# A two-scale computational approach suitable for parallel computing

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# A TWO-SCALE COMPUTATIONAL APPROACH SUITABLE FOR PARALLEL COMPUTING

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**Abstract.** *This paper describes a micro-macro computational strategy for the analysis of structures which are described up to the "micro" level, such as composite structures. This approach splits the solution in a macroscopic part and an additive micro complement. Moreover, the resulting algorithm is related to a 2-level "mixed" domain decomposition method, involving an homogenization procedure in order to define the macro-scale problem. It produces an efficient algorithm for solving heterogeneous large-size finite element problems.*

# 1 Introduction

Structural analysis has become more heavily concerned with material models which are described up to a scale smaller than the macroscopic structural level (composite materials, for instance). For linear analysis, the treatment of such two-level problems is currently performed with techniques that take into account homogenization, coupled with a local reanalysis. The most mastered technique is probably the one initiated by Sanchez-Palencia for periodic media [8]. Of course, a "constraint" in the use of this method lies in the fact that the ratio between the small scale and the large scale has to be small.

The objectives of the micro-macro approach proposed herein are to avoid several of the limitations in the classical homogenization techniques, and to be suited to the currently most powerful computer resources, i.e. parallel architecture computers. This iterative strategy has a strong mechanical base; it is built upon remarkable properties satisfied by structural models described up to the micro scale. For the sake of simplicity, it is described here for linear elastic problems.

The first step is the decomposition of the structure in an assembly of simple constituents: sub-structures and interfaces. For instance, a sub-structure may gather one or several cells of composite structures and an interface represents the behavior of the liaison between two sub-structures. Each of these components possesses its own variables and equations. An interface transfers both a distribution of displacement and a distribution of forces. Since each of these components possesses its own variables and equations, both the displacements and the forces on the interfaces are the unknowns. The resulting approach is then qualified as a "mixed" sub-structuring technique [5]. The originality of this method lies in the splitting of the unknowns into macro quantities and an additive micro complement.

The second step of the micro-macro strategy is the use of the so-called LATIN method on the problem to be solved, expressed as an assembly of sub-structures and interfaces. The LATIN method is a non-incremental iterative computational strategy. It deals with the entire studied time interval [5]. For linear problems, the strategy involves a numerical parameter that can be interpreted as an interface stiffness. At each iteration, one has to solve a "macro" problem, defined on the entire structure, along with a family of linear problems, independent of each sub-structure and interface. These are the "micro" problems; they are expected to described the short length effects of the solution, whereas the "macro" problem is related to the homogenized structure.

This approach does not require any specific treatment for boundary areas and concerns structures that are not required to be periodic media. The homogenization procedure is automatically performed within the algorithm. Moreover, this approach is well suited to parallel-architecture computers since the underlying algorithm is a 2-level domain decomposition approach [3, 7, 6]. A first version suited to weakly heterogeneous structures has been reported in [1]. This version is a priori less efficient than the micro-macro computational strategy, but is nevertheless comparable to the FETI domain decomposition method [3]. Numerical example in the case of heterogeneous linear elasticity and perfect interfaces illustrate the possibilities of this method.

## 2 Reference problem and its re-formulation

The reference problem is related to the quasi-static behavior of a structure denoted by $\Omega$, for small perturbations and isothermal evolution. The loadings are:

- a prescribed displacement $\underline{U}_d$ on a first part of the boundary $\partial_1\Omega$,
- a prescribed traction force $\underline{F}_d$ on the complementary part of the boundary $\partial_2\Omega$,
- a prescribed body force $\underline{f}_d$ on $\Omega$.

For the sake of simplicity, only the case of linear elasticity will be described herein. Therefore, only the final configuration is of interest, and time is no longer taken into account. The non-linear case is described in [5].

The current state of the structure is given by the stress field $\boldsymbol{\sigma}$ at each point $\underline{M}$ of $\Omega$ and the displacement field $\underline{U}$. $\boldsymbol{\sigma}$ is searched in the corresponding space $\mathcal{S}$, while $\underline{U}$ is searched in $\mathcal{U}$.

The problem to be solved then is to find $\mathbf{s} = (\underline{U}, \boldsymbol{\sigma})$ in $\mathcal{U} \times \mathcal{S}$, which satisfies:

- kinematic equations: $\underline{U} \in \mathcal{U}, \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\underline{U}), \quad \underline{U}\,|_{\partial_1\Omega} = \underline{U}_d$
- equilibrium equations: $\boldsymbol{\sigma} \in \mathcal{S}$,

$$\forall\, \underline{U}^\star \in \mathcal{U}_0 \qquad \int_\Omega \mathrm{Tr}[\boldsymbol{\sigma}\boldsymbol{\varepsilon}(\underline{U}^\star)]d\Omega = \int_\Omega \underline{f}_d \cdot \underline{U}^\star d\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{U}^\star dS$$

- the constitutive relation: $\boldsymbol{\sigma} = \mathbf{K}\boldsymbol{\varepsilon}$, where $\mathbf{K}(\underline{M})$ is Hooke's tensor, characteristic of the local material behavior.

