



HAL
open science

Compact operators that commute with a contraction

Karim Kellay, Mohamed Zarrabi

► **To cite this version:**

Karim Kellay, Mohamed Zarrabi. Compact operators that commute with a contraction. Integral Equations and Operator Theory, 2009, 65 (4), pp.543-550. hal-00322683

HAL Id: hal-00322683

<https://hal.science/hal-00322683>

Submitted on 18 Sep 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COMPACT OPERATORS THAT COMMUTE WITH A CONTRACTION

K. KELLAY AND M. ZARRABI

ABSTRACT. Let T be a C_0 -contraction on a separable Hilbert space. We assume that $I_H - T^*T$ is compact. For a function f holomorphic in the unit disk \mathbb{D} and continuous on $\overline{\mathbb{D}}$, we show that $f(T)$ is compact if and only if f vanishes on $\sigma(T) \cap \mathbb{T}$, where $\sigma(T)$ is the spectrum of T and \mathbb{T} the unit circle. If f is just a bounded holomorphic function on \mathbb{D} we prove that $f(T)$ is compact if and only if $\lim_{n \rightarrow \infty} T^n f(T) = 0$.

1. INTRODUCTION

Let H be a separable Hilbert space, and $\mathcal{L}(H)$ the space of all bounded operators on H . For $T \in \mathcal{L}(H)$, we denote by $\sigma(T)$ the spectrum of T . The Hardy space H^∞ is the set of all bounded and holomorphic functions on \mathbb{D} . The spectrum of an inner function θ is defined by

$$\sigma(\theta) = \text{clos } \theta^{-1}(0) \cup \text{supp } \mu,$$

where μ is the singular measure associated to the singular part of θ and $\text{supp } \mu$ is the closed support of μ (see [13], p. 63). A contraction T on H is called a C_0 -contraction (or in class C_0) if it is completely nonunitary and there exists a nonzero function $\theta \in H^\infty$ such that $\theta(T) = 0$, thus there exists a minimal inner function m_T that annihilates T , i.e $m_T(T) = 0$, and we have $\sigma(T) = \sigma(m_T)$ (see [11] p. 117 and [13], p. 71-72). A contraction T is said essentially unitary if $I_H - T^*T$ is compact, where I_H is the identity map on H .

Let T be a C_0 -contraction on H , and let $H^\infty(T) = \{f(T) : f \in H^\infty\}$. $H^\infty(T)$ is clearly a subspace of the commutant $\{T\}' = \{A \in \mathcal{L}(H) : AT = TA\}$. In this note we study the question of when $H^\infty(T)$ contains a nonzero compact operator. B. Sz-Nagy [12], proved that $\{T\}'$ contains always a nonzero compact operator, but there exists a C_0 -contraction T such that zero is the unique compact operator contained in $H^\infty(T)$ (see section 4). Nordgreen [15] proved that if T is an essentially unitary C_0 -contraction

2000 *Mathematics Subject Classification.* primary 47B05; secondary 30H05.

Key words and phrases. compact operators, essentially unitary, commutant.

The research of the first author was supported in part by ANR Dynop.

then $H^\infty(T)$ contains a nonzero compact operator. There are also results about the existence of smooth operators (finite rank, Schatten–von Neuman operators) in $H^\infty(T)$ (see [18]). It is also shown in the Atzmon’s paper [2], that if T is a cyclic completely nonunitary contraction such that $\sigma(T) = \{1\}$ and

$$\log \|T^{-n}\| = O(\sqrt{n}), \quad n \rightarrow \infty, \quad (1)$$

then $T - I_H$ is compact. Our hope in this paper is to establish results of this kind for more general contractions. Section 2 is devoted to the study of the compactness of $f(T)$ when f is in the disk algebra. We recall that the disk algebra $\mathcal{A}(\mathbb{D})$ is the space of all functions that are holomorphic in \mathbb{D} and continuous on \mathbb{T} . We show (Theorem 2.1), that, if $f \in \mathcal{A}(\mathbb{D})$ and if T is a C_0 -contraction which is essentially unitary, then $f(T)$ is compact if and only if f vanishes on $\sigma(T) \cap \mathbb{T}$. The main tool used in the proof of this result is the Beurling-Rudin theorem about the characterization of the closed ideals of $\mathcal{A}(\mathbb{D})$. For a large class of C_0 -contractions we show (Proposition 2.5) that the condition “ T is essentially unitary” is necessary in the above result. As corollary, we obtain that if T is a contraction that is annihilated by a nonzero function in $\mathcal{A}(\mathbb{D})$ and if T is cyclic (or, more generally, of finite multiplicity) then $f(T)$ is compact whenever $f \in \mathcal{A}(\mathbb{D})$ and f vanishes on $\sigma(T) \cap \mathbb{T}$. We notice that an invertible contraction with spectrum reduced to a single point and satisfying condition (1) is necessarily annihilated by a nonzero function in $\mathcal{A}(\mathbb{D})$ (see [1]).

