\lambda \in (\lambda_E^+, \lambda_F^-)$ one has*

$$\lim_{n\to+\infty} e^{-\lambda.n} d(f^n(x), f^n(y)) = +\infty.$$

One now finishes the proof of Proposition 8.9.

End of the proof of Proposition 8.9. By corollary 8.12, for any $\lambda > \lambda_E^+$, for μ-a.e. point x and any $y \in D_x$, we have shown

$$d(f^n(x), f^n(y)).e^{-\lambda.n} \underset{n\to+\infty}{\longrightarrow} 0. \tag{12}$$

With Lemma 8.7 above, one obtains the properties (1) and (2) of Proposition 8.9 by defining $W^E_{\text{loc}}(x) = D_x$.

For some $\lambda \leq 0$ contained in $(\lambda_E^+, \lambda_F^-)$ and at μ-a.e. point $x \in M$, one now considers any point $y \in M$ which satisfies (12). In particular $d(f^n(x), f^n(y))$ goes to 0 as n goes to $+\infty$. We are not in the second case of Lemma 8.13, hence, there is a forward iterate $f^n(y)$ of y that belongs to $W^E_{\rm loc}(f^n(x))$. One deduces that the two following sets coincide:

$$\left\{ y \in M,\ d(f^n(x), f^n(y)).e^{-\lambda.n} \underset{n\to+\infty}{\longrightarrow} 0 \right\} = \bigcup_{n\geq 0} f^{-n}\left(W^E_{\rm loc}(f^n(x))\right).$$

This set does not depend on the choice of $\lambda \leq 0$ in $(\lambda_E^+, \lambda_F^-)$ and will be denoted by $W^E(x)$.

Let us now remark that the forward iterates of any local manifold $W^E_{\rm loc}(f^m(x))$ have a diameter which goes to 0. Since the local manifolds at infinitely many iterates of x have a radius uniformly bounded away from zero, one deduces that any finite union $\bigcup_{0\leq k\leq m} f^{-k}\left(W^E_{\rm loc}(f^k(x))\right)$ is contained in an embedded manifold $f^{-n}\left(W^E_{\rm loc}(f^n(x))\right)$ for some large n. This implies that $W^E(x) = \bigcup_{n\geq 0} f^{-n}\left(W^E_{\rm loc}(f^n(x))\right)$ is an injectively immersed submanifold. □

The end of the section will be devoted to the proof of Lemma 8.13. We choose δ_2 such that $\mu\{x \in M, L^E(x) > \delta_2\} > 0$. Hence for μ-almost every point x, there exist infinitely many forward iterates $f^n(x)$ such that $L^E(f^n(x)) > \delta$.

We first consider some $\lambda \in (\lambda_E^+, \lambda_F^-)$. The proof uses an invariant cone-field defined in a neighborhood of $\operatorname{Supp}\mu$. More precisely we will use the following classical result:

Lemma 8.14. *Consider any $\lambda' \in (\lambda, \lambda_F^-)$. Then there are*

- *an integer $n_0 > 0$,*
- *a neighborhood U_0 of $\operatorname{Supp}(\mu)$*
- *two continuous bundles $T_xM = E_0(x) \oplus F_0(x)$ for $x \in U_0$ such that $E_0(x) = E(x)$ and $F_0(x) = F(x)$ for $x \in \operatorname{Supp}(\mu)$; for every $K > 0$ we denote by $\mathcal{C}^F_K$ the cone field defined for $x \in U_0$ by*

$$\mathcal{C}^F_K(x) = \{v = v_1 + v_2 \in T_xM = E_0(x) \oplus F_0(x),\ \|v_1\| \leq K\|v_2\|\},$$

- *two positive numbers $0 < b < a$ (hence, for every $x \in U_0$, one has $\mathcal{C}^F_b(x) \subset \mathcal{C}^F_a(x)$),*

such that one has the following properties:

- *for every $x \in U_0$ and every $v \in \mathcal{C}^F_a(x)$ one has $\|Df^{n_0}(v)\| \geq e^{\lambda'.n_0}\|v\|$;*
- *for every $x \in U_0$ such that $f^{n_0}(x) \in U_0$ one has $Df^{n_0}(\mathcal{C}^F_a(x)) \subset \mathcal{C}^F_b(f^{n_0}(x))$.*

We consider $r_0 > 0$ such that any two points $x, y \in M$ with $d(x, y) \leq r_0$ are joined by a unique geodesic segment of length bounded by r_0, which we denote by $[x, y]_{geo}$. Notice that the length $\ell([x, y]_{geo})$ is precisely $d(x, y)$.

