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# Free brace algebras are free pre-Lie algebras

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ABSTRACT. Let  $\mathfrak{g}$  be a free brace algebra. This structure implies that  $\mathfrak{g}$  is also a pre-Lie algebra and a Lie algebra. It is already known that  $\mathfrak{g}$  is a free Lie algebra. We prove here that  $\mathfrak{g}$  is also a free pre-Lie algebra, using a description of  $\mathfrak{g}$  with the help of planar rooted trees, a permutative product, and manipulations on the Poincaré-Hilbert series of  $\mathfrak{g}$ .

KEYWORDS. Pre-Lie algebras, brace algebras.

AMS CLASSIFICATION. 17A30, 05C05, 16W30.

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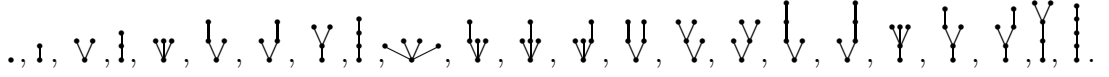
## Introduction

Let  $\mathcal{D}$  be a set. The Connes-Kreimer Hopf algebra of rooted trees  $\mathcal{H}_R^{\mathcal{D}}$  is introduced in [5] in the context of Quantum Field Theory and Renormalization. It is a graded, connected, commutative, non-cocommutative Hopf algebra. If the characteristic of the base field is zero, the Cartier-Quillen-Milnor-Moore theorem insures that its dual  $(\mathcal{H}_R^{\mathcal{D}})^*$  is the enveloping algebra of a Lie algebra, based on rooted trees (note that  $(\mathcal{H}_R^{\mathcal{D}})^*$  is isomorphic to the Grossman-Larson Hopf algebra [10, 11], as proved in [12, 16]). This Lie algebra admits an operadic interpretation: it is the free pre-Lie algebra  $\mathcal{PL}(\mathcal{D})$  generated by  $\mathcal{D}$ , as shown in [4]; recall that a (left) pre-Lie algebra, also called a Vinberg algebra or a left-symmetric algebra, is a vector space  $V$  with a product  $\circ$  satisfying:

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$



3. Planar rooted trees with weight smaller than 5:



4. Planar rooted trees decorated by  $\mathcal{D}$  with weight smaller than 4:

$$\begin{aligned} \bullet_a, a \in \mathcal{D}, \quad \mathbf{1}_a^b, (a, b) \in \mathcal{D}^2, \quad {}^b\mathbf{V}_a^c, \mathbf{1}_a^c, (a, b, c) \in \mathcal{D}^3, \\ {}^b\mathbf{V}_a^d, {}^c\mathbf{1}_a^d, {}^b\mathbf{V}_a^c, {}^c\mathbf{V}_a^d, \mathbf{1}_a^d, (a, b, c, d) \in \mathcal{D}^4. \end{aligned}$$

Let  $t_1, \dots, t_n$  be elements of  $\mathcal{T}^{\mathcal{D}}$  and let  $d \in \mathcal{D}$ . We denote by  $B_d(t_1 \dots t_n)$  the rooted tree obtained by grafting  $t_1, \dots, t_n$  on a common root decorated by  $d$ . For example,  $B_d(\mathbf{1}_a^b \bullet_c) = {}^b\mathbf{1}_a^c$ . This application  $B_d$  can be extended in an operator:

$$B_d : \begin{cases} K[\mathcal{T}^{\mathcal{D}}] & \longrightarrow K\mathcal{T}^{\mathcal{D}} \\ t_1 \dots t_n & \longrightarrow B_d(t_1 \dots t_n), \end{cases}$$

where  $K[\mathcal{T}^{\mathcal{D}}]$  is the polynomial algebra generated by  $\mathcal{T}^{\mathcal{D}}$  over  $K$  and  $K\mathcal{T}^{\mathcal{D}}$  is the  $K$ -vector space generated by  $\mathcal{T}^{\mathcal{D}}$ . This operator is monic, and moreover  $K\mathcal{T}^{\mathcal{D}}$  is the direct sum of the images of the  $B_d$ 's,  $d \in \mathcal{D}$ .

