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Free brace algebras are free prelie algebras

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ABSTRACT. Let \mathfrak{g} be a free brace algebra. This structure implies that \mathfrak{g} is also a prelie algebra and a Lie algebra. It is already known that \mathfrak{g} is a free Lie algebra. We prove here that \mathfrak{g} is also a free prelie algebra, using a description of \mathfrak{g} with the help of planar rooted trees, a permutative product, and manipulations on the Poincaré-Hilbert series of \mathfrak{g} .

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Introduction

Let \mathcal{D} be a set. The Connes-Kreimer Hopf algebra of rooted trees $\mathcal{H}_R^{\mathcal{D}}$ is introduced in [2] in the context of Quantum Field Theory and Renormalization. It is a graded, connected, commutative, non-cocommutative Hopf algebra. If the characteristic base field is zero, the Cartier-Quillen-Milnor-Moore theorem insures that its dual $(\mathcal{H}_R^{\mathcal{D}})^*$ is the enveloping algebra of a Lie algebra, based on rooted trees (note that $(\mathcal{H}_R^{\mathcal{D}})^*$ is isomorphic to the Grossman-Larson Hopf algebra [7, 8], as proved in [9, 13]). This Lie algebra admits an operadic interpretation: it is the free prelie algebra $\mathcal{PL}(\mathcal{D})$ generated by \mathcal{D} , as shown in [1]; recall that a (left) prelie algebra, also called a left Vinberg algebra or a left-symmetric algebra, is a vector space V with a product \circ satisfying:

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$

A non-commutative version of these objects is introduced in [5, 10]. Replacing rooted trees by planar rooted trees, a Hopf algebra $\mathcal{H}_{PR}^{\mathcal{D}}$ is constructed. This self-dual Hopf algebra is isomorphic to the Loday-Ronco free dendriform algebra based on planar binary trees [12], so by Maria Ronco's dendriform Milnor-Moore theorem [14], the space of its primitive elements, or equivalently the space of the primitive elements of its dual, admits a structure of brace algebra, described in terms of trees in [4] by graftings of planar forests on planar trees, and is in fact

the free brace algebra $\mathcal{B}r(\mathcal{D})$ generated by \mathcal{D} . This structure implies also a structure of prelie algebra on $\mathcal{B}r(\mathcal{D})$.

As a summary, the brace structure of $\mathcal{B}r(\mathcal{D})$ implies a prelie structure on $\mathcal{B}r(\mathcal{D})$, which implies a Lie structure on $\mathcal{B}r(\mathcal{D})$. It is already proved, using combinatorial methods on trees, that $\mathcal{P}\mathcal{L}(\mathcal{D})$ and $\mathcal{B}r(\mathcal{D})$ are free Lie algebras in characteristic zero [4]. A remaining question was the structure of $\mathcal{B}r(\mathcal{D})$ as a prelie algebra. The aim of the present text is to prove that $\mathcal{B}r(\mathcal{D})$ is a free prelie algebra. We use for this the notion of permutative algebra [11] and a manipulation of formal series. As a corollary, we prove that the abelianisation of $\mathcal{H}_{PR}^{\mathcal{D}}$ (which is not $\mathcal{H}_R^{\mathcal{D}}$), is isomorphic to a Hopf algebra $\mathcal{H}_R^{\mathcal{D}'}$ for a good choice of \mathcal{D}' . This implies that $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$ is a cofree coalgebra and we recover in a different way the result of freeness of $\mathcal{B}r(\mathcal{D})$ as a Lie algebra in characteristic zero.

Notations. We denote by K a commutative field of any characteristic. All objects (vector spaces, algebras...) will be taken over K .

1 A description of free prelie and brace algebras

1.1 Rooted trees and planar rooted trees

Definition 1

1. A *rooted tree* t is a finite graph, without loops, with a special vertex called *root* of t . The *weight* of t is the number of its vertices. The set of rooted trees will be denoted by \mathcal{T} .
2. A *planar rooted tree* t is a rooted tree given an imbedding in the plane. the set of planar rooted trees will be denoted by \mathcal{T}_P .
3. Let \mathcal{D} be a nonempty set. A rooted tree decorated by \mathcal{D} is a planar tree with an application from the set of its vertices into \mathcal{D} . The set of rooted trees decorated by \mathcal{D} will be denoted by $\mathcal{T}^{\mathcal{D}}$.
4. Let \mathcal{D} be a nonempty set. A planar rooted tree decorated by \mathcal{D} is a planar tree with an application from the set of its vertices into \mathcal{D} . The set of planar rooted trees decorated by \mathcal{D} will be denoted by $\mathcal{T}_P^{\mathcal{D}}$.