The first step of the micro/macro strategy is the re-formulation of the problem in terms of a decomposition of the structure into an assembly of simple constituents: sub-structures and interfaces [4] (see Figure 1). Each of these components possesses its own variables and equations.

A sub-structure $\Omega_E$, $E \in \mathbf{E}$, is submitted to the action of its environment (its neighboring interfaces): a force field $\underline{F}_E$ and a displacement field $\underline{W}_E$ on its boundary $\partial\Omega_E$.

An interface $\Gamma_{EE'}$ between sub-structures $E$ and $E'$ transfers both the displacement field and force field on each side: $\underline{W}_E$, $\underline{W}_{E'}$ and $\underline{F}_E$, $\underline{F}_{E'}$. The corresponding spaces then are $\mathcal{W}_{EE'}$ and $\mathcal{F}_{EE'}$. Extended to all of the interfaces, they become $\mathcal{W}$ and $\mathcal{F}$.

Since both the displacements and forces on the interfaces are the unknowns, the resulting approach is a "mixed" sub-structuring technique, as opposed to the primal sub-structuring [7, 6], or dual approach [3].

The solution to the reference problem,

$$\mathbf{s} = \bigcup_{E \in \mathbf{E}} \mathbf{s}_E \quad \text{with} \quad \mathbf{s}_E = (\underline{U}_E, \underline{W}_E, \boldsymbol{\sigma}_E, \underline{F}_E)$$

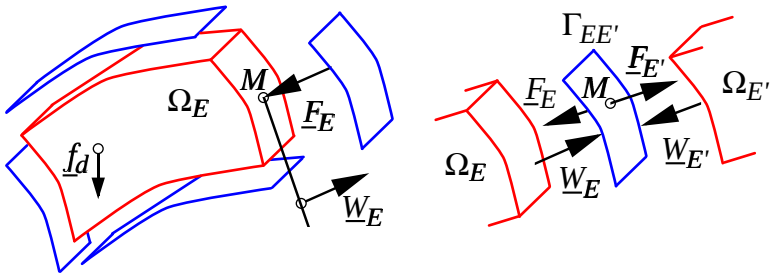


Figure 1: Sub-structures and interfaces

with the corresponding space being $\mathbf{S}$, must satisfy an initial set of equations, $\mathbf{A_d}$, in order to be admissible on each sub-structure:

- kinematic equations: $\underline{U}_E \in \mathcal{U}_E, \quad \boldsymbol{\varepsilon}_E = \boldsymbol{\varepsilon}(\underline{U}_E), \quad \underline{U}_E \mid_{\partial\Omega_E} = \underline{W}_E$
- equilibrium equations: $\boldsymbol{\sigma}_E \in \mathcal{S}_E$,

$$\forall \underline{U}^\star \in \mathcal{U}_E \qquad \int_{\Omega_E} \mathrm{Tr}[\boldsymbol{\sigma}_E \boldsymbol{\varepsilon}(\underline{U}^\star)] d\Omega = \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^\star d\Omega + \int_{\partial\Omega_E} \underline{F}_E \cdot \underline{U}^\star dS$$

In addition, $\mathbf{s}$ must also satisfy a second set of equations, $\mathbf{\Gamma}$, in order to verify the material and interface behaviors:

- constitutive relation: $\boldsymbol{\sigma}_E = \mathbf{K}\boldsymbol{\varepsilon}_E$,
- interface behavior: for instance, with a perfect interface, the transmission conditions are $\underline{W}_E = \underline{W}_{E'}$, $\underline{F}_E + \underline{F}_{E'} = 0$ and the boundary conditions on $\partial_1\Omega$ and $\partial_2\Omega$.

The regularity required for displacement field $\underline{U}_E$ and stress field $\boldsymbol{\sigma}_E$ is the classical one; for instance, with a tridimensional analysis, $\mathcal{U}_E = [H^1(\Omega_E)]^3$ and $\mathcal{S}_E = [L^2(\Omega_E)]^6$.

Such a sub-structuring technique is well suited to the case of periodic structures [2], but this approach has not been used herein: boundary areas and interior areas are treated in the same way.

## 3 Description on the micro and macro scales

### 3.1 A general displacement-oriented description

The state of the structure is expected to possess two parts, related to the micro scale, denoted by $m$, and to the macro scale $M$, each with a different length of variation [1].