Similarly, let  $t_1, \dots, t_n$  be elements of  $\mathcal{T}_P^{\mathcal{D}}$  and let  $d \in \mathcal{D}$ . We denote by  $B_d(t_1 \dots t_n)$  the planar rooted tree obtained by grafting  $t_1, \dots, t_n$  in this order from left to right on a common root decorated by  $d$ . For example,  $B_d(\mathbf{1}_b^c \bullet_a) = {}^c\mathbf{1}_a^d$  and  $B_d(\bullet_a \mathbf{1}_b^c) = {}^a\mathbf{1}_a^b$ . This application  $B_d$  can be extended in an operator:

$$B_d : \begin{cases} K\langle \mathcal{T}_P^{\mathcal{D}} \rangle & \longrightarrow K\mathcal{T}_P^{\mathcal{D}} \\ t_1 \dots t_n & \longrightarrow B_d(t_1 \dots t_n), \end{cases}$$

where  $K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$  is the free associative algebra generated by  $\mathcal{T}_P^{\mathcal{D}}$  over  $K$  and  $K\mathcal{T}_P^{\mathcal{D}}$  is the  $K$ -vector space generated by  $\mathcal{T}_P^{\mathcal{D}}$ . This operator is monic, and moreover  $K\mathcal{T}_P^{\mathcal{D}}$  is the direct sum of the images of the  $B_d$ 's,  $d \in \mathcal{D}$ .

## 1.2 Free pre-Lie algebras

**Definition 2** A (left) pre-Lie algebra is a couple  $(A, \circ)$  where  $A$  is a vector space and  $\circ : A \otimes A \longrightarrow A$  satisfying the following relation: for all  $x, y, z \in A$ ,

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$

Let  $\mathcal{D}$  be a set. A description of the free pre-Lie algebra  $\mathcal{PL}(\mathcal{D})$  generated by  $\mathcal{D}$  is given in [4]. As a vector space, it has a basis given by  $\mathcal{T}^{\mathcal{D}}$ , and its pre-Lie product is given, for all  $t_1, t_2 \in \mathcal{T}^{\mathcal{D}}$ , by:

$$t_1 \circ t_2 = \sum_{s \text{ vertex of } t_2} \text{grafting of } t_1 \text{ on } s.$$

For example:

$$\bullet_a \circ {}^b\mathbf{V}_d^c = {}^a\mathbf{V}_d^c + {}^b\mathbf{1}_d^c + {}^b\mathbf{V}_d^a = {}^a\mathbf{V}_d^c + {}^a\mathbf{1}_d^c + {}^a\mathbf{V}_d^b.$$

In other terms, the pre-Lie product can be inductively defined by:

$$\begin{cases} t \circ \bullet_a & \longrightarrow B_d(t), \\ t \circ B_d(t_1 \dots t_n) & \longrightarrow B_d(tt_1 \dots t_n) + \sum_{i=1}^n B_d(t_1 \dots (t \circ t_i) \dots t_n). \end{cases}$$

**Lemma 3** *Let  $\mathcal{D}$  a set. We suppose that  $\mathcal{D}$  has a gradation  $(\mathcal{D}(n))_{n \in \mathbb{N}}$  such that, for all  $n \in \mathbb{N}$ ,  $\mathcal{D}(n)$  is finite set of cardinality denoted by  $d_n$ , and  $\mathcal{D}(0)$  is empty. We denote by  $F_{\mathcal{D}}(x)$  the Poincaré-Hilbert series of this set:*

$$F_{\mathcal{D}}(x) = \sum_{n=1}^{\infty} d_n x^n.$$

*This gradation induces a gradation  $(\mathcal{P}\mathcal{L}(\mathcal{D})(n))_{n \in \mathbb{N}}$  of  $\mathcal{P}\mathcal{L}(\mathcal{D})$ . Moreover, for all  $n \geq 0$ ,  $\mathcal{P}\mathcal{L}(\mathcal{D})(n)$  is finite-dimensional. We denote by  $t_n^{\mathcal{D}}$  its dimension. Then the Poincaré-Hilbert series of  $\mathcal{P}\mathcal{L}(\mathcal{D})$  satisfies:*

$$F_{\mathcal{P}\mathcal{L}(\mathcal{D})}(x) = \sum_{n=1}^{\infty} t_n^{\mathcal{D}} x^n = \frac{F_{\mathcal{D}}(x)}{\prod_{i=1}^{\infty} (1 - x^i)^{t_i^{\mathcal{D}}}}.$$