In section 3, we are interested in the compactness of $f(T)$ when $f \in H^\infty$. Let H^2 be the usual Hardy space on \mathbb{D} , θ being an inner function and $K_\theta = H^2 \ominus \theta H^2$. We consider the model operator $T_\theta : K_\theta \rightarrow K_\theta$ given by $T_\theta g = P_\theta(zg)$, where P_θ stands the orthogonal projection on K_θ . Notice that $m_{T_\theta} = \theta$ and the commutant $\{T_\theta\}'$ of T_θ coincide with $H^\infty(T_\theta)$. Hartman [7] and Sarason [17] gave a complete characterization of compact operators commuting with T_θ (see [13] p. 182). They showed that

$$T_\theta^n f(T_\theta) \rightarrow 0 \iff f(T_\theta) \text{ is compact} \iff f\bar{\theta} \in H^\infty + C(\mathbb{T}).$$

For C_{00} contractions, that is contractions T such that $T^n x \rightarrow 0$ and $T^{*n} x \rightarrow 0$ for every $x \in H$, Muhly gave in [10] (see also [16]) a characterization of functions $f \in H^\infty$ such that $f(T)$ is compact, in term of the characteristic function of T . With the help of the corona theorem, we show (Theorem 3.4) that if T is an essentially unitary C_0 -contraction, then $f(T)$ ($f \in H^\infty$) is compact if and only if $T^n f(T) \rightarrow 0$. We obtain in particular that if $\lim_{r \rightarrow 1^-} f(rz) = 0$ for every $z \in \sigma(T) \cap \mathbb{T}$, then $f(T)$ is compact.

2. COMPACTNESS OF $f(T)$ WITH f IN THE DISK ALGEBRA

Let T be a C_0 -contraction on H . We will introduce some definitions and results we will need later. We call $\lambda \in \sigma(T)$ a normal eigenvalue if it is an isolated point of $\sigma(T)$ and if the corresponding Riesz projection has a finite rank. We set $\sigma_{np}(T)$ the set of all normal eigenvalues of T . The weakly continue spectrum of T is defined by $\sigma_{wc}(T) = \sigma(T) \setminus \sigma_{np}(T)$ (see [14], p. 113).

Let us suppose furthermore that T is essentially unitary. Since $\mathbb{D} \setminus \sigma(T) \neq \emptyset$, there exists a unitary operator U and a compact operator K such that $T = U + K$ (see [14] p. 115) and we have $\sigma_{wc}(T) = \sigma_{wc}(U) \subset \mathbb{T}$. Since T is in class C_0 the set of eigenvalues of T is $\sigma(T) \cap \mathbb{D}$ (see [13] p. 72). In particular we have $\sigma_{np}(T) \subset \sigma(T) \cap \mathbb{D}$. It follows that $\sigma_{wc}(T) = \sigma(T) \cap \mathbb{T}$ and $\sigma_{np}(T) = \sigma(T) \cap \mathbb{D}$. We deduce also from the above observations that if T is in class C_0 then T is essentially unitary if and only if T^* is too.

Let \mathcal{I} be a closed ideal of $\mathcal{A}(\mathbb{D})$. We denote by $S_{\mathcal{I}}$ the inner factor of \mathcal{I} , that is the greatest inner common divisor of all nonzero functions in \mathcal{I} (see [8] p. 85), and we set $Z(\mathcal{I}) = \bigcap_{f \in \mathcal{I}} \{\zeta \in \mathbb{T} : f(\zeta) = 0\}$. For $E \subset \mathbb{T}$ we set $\mathcal{J}(E) = \{f \in \mathcal{A}(\mathbb{D}) : f|_E = 0\}$. We shall need the Beurling-Rudin theorem (see [8] p. 85) about the structure of closed ideals of $\mathcal{A}(\mathbb{D})$, which states that every closed ideal $\mathcal{I} \subset \mathcal{A}(\mathbb{D})$ has the form

$$\mathcal{I} = S_{\mathcal{I}}H^{\infty} \cap \mathcal{J}(Z(\mathcal{I})).$$

Theorem 2.1. *Let T be an essentially unitary C_0 -contraction and let $f \in \mathcal{A}(\mathbb{D})$. The following assertions are equivalents.*

- (1) $f(T)$ is compact.
- (2) $f = 0$ on $\sigma(T) \cap \mathbb{T}$.