The first step is the description of forces and displacements on the interfaces for both scales. For an interface $\Gamma_{EE'}$ from the sub-structure $E$, the force $\underline{F}_E \mid_{\Gamma_{EE'}}$ and the displacement $\underline{W}_E \mid_{\Gamma_{EE'}}$ are split into:

$$\underline{F}_E = \underline{F}_E^M + \underline{F}_E^m \qquad \underline{W}_E = \underline{W}_E^M + \underline{W}_E^m$$

The description consists of defining a projection operator $\boldsymbol{\pi}_{\Gamma_{EE'}}$ such that $\underline{W}_E^M \mid_{\Gamma_{EE'}} = \boldsymbol{\pi}_{\Gamma_{EE'}} \underline{W}_E \mid_{\Gamma_{EE'}}$. The micro and macro spaces for the displacement on $\Gamma_{EE'}$ are $\mathcal{W}_{EE'}^m$ and $\mathcal{W}_{EE'}^M$, respectively. The corresponding forces arise from the contribution of work on the interface $\Gamma_{EE'}$ of sub-structure $E$:

$$(\underline{F}_E, \underline{W}_E)_{\Gamma_{EE'}} \equiv \int_{\Gamma_{EE'}} \underline{F}_E \cdot \underline{W}_E dS = (\underline{F}_E^m, \underline{W}_E^m)_{\Gamma_{EE'}} + (\underline{F}_E^M, \underline{W}_E^M)_{\Gamma_{EE'}} \qquad (1)$$

Another possible description consists of defining a projection which operates on forces, and the associated displacements will be defined thanks to the same duality.

Extended to all of the interfaces of sub-structures, the previous splittings leads to: $\mathcal{W} = \mathcal{W}^M + \mathcal{W}^m$ and $\mathcal{F} = \mathcal{F}^M + \mathcal{F}^m$. Other major choices must also be stated: the "micro" displacements and forces related to sub-structures $E$ and $E'$ with a common interface $\Gamma_{EE'}$ do not have to satisfy the transmission conditions. On the contrary, the "macro" quantities satisfy these conditions in a weak sense: $(\underline{W}^M, \underline{F}^M)$ must belong to $\mathcal{W}_{\text{ad}}^M \times \mathcal{F}_{\text{ad}}^M$; $\mathcal{W}_{\text{ad}}^M$ and $\mathcal{F}_{\text{ad}}^M$ will be specified further.

Briefly, the state of the structure $\mathbf{s}$, is given by micro and macro quantities related to $(\underline{W}^M, \underline{F}^M) \in \mathcal{W}_{\text{ad}}^M \times \mathcal{F}_{\text{ad}}^M$ and $(\underline{W}^m, \underline{F}^m) \in \mathcal{W}^m \times \mathcal{F}^m$.

### 3.2 A micro/macro description based on hierarchical element or super-element

An approximation is added to the description of micro and macro quantities, because the micro/macro description is built on a finite element discretization of the problem. Let us consider the case where different meshes are used for each scale and, for purposes of simplification, when the discretization spaces are embedded, as shown for a 2D analysis (and then 1D interfaces) in Figure 2.

The representation of the displacements on the two scales is performed on the hierarchical basis for macro and micro variables [9]. We will now denote with a subscript $h$ the different spaces already mentioned, since they are strongly related to the hierarchical basis arising from the discretization.

Let us consider a common boundary $\Gamma_{EE'}$ to sub-structures $E$ and $E'$. The displacement $\underline{W}_E$ on $\Gamma_{EE'}$ is:

$$\underline{W}_E^M \mid_{\Gamma_{EE'}} = \sum_{i=1}^{m} \underline{W}_E^M(\underline{X}_i)\varphi_i \mid_{\Gamma_{EE'}} \qquad \underline{W}_E^m \mid_{\Gamma_{EE'}} = \sum_{j=m+1}^{n} \underline{W}_E^m(\underline{X}_j)\varphi_j \mid_{\Gamma_{EE'}}$$

with $\underline{W}_E^m(\underline{X}_i) = 0$ for $i \in 1, \ldots, m$, and $\varphi$ the hierarchical basis functions, see Figure 2. These relations serve to define $\mathcal{W}_h^M$ and $\mathcal{W}_h^m$. The corresponding projector $\boldsymbol{\pi}_{\Gamma_{EE'}}$ is not orthogonal with respect to the standard scalar product. With the duality properties in (1), one obtains $\mathcal{F}_h^M$ and $\mathcal{F}_h^m$. In particular, $\underline{F}_E^M \mid_{\Gamma_{EE'}} = \boldsymbol{\pi}_{\Gamma_{EE'}}^T \underline{F}_E \mid_{\Gamma_{EE'}}$ and $\underline{F}_E^m \mid_{\Gamma_{EE'}} = (\mathbf{1} - \boldsymbol{\pi}_{\Gamma_{EE'}}^T) \underline{F}_E \mid_{\Gamma_{EE'}}$. The superscript $T$ denotes the transposition associated to the symmetric form (1). Let us also denote the extension of $\boldsymbol{\pi}_{\Gamma_{EE'}}$ to a set of interfaces by $\boldsymbol{\pi}$.