Examples.

1. Rooted trees with weight smaller than 5:

$$., \mathbf{1}, \mathbf{V}, \mathbf{1}, \Psi, \mathbf{V}, \mathbf{Y}, \mathbf{1}, \mathbf{V}, \mathbf{V}, \mathbf{V}, \mathbf{V}, \mathbf{Y}, \Psi, \mathbf{Y}, \mathbf{1}, \mathbf{1}.$$

2. Rooted trees decorated by \mathcal{D} with weight smaller than 4:

$$\bullet_a, a \in \mathcal{D}, \quad \mathbf{1}_a^b, (a, b) \in \mathcal{D}^2, \quad {}^b\mathbf{V}_a^c = {}^c\mathbf{V}_a^b, \mathbf{1}_a^c, (a, b, c) \in \mathcal{D}^3,$$

$${}^b\mathbf{V}_a^c = {}^b\mathbf{V}_a^d = {}^c\mathbf{V}_a^d = {}^c\mathbf{V}_a^b = {}^d\mathbf{V}_a^c = {}^d\mathbf{V}_a^b, {}^c\mathbf{Y}_a^d, {}^c\mathbf{Y}_a^d = {}^d\mathbf{Y}_a^c, \mathbf{1}_a^d, (a, b, c, d) \in \mathcal{D}^4.$$

3. Planar rooted trees with weight smaller than 5:

$$., \mathbf{1}, \mathbf{V}, \mathbf{1}, \Psi, \mathbf{V}, \mathbf{V}, \mathbf{Y}, \mathbf{1}, \mathbf{V}, \mathbf{V}, \mathbf{V}, \mathbf{V}, \mathbf{Y}, \Psi, \mathbf{Y}, \mathbf{1}, \mathbf{1}, \mathbf{1}.$$

4. Planar rooted trees decorated by \mathcal{D} with weight smaller than 4:

$$\begin{aligned} \bullet_a, a \in \mathcal{D}, \quad \mathbf{1}_a^b, (a, b) \in \mathcal{D}^2, \quad {}^b\mathbf{V}_a^c, \mathbf{1}_a^c, (a, b, c) \in \mathcal{D}^3, \\ {}^b\mathbf{V}_a^c, {}^c\mathbf{1}_a^d, {}^b\mathbf{V}_a^d, {}^c\mathbf{V}_a^d, \mathbf{1}_a^d, (a, b, c, d) \in \mathcal{D}^4. \end{aligned}$$

Let t_1, \dots, t_n be elements of $\mathcal{T}^{\mathcal{D}}$ and let $d \in \mathcal{D}$. We denote by $B_d(t_1 \dots t_n)$ the rooted tree obtained by grafting t_1, \dots, t_n on a common root decorated by d . For example, $B_a(\mathbf{1}_b^c \bullet_d) = {}^c\mathbf{1}_a^d$. This application B_d can be extended in an operator:

$$B_d : \begin{cases} K[\mathcal{T}^{\mathcal{D}}] & \longrightarrow K\mathcal{T}^{\mathcal{D}} \\ t_1 \dots t_n & \longrightarrow B_d(t_1 \dots t_n), \end{cases}$$

where $K[\mathcal{T}^{\mathcal{D}}]$ is the polynomial algebra generated by $\mathcal{T}^{\mathcal{D}}$ over K and $K\mathcal{T}^{\mathcal{D}}$ is the K -vector space generated by $\mathcal{T}^{\mathcal{D}}$. This operator is monic, and moreover $K\mathcal{T}^{\mathcal{D}}$ is the direct sum of the images of the B_d 's, $d \in \mathcal{D}$.

Similarly, let t_1, \dots, t_n be elements of $\mathcal{T}_P^{\mathcal{D}}$ and let $d \in \mathcal{D}$. We denote by $B_d(t_1 \dots t_n)$ the planar rooted tree obtained by grafting t_1, \dots, t_n in this order from left to right on a common root decorated by d . For example, $B_a(\mathbf{1}_b^c \bullet_d) = {}^c\mathbf{1}_a^d$ and $B_a(\bullet_d \mathbf{1}_b^c) = {}^d\mathbf{V}_a^b$. This application B_d can be extended in an operator:

$$B_d : \begin{cases} K\langle \mathcal{T}_P^{\mathcal{D}} \rangle & \longrightarrow K\mathcal{T}_P^{\mathcal{D}} \\ t_1 \dots t_n & \longrightarrow B_d(t_1 \dots t_n), \end{cases}$$

where $K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$ is the free algebra generated by $\mathcal{T}_P^{\mathcal{D}}$ over K and $K\mathcal{T}_P^{\mathcal{D}}$ is the K -vector space generated by $\mathcal{T}_P^{\mathcal{D}}$. This operator is monic, and moreover $K\mathcal{T}_P^{\mathcal{D}}$ is the direct sum of the images of the B_d 's, $d \in \mathcal{D}$.