For the proof of this theorem we need the following lemma.

Lemma 2.2. *Let T_1, T_2 be two contractions on H such that $T_1 - T_2$ is compact and $f \in \mathcal{A}(\mathbb{D})$. Then $f(T_1)$ is compact if and only if $f(T_2)$ is too.*

Proof. There exists a sequence $(P_n)_n$ of polynomials such that $\|f - P_n\|_{\infty} \rightarrow 0$, where $\|\cdot\|_{\infty}$ is the supremum norm on \mathbb{T} . We set $R_n = f - P_n$. Note that for every n , $P_n(T_2) - P_n(T_1)$ is compact. On the other hand, for $i = 1, 2$, by the von Neumann inequality, we have $\|R_n(T_i)\| \leq \|R_n\|_{\infty}$. So $\|R_n(T_i)\| \rightarrow 0$. It follows that

$$\begin{aligned} f(T_2) - f(T_1) &= \lim_{n \rightarrow +\infty} (P_n(T_2) - P_n(T_1) + R_n(T_2) - R_n(T_1)) \\ &= \lim_{n \rightarrow +\infty} (P_n(T_2) - P_n(T_1)). \end{aligned}$$

Thus $f(T_2) - f(T_1)$ is compact, which finishes the proof. \square

Proof of Theorem 2.1. (1) \Rightarrow (2) : Let \mathcal{B}_T denote the maximal commutative Banach algebra that contains I_H and T . We have $\sigma(T) = \sigma_{\mathcal{B}_T}(T)$, where $\sigma_{\mathcal{B}_T}(T)$ is the spectrum of T in \mathcal{B}_T . Let $\lambda \in \sigma(T)$, there exists a character χ_λ on \mathcal{B}_T such that $\chi_\lambda(T) = \lambda$. Since the set of polynomials is dense in $\mathcal{A}(\mathbb{D})$,

$$\chi_\lambda(f(T)) = f(\chi_\lambda(T)) = f(\lambda), \quad f \in \mathcal{A}(\mathbb{D}).$$

Let now $f \in \mathcal{A}(\mathbb{D})$ such that $f(T)$ be compact and let $\lambda \in \sigma(T) \cap \mathbb{T}$. We have

$$|f(\lambda)| = |\lambda^n f(\lambda)| = |\chi_\lambda(T^n f(T))| \leq \|T^n f(T)\|. \quad (2)$$

Since T is in class C_0 , $T^n x \rightarrow 0$ whenever $x \in H$, (see [11] Proposition III.4.1). Thus for every compact set $C \subset H$,

$$\lim_{n \rightarrow \infty} \sup_{x \in C} \|T^n x\| = 0.$$

For $C = \overline{f(T)(\mathbb{B})}$, where $\mathbb{B} = \{x \in H : \|x\| \leq 1\}$, we get $T^n f(T) \rightarrow 0$. Then it follows from (2) that $f(\lambda) = 0$.

(2) \Rightarrow (1) : Without loss of generality, we may assume that $\sigma(T) \cap \mathbb{T}$ is of Lebesgue measure zero. We set

$$\mathcal{I} = \{f \in \mathcal{A}(\mathbb{D}) : f(T) \text{ compact}\};$$

\mathcal{I} is a closed ideal of $\mathcal{A}(\mathbb{D})$. We have to prove that $\mathcal{I} = \mathcal{J}(\sigma(T) \cap \mathbb{T})$. By the Beurling–Rudin theorem, it suffice to show that $Z(\mathcal{I}) = \sigma(T) \cap \mathbb{T}$ and $S_{\mathcal{I}} = 1$. In the proof of the implication ((1) \Rightarrow (2)) we have seen that $\mathcal{I} \subset \mathcal{J}(\sigma(T) \cap \mathbb{T})$, which implies $\sigma(T) \cap \mathbb{T} \subset Z(\mathcal{I})$. It remains to show that $S_{\mathcal{I}} = 1$ and $Z(\mathcal{I}) \subset \sigma(T) \cap \mathbb{T}$.