In order to define $\mathcal{W}^M_{\mathrm{ad},h}$, we have elected to enforce the exact transmission and boundary conditions for the macro displacements $\underline{W}^M$ only:

$$\mathcal{W}^M_{\mathrm{ad},h} = \left\{ \underline{W} \in \mathcal{W}_h \quad | \quad \underline{W}^M_E \,|_{\Gamma_{EE'}} = \underline{W}^M_{E'} \,|_{\Gamma_{EE'}}, \quad \underline{W}^M_E \,|_{\partial_1\Omega} = \boldsymbol{\pi}\underline{U}_d \right\}$$

In this case, $\mathcal{F}^M_{\mathrm{ad},h} = \mathcal{F}^M_h$.

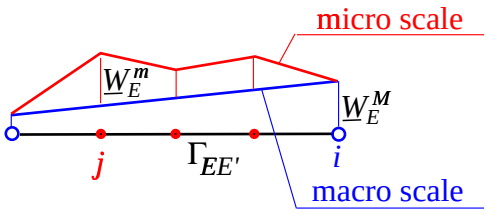


Figure 2: Description of macro and hierarchical micro displacements on an interface

## 4 Computational micro/macro strategy: basis aspects

In order to solve the problem related to the assembly of sub-structures and interfaces, a strategy is developed with the LATIN method [4]. For the linear elasticity case, the corresponding duality changes: it is now a work-based duality and no longer a dissipative one.

The LATIN method is a non-incremental iterative strategy [4]. It successively builds an element $\mathbf{s}$ of the space of admissible fields, $\mathbf{A_d}$ (kinematic and equilibrium equations on each sub-structure), and an element of the second set $\boldsymbol{\Gamma}$ (constitutive relation and interface behavior) within each iteration. At iteration $n$, the element $\hat{\mathbf{s}}_{n+1/2}$ of $\boldsymbol{\Gamma}$ is defined at the local stage from a previous element $\mathbf{s}_n$ of $\mathbf{A_d}$, using the search direction $\mathbf{E}^+$. Then, the next element $\mathbf{s}_{n+1}$ of $\mathbf{A_d}$ is built using a second search direction $\mathbf{E}^-$, see Figure 3. These search directions are the parameters of the method.

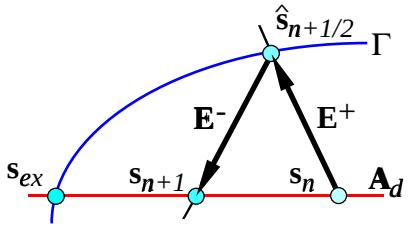


Figure 3: One iteration of the LATIN method

### 4.1 Local stage at iteration $n$

At this stage, the material behavior, as well as the interface behavior, are satisfied. The problem consists of finding $\hat{\mathbf{s}}_{n+1/2} \in \mathbf{\Gamma}$, given $\mathbf{s}_n \in \mathbf{A_d}$. Moreover, $\hat{\mathbf{s}}_{n+1/2} - \mathbf{s}_n$ must belong to the search direction $\mathbf{E}^+$. This last one is, for all sub-structure $E$:

$$(\hat{\boldsymbol{\sigma}}_{En+1/2} - \boldsymbol{\sigma}_{En}) + \mathbf{K}(\hat{\boldsymbol{\varepsilon}}_{En+1/2} - \boldsymbol{\varepsilon}_{En}) = 0$$

and for each interface:

$$\forall \underline{W}^{m\star} \in \mathcal{W}^m_{EE'} \quad (\underline{\hat{F}}^m_{E\,n+1/2} - \underline{F}^m_{E\,n}, \underline{W}^{m\star})_{\Gamma_{EE'}} - (k^m(\underline{\hat{W}}^m_{E\,n+1/2} - \underline{W}^m_{E\,n}), \underline{W}^{m\star})_{\Gamma_{EE'}} = 0$$

with a similar equation for the macro scale (with superscript $M$ rather than $m$).