**Proof.** The formal series of the space  $K[\mathcal{T}^{\mathcal{D}}]$  is given by:

$$F(x) = \prod_{i=1}^{\infty} \frac{1}{(1 - x^i)^{t_i^{\mathcal{D}}}}.$$

Moreover, for all  $d \in \mathcal{D}(n)$ ,  $B_d$  is homogeneous of degree  $n$ , so the Poincaré-Hilbert series of  $Im(B_d)$  is  $x^n F(x)$ . As  $\mathcal{P}\mathcal{L}(\mathcal{D}) = K\mathcal{T}^{\mathcal{D}} = \bigoplus Im(B_d)$  as a graded vector space, its Poincaré-Hilbert formal series is:

$$F_{\mathcal{P}\mathcal{L}(\mathcal{D})}(x) = F(x) \sum_{n=1}^{\infty} d_n x^n = F(x) F_{\mathcal{D}}(x),$$

which gives the announced result. □

### 1.3 Free brace algebras

**Definition 4** [1, 2, 18] A brace algebra is a couple  $(A, \langle \rangle)$  where  $A$  is a vector space and  $\langle \rangle$  is a family of operators  $A^{\otimes n} \rightarrow A$  defined for all  $n \geq 2$ :

$$\begin{cases} A^{\otimes n} & \longrightarrow & A \\ a_1 \otimes \dots \otimes a_n & \longrightarrow & \langle a_1, \dots, a_{n-1}; a_n \rangle, \end{cases}$$

with the following compatibilities: for all  $a_1, \dots, a_m, b_1, \dots, b_n, c \in A$ ,

$$\langle a_1, \dots, a_m; \langle b_1, \dots, b_n; c \rangle \rangle = \sum \langle \langle A_0, \langle A_1; b_1 \rangle, A_2, \langle A_3; b_2 \rangle, A_4, \dots, A_{2n-2}, \langle A_{2n-1}; b_n \rangle, A_{2n}; c \rangle \rangle,$$

where this sum runs over partitions of the ordered set  $\{a_1, \dots, a_m\}$  into (possibly empty) consecutive intervals  $A_0 \sqcup \dots \sqcup A_{2n}$ . We use the convention  $\langle a \rangle = a$  for all  $a \in A$ .

For example, if  $A$  is a brace algebra and  $a, b, c \in A$ :

$$\langle a; \langle b; c \rangle \rangle = \langle a, b; c \rangle + \langle b, a; c \rangle + \langle \langle a; b \rangle; c \rangle.$$

As an immediate corollary,  $(A, \langle -; - \rangle)$  is a pre-Lie algebra. Here is another example of relation in a brace algebra: for all  $a, b, c, d \in A$ ,

$$\langle a, b; \langle c; d \rangle \rangle = \langle a, b, c; d \rangle + \langle a, \langle b; c \rangle; d \rangle + \langle \langle a, b; c \rangle; d \rangle + \langle a, c, b; d \rangle + \langle \langle a; c \rangle, b; d \rangle + \langle c, a, b; d \rangle.$$

Let  $\mathcal{D}$  be a set. A description of the free brace algebra  $\mathcal{B}r(\mathcal{D})$  generated by  $\mathcal{D}$  is given in [2, 9]. As a vector space, it has a basis given by  $\mathcal{T}_P^{\mathcal{D}}$  and the brace structure is given, for all  $t_1, \dots, t_n \in \mathcal{T}_P^{\mathcal{D}}$ , by:

$$\langle t_1, \dots; t_n \rangle = \sum \text{graftings of } t_1 \dots t_{n-1} \text{ over } t_n.$$

Note that for any vertex  $s$  of  $t_n$ , there are several ways of grafting a planar tree on  $s$ . For example:

$$\langle \cdot_a, \cdot_b; \mathfrak{!}_d^c \rangle = {}^a\mathfrak{V}_d^c + {}^a\mathfrak{V}_d^b + {}^a\mathfrak{V}_d^b + \mathfrak{Y}_d^c + {}^a\mathfrak{V}_d^b + {}^c\mathfrak{V}_d^b.$$

As a consequence, the pre-Lie product of  $\mathcal{B}r(\mathcal{D})$  can be inductively defined in this way:

$$\begin{cases} \langle t; \cdot_a \rangle & \longrightarrow B_d(t), \\ \langle t; B_d(t_1 \dots t_n) \rangle & \longrightarrow \sum_{i=0}^n B_d(t_1 \dots t_i t t_{i+1} \dots t_n) + \sum_{i=1}^n B_d(t_1 \dots t_{i-1} \langle t; t_i \rangle t_{i+1} \dots t_n). \end{cases}$$