1.2 Free prelie algebras

Definition 2 A (left) prelie algebra is a couple (A, \circ) where A is a vector space and $\circ : A \otimes A \longrightarrow A$ satisfying the following relation: for all $x, y, z \in A$,

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$

Let \mathcal{D} be a set. A description of the free prelie algebra $\mathcal{PL}(\mathcal{D})$ generated by \mathcal{D} is given in [1]. As a vector space, it has a basis given by $\mathcal{T}^{\mathcal{D}}$, and its prelie product is given, for all $t_1, t_2 \in \mathcal{T}^{\mathcal{D}}$, by:

$$t_1 \circ t_2 = \sum_{s \text{ vertex of } t_2} \text{grafting of } t_1 \text{ on } s.$$

For example:

$$\bullet_a \circ {}^b\mathbf{V}_d^c = {}^b\mathbf{V}_d^c + {}^a\mathbf{V}_d^c + {}^b\mathbf{V}_d^a = {}^b\mathbf{V}_d^c + {}^a\mathbf{V}_d^c + {}^a\mathbf{V}_d^b.$$

In other terms, the prelie product can be inductively defined by:

$$\begin{cases} t \circ \bullet_a & \longrightarrow B_d(t), \\ t \circ B_d(t_1 \dots t_n) & \longrightarrow B_d(tt_1 \dots t_n) + \sum_{i=1}^n B_d(t_1 \dots (t \circ t_i) \dots t_n). \end{cases}$$

Lemma 3 Let \mathcal{D} a set. We suppose that \mathcal{D} has a gradation $(\mathcal{D}(n))_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $\mathcal{D}(n)$ is finite set of cardinal denoted by d_n , and $\mathcal{D}(0)$ is empty. We denote by $F_{\mathcal{D}}(x)$ the Poincaré-Hilbert series of this set:

$$F_{\mathcal{D}}(x) = \sum_{n=1}^{\infty} d_n x^n.$$

This gradation induces a gradation $(\mathcal{P}\mathcal{L}(\mathcal{D})(n))_{n \in \mathbb{N}}$ of $\mathcal{P}\mathcal{L}(\mathcal{D})$. Moreover, for all $n \geq 0$, $\mathcal{P}\mathcal{L}(\mathcal{D})(n)$ is finite-dimensional. We denote by t_n its dimension. Then the Poincaré-Hilbert series of $\mathcal{P}\mathcal{L}(\mathcal{D})$ satisfies:

$$F_{\mathcal{P}\mathcal{L}(\mathcal{D})}(x) = \sum_{n=1}^{\infty} t_n x^n = \frac{F_{\mathcal{D}}(x)}{\prod_{i=1}^{\infty} (1 - x^i)^{t_i}}.$$

Proof. The formal series of the space $K[\mathcal{T}^{\mathcal{D}}]$ is given by:

$$F(x) = \prod_{i=1}^{\infty} \frac{1}{(1 - x^i)^{t_i}}.$$

Moreover, for all $d \in \mathcal{D}(n)$, B_d is homogeneous of degree n , so the Poincaré-Hilbert series of $Im(B_d)$ is $x^n F(x)$. As $\mathcal{P}\mathcal{L}(\mathcal{D}) = K\mathcal{T}^{\mathcal{D}} = \bigoplus Im(B_d)$ as a graded vector space, its Poincaré-Hilbert formal series is:

$$F_{\mathcal{P}\mathcal{L}(\mathcal{D})}(x) = F(x) \sum_{n=1}^{\infty} d_n x^n = F(x) F_{\mathcal{D}}(x),$$

which gives the announced result. □

1.3 Free brace algebras

Definition 4 A brace algebra is a couple $(A, \langle \rangle)$ where A is a vector space and $\langle \rangle$ is a family of operators $A^{\otimes n} \rightarrow A$ defined for all $n \geq 2$, with the following compatibilities: for all $a_1, \dots, a_m, b_1, \dots, b_n, c \in A$,

$$\langle a_1, \dots, a_m, \langle b_1, \dots, b_m, c \rangle \rangle = \sum \langle \langle X_1 \rangle, \dots, \langle X_r \rangle, c \rangle,$$

where this sum runs over the couple (σ, X) , where σ is a (m, n) -shuffle and X is a σ -admissible partition, that is to say:

1. $X = (X_1, \dots, X_r)$ is an ordered partition of $\{a_1, \dots, a_m, b_1, \dots, b_n\}$.
2. $X_1^{\otimes} \dots \otimes X_r^{\otimes} = \sigma.(a_1 \otimes \dots \otimes a_m \otimes b_1 \otimes \dots \otimes b_n)$, where S_{n+m}^S acts on $A^{\otimes(n+m)}$ by permutations of the indices.
3. For all i , X_i is not empty, and, if X_i is not a singleton, it contains only a b_j and this b_j is its last element.

We use the convention $\langle a \rangle = a$ for all $a \in A$.

For example, if A is a brace algebra and $a, b, c \in A$:

$$\langle a, \langle b, c \rangle \rangle = \langle a, b, c \rangle + \langle b, a, c \rangle + \langle \langle a, b \rangle, c \rangle.$$

As an immediate corollary, $(A, \langle -, - \rangle)$ is a prelie algebra. Here is another example of relation in a brace algebra: for all $a, b, c, d \in A$,

$$\langle a, b, \langle c, d \rangle \rangle = \langle a, b, c, d \rangle + \langle a, \langle b, c \rangle, d \rangle + \langle \langle a, b, c \rangle, d \rangle + \langle a, c, b, d \rangle + \langle \langle a, c \rangle, b, d \rangle + \langle c, a, b, d \rangle.$$

Let \mathcal{D} be a set. We gave in [6] a description of the free brace algebra $\mathcal{B}r(\mathcal{D})$ defined by \mathcal{D} . As a vector space, it has a basis given by $\mathcal{T}_P^{\mathcal{D}}$ and the brace structure is given, for all $t_1, \dots, t_n \in \mathcal{T}_P^{\mathcal{D}}$, by:

$$\langle t_1, \dots, t_n \rangle = \sum \text{graftings of } t_{n-1} \dots t_1 \text{ over } t_n.$$

Note that for any vertex s of t_n , there are several ways of grafting a planar tree on s . For example:

$$\langle \cdot_a, \cdot_b, \mathbf{!}_d^c \rangle = {}^a\mathbf{V}_d^c + {}^a\mathbf{V}_d^b + {}^a\mathbf{V}_d^b + \mathbf{Y}_d^c + {}^c\mathbf{V}_d^b + {}^c\mathbf{V}_d^b.$$

As a consequence, the prelie product of $\mathcal{B}r(\mathcal{D})$ can be inductively defined in this way:

$$\left\{ \begin{array}{l} \langle t, \cdot_d \rangle \longrightarrow B_d(t), \\ \langle t, B_d(t_1 \dots t_n) \rangle \longrightarrow \sum_{i=0}^n B_d(t_1 \dots t_i t t_{i+1} \dots t_n) + \sum_{i=1}^n B_d(t_1 \dots t_{i-1} \langle t, t_i \rangle t_{i+1} \dots t_n). \end{array} \right.$$

Proposition 5 $\mathcal{B}r(\mathcal{D})$ is the free brace algebra generated by \mathcal{D} .

Proof. The brace algebra $\mathcal{B}r(\mathcal{D})$ is the space of primitive elements of the free dendriform algebra generated by \mathcal{D} , described in terms of planar rooted trees. By the dendriform Milnor-Moore theorem [14], it is the free brace algebra generated by \mathcal{D} . \square

Lemma 6 Let \mathcal{D} a set, with the hypotheses and notations of lemma 3. The gradation of \mathcal{D} induces a gradation $(\mathcal{B}r(\mathcal{D})(n))_{n \in \mathbb{N}}$ of $\mathcal{B}r(\mathcal{D})$. Moreover, for all $n \geq 0$, $\mathcal{B}r(\mathcal{D})(n)$ is finite-dimensional. Then the Poincaré-Hilbert series of $\mathcal{B}r(\mathcal{D})$ is:

$$F_{\mathcal{B}r(\mathcal{D})}(x) = \sum_{n=0}^{\infty} t'_n x^n = \frac{1 - \sqrt{1 - 4F_{\mathcal{D}}(x)}}{2}.$$