Since T is in class C_0 and $I_H - T^*T$ is compact, by the observation in the beginning of this section $T = U + K$, where U is unitary and K is compact. Moreover we have $\sigma_{wc}(U) = \sigma_{wc}(T) = \sigma(T) \cap \mathbb{T}$ ([14] p. 115), and since $\sigma_{np}(U)$ is countable, we see that $\sigma(U)$ is a subset of \mathbb{T} of Lebesgue measure zero. By the Fatou theorem ([8] p. 80), there exists a nonzero outer function $f \in \mathcal{A}(\mathbb{D})$ which vanishes exactly on $\sigma(U)$. We have $f(U) = 0$ since U is unitary. By Lemma 2.2, $f(T)$ is compact. This shows that $S_{\mathcal{I}} = 1$ and $Z(\mathcal{I}) \subset \sigma(U)$. We shall now show that $Z(\mathcal{I}) \subset \sigma_{wc}(U)$.

Let $\lambda \in \sigma_{np}(U)$; λ is an isolated point in $\sigma(U)$ and $\text{Ker}(U - \lambda I_H)$ is of finite dimension. There exists $g \in \mathcal{A}(\mathbb{D})$ with $g(\lambda) \neq 0$ and $f|_{\sigma(U) \setminus \{\lambda\}} = 0$. Since $(z - \lambda)f(z) = 0$, $z \in \sigma(U)$ and U unitary, $(U - \lambda)f(U) = 0$. So $f(U)(H) \subset \text{Ker}(U - \lambda I_H)$. So $f(U)$ is of finite rank, thus $f(U)$ is compact and by Lemma 2.2, $f(T)$ is compact. Hence $\lambda \notin Z(\mathcal{I})$. We deduce that $Z(\mathcal{I}) \subset \sigma_{wc}(U) = \sigma_{wc}(T) = \sigma(T) \cap \mathbb{T}$, which finishes the proof.

Let $T \in \mathcal{L}(H)$. The spectral multiplicity of T is the cardinal number given by the formula

$$\mu_T = \inf \text{card } L,$$

where $\text{card } L$ is the cardinal of L and where the infimum is taken over all nonempty sets $L \subset H$ such that $\text{span}\{T^n L; n \geq 0\}$ is dense in H . Notice that $\mu_T = 1$ means that T is cyclic.

Corollary 2.3. *Let T be a contraction on H with $\mu_T < +\infty$. Assume that there exists a nonzero function $\varphi \in \mathcal{A}(\mathbb{D})$ such that $\varphi(T) = 0$. Then $f(T)$ is compact for every function $f \in \mathcal{A}(\mathbb{D})$ that vanishes on $\sigma(T) \cap \mathbb{T}$.*

Proof. There exists two orthogonal Hilbert subspaces H_u and H_0 that are invariant by T , such that $H = H_u \oplus H_0$, $T_u = T|_{H_u}$ is unitary and $T_0 = T|_{H_0}$ is completely nonunitary (see [11], Theorem 3.2, p. 9 or [13], p. 7). T_0 is clearly in class C_0 and we have $\mu_{T_0} < +\infty$. By Proposition 4.3 of [4], $I_{H_0} - T_0^* T_0$ is compact.

Let $f \in \mathcal{A}(\mathbb{D})$, with $f|_{\sigma(T) \cap \mathbb{T}} = 0$. Since $\sigma(T_0) \subset \sigma(T)$, it follows from Theorem 2.1 that $f(T_0)$ is compact. Now, since T_u is unitary and $\sigma(T_u) \subset \sigma(T) \cap \mathbb{T}$, we get $f(T_u) = 0$. Thus $f(T)$ is compact. \square

Remark 2.4. *Let T be a cyclic contraction satisfying condition (1) and with finite spectrum, $\sigma(T) = \{\lambda_1, \dots, \lambda_n\}$. By Theorem 2 of [1], there exist a function $f = \sum_{n \geq 0} a_n z^n$, $f \neq 0$, such that $\sum_{n \geq 0} |a_n| < +\infty$ and $f(T) = 0$. Then, it follows from Corollary 2.3 that $(T - \lambda_1) \dots (T - \lambda_n)$ is compact. Thus we find the result Corollary 4.3 of [2], mentioned in the introduction.*

Now we finish this section by showing that the hypothesis "essentially unitary" in Theorem 2.1 is necessary. Let us first make some observations. An operator $T \in \mathcal{L}(H)$ is called essentially normal if $TT^* - T^*T$ is compact. Notice that if T is a C_0 -contraction which is essentially unitary then T^* is essentially unitary too. Hence T is essentially normal since $I_H - T^*T$ and $I_H - TT^*$ are both compacts. Notice also that Theorem 2.1 is of interest in the case of contractions T such that $\sigma(T) \cap \mathbb{T}$ is of Lebesgue measure zero.