$\mathbf{K}$ is the Hooke's tensor. $k^m$ and $k^M$ are two parameters of the method which are null or positive scalars. $k^m$ is only related to micro quantities and to interface characteristics, while $k^M$ is related to structural characteristics, [1].

For a perfect interface, $\mathbf{\Gamma}$ contains the transmission conditions: $\underline{W}^m_E = \underline{W}^m_{E'} \qquad \underline{W}^M_E = \underline{W}^M_{E'}$ and $\forall \underline{W}^{m\star} \in \mathcal{W}^m_{EE'}, \forall \underline{W}^{M\star} \in \mathcal{W}^M_{EE'}$ $(\underline{F}^m_E + \underline{F}^m_{E'}, \underline{W}^{m\star})_{\Gamma_{EE'}} = (\underline{F}^M_E + \underline{F}^M_{E'}, \underline{W}^{M\star})_{\Gamma_{EE'}} = 0$ as well as the boundary conditions on $\partial_1\Omega$ and $\partial_2\Omega$.

### 4.2 Linear stage at iteration $n$

The problem now is to find $\mathbf{s}_{n+1} \in \mathbf{A_d}$, given $\hat{\mathbf{s}}_{n+1/2} \in \mathbf{\Gamma}$. For each sub-structure $E$, the stress field must balance the forces on interfaces:

$$\begin{aligned} &\boldsymbol{\sigma}_E \in \mathcal{S}_E, \quad \underline{F}^M \in \mathcal{F}^M_{\rm ad} = \mathcal{F}^M, \quad \underline{F}^m \in \mathcal{F}^m, \\ &\forall \underline{U}^\star \in \mathcal{U}_E \int_{\Omega_E} \mathrm{Tr}[\boldsymbol{\sigma}_E \boldsymbol{\varepsilon}(\underline{U}^\star)] d\Omega = \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^\star d\Omega + \int_{\partial\Omega_E} (\underline{F}^M_E \cdot \underline{U}^\star + \underline{F}^m_E \cdot \underline{U}^\star) dS \end{aligned} \quad (2)$$

The displacement field must be compatible with the displacements on interfaces:

$$\underline{U}_E \in \mathcal{U}_E, \quad \underline{W}^M \in \mathcal{W}^M_{\rm ad}, \quad \underline{W}^m \in \mathcal{W}^m, \quad \underline{U}_E \mid_{\partial\Omega_E} = (\underline{W}^M_E + \underline{W}^m_E) \mid_{\partial\Omega_E}$$

In the previous conditions, note that $(\underline{W}^M, \underline{F}^M)$ has been imposed to belong to $\mathcal{W}^M_{\rm ad} \times \mathcal{F}^M_{\rm ad}$.

$\mathbf{s}_{n+1}$ is also defined with the search direction $\mathbf{E}^-$: for each sub-structure $E$:

$$(\boldsymbol{\sigma}_{En+1} - \hat{\boldsymbol{\sigma}}_{En+1/2}) - \mathbf{K}(\boldsymbol{\varepsilon}_{En+1} - \hat{\boldsymbol{\varepsilon}}_{En+1/2}) = 0$$

and for each interface $\Gamma_{EE'}$ and sub-structure $E$:

$$\begin{aligned} &\forall \underline{W}^{m\star} \in \mathcal{W}^m_{EE'} \\ &\qquad (\underline{F}^m_{E\,n+1} - \underline{\hat{F}}^m_{E\,n+1/2}, \underline{W}^{m\star})_{\Gamma_{EE'}} + (k^m(\underline{W}^m_{E\,n+1} - \underline{\hat{W}}^m_{E\,n+1/2}), \underline{W}^{m\star})_{\Gamma_{EE'}} = 0 \end{aligned} \quad (3)$$

Concerning the macro quantities, the search direction must be global:

$$\forall \underline{W}^{M\star} \in \mathcal{W}^M_{\text{ad},0}$$
$$\sum_{\Gamma_{EE'}} \sum_{E} (\underline{F}^M_{E\ n+1} - \underline{\hat{F}}^M_{E\ n+1/2}, \underline{W}^{M\star})_{\Gamma_{EE'}} + (k^M(\underline{W}^M_{E\ n+1} - \underline{\hat{W}}^M_{E\ n+1/2}), \underline{W}^{M\star})_{\Gamma_{EE'}} = 0 \quad (4)$$

The resulting problem is then split into two kinds of sub-problems: a global macro problem and a micro problem on each sub-structure. In the following discussions, subscripts $n+1/2$ and $n+1$ will be omitted.