**Proposition 5**  $\mathcal{B}r(\mathcal{D})$  is the free brace algebra generated by  $\mathcal{D}$ .

**Proof.** From [2, 9]. □

**Lemma 6** Let  $\mathcal{D}$  a set, with the hypotheses and notations of lemma 3. The gradation of  $\mathcal{D}$  induces a gradation  $(\mathcal{B}r(\mathcal{D})(n))_{n \in \mathbb{N}}$  of  $\mathcal{B}r(\mathcal{D})$ . Moreover, for all  $n \geq 0$ ,  $\mathcal{B}r(\mathcal{D})(n)$  is finite-dimensional. Then the Poincaré-Hilbert series of  $\mathcal{B}r(\mathcal{D})$  is:

$$F_{\mathcal{B}r(\mathcal{D})}(x) = \sum_{n=1}^{\infty} t_n^{\mathcal{D}} x^n = \frac{1 - \sqrt{1 - 4F_{\mathcal{D}}(x)}}{2}.$$

**Proof.** The Poincaré-Hilbert formal series of  $K\langle \mathcal{T}_{\mathcal{P}}^{\mathcal{D}} \rangle$  is given by:

$$F(x) = \frac{1}{1 - F_{\mathcal{B}r(\mathcal{D})}(x)}.$$

Moreover, for all  $d \in \mathcal{D}(n)$ ,  $B_d$  is homogeneous of degree  $n$ , so the Poincaré-Hilbert series of  $Im(B_d)$  is  $x^n F(x)$ . As  $\mathcal{B}r(\mathcal{D}) = K\mathcal{T}_{\mathcal{P}}^{\mathcal{D}} = \bigoplus Im(B_d)$  as a graded vector space, its Poincaré-Hilbert formal series is:

$$F_{\mathcal{B}r(\mathcal{D})}(x) = F(x) \sum_{n=1}^{\infty} d_n x^n = F(x) F_{\mathcal{D}}(x).$$

As a consequence,  $F_{\mathcal{B}r(\mathcal{D})}(x) - F_{\mathcal{B}r(\mathcal{D})}(x)^2 = F_{\mathcal{D}}(x)$ , which implies the announced result. □

## 2 A non-associative permutative product on $\mathcal{B}r(\mathcal{D})$

### 2.1 Definition and recalls

The following definition is introduced in [14]:

**Definition 7** A (left) non-associative permutative algebra is a couple  $(A, \star)$ , where  $A$  is a vector space and  $\star : A \otimes A \longrightarrow A$  satisfies the following property: for all  $x, y, z \in A$ ,

$$x \star (y \star z) = y \star (x \star z).$$

Let  $\mathcal{D}$  be a set. A description of the free non-associative permutative algebra  $\mathcal{NAPerm}(\mathcal{D})$  generated by  $\mathcal{D}$  is given in [14]. As a vector space,  $\mathcal{NAPerm}(\mathcal{D})$  is equal to  $K\mathcal{T}^{\mathcal{D}}$ . The non-associative permutative product is given in this way: for all  $t_1 \in \mathcal{T}^{\mathcal{D}}$ ,  $t_2 = B_d(F_2) \in \mathcal{T}^{\mathcal{D}}$ ,

$$t_1 \star t_2 = B_d(t_1 F_2).$$

In other terms,  $t_1 \star t_2$  is the tree obtained by grafting  $t_1$  on the root of  $t_2$ . As  $\mathcal{NAPerm}(\mathcal{D}) = \mathcal{P}\mathcal{L}(\mathcal{D})$  as a vector space, lemma 3 is still true when one replaces  $\mathcal{P}\mathcal{L}(\mathcal{D})$  by  $\mathcal{NAPerm}(\mathcal{D})$ .

## 2.2 Permutative structures on planar rooted trees

Let us fix now a non-empty set  $\mathcal{D}$ . We define the following product on  $\mathcal{B}r(\mathcal{D}) = K\mathcal{T}_P^{\mathcal{D}}$ : for all  $t \in \mathcal{T}_P^{\mathcal{D}}$ ,  $t' = B_d(t_1 \dots t_n) \in \mathcal{T}_P^{\mathcal{D}}$ ,

$$t \star t' = \sum_{i=0}^n B_d(t_1 \dots t_i t t_{i+1} \dots t_n).$$

**Proposition 8**  $(\mathcal{B}r(\mathcal{D}), \star)$  is a non-associative permutative algebra.

**Proof.** Let us give  $K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$  its shuffle product: for all  $t_1, \dots, t_{m+n} \in \mathcal{T}_P^{\mathcal{D}}$ ,

$$(t_1 \dots t_m) * (t_{m+1} \dots t_{m+n}) = \sum_{\sigma \in Sh(m,n)} t_{\sigma^{-1}(1)} \dots t_{\sigma^{-1}(m+n)},$$