Proposition 2.5. *Let $T \in \mathcal{L}(H)$ be a C_0 -contraction which is essentially normal and such that $\sigma(T) \cap \mathbb{T}$ is of Lebesgue measure zero. Assume that for every $f \in \mathcal{A}(\mathbb{D})$ vanishing on $\sigma(T) \cap \mathbb{T}$, $f(T)$ is compact. Then T is essentially unitary.*

Proof. Denote by $\mathcal{K}(H)$ the set of all compact operators on H and by $\pi : \mathcal{L}(H) \rightarrow \mathcal{L}(H)/\mathcal{K}(H)$ the canonical surjection. The essential spectrum $\sigma_{ess}(T)$ of T is defined as the spectrum of $\pi(T)$ in the Banach algebra $\mathcal{L}(H)/\mathcal{K}(H)$.

Let $\lambda \in \sigma(T) \cap \mathbb{D}$. By Fatou theorem [8], there exists a non zero outer function $f \in \mathcal{A}(\mathbb{D})$ such that $f|_{\sigma(T) \cap \mathbb{T}} = 0$. By hypothesis $f(T)$ is compact. The function $z - \lambda$ and f have no common zero in $\overline{\mathbb{D}}$. So there exists two functions g_1 and g_2 in $\mathcal{A}(\mathbb{D})$ such that $(z - \lambda)g_1 + fg_2 = 1$. Thus $(T - \lambda)g_1(T) + f(T)g_2(T) = I_H$, which shows that $\pi(T) - \lambda$ is invertible in $\mathcal{L}(H)/\mathcal{K}(H)$. Hence $\sigma_{ess}(T) \subset \sigma(T) \cap \mathbb{T}$.

By Rudin-Carleson-Bishop theorem (see [8] p. 81), there exists a function $h \in \mathcal{A}(\mathbb{D})$ such that $\bar{z} = h(z)$, $z \in \sigma(T) \cap \mathbb{T}$. Since $\pi(T)$ is a normal element in the C^* algebra $\mathcal{L}(H)/\mathcal{K}(H)$, we get $\pi(T)^* = h(\pi(T))$. On the other hand we have $1 - h(z)z = 0$ on $\sigma(T) \cap \mathbb{T}$, which implies that $\pi(I_H) - \pi(T)^*\pi(T) = \pi(I_H) - h(\pi(T))\pi(T) = 0$. Therefore $I_H - T^*T$ is compact. \square

3. THE CASE OF $f(T)$ FOR $f \in H^\infty$

In this section we are interested in the compactness of $f(T)$ when $f \in H^\infty$. As a consequence of Theorem 2.1 we prove the following result which was first established by Moore–Nordgren in [9], Theorem 1. The proof given in [9] uses a result of Muhly [10] (see Remark 1 below), we give here a simple proof.

Lemma 3.1. *Let T be an essentially unitary C_0 -contraction on H , θ be an inner function that divide m_T (i.e $m_T/\theta \in H^\infty$) and such that $\sigma(\theta) \cap \mathbb{T}$ is of Lebesgue measure zero. Let $\psi \in \mathcal{A}(\mathbb{D})$ be such that $\psi|_{\sigma(\theta) \cap \mathbb{T}} = 0$. If $\phi = \psi m_T/\theta$, then $\phi(T)$ is compact.*

In particular the commutant $\{T\}'$ contains a nonzero compact operator.

Proof. Let $\Theta = m_T/\theta$ and $T_1 = T|_{\overline{\Theta(T)H}}$ be the restriction of T to $\overline{\Theta(T)H}$; T_1 is a C_0 -contraction with $m_{T_1} = \theta$. Moreover $I_{H_1} - T_1^*T_1 = P_{H_1}(I_H - T^*T)|_{H_1}$ is compact, where P_{H_1} is the orthogonal projection from H onto H_1 . By Theorem 2.1, $\psi(T_1)$ is compact and thus $\phi(T) = \psi(T)\Theta(T) = \psi(T_1)\Theta(T)$ is also compact. \square

Lemma 3.2. *Let T be an essentially unitary C_0 -contraction on H , θ be an inner function that divide m_T and such that $\sigma(\theta) \cap \mathbb{T}$ is of Lebesgue measure zero. Let $f \in H^\infty$ be such that $\lim_{n \rightarrow +\infty} T^n f(T) = 0$. If $\phi = f m_T/\theta$, then $\phi(T)$ is compact.*