Lemma 8.15. *Given any $r \in (0, r_0)$, there is a neighborhood $U_1 \subset U_0$ of $\operatorname{Supp}(\mu)$, and $\delta_3 \in (0, r)$ with the following property.*

Consider $y, z \in U_1$ with $d(y, z) < \delta_3$, such that the segment $[y, z]_{geo}$ is contained in U_0 and is tangent to $\mathcal{C}^F_a$. Then:

$$f^{n_0}(y) \in U_1 \Longrightarrow \begin{cases} d(f^{n_0}(y), f^{n_0}(z)) < r, \\ [f^{n_0}(y), f^{n_0}(z)]_{geo} \subset U_1 \text{ and is tangent to } \mathcal{C}^F_a, \\ d(f^{n_0}(y), f^{n_0}(z)) \geq e^{\lambda.n_0} d(y, z). \end{cases}$$

Idea of the proof. The proof of Lemma 8.15 follows from the invariance of the conefield $\mathcal{C}_a^F$ and from the fact that, for δ_3 small enough, the segment $f^{n_0}([y,z]_{geo})$ is very close (in the C^1-topology) to the geodesic segment $[f^{n_0}(y), f^{n_0}(z)]_{geo}$; in particular the ratio $\frac{d(f^{n_0}(y),f^{n_0}(z))}{\ell(f^{n_0}([y,z]_{geo})}$, where $\ell(f^{n_0}([y,z]_{geo})$ is the length of the segment $f^{n_0}([y,z]_{geo})$, is almost 1. □

Finally, the next lemma defines a kind of projection on $\widehat{D_x^E}$ of the points close enough to x:

Lemma 8.16. *For $r > 0$ small enough, there is $C_1 > 0$ and $\delta_4 \in (0, \delta_3)$ such that, for every $x \in \mathrm{Supp}(\mu)$, for every $y \in M$ with $d(x,y) \leq \delta_4$, one has:*

- *there is $z \in \widehat{D_x^E}$ such that $[y,z]_{geo}$ is tangent to $\mathcal{C}_a^F$ and $d(y,z) < r$;*
- *for every $z \in \widehat{D_x^E}$ such that $[y,z]_{geo}$ is tangent to $\mathcal{C}_a^F$ and $d(y,z) < r$, one has:*

$$d(y,z) < C_1 d(x,y).$$

Idea of the proof. The proof follows from the compactness of $\mathrm{Supp}(\mu)$ and from the fact that the familly $\{\widehat{D_x^E}\}_{x\in\mathrm{Supp}(\mu)}$ is a continuous family for the C^1 topology, hence is a compact family of C^1-disks. □

One can choose the constant $\delta_4 > 0$ small enough so that $e^{\lambda.n_0}.C_1.\delta_4 < r$. In particular, from Lemma 8.15, the segment $[y,z]_{geo}$ in Lemma 8.16 also satisfies $d(f^{n_0}(y), f^{n_0}(z)) < r$. One also can always assume that $\delta_3 + \delta_4 < \lambda_2$.

Proof of Lemma 8.13. Let us assume that the first case of the lemma does not occur and fix some $\lambda \in (\lambda_E^+, \lambda_F^-)$. One may choose an iterate $f^n(x)$ such that $L^E(f^n(x)) > \delta_2$ and $d(f^{n+k}(x), f^{n+k}(y)) < \delta_4$ for every $k \geq 0$. By Lemma 8.16, there is $z \in \widehat{D_x^E}$ such that the segment $[f^n(y), z]_{geo}$ is tangent to $\mathcal{C}_a^F$ and has length bounded by δ_3. The distance $d(f^n(x), z)$ is thus less than λ_2 and z belongs to $D_{f^n(x)}$. Since we are not in the first case of the lemma, one has $f^n(y) \neq z$. Hence, by Corollary 8.12 one has

$$\lim_{k\to+\infty} e^{-\lambda.k} d(f^k(z), f^{n+k}(x)) = 0.$$

One verifies by induction that, for every $k > 0$, the segment $[f^{n+k}(y), f^k(z)]_{geo}$ is tangent to $\mathcal{C}_a^F$ and has length bounded by δ_3. All the iterates $f^{n+k}(y)$ are contained in U_1, so Lemma 8.15 implies that

$$\lim_{k\to+\infty} e^{-\lambda.k} d(f^{n+k}(y), f^k(z)) = +\infty.$$

So $\lim_{k\to+\infty} e^{-\lambda.k} d(f^{n+k}(y), f^{n+k}(x)) = +\infty$, which gives the second case of the lemma as required. □