where  $Sh(m,n)$  is the set of permutations of  $S_{m+n}$  which are increasing on  $\{1, \dots, m\}$  and  $\{m+1, \dots, m+n\}$ . It is well known that  $*$  is an associative, commutative product. For example, for all  $t, t_1, \dots, t_n \in \mathcal{T}_P^{\mathcal{D}}$ :

$$t * (t_1 \dots t_n) = \sum_{i=0}^n t_1 \dots t_i t t_{i+1} \dots t_n.$$

As a consequence, for all  $x \in K\mathcal{T}_P^{\mathcal{D}}$ ,  $y \in K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$ ,  $d \in \mathcal{D}$ :

$$x \star B_d(y) = B_d(x * y). \quad (1)$$

Let  $t_1, t_2, t_3 = B_d(F_3) \in \mathcal{T}_P^{\mathcal{D}}$ . Then, using (1):

$$\begin{aligned} t_1 \star (t_2 \star t_3) &= t_1 \star B_d(t_2 * F_3) \\ &= B_d(t_1 * (t_2 * F_3)) \\ &= B_d((t_1 * t_2) * F_3) \\ &= B_d((t_2 * t_1) * F_3) \\ &= B_d(t_2 * (t_1 * F_3)) \\ &= t_2 \star (t_1 \star t_3). \end{aligned}$$

So  $\star$  is a non-associative permutative product on  $\mathcal{B}r(\mathcal{D})$ . □

## 2.3 Freeness of $\mathcal{B}r(\mathcal{D})$ as a non-associative permutative algebra

We now assume that  $\mathcal{D}$  is finite, of cardinality  $D$ . We can then assume that  $\mathcal{D} = \{1, \dots, D\}$ .

**Theorem 9**  $(\mathcal{B}r(\mathcal{D}), \star)$  is a free non-associative permutative algebra.

**Proof.** We graduate  $\mathcal{D}$  by putting  $\mathcal{D}(1) = \mathcal{D}$ . Then  $\mathcal{B}r(\mathcal{D})$  is graded, the degree of a tree  $t \in \mathcal{T}_P^{\mathcal{D}}$  being the number of its vertices. By lemma 6, as the Poincaré-Hilbert series of  $\mathcal{D}$  is  $F_{\mathcal{D}}(x) = Dx$ , the Poincaré-Hilbert series of  $\mathcal{B}r(\mathcal{D})$  is:

$$F_{\mathcal{B}r(\mathcal{D})}(x) = \sum_{i=1}^{\infty} t_i^{\mathcal{D}} x^i = \frac{1 - \sqrt{1 - 4Dx}}{2}. \quad (2)$$

We consider the following isomorphism of vector spaces:

$$B : \begin{cases} (K\langle \mathcal{T}_P^{\mathcal{D}} \rangle)^d & \longrightarrow \mathcal{B}r(\mathcal{D}) \\ (F_1, \dots, F_D) & \longrightarrow \sum_{i=1}^d B_i(F_i). \end{cases}$$

Let us fix a graded complement  $V$  of the graded subspace  $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})$  in  $\mathcal{B}r(\mathcal{D})$ . Because  $\mathcal{B}r(\mathcal{D})$  is a graded and connected (that is to say  $\mathcal{B}r(\mathcal{D})(0) = (0)$ ),  $V$  generates  $\mathcal{B}r(\mathcal{D})$  as a non-associative permutative algebra. By (1),  $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D}) = B((\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle)^{\mathcal{D}})$ .

Let us then consider  $\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$ , that is to say the ideal of  $(K\langle \mathcal{T}_P^{\mathcal{D}} \rangle, *)$  generated by  $\mathcal{T}_P^{\mathcal{D}}$ . It is known that  $(K\langle \mathcal{T}_P^{\mathcal{D}} \rangle, *)$  is isomorphic to a symmetric algebra (see [17]). Hence, there exists a graded subspace  $W$  of  $K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$ , such that  $(K\langle \mathcal{T}_P^{\mathcal{D}} \rangle, *) \approx S(W)$  as a graded algebra. We can assume that  $W$  contains  $K\mathcal{T}_P^{\mathcal{D}}$ . As a consequence:

$$\frac{K\langle \mathcal{T}_P^{\mathcal{D}} \rangle}{\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle} \approx \frac{S(W)}{S(W)\mathcal{T}_P^{\mathcal{D}}} \approx S\left(\frac{W}{K\mathcal{T}_P^{\mathcal{D}}}\right). \quad (3)$$