Proof. The Poincaré-Hilbert formal series of $K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$ is given by:

$$F(x) = \frac{1}{1 - F_{\mathcal{B}r(\mathcal{D})}(x)}.$$

Moreover, for all $d \in \mathcal{D}(n)$, B_d is homogeneous of degree n , so the Poincaré-Hilbert series of $Im(B_d)$ is $x^n F(x)$. As $\mathcal{B}r(\mathcal{D}) = K\mathcal{T}_P^{\mathcal{D}} = \bigoplus Im(B_d)$ as a graded vector space, its Poincaré-Hilbert formal series is:

$$F_{\mathcal{B}r(\mathcal{D})}(x) = F(x) \sum_{n=1}^{\infty} d_n x^n = F(x) F_{\mathcal{D}}(x).$$

As a consequence, $F_{\mathcal{B}r(\mathcal{D})}(x) - F_{\mathcal{B}r(\mathcal{D})}(x)^2 = F_{\mathcal{D}}(x)$, which implies the announced result. \square

2 A permutative product on $\mathcal{B}r(\mathcal{D})$

2.1 Definition and recalls

The following definition is introduced in [11]:

Definition 7 A (left) permutative algebra is a couple (A, \star) , where A is a vector space and $\star : A \otimes A \longrightarrow A$ satisfies the following property: for all $x, y, z \in A$,

$$x \star (y \star z) = y \star (x \star z).$$

Let \mathcal{D} be a set. A description of the free permutative algebra $\mathcal{P}erm(\mathcal{D})$ generated by \mathcal{D} is given in [11]. As a vector space, $\mathcal{P}erm(\mathcal{D})$ is equal to $K\mathcal{T}^{\mathcal{D}}$. The permutative product is given in this way: for all $t_1 \in \mathcal{T}^{\mathcal{D}}$, $t_2 = B_d(F_2) \in \mathcal{T}^{\mathcal{D}}$,

$$t_1 \star t_2 = B_d(t_1 F_2).$$

In other terms, $t_1 \star t_2$ is the tree obtained by grafting t_1 on the root of t_2 . As $\mathcal{P}erm(\mathcal{D}) = \mathcal{P}\mathcal{L}(\mathcal{D})$ as a vector space, lemma 3 is still true when one replaces $\mathcal{P}\mathcal{L}(\mathcal{D})$ by $\mathcal{P}erm(\mathcal{D})$.

2.2 Permutative structures on planar rooted trees

Let us fix now a non-empty set \mathcal{D} . We define the following product on $\mathcal{B}r(\mathcal{D}) = K\mathcal{T}_P^{\mathcal{D}}$: for all $t \in \mathcal{T}_P^{\mathcal{D}}$, $t' = B_d(t_1 \dots t_n) \in \mathcal{T}_P^{\mathcal{D}}$,

$$t \star t' = \sum_{i=0}^n B_d(t_1 \dots t_i t t_{i+1} \dots t_n).$$

Proposition 8 $(\mathcal{B}r(\mathcal{D}), \star)$ is a permutative algebra.

Proof. Let us give $K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$ its shuffle product: for all $t_1, \dots, t_{m+n} \in \mathcal{T}_P^{\mathcal{D}}$,

$$(t_1 \dots t_m) * (t_{m+1} \dots t_{m+n}) = \sum_{\sigma \in Sh(m,n)} t_{\sigma^{-1}(1)} \dots t_{\sigma^{-1}(m+n)},$$

where $Sh(m,n)$ is the set of permutations of S_{m+n} which are increasing on $\{1, \dots, m\}$ and $\{m+1, \dots, m+n\}$. It is well known that $*$ is an associative, commutative product. For example, for all $t, t_1, \dots, t_n \in \mathcal{T}_P^{\mathcal{D}}$:

$$t * (t_1 \dots t_n) = \sum_{i=0}^n t_1 \dots t_i t t_{i+1} \dots t_n.$$

As a consequence, for all $x \in K\mathcal{T}_P^{\mathcal{D}}$, $y \in K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$, $d \in \mathcal{D}$:

$$x \star B_d(y) = B_d(x * y). \quad (1)$$

Let $t_1, t_2, t_3 = B_d(F_3) \in \mathcal{T}_P^{\mathcal{D}}$. Then, using (1):

$$\begin{aligned} t_1 \star (t_2 \star t_3) &= t_1 \star B_d(t_2 * F_3) \\ &= B_d(t_1 * (t_2 * F_3)) \\ &= B_d((t_1 * t_2) * F_3) \\ &= B_d((t_2 * t_1) * F_3) \\ &= B_d(t_2 * (t_1 * F_3)) \\ &= t_2 \star (t_1 \star t_3). \end{aligned}$$

So \star is a permutative product on $\mathcal{B}r(\mathcal{D})$. □

2.3 Freeness of $\mathcal{B}r(\mathcal{D})$ as a permutative algebra

We now assume that \mathcal{D} is finite, of cardinal d . We can then assume that $\mathcal{D} = \{1, \dots, d\}$.