# 5 A micro/macro strategy based on a hierarchical finite element description

Let us first consider the hierarchical finite element description. In order to easily solve the discretized problem, the prolongation of both the macro and micro parts of the displacement inside each sub-structure $E$ is defined according to the hierarchical basis of the meshes, as in Figure 4. An additional approximation is then introduced, after that

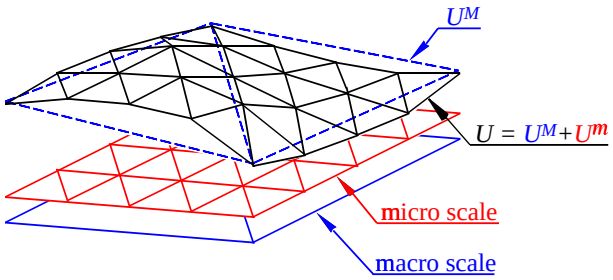


Figure 4: Macro and hierarchical micro displacements on a sub-structure

related to the discretization on interfaces, and a subscript $h$ will be added in order to recall these additions. In particular, the interior micro and macro displacement fields will be the displacements of $\mathcal{U}^m_{E,h}$ and $\mathcal{U}^M_{E,h}$, respectively. Therefore, the displacement $\underline{U}$ is split into: $\underline{U} = \underline{U}^m + \underline{U}^M$ with $\underline{U}_E \in \mathcal{U}_{E,h}$, $\underline{U}^m_E \in \mathcal{U}^m_{E,h}$ and $\underline{U}^M_E \in \mathcal{U}^M_{E,h}$.

## 5.1 Micro-scale problem

This problem is defined on a sub-structure $E$ and is related to quantities defined on the micro subspaces, with all macro quantities considered as fixed. Let us recall that $\hat{\mathbf{s}}$ is given. Using the search direction on micro quantities, independent of each sub-structure,

the problem is:

$$\underline{U}_E^m \in \mathcal{U}_{E,h}^m, \quad \underline{W}_E^m \in \mathcal{W}_{EE',h}^m, \quad \underline{U}_E^m \mid_{\partial\Omega_E} = \underline{W}_E^m \mid_{\partial\Omega_E},$$
$$\forall \underline{U}^{m\star} \in \mathcal{U}_{E,h}^m \quad \int_{\Omega_E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\underline{U}_E^m)\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{m\star})]d\Omega + \int_{\partial\Omega_E} \underline{U}_E^m \cdot k^m \underline{U}^{m\star} dS =$$
$$= \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^{m\star} d\Omega + \int_{\partial\Omega_E} (\hat{\underline{F}}_E^m + k^m \hat{\underline{W}}_E^m) \cdot \underline{U}^{m\star} dS - \int_{\Omega_E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\underline{U}^M) \mid_{\Omega_E} \mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{m\star})]d\Omega \quad (5)$$

Due to the definition of $\boldsymbol{\pi}$ used herein, this problem possesses a unique solution when $k^m \geq k > 0$ ($k$ is a constant).

In any case, the micro-scale problem (5) is linear. Its solution depends separately on:

- $\underline{f}_d \mid_{\Omega_E}$ and $(\hat{\underline{F}}_E^m + k^m \hat{\underline{W}}_E^m) \mid_{\partial\Omega_E}$,
- $\boldsymbol{\varepsilon}(\underline{U}^M) \mid_{\Omega_E}$

Therefore, this solution can be written as: $\underline{U}_E^m = \hat{\underline{U}}_{E,d}^m + \tilde{\underline{U}}_E^m$ where $\tilde{\underline{U}}_E^m$ is related to $\boldsymbol{\varepsilon}(\underline{U}^M) \mid_{\Omega_E}$ and where $\hat{\underline{U}}_{E,d}^m$ involves only known quantities at this stage. More precisely, we have:

$$\mathbf{K}\tilde{\underline{U}}_E^m = -\mathbf{H}_E \boldsymbol{\varepsilon}(\underline{U}^M) \mid_{\Omega_E}$$

where $\mathbf{H}_E$ is a linear symmetric and positive definite operator.

### 5.2 Macro-scale problem

The macro-scale problem is related to macro quantities defined on the entire structure and arises from the macro-search direction:

$$\forall \underline{W}^{M\star} \in \mathcal{W}_{\mathbf{ad},0,h}^M$$
$$\sum_{E\in\mathbf{E}} \int_{\partial\Omega_E} (\underline{F}_E^M - \hat{\underline{F}}_E^M) \cdot \underline{W}^{M\star} dS + \int_{\partial\Omega_E} (\underline{W}_E^M - \hat{\underline{W}}_E^M) \cdot k^M \underline{W}^{M\star} dS = 0 \quad (6)$$

and $\underline{W}^M \in \mathcal{W}_{\mathbf{ad},h}^M$, $\underline{W}_E^M \mid_{\partial\Omega_E} = \underline{U}_E^M \mid_{\partial\Omega_E}$, $\underline{U}^M \in \mathcal{U}_h^M$.