We denote by  $w_i$  the dimension of  $W(i)$  for all  $i \in \mathbb{N}$ . Then, the Poincaré-Hilbert formal series of  $S\left(\frac{W}{K\mathcal{T}_P^{\mathcal{D}}}\right)$  is:

$$F_{S\left(\frac{W}{K\mathcal{T}_P^{\mathcal{D}}}\right)}(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{w_i-t_i^{\mathcal{D}}}}. \quad (4)$$

Moreover, the Poincaré-Hilbert formal series of  $K\langle \mathcal{T}_P^{\mathcal{D}} \rangle \approx S(W)$  is, by (2):

$$F_{S(W)}(x) = \frac{1}{1-F_{\mathcal{B}r(\mathcal{D})}(x)} = \frac{1-\sqrt{1-4Dx}}{2Dx} = \frac{F_{\mathcal{B}r(\mathcal{D})}(x)}{Dx} = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{w_i}}. \quad (5)$$

So, from (3), using (4) and (5), the Poincaré-Hilbert series of  $\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$  is:

$$\begin{aligned} F_{\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle}(x) &= F_{S(W)}(x) - F_{S\left(\frac{W}{K\mathcal{T}_P^{\mathcal{D}}}\right)}(x) \\ &= \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{w_i}} \left(1 - \prod_{i=1}^{\infty} (1-x^i)^{t_i^{\mathcal{D}}}\right) \\ &= \frac{F_{\mathcal{B}r(\mathcal{D})}(x)}{Dx} \left(1 - \prod_{i=1}^{\infty} (1-x^i)^{t_i^{\mathcal{D}}}\right). \end{aligned}$$

As  $B$  is homogeneous of degree 1, the Poincaré-Hilbert formal series of  $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})$  is:

$$F_{\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})}(x) = Dx F_{\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle}(x) = F_{\mathcal{B}r(\mathcal{D})}(x) \left(1 - \prod_{i=1}^{\infty} (1-x^i)^{t_i^{\mathcal{D}}}\right).$$

Finally, the Poincaré-Hilbert formal series of  $V$  is:

$$F_V(x) = F_{\mathcal{B}r(\mathcal{D})}(x) - F_{\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})}(x) = F_{\mathcal{B}r(\mathcal{D})}(x) \prod_{i=1}^{\infty} (1-x^i)^{t_i^{\mathcal{D}}}.$$

Let us now fix a basis  $(v_i)_{i \in I}$  of  $V$ , formed of homogeneous elements. There is a unique epimorphism of non-associative permutative algebras:

$$\Theta : \begin{cases} \mathcal{NAPerm}(I) & \longrightarrow \mathcal{B}r(\mathcal{D}) \\ \bullet_i & \longrightarrow v_i. \end{cases}$$

We give to  $i \in I$  the degree of  $v_i \in \mathcal{B}r(\mathcal{D})$ . With the induced gradation of  $\mathcal{NAPerm}(I)$ ,  $\Theta$  is a graded epimorphism. In order to prove that it is an isomorphism, it is enough to prove that the Poincaré-Hilbert series of  $\mathcal{NAPerm}(I)$  and  $\mathcal{B}r(\mathcal{D})$  are equal. By lemma 3, the formal series of  $\mathcal{NAPerm}(I)$ , or, equivalently, of  $\mathcal{PL}(I)$ , is:

$$F_{\mathcal{NAPerm}(I)}(x) = \sum_{n=1}^{\infty} t_i^{\mathcal{D}} x^i = \frac{F_V(x)}{\prod_{i=1}^{\infty} (1-x^i)^{t_i^{\mathcal{D}}}} = F_{\mathcal{B}r(\mathcal{D})}(x) \prod_{i=1}^{\infty} (1-x^i)^{t_i^{\mathcal{D}}-t_i^{\mathcal{D}}}. \quad (6)$$



Let us prove inductively that  $t_n = t'_n$  for all  $n \in \mathbb{N}$ . It is immediate if  $n = 0$ , as  $t_0 = t'_0 = 0$ . Let us assume that  $t'_i = t_i$  for all  $i < n$ . Then:

$$\prod_{i=1}^{\infty} (1 - x^i)^{t_i - t'_i} = 1 + \mathcal{O}(x^n).$$

As  $t'_0 = 0$ , the coefficient of  $x^n$  in (6) is  $t_n = t'_n$ . So  $F_{\mathcal{NAPerm}(I)}(x) = F_{S(W)}(x)$ , and  $\Theta$  is an isomorphism.  $\square$