9 Irregular Points

9.1 Irregular$^+$ points of generic diffeomorphisms

Theorem 3.15 is a direct consequence of the two following results:

Proposition 9.1. *Let p be a hyperbolic periodic saddle of a diffeomorphism f, whose homoclinic class $H(p)$ is not reduced to the orbit of p (i.e., p has transverse homoclinic points). Let $K = \overline{W^s(\mathcal{O}(p))}$ be the closure of the stable manifold of the orbit of p. Then generic points in K are irregular$^+$.*

Proposition 9.2. *Let p and q be hyperbolic periodic saddles of a diffeomorphism f which are homoclinically related, and hence whose homoclinic classes $H(p)$ and $H(q)$ coincide. Assume that the largest Lyapunov exponents of p and q are different. Let $K = \overline{W^s(\mathcal{O}(p))}$ be the closure of the stable manifold of the orbit of p. Then generic points in K are Lyapunov-irregular$^+$.*

Proof of Theorem 3.15. Let f be a C^1-generic tame diffeomorphism and x be a generic point of M. Assume that $\omega(x)$ is not a sink. Then, according to [MP, BC], $\omega(x)$ is a non-trivial attracting homoclinic class $H(p)$, and hence x belongs to the basin of this attracting class. Furthermore $W^s(p)$ is dense in the open set $W^s(H(p))$. So x belongs to the interior of closure $\overline{W^s(p)}$. As a consequence, a generic point x of M which belongs to $W^s(H(p))$ is a generic point in $\overline{W^s(p)}$. As $H(p)$ is non-trivial and f is generic, there is a hyperbolic periodic point $q \notin \mathcal{O}(p)$ homoclinically related to p such that the largest Lyapunov exponents of p and q are distinct.

Now, Propositions 9.1 and 9.2 imply that x is irregular$^+$ and Lyapunov irregular$^+$, respectively. □

We now need only prove Propositions 9.1 and 9.2.

9.2 Proofs of Propositions 9.1 and 9.2

The two propositions are consequences of three lemmas, the first two of which are classical results from hyperbolic theory:

Lemma 9.3. *If K is a hyperbolic basic set there is $k \in \mathbb{N}$ and a compact subset $K_0 \subset K$ such that K is the disjoint union $K_0 \cup f(K_0) \cup \cdots \cup f^{k-1}(K_0)$ and K_0 is invariant by f^k and is a topologically mixing basic set of f^k.*

Lemma 9.4. *Let K be a topologically mixing hyperbolic basic set. Then for any point $z \in K$ one has*

$$\overline{W^s(z)} = \overline{W^s(K)}.$$

The third lemma requires a proof:

Lemma 9.5. *Let K be a non-trivial hyperbolic basic set. Then generic points in K are irregular$^+$.*

Proof. Consider a continuous map $\phi\colon M \to \mathbb{R}$ which equals 0 on a periodic orbit $\gamma_0 \subset K$ and 3 on a periodic orbit $\gamma_1 \subset K$.

One denotes

$$O_n = \{x \in K | \exists m_1 > n, \frac{1}{m_1} \sum_{i=0}^{m_1-1} \phi(f^i(x)) < 1 \text{ and } \exists m_2 > n, \frac{1}{m_2} \sum_{i=0}^{m_2-1} \phi(f^i(x)) > 2\}.$$

O_n is open in K and one easily verifies that O_n is dense. For that one considers a fine Markov partition of K and given any point $z \in K$ one considers a point x whose itinerary coincides with that of z an arbitrarily large number of periods (so that the point x is arbitrarily close to z), then with the itinerary of γ_1 an arbitrarily large number of periods (larger that n)(so that the average $\frac{1}{m_1}\sum_{i=0}^{m_1-1}\phi(f^i(x))$ will be as close to 0 as we want), and next with the itinerary of γ_2 an arbitrarily large number (so that the average $\frac{1}{m_2}\sum_{i=0}^{m_2-1}\phi(f^i(x))$ will be close to 3).