Moreover, for each sub-structure:

$$\forall \underline{U}^{M\star} \in \mathcal{U}_{E,h}^M \quad \int_{\Omega_E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\underline{U}_E^M)\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{M\star})]d\Omega =$$
$$= \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^{M\star} d\Omega + \int_{\partial\Omega_E} \underline{F}_E^M \cdot \underline{U}^{M\star} dS - \int_{\Omega_E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\underline{U}_E^m)\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{M\star})]d\Omega \quad (7)$$

Using (6) and (7), the displacement-oriented formulation of the macro-scale problem then becomes: find $\underline{U}^M \in \mathcal{U}_h^M$ such that

$$\forall \underline{U}^{M\star} \in \mathcal{U}_{0,h}^M \quad \sum_{E\in\mathbf{E}} \int_{\Omega_E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\underline{U}_E^M)\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{M\star})]d\Omega + \\ + \int_{\partial\Omega_E} \underline{U}_E^M \cdot k^M \underline{U}^{M\star} dS = \sum_{E\in\mathbf{E}} \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^{M\star} d\Omega + \int_{\partial\Omega_E} (\hat{\underline{F}}_E^M + k^M \hat{\underline{W}}_E^M) \cdot \underline{U}^{M\star} dS + \\ - \int_{\Omega_E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\underline{U}_E^m)\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{M\star})]d\Omega \quad (8)$$

Using the operator $\mathbf{H}_E$, this becomes: find $\underline{U}^M \in \mathcal{U}_h^M$ such that

$$\forall \underline{U}^{M\star} \in \mathcal{U}_{0,h}^M \quad \sum_{E\in\mathbf{E}} \int_{\Omega_E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\underline{U}_E^M)(\mathbf{K}-\mathbf{H}_E)\boldsymbol{\varepsilon}(\underline{U}^{M\star})]d\Omega + \\ + \int_{\partial\Omega_E} \underline{U}_E^M \cdot k^M \underline{U}^{M\star} dS = \sum_{E\in\mathbf{E}} \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^{M\star} d\Omega + \int_{\partial\Omega_E} (\hat{\underline{F}}_E^M + k^M \hat{\underline{W}}_E^M) \cdot \underline{U}^{M\star} dS + \\ - \int_{\Omega_E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\hat{\underline{U}}_{E,d}^m)\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{M\star})]d\Omega \quad (9)$$

The operator $(\mathbf{K}-\mathbf{H}_E)$ is the homogenized Hooke's operator. It depends both upon the material characteristics of the sub-structure $E$ and the choice of the micro/macro description. $(\mathbf{K}-\mathbf{H}_E)$ is still a symmetric positive definite operator.

The problem in (8) is thereby a standard finite-element discretized problem with $k^M = 0$ for which a unique solution exist. For $k^M > 0$, rigidities associated with the interfaces are added and the problem has a unique solution again.

Once $\underline{U}^M$ has been determined, one obtains $\underline{W}_E^M = \underline{U}_E^M \mid_{\partial\Omega_E}$ and $\underline{F}_E^M$ with the search direction

$$\forall \underline{W}^{M\star} \in \mathcal{W}_{EE'}^M \quad (\underline{F}_E^M, \underline{W}^{M\star})_{\Gamma_{EE'}} = (\hat{\underline{F}}_E^M + k^M \hat{\underline{W}}_E^M - k^M \underline{W}_E^M, \underline{W}^{M\star})_{\Gamma_{EE'}}$$

$\underline{U}^M$ is then needed to compute the generalized loading on the micro scale, and thus, the micro-scale corrections $\underline{U}_E^m$.

Both problems (5) and (9) involve linear operators which can be factorized once in the case of linear elasticity problems, while the right hand sides are iteration-dependent.

## 6 Convergence

Following the convergence proof of the mono-level strategy given in [4], with standard assumptions for elasticity, convergence is reached if the search directions are such that: $\infty > k_2 \geq k^m \mid_{\Gamma_{EE'}} \geq k_1 > 0$ and $\infty > k_2 \geq k^M \mid_{\Gamma_{EE'}} \geq 0$ (where $k_1$ and $k_2$ are constants).