Theorem 9 $(\mathcal{B}r(\mathcal{D}), \star)$ is a free permutative algebra.

Proof. We graduate \mathcal{D} by putting $\mathcal{D}(1) = \mathcal{D}$. Then $\mathcal{B}r(\mathcal{D})$ is graded, the degree of a tree $t \in \mathcal{T}_P^{\mathcal{D}}$ being the number of its vertices. By lemma 6, as the Poincaré-Hilbert series of \mathcal{D} is $F_{\mathcal{D}}(x) = dx$, the Poincaré-Hilbert series of $\mathcal{B}r(\mathcal{D})$ is:

$$F_{\mathcal{B}r(\mathcal{D})}(x) = \sum_{i=1}^{\infty} t'_i x^i = \frac{1 - \sqrt{1 - 4dx}}{2}. \quad (2)$$

We consider the following isomorphism of vector spaces:

$$B : \begin{cases} (K\langle \mathcal{T}_P^{\mathcal{D}} \rangle)^d & \longrightarrow \mathcal{B}r(\mathcal{D}) \\ (F_1, \dots, F_d) & \longrightarrow \sum_{i=1}^d B_i(F_i). \end{cases}$$

Let us fix a minimal vector space of generators of $\mathcal{B}r(\mathcal{D})$. Because $\mathcal{B}r(\mathcal{D})$ is a graded, connected permutative algebra (that is to say $\mathcal{B}r(\mathcal{D})(0) = (0)$), we can choose a graded complement V of the graded subspace $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})$ in $\mathcal{B}r(\mathcal{D})$. By (1), $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D}) = B((\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle)^d)$.

Let us then consider $\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$, that is to say the ideal of $(K\langle \mathcal{T}_P^{\mathcal{D}} \rangle, *)$ generated by $\mathcal{T}_P^{\mathcal{D}}$. It is well known that $(K\langle \mathcal{T}_P^{\mathcal{D}} \rangle, *)$ is isomorphic to a symmetric algebra: hence, there exists a graded subspace W of $K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$, such that $(K\langle \mathcal{T}_P^{\mathcal{D}} \rangle, *) \approx S(W)$ as a graded algebra. We can assume that W contains $K\mathcal{T}_P^{\mathcal{D}}$. As a consequence:

$$\frac{K\langle \mathcal{T}_P^{\mathcal{D}} \rangle}{\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle} \approx \frac{S(W)}{S(W)\mathcal{T}_P^{\mathcal{D}}} \approx S\left(\frac{W}{K\mathcal{T}_P^{\mathcal{D}}}\right). \quad (3)$$

We denote by w_i the dimension of $W(i)$ for all $i \in \mathbb{N}$. Then, the Poincaré-Hilbert formal series of $S\left(\frac{W}{K\mathcal{T}_P^{\mathcal{D}}}\right)$ is:

$$F_{S\left(\frac{W}{K\mathcal{T}_P^{\mathcal{D}}}\right)}(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{w_i-t'_i}}. \quad (4)$$

Moreover, the Poincaré-Hilbert formal series of $K\langle \mathcal{T}_P^{\mathcal{D}} \rangle \approx S(W)$ is, by (2):

$$F_{S(W)}(x) = \frac{1}{1-F_{\mathcal{B}r(\mathcal{D})}(x)} = \frac{1-\sqrt{1-4dx}}{2dx} = \frac{F_{\mathcal{B}r(\mathcal{D})}(x)}{dx} = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{w_i}}. \quad (5)$$

So, from (3), using (4) and (5), the Poincaré-Hilbert series of $\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$ is:

$$\begin{aligned} F_{\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle}(x) &= F_{S(W)}(x) - F_{S\left(\frac{W}{K\mathcal{T}_P^{\mathcal{D}}}\right)}(x) \\ &= \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{w_i}} \left(1 - \prod_{i=1}^{\infty} (1-x^i)^{t'_i}\right) \\ &= \frac{F_{\mathcal{B}r(\mathcal{D})}(x)}{dx} \left(1 - \prod_{i=1}^{\infty} (1-x^i)^{t'_i}\right). \end{aligned}$$