Proof. By the Rudin-Carleson-Bishop theorem, for every nonnegative integers n , there exists $h_n \in \mathcal{A}(\mathbb{D})$ such that $\bar{z}^n = h_n(z)$, $z \in \sigma(\theta) \cap \mathbb{T}$ and $\|h_n\|_\infty = 1$, where $\|\cdot\|_\infty$ is the supremum norm on \mathbb{T} (see [8] p. 81). We have, for every n , $1 - z^n h_n(z) = 0$, $z \in \sigma(\theta) \cap \mathbb{T}$, then by Lemma 3.1, $(I_H - T^n h_n(T))(m_T/\theta)(T)$ is compact. It follows that $\phi(T) - T^n f(T) h_n(T) (m_T/\theta)(T)$ is also compact. Since

$$\|T^n f(T) h_n(T) (m_T/\theta)(T)\| \leq \|T^n f(T)\| \longrightarrow 0,$$

we deduce that $\phi(T)$ is compact. \square

We need the following lemma about inner functions, which is in fact contained in the proof of the main result of [15]. For the completeness we include here its proof.

Lemma 3.3. *Let Θ an inner function. There exists a sequence $(\theta_n)_n$ of inner functions such that for each n , θ_n divides Θ , $\sigma(\theta_n) \cap \mathbb{T}$ is of Lebesgue measure zero and for every $z \in \mathbb{D}$, $\lim_{n \rightarrow +\infty} \theta_n(z) = \Theta(z)$.*

Proof. Let B_n be the Blaschke product constructed with the zeros of Θ contained in the disk $\{|z| \leq 1 - 1/n\}$, each zero of Θ repeated according to its multiplicity.

Let ν be the singular measure defining the singular part of Θ . There exists $F \subset \mathbb{T}$ of Lebesgue measure Zero such that $\nu(F) = \nu(\mathbb{T})$. There exists a sequence $(K_n)_n$ of compact subsets of F such that $\lim_{n \rightarrow \infty} \nu(K_n) = \nu(F)$. For every n , let ν_n be the measure on \mathbb{T} defined by $\nu_n(E) = \nu(E \cap K_n)$. Denote by S_n the singular inner function associated to the measure ν_n . It suffice now to take $\theta_n = B_n S_n$. \square

We are now able to prove the main result of this section.

Theorem 3.4. *Let T be an essentially unitary C_0 -contraction on H . Let $f \in H^\infty$. Then the following assertions are equivalents.*

- (1) $T^n f(T) \rightarrow 0$.
- (2) $f(T)$ is compact.

Proof. (1) \Rightarrow (2) : Let $\Theta = m_T$ and let $(\theta_n)_n$ be the sequence of inner functions given by Lemma 3.3. For every n , we set $\varphi_n = m_T/\theta_n$. Since $(\varphi_n)_n$ is a bounded sequence in H^∞ and $\varphi_n(z) \rightarrow 1$ ($z \in \mathbb{D}$), $(\varphi_n)_n$ converges to 1 uniformly on the compacts of \mathbb{D} . Then for every k , there exists a nonnegative integer n_k such that $|\varphi_{n_k}(z)| \geq e^{-1}$ for $|z| \leq \frac{k}{k+1}$. Clearly the sequence $(n_k)_k$ may be chosen to be strictly increasing. Moreover for $|z| \geq \frac{k}{k+1}$, we have $|z^k| \geq e^{-1}$. So

$$e^{-1} \leq |z^k| + |\varphi_{n_k}(z)| \leq 2, \quad z \in \mathbb{D}.$$

By the corona theorem ([13], p. 66), there exists two functions h_1 and h_2 in H^∞ such that

$$z^k h_1 + \varphi_{n_k} h_2 = 1 \quad \text{and} \quad |h_1|, |h_2| \leq C,$$

where C is an absolute constant. Thus we get

$$T^k f(T) h_1(T) + f(T) \varphi_{n_k}(T) h_2(T) = f(T),$$

and

$$\|T^k f(T) h_1(T)\| \leq C \|T^k f(T)\| \rightarrow 0.$$

Consequently, $f(T) = \lim_{k \rightarrow \infty} f(T)\varphi_{n_k}(T)h_2(T)$ in the $\mathcal{L}(H)$ norm. Finally $f(T)$ is compact since by Lemma 3.2, for every k , $f(T)\varphi_{n_k}(T)h_2(T)$ is compact.