Any point in the intersection $R_0 = \bigcap_0^\infty O_n$ is irregular$^+$. □

Proof of Proposition 9.1. Let p be a periodic saddle point whose homoclinic class $H(p)$ is not trivial and consider two periodic orbits $\gamma_0 \neq \gamma_1$ homoclinically related to p. We fix a continuous function $\phi\colon M \to \mathbb{R}$ such that $\phi(\gamma_0) = 0$ and $\phi(\gamma_1) = 3$.

We define

$$W_n = \{x \in \overline{W^s(\mathcal{O}((p))} | \exists m_1 > n, \frac{1}{m_1} \sum_{i=0}^{m_1-1} \phi(f^i(x)) < 1 \text{ and } \exists m_2 > n, \frac{1}{m_2} \sum_{i=0}^{m_2-1} \phi(f^i(x)) > 2\}.$$

Again, this is an open set of $\overline{W^s(\mathcal{O}(p))}$ and any point in $G_0 = \bigcap_0^\infty W_n$ is irregular$^+$. It remains to prove that W_n is dense.

We consider a hyperbolic basic set $K \subset H(p)$ containing γ_0 and γ_1. Let x be a point in the residual subset R_0 built in the proof of Lemma 9.5. Notice that the stable manifold $W^s(f^i(x))$ is contained in W_n for every $n > 0$ and every $i \in \mathbb{Z}$. By lemmas 9.3 and 9.4, the stable manifold of the orbit of x is dense in $\overline{W^s(\mathcal{O}(p))}$. So the open sets W_n are dense and G_0 is residual, concluding the proof of Proposition 9.1.

□

The proof of Proposition 9.2 is more delicate because, a priori, points in the stable manifold of a Lyapunov irregular$^+$ point may be Lyapunov regular. For this reason we will follow a more subtle strategy.

Proof of Proposition 9.2. Let p be a periodic saddle point point whose homoclinic class $H(p)$ is not trivial and consider a periodic point q homoclinically related to p. We assume that p and q have distinct largest Lyapunov exponents $0 < \lambda_p < \alpha < \beta < \lambda_q$, for some positive numbers α, β.

We define

$$U_n = \{x \in \overline{W^s(\mathcal{O}(p))} | \exists m_1, m_2 > n, \frac{1}{m_1} \log \|D(f^{m_1}(x)\| < \alpha \text{ and } \frac{1}{m_2} \log \|D(f^{m_2}(x)\| > \beta\}.$$

The set U_n is open and any point in $G_1 = \bigcap_0^\infty U_n$ is Lyapunov irregular$^+$. In order to prove Proposition 9.2 it suffices to prove that the U_n are dense in $\overline{W^s(\mathcal{O}(p))}$. For that we consider a basic set K containing the orbits of p and q and we will prove

Claim 9.6. *There is a point $x \in K$ whose stable manifold $W^s(x)$ is contained in U_n.*

Now Lemmas 9.3 and 9.4 implies that U_n is dense in $\overline{W^s(\mathcal{O}(p))}$, concluding the proof of Proposition 9.2. It remains to prove the claim. □

Proof of the claim. We fix a Markov partition generating K and we denote by a and b the itineraries of p and q in terms of this partition. We denote by T_{pq} (resp. T_{qp}) some itinerary from the rectangle containing p (resp. q) to the rectangle containing q (resp. p). We denote by $\ell(m)$ the length of a word m.

We consider the positively infinite word

$$a^{t_0} T_{pq} b^{t_1} T_{qp} a^{t_2} \dots a^{t_i} T_{pq} b^{t_{i+1}} T_{qp} a^{t_{i+2}} \dots,$$

with

$$\lim_{i \to +\infty} \frac{t_i}{\sum_{j=0}^{i-1} t_j} = +\infty.$$

Let y be a point of K having this itinerary and $z \in W^s(y)$. We will show that $z \in U_n$. After some time, the point z has the same itinerary as the point y. We look at the successive stays of the orbit of z close to p and q.

We fix a non-decreasing sequence of positive integers $k_n \to +\infty$ satisfying $\lim_{n\to\infty} \frac{k_n}{n} = 0$.