### 3 Freeness of $\mathcal{B}r(\mathcal{D})$ as a pre-Lie algebra

#### 3.1 Main theorem

**Theorem 10** *Let  $\mathcal{D}$  be a finite set. Then  $\mathcal{B}r(\mathcal{D})$  is a free pre-Lie algebra.*

**Proof.** We give a  $\mathbb{N}^2$ -gradation on  $\mathcal{B}r(\mathcal{D})$  in the following way:

$$\mathcal{B}r(\mathcal{D})(k, l) = \text{Vect}(t \in \mathcal{T}_P^{\mathcal{D}} / t \text{ has } k \text{ vertices and the fertility of its root is } l).$$

The following points are easy:

1. For all  $i, j, k, l \in \mathbb{N}$ ,  $\mathcal{B}r(\mathcal{D})(i, j) \star \mathcal{B}r(\mathcal{D})(k, l) \subseteq \mathcal{B}r(\mathcal{D})(i + k, l + 1)$ .
2. For all  $i, j, k, l \in \mathbb{N}$ ,  $t_1 \in \mathcal{B}r(\mathcal{D})(i, j)$ ,  $t_2 \in \mathcal{B}r(\mathcal{D})(k, l)$ ,  $\langle t_1; t_2 \rangle - t_1 \star t_2 \in \mathcal{B}r(\mathcal{D})(i + k, l)$ .

Let us fix a complement  $V$  of  $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})$  in  $\mathcal{B}r(\mathcal{D})$  which is  $\mathbb{N}^2$ -graded. Then  $\mathcal{B}r(\mathcal{D})$  is isomorphic as a  $\mathbb{N}$ -graded non-associative permutative algebra to  $\mathcal{NAPerm}(V)$ , the free non-associative permutative algebra generated by  $V$ .

Let us prove that  $V$  also generates  $\mathcal{B}r(\mathcal{D})$  as a pre-Lie algebra. As  $\mathcal{B}r(\mathcal{D})$  is  $\mathbb{N}$ -graded, with  $\mathcal{B}r(\mathcal{D})(0)$ , it is enough to prove that  $\mathcal{B}r(\mathcal{D}) = V + \langle \mathcal{B}r(\mathcal{D}); \mathcal{B}r(\mathcal{D}) \rangle$ . Let  $x \in \mathcal{B}r(\mathcal{D})(k, l)$ , let us show that  $x \in V + \langle \mathcal{B}r(\mathcal{D}); \mathcal{B}r(\mathcal{D}) \rangle$  by induction on  $l$ . If  $l = 0$ , then  $t \in \mathcal{B}r(\mathcal{D})(1) = V(1)$ . If  $l = 1$ , we can suppose that  $x = B_d(t)$ , where  $t \in \mathcal{T}_P^{\mathcal{D}}$ . Then  $x = \langle t; \cdot_a \rangle \in \langle \mathcal{B}r(\mathcal{D}); \mathcal{B}r(\mathcal{D}) \rangle$ . Let us assume the result for all  $l' < l$ . As  $V$  generates  $(\mathcal{B}r(\mathcal{D}), \star)$ , we can write  $x$  as:

$$x = x' + \sum_i x_i \star y_i,$$

where  $x' \in V$  and  $x_i, y_i \in \mathcal{B}r(\mathcal{D})$ . By the first point, we can assume that:

$$\sum_i x_i \otimes y_i \in \bigoplus_{i+j=k} \mathcal{B}r(\mathcal{D})(i) \otimes \mathcal{B}r(\mathcal{D})(j, l - 1).$$

So, by the second point:

$$\begin{aligned} x - x' - \sum_i \langle x_i; y_i \rangle &= \sum_i x_i \star y_i - \langle x_i; y_i \rangle \\ &\in \sum_{i+j=k} \mathcal{B}r(\mathcal{D})(i + j, l - 1) \\ &\in V + \langle \mathcal{B}r(\mathcal{D}); \mathcal{B}r(\mathcal{D}) \rangle, \end{aligned}$$

by the induction hypothesis. So  $x \in V + \langle \mathcal{B}r(\mathcal{D}); \mathcal{B}r(\mathcal{D}) \rangle$ .