As B is homogeneous of degree 1, the Poincaré-Hilbert formal series of $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})$ is:

$$F_{\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})}(x) = dx F_{\mathcal{T}_P^{\mathcal{D}} * K\langle \mathcal{T}_P^{\mathcal{D}} \rangle}(x) = F_{\mathcal{B}r(\mathcal{D})}(x) \left(1 - \prod_{i=1}^{\infty} (1-x^i)^{t'_i}\right).$$

Finally, the Poincaré-Hilbert formal series of V is:

$$F_V(x) = F_{\mathcal{B}r(\mathcal{D})}(x) - F_{\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})}(x) = F_{\mathcal{B}r(\mathcal{D})}(x) \prod_{i=1}^{\infty} (1-x^i)^{t'_i}.$$

Let us now fix a basis $(v_i)_{i \in I}$ of V , formed of homogeneous elements. There is a unique epimorphism of permutative algebras:

$$\Theta : \begin{cases} \mathcal{P}erm(I) & \longrightarrow \mathcal{B}r(\mathcal{D}) \\ \bullet_i & \longrightarrow v_i. \end{cases}$$

We give to $i \in I$ the degree of $v_i \in \mathcal{B}r(\mathcal{D})$. With the induced gradation of $\mathcal{P}erm(I)$, Θ is a graded epimorphism. In order to prove that it is an isomorphism, it is enough to prove that the Poincaré-Hilbert series of $\mathcal{P}erm(I)$ and $\mathcal{B}r(\mathcal{D})$ are equal. By lemma 3, the formal series of $\mathcal{P}erm(I)$, or, equivalently, of $\mathcal{P}\mathcal{L}(I)$, is:

$$F_{\mathcal{P}erm(I)}(x) = \sum_{n=1}^{\infty} t_i x^i = \frac{F_V(x)}{\prod_{i=1}^{\infty} (1-x^i)^{t_i}} = F_{\mathcal{B}r(\mathcal{D})}(x) \prod_{i=1}^{\infty} (1-x^i)^{t'_i-t_i}. \quad (6)$$

Let us prove inductively that $t_n = t'_n$ for all $n \in \mathbb{N}$. It is immediate if $n = 0$, as $t_0 = t'_0 = 0$. Let us assume that $t_i = t'_i$ for all $i < n$. Then:

$$\prod_{i=1}^{\infty} (1 - x^i)^{t_i - t'_i} = 1 + \mathcal{O}(x^n).$$

As $t'_0 = 0$, the coefficient of x^n in (6) is $t_n = t'_n$. So $F_{\mathcal{P}erm(I)}(x) = F_{S(W)}(x)$, and Θ is an isomorphism. \square

3 Freeness of $\mathcal{B}r(\mathcal{D})$ as a prelie algebra

3.1 Main theorem

Theorem 10 *Let \mathcal{D} be a finite set. Then $\mathcal{B}r(\mathcal{D})$ is a free prelie algebra.*

Proof. We give a \mathbb{N}^2 -gradation on $\mathcal{B}r(\mathcal{D})$ in the following way:

$$\mathcal{B}r(\mathcal{D})(k, l) = \text{Vect}(t \in \mathcal{T}_P^{\mathcal{D}} / t \text{ has } k \text{ vertices and the fertility of its root is } l).$$

The following points are easy:

1. For all $i, j, k, l \in \mathbb{N}$, $\mathcal{B}r(\mathcal{D})(i, j) \star \mathcal{B}r(\mathcal{D})(k, l) \subseteq \mathcal{B}r(\mathcal{D})(i + k, l + 1)$.
2. For all $i, j, k, l \in \mathbb{N}$, $t_1 \in \mathcal{B}r(\mathcal{D})(i, j)$, $t_2 \in \mathcal{B}r(\mathcal{D})(k, l)$, $\langle t_1, t_2 \rangle - t_1 \star t_2 \in \mathcal{B}r(\mathcal{D})(i + k, l)$.

Let us fix a complement V of $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})$ in $\mathcal{B}r(\mathcal{D})$ which is \mathbb{N}^2 -graded. By the first point, this is possible. Then $\mathcal{B}r(\mathcal{D})$ is isomorphic as a \mathbb{N} -graded permutative algebra to $\mathcal{P}erm(V)$, the free permutative algebra generated by V .