We consider the point $z_i = f^{r_i}(z)$ with

$$r_{2i} = k_{2i}.\ell(a) \ + \ \ell(a).\sum_0^{i-1} t_{2j} \ + \ \ell(b).\sum_0^{i-1} t_{2j+1} \ + \ i.\left(\ell(T_{pq}) + \ell(T_{qp})\right)$$

and

$$r_{2i+1} = k_{2i+1}.\ell(b) \ + \ \ell(a).\sum_{0}^{i} t_{2j} \ + \ \ell(b).\sum_{0}^{i-1} t_{2j+1} \ + \ T_{pq} \ + \ i.\left(\ell(T_{pq}) + \ell(T_{qp})\right).$$

One easily verifies that for any neighborhoods U_p of the orbit of p and U_q of the orbit of q, there is i_0 such that for $i \geq i_0$ one has :

$$z_{2i}, f(z_{2i}), \ldots, f^{(t_{2i}-k_{2i})\cdot\ell(a)}(z_{2i}) \in U_p,$$

and

$$z_{2i+1}, f(z_{2i+1}), \ldots, f^{(t_{2i+1}-k_{2i+1})\cdot\ell(b)}(z_{2i+1}) \in U_q.$$

We write $s_{2i} = (t_{2i} - k_{2i}) \cdot \ell(a)$ and $s_{2i+1} = (t_{2i+1} - k_{2i+1}) \cdot \ell(b)$.
One deduces that

$$\lim_{i\to+\infty} \frac{1}{s_{2i}} \log \|Df^{s_{2i}}(z_{2i})\| = \lambda_p < \alpha, \text{ and } \lim_{i\to+\infty} \frac{1}{s_{2i+1}} \log \|Df^{s_{2i+1}}(z_{2i+1})\| = \lambda_q > \beta.$$

Furthermore, $\lim_{i\to\infty} \frac{s_i}{r_i} = +\infty$. As a consequence, one obtains that the norms $\|Df^{r_i}(z)\|$ and $\|(Df^{r_i}(z))^{-1}\|$ are very small in comparison with $(\frac{\alpha}{\lambda_p})^{s_i}$ and $(\frac{\lambda_q}{\beta})^{s_i}$. One deduces that, for i large enough, one has

$$\frac{1}{r_{2i} + s_{2i}} \log \|Df^{r_{2i}+s_{2i}}(z)\| < \alpha, \text{ and } \frac{1}{r_{2i+1} + s_{2i+1}} \log \|Df^{r_{2i+1}+s_{2i+1}}(z)\| > \beta.$$

We proved $z \in U_n$ for every n, concluding the proof of the claim. □

9.3 Generic points of generic diffeomorphisms are irregular

Let f be a C^1-generic diffeomorphism: by [BC] the chain-recurrent set coincides with the non-wandering set of f; moreover for each connected component U of $\operatorname{Int}\Omega(f)$, there exists a periodic orbit $\mathcal{O}$ whose homoclinic class is non-trivial and such that the closure K of $W^s(\mathcal{O})$ contains U. Let us consider a generic point $x \in M$. Two cases occurs.

- Either x belongs to $M \setminus \Omega(f)$ and in this case, it is non-recurrent. So Conley theory [C] implies that the omega- and the alpha-limit sets of x are contained in different chain recurrence classes of f which are disjoint compact sets. This implies that x is irregular: the positive and the negative averages along the orbit of x of a continuous map φ with $\varphi(\alpha(x)) = 0$ and $\varphi(\omega(x)) = 1$ will converge to 0 and 1 respectively.
- Or x belongs to $\operatorname{Int}(\Omega(f))$ and is generic in the closure K of $W^s(\mathcal{O})$ for a periodic orbit $\mathcal{O}$ having a non-trivial homoclinic class. By proposition 9.1, such a point x is irregular$^+$.

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Flavio Abdenur (flavio@mat.puc-rio.br)
Departamento de Matemática, PUC-Rio de Janeiro
22460-010 Rio de Janeiro RJ, Brazil

Christian Bonatti (bonatti@u-bourgogne.fr)
CNRS - Institut de Mathématiques de Bourgogne, UMR 5584
BP 47 870, 21078 Dijon Cedex, France

Sylvain Crovisier (crovisie@math.univ-paris13.fr)
CNRS - LAGA, UMR 7539, Université Paris 13
99, Av. J.-B. Clément, 93430 Villetaneuse, France