Hence, there is an homogeneous epimorphism:

$$\begin{cases} \mathcal{P}\mathcal{L}(V) &\longrightarrow \mathcal{B}r(\mathcal{D}) \\ v \in V &\longrightarrow v. \end{cases}$$

As  $\mathcal{P}\mathcal{L}(V)$ ,  $\mathcal{NAPerm}(V)$  and  $\mathcal{B}r(\mathcal{D})$  have the same Poincaré-Hilbert formal series, this is an isomorphism.  $\square$

We now give the number of generators of  $\mathcal{B}r(\mathcal{D})$  in degree  $n$  when  $\text{card}(\mathcal{D}) = D$  for small values of  $n$ , computed using lemmas 3 and 6:

1. For  $n = 1$ ,  $D$ .
2. For  $n = 2$ ,  $0$ .
3. For  $n = 3$ ,  $\frac{D^2(D-1)}{2}$ .
4. For  $n = 4$ ,  $\frac{D^2(2D-1)(2D+1)}{3}$ .
5. For  $n = 5$ ,  $\frac{D^2(31D^3 - 2D^2 - 3D - 2)}{8}$ .
6. For  $n = 6$ ,  $\frac{D^2(356D^4 - 20D^3 - 5D^2 + 5D - 6)}{30}$ .
7. For  $n = 7$ ,  $\frac{D^2(5441D^5 - 279D^4 - 91D^3 - 129D^2 - 22D - 24)}{144}$ .

### 3.2 Corollaries

**Corollary 11** *Let  $\mathcal{D}$  be any set. Then  $\mathcal{B}r(\mathcal{D})$  is a free pre-Lie algebra.*

**Proof.** We graduate  $\mathcal{B}r(\mathcal{D})$  by putting all the  $\bullet_d$ 's homogeneous of degree 1. Let  $V$  be a graded complement of  $\langle \mathcal{B}r(\mathcal{D}), \mathcal{B}r(\mathcal{D}) \rangle$ . There exists an epimorphism of graded pre-Lie algebras:

$$\Theta : \begin{cases} \mathcal{P}\mathcal{L}(V) & \longrightarrow & \mathcal{B}r(\mathcal{D}) \\ \bullet_v & \longrightarrow & v. \end{cases}$$

Let  $x$  be in the kernel of  $\Theta$ . There exists a finite subset  $\mathcal{D}'$  of  $\mathcal{D}$ , such that all the decorations of the vertices of the trees appearing in  $x$  belong to  $\mathcal{B}r(\mathcal{D}')$ . By the preceding theorem, as  $\mathcal{B}r(\mathcal{D}')$  is a free pre-Lie algebra,  $x = 0$ . So  $\Theta$  is an isomorphism.  $\square$

**Corollary 12** *Let  $\mathcal{D}$  be a graded set, satisfying the conditions of lemma 3. There exists a graded set  $\mathcal{D}'$ , such that  $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$  is isomorphic, as a graded Hopf algebra, to  $\mathcal{H}_R^{\mathcal{D}'}$ .*

**Proof.**  $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$  is isomorphic, as a graded Hopf algebra, to  $\mathcal{U}(\mathcal{B}r(\mathcal{D}))^*$ . For a good choice of  $\mathcal{D}'$ ,  $\mathcal{B}r(\mathcal{D})$  is isomorphic to  $\mathcal{P}\mathcal{L}(\mathcal{D}')$  as a pre-Lie algebra, so also as a Lie algebra. So  $\mathcal{U}(\mathcal{B}r(\mathcal{D}))$  is isomorphic to  $\mathcal{U}(\mathcal{P}\mathcal{L}(\mathcal{D}'))$ . Dually,  $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$  is isomorphic to  $\mathcal{H}_R^{\mathcal{D}'}$ .  $\square$

**Corollary 13** *Let  $\mathcal{D}$  be graded set, satisfying the conditions of lemma 3. Then  $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$  is a cofree coalgebra. Moreover,  $\mathcal{B}r(\mathcal{D})$  is free as a Lie algebra.*

**Proof.** It is proved in [7] that  $(\mathcal{H}_R^{\mathcal{D}'})^*$  is a free algebra, so  $\text{Prim}((\mathcal{H}_R^{\mathcal{D}'})^*) = \mathcal{P}\mathcal{L}(\mathcal{D}')$  is a free Lie algebra and  $\mathcal{H}_R^{\mathcal{D}'}$  is a cofree coalgebra. So  $\text{Prim}((\mathcal{H}_{PR}^{\mathcal{D}})^*) = \mathcal{B}r(\mathcal{D})$  is a free Lie algebra and  $\mathcal{H}_{PR}^{\mathcal{D}}$  is a cofree coalgebra.  $\square$

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