(2) \Rightarrow (1) : see the proof of Theorem 2.1. \square

Let T be a contraction on H . It is shown by Esterle, Strouse and Zouakia in [5], that if $f \in \mathcal{A}(\mathbb{D})$, then $\lim_{n \rightarrow \infty} T^n f(T) = 0$ if and only if f vanishes on $\sigma(T) \cap \mathbb{T}$. So Theorem 3.4 implies Theorem 2.1. Now, if T is completely non unitary, Bercovici showed in [3] that if $f \in H^\infty$ and $\lim_{r \rightarrow 1^-} f(rz) = 0$, for every $z \in \sigma(T) \cap \mathbb{T}$, then $\lim_{n \rightarrow \infty} T^n f(T) = 0$. So it follows immediately from this fact and Theorem 3.4 the following result.

Corollary 3.5. *Let T be an essentially unitary C_0 -contraction on H . Let $f \in H^\infty$. If for every $z \in \sigma(T) \cap \mathbb{T}$, $\lim_{r \rightarrow 1^-} f(rz) = 0$, then $f(T)$ is compact.*

4. REMARKS

1. As in Corollary 2.3, Theorem 3.4 holds for a C_0 -contraction such that $\mu_T < +\infty$.

2. Let \mathcal{H} be a separable Hilbert space and $L^2(\mathcal{H})$ the space of weakly measurable functions, \mathcal{H} -valued, norm square integrable functions on \mathbb{T} . We denote by $H^2(\mathcal{H})$ the space of functions in $L^2(\mathcal{H})$ whose negatively indexed Fourier coefficients vanish. The space of bounded weakly measurable function on \mathbb{T} taking values in $\mathcal{L}(\mathcal{H})$ is $L^\infty(\mathcal{L}(\mathcal{H}))$, and the subspace of $L^\infty(\mathcal{L}(\mathcal{H}))$ consisting of those functions whose negatively indexed Fourier coefficients vanish is $H^\infty(\mathcal{L}(\mathcal{H}))$. A function $\Theta \in H^\infty(\mathcal{L}(\mathcal{H}))$ is called inner if it is unitary valued almost everywhere. The shift S on $H^2(\mathcal{H})$ is given by $SF(z) = zF(z)$. If $\Theta \in H^\infty(\mathcal{L}(\mathcal{H}))$ is an inner function, $\Theta H^2(\mathcal{H})$ is an invariant subspace for S . Let $\mathcal{K}(\Theta)$ be the orthogonal complement of $\Theta H^2(\mathcal{H})$ and $P_{\mathcal{K}(\Theta)}$ be the orthogonal projection on $\mathcal{K}(\Theta)$. The compression $S(\Theta)$ of the shift S to a subspace $\mathcal{K}(\Theta)$ is given by

$$S(\Theta)F = P_{\mathcal{K}(\Theta)}(SF),$$

If T is a C_0 -contraction on a Hilbert H , then there exists a Hilbert space \mathcal{H} and inner functions $\Theta \in H^\infty(\mathcal{L}(\mathcal{H}))$ such that T is unitarily equivalent to $S(\Theta)$.

Muhly showed in [10] that for $f \in H^\infty$, the operator $f(T)$ is compact if and only if

$$f\Theta^* \in H^\infty(\mathcal{L}(\mathcal{H})) + C(S_\infty(\mathcal{H})),$$

where $C(S_\infty(\mathcal{H}))$ is the space of continuous functions on \mathbb{T} that takes values in $S_\infty(\mathcal{H})$, the space of compact operators on \mathcal{H} .

Suppose now that the contraction T satisfies the hypothesis of Theorem 3.4. Since $H^\infty + C(\mathbb{T}) = \text{Clos}_{L^\infty(\mathbb{T})} \{\bar{z}^n H^\infty : n \geq 0\}$ ([13] p. 183), we get for every $f \in H^\infty$,

$$\begin{aligned} \|T^n f(T)\| &= \inf_{h \in H^\infty} \|T^n f(T) + m_T(T)h(T)\| \leq \inf_{h \in H^\infty} \|z^n f(z) + m_T(z)h(z)\|_\infty \\ &= \inf_{h \in H^\infty} \|f(z)\overline{m_T}(z) + \bar{z}^n h(z)\|_\infty = \text{dist}(f\overline{m_T}, \bar{z}^n H^\infty). \end{aligned}$$

So if $f\overline{m_T} \in H^\infty + C(\mathbb{T})$, then $T^n f(T) \rightarrow 0$ and by Theorem 3.4, $f(T)$ is compact. We do not know if the converse is true that is $f(T)$ compact implies that $f\overline{m_T} \in H^\infty + C(\mathbb{T})$.