In particular, if $\mathbf{s}_{ex}$ denotes the solution to the reference problem (which belongs to both $\mathbf{A_d}$ and $\mathbf{\Gamma}$, see Figure 3), $\lim_{n\to\infty} \|\mathbf{s}_n - \mathbf{s}_{ex}\| = 0$ and $\lim_{n\to\infty} \|\hat{\mathbf{s}}_{n+1/2} - \mathbf{s}_{ex}\| = 0$, where (in the case $k^M > 0$):

$$\|\mathbf{s}\|^2 = \sum_{E\in\mathbf{E}} \int_{\Omega_E} (\mathrm{Tr}[\boldsymbol{\sigma}_E \mathbf{K}^{-1} \boldsymbol{\sigma}_E] + \mathrm{Tr}[\boldsymbol{\varepsilon}_E \mathbf{K} \boldsymbol{\varepsilon}_E]) d\Omega +$$
$$+ \int_{\partial\Omega_E} (\underline{F}_E^m \cdot k^{m-1} \underline{F}_E^m + \underline{W}_E^m \cdot k^m \underline{W}_E^m + \underline{F}_E^M \cdot k^{M-1} \underline{F}_E^M + \underline{W}_E^M \cdot k^M \underline{W}_E^M) dS$$

## 7 Example of a heterogeneous structure

In order to illustrate the proposed micro/macro computational strategy, a test example is described in Figure 5. It concerns a tridimensional cantilever structure which is weakened with multi-perforations in the transverse direction and submitted to a terminal parabolic load.

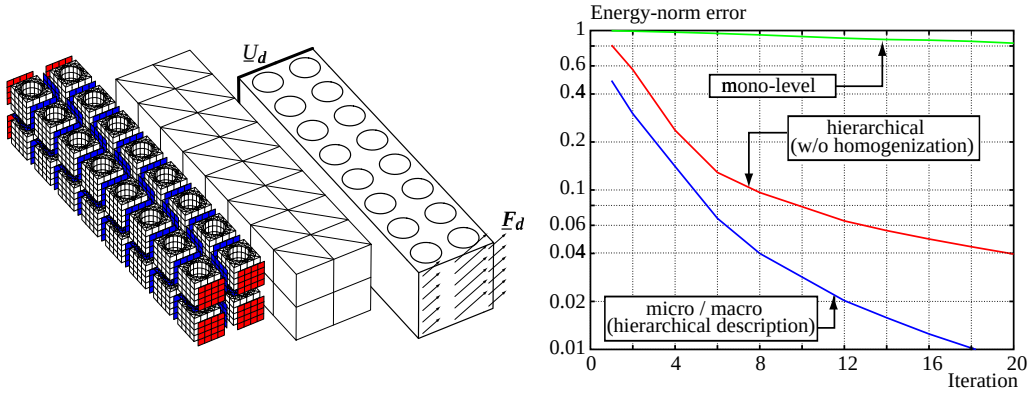


Figure 5: Discretizations of the micro and macro scales, the considered problem and the convergence curves

The Mises equivalent stress field of the solution is illustrated in Figure 6, where localized high gradient areas are shown.

Figure 5 also displays the error with respect to the standard discretized reference problem (96 819 degrees of freedom). It is compared to the error obtained using the hierarchical approach without homogenization [1], as well as to the error obtained with the former mono-level approach without the macro scale [4]. In terms of the iteration count, the level of efficiency is quite high.

The search directions in this example are $k^m = \frac{E}{0.6l}$ and $k^M = 0$; $E$ is the average Young's modulus according to the mixing law, and $l$ is the length of a side of interface.

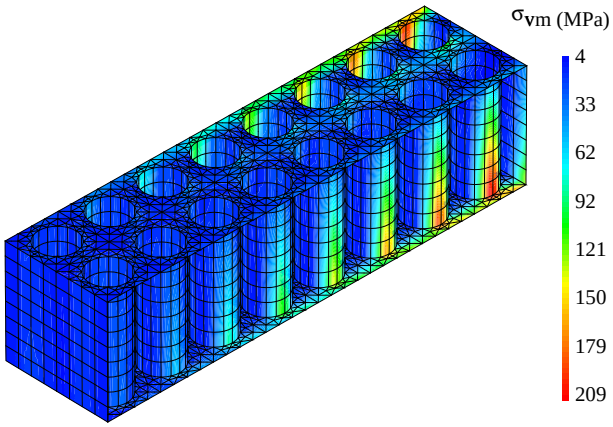


Figure 6: Equivalent stress field level of the solution

## 8 Conclusion

The technique proposed herein belongs to the set of structural analysis with homogenization. It uses a formulation on both the micro and macro scales within the LATIN method. A key point is the description of the micro and macro quantities on the interfaces between sub-structures. Several choices are possible and one of them, which arise from the hierarchical partitioning of displacement have been detailed.

Moreover, this approach leads to a parallel and mechanical approach which is related to domain decomposition methods, and well suited to parallel architecture computers: the underlying algorithm can be interpreted as a "mixed" and 2-level domain decomposition method.

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