3. The following remark gives an example of a C_0 -contraction T such that zero is the unique compact operator contained in $H^\infty(T)$. Let S be any C_0 -contraction on H . We consider the Hilbert space

$$\ell^2(H) = \left\{ x = (x_n)_{n \geq 1} \subset H : \|x\|_2 = \left(\sum_{n \geq 1} \|x_n\|^2 \right)^{\frac{1}{2}} < +\infty \right\}.$$

and the operator T defined on $\ell^2(H)$ by the formula : $Tx = (Sx_n)_n$, $x = (x_n)_n$. Clearly T is a C_0 -contraction with $m_T = m_S$. We claim that for every $f \in H^\infty$ such that $f(T) \neq 0$, $f(T)$ is not compact. Indeed, let $v \in H$ with norm 1 and such that $f(S)v \neq 0$ and for $n \geq 1$, let $x(n) \in \ell^2(H)$ defined by $x(n)_k = v$ if $k = n$ and $x(n)_k = 0$ if $k \neq n$. Since for every k the projection $x \rightarrow x_k$ is continue, zero is the unique limit of any convergent subsequence of $(f(T)x(n))_n$. On the other hand, for every n , $\|f(T)x(n)\|_2 = \|f(S)v\|$. So all subsequences of $(f(T)x(n))_n$ diverge.

REFERENCES

- [1] A. Atzmon, Operators which are annihilated by analytic functions and invariant subspaces, Acta Math. 144 (1980), 27–63.
- [2] A. Atzmon, Unicellular and nonunicellular dissipative operators. Acta Sci. Math. (Szeged) 57 (1993), no. 1-4, 45–54.
- [3] H. Bercovici, On the iterates of a completely nonunitary contraction, in: Topics in Operator Theory: Ernest D. Hellinger memorial volume, Operator Theory Adv. Appl. 48, Birkhäuser, Basel, (1990), 185–188.
- [4] H. Bercovici; D. Voiculescu. Tensor operations on characteristic functions of C_0 contractions. Acta Sci. Math. (Szeged) 39 (1977), no. 3-4, 205–231
- [5] J. Esterle; E. Strouse; F. Zouakia. Theorems of Katznelson-Tzafriri type for contractions. J. Funct. Anal. 94 (1990), no. 2, 273–287.
- [6] L. E. Isaev, A certain class of operators with spectrum concentrated at zero. (Russian) Dokl. Akad. Nauk SSSR 178 (1968) 783–785.
- [7] P. Hartman, On completely continuous Hankel matrices. Proc. Amer. Math. Soc. 9 (1958), 862–866.
- [8] K. Hoffman, Banach spaces of analytic functions. Prentice-Hall, Inc., Englewood Cliffs, N. J. 1962

- [9] B. Moore III; E. Nordgren, On transitive algebras containing C_0 operators. *Indiana Univ. Math. J.* 24 (1974/75), 777–784.
- [10] P. Muhly, Compact operators in the commutant of a contraction, *J. Funct. Anal.* 8 (1971) 197–224
- [11] B. Sz.-Nagy; C. Foias, *Analyse harmonique des opérateurs de l’espace de Hilbert*, Masson, 1967.
- [12] B. Sz.-Nagy, On a property of operators of class C_0 , *Acta Sci. Math. (Szeged)* 36 (1974), 219–220.
- [13] N. K. Nikolski, *Treatise on the Shift Operator*, Springer, Heidelberg, 1986.
- [14] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 2. Model operators and systems. Mathematical Surveys and Monographs, 93.* American Mathematical Society, Providence, RI, 2002.
- [15] E. A. Nordgren, Compact operators in the algebra generated by essentially unitary C_0 operators, *Proc. Amer. Math. Soc.* 51 (1975), 159–162.
- [16] L. Page, Bounded and compact vectorial Hankel operators. *Trans. Amer. Math. Soc.* 150 (1970) 529–539.
- [17] D. Sarason, Generalized interpolation in H^∞ . *Trans. Amer. Math. Soc.* 127 (1967), 179–203.
- [18] P. Vitse, Smooth operators in the commutant of a contraction. *Studia Math.* 155 (2003), no. 3, 241–263.

CMI, LATP, UNIVERSITÉ DE PROVENCE, 39, RUE F. JOLIOT-CURIE, 13453 MARSEILLE CEDEX 13, FRANCE

E-mail address: `kellay@cmi.univ-mrs.fr`

UNIVERSITÉ DE BORDEAUX, UMR 5251, 351, COURS DE LA LIBÉRATION, F-33405 TALENCE CEDEX, FRANCE

E-mail address: `Mohamed.Zarrabi@math.u-bordeaux1.fr`