Let us prove that V also generates $\mathcal{B}r(\mathcal{D})$ as a prelie algebra. As $\mathcal{B}r(\mathcal{D})$ is \mathbb{N} -graded, with $\mathcal{B}r(\mathcal{D})(0)$, it is enough to prove that $\mathcal{B}r(\mathcal{D}) = V + \langle \mathcal{B}r(\mathcal{D}), \mathcal{B}r(\mathcal{D}) \rangle$. Let $x \in \mathcal{B}r(\mathcal{D})(k, l)$, let us show that $x \in V + \langle \mathcal{B}r(\mathcal{D}), \mathcal{B}r(\mathcal{D}) \rangle$ by induction on l . If $l = 0$, then $t \in \mathcal{B}r(\mathcal{D})(1) = V(1)$. If $l = 1$, we can suppose that $x = B_d(t)$, where $t \in \mathcal{T}_P^{\mathcal{D}}$. Then $x = \langle t, \cdot_a \rangle \in \langle \mathcal{B}r(\mathcal{D}), \mathcal{B}r(\mathcal{D}) \rangle$. Let us assume the result for all $l' < l$. As V generates $(\mathcal{B}r(\mathcal{D}), \star)$, we can write x as:

$$x = x' + \sum_i x_i \star y_i,$$

where $x' \in V$ and $x_i, y_i \in \mathcal{B}r(\mathcal{D})$. By the first point, we can assume that:

$$\sum_i x_i \otimes y_i \in \bigoplus_{i+j=k} \mathcal{B}r(\mathcal{D})(i) \otimes \mathcal{B}r(\mathcal{D})(j, l - 1).$$

So, by the second point:

$$\begin{aligned} x - x' - \sum_i \langle x_i, y_i \rangle &= \sum_i x_i \star y_i - \langle x_i, y_i \rangle \\ &\in \sum_{i+j=k} \mathcal{B}r(\mathcal{D})(i + j, l - 1) \\ &\in V + \langle \mathcal{B}r(\mathcal{D}), \mathcal{B}r(\mathcal{D}) \rangle, \end{aligned}$$

by the induction hypothesis. So $x \in V + \langle \mathcal{B}r(\mathcal{D}), \mathcal{B}r(\mathcal{D}) \rangle$.

Hence, there is an homogeneous epimorphism:

$$\begin{cases} \mathcal{P}\mathcal{L}(V) &\longrightarrow \mathcal{B}r(\mathcal{D}) \\ v \in V &\longrightarrow v. \end{cases}$$

As $\mathcal{P}\mathcal{L}(V)$, $\mathcal{P}erm(V)$ and $\mathcal{B}r(\mathcal{D})$ have the same Poincaré-Hilbert formal series, this is an isomorphism. \square

We now give the number of generators of $\mathcal{B}r(\mathcal{D})$ in degree n when $\text{card}(\mathcal{D}) = d$ for small values of n , computed using lemmas 3 and 6:

1. For $n = 1$, d .
2. For $n = 2$, 0 .
3. For $n = 3$, $\frac{d^2(d-1)}{2}$.
4. For $n = 4$, $\frac{d^2(2d-1)(2d+1)}{3}$.
5. For $n = 5$, $\frac{d^2(31d^3 - 2d^2 - 3d - 2)}{8}$.
6. For $n = 6$, $\frac{d^2(356d^4 - 20d^3 - 5d^2 + 5d - 6)}{30}$.
7. For $n = 7$, $\frac{d^2(5441d^5 - 279d^4 - 91d^3 - 129d^2 - 22d - 24)}{144}$.

3.2 Corollaries

Corollary 11 *Let \mathcal{D} be any set. Then $\mathcal{B}r(\mathcal{D})$ is a free prelie algebra.*

Proof. We graduate $\mathcal{B}r(\mathcal{D})$ by putting all the \cdot_d 's homogeneous of degree 1. Let V be a graded complement of $\langle \mathcal{B}r(\mathcal{D}), \mathcal{B}r(\mathcal{D}) \rangle$. There exists an epimorphism of graded prelie algebras:

$$\Theta : \begin{cases} \mathcal{P}\mathcal{L}(V) & \longrightarrow & \mathcal{B}r(\mathcal{D}) \\ \cdot_v & \longrightarrow & v. \end{cases}$$

Let x be in the kernel of Θ . There exists a finite subset \mathcal{D}' of \mathcal{D} , such that all the decorations of the vertices of the trees appearing in x belong to $\mathcal{B}r(\mathcal{D}')$. By the preceding theorem, as $\mathcal{B}r(\mathcal{D}')$ is a free prelie algebra, $x = 0$. So Θ is an isomorphism. \square

Corollary 12 *Let us assume that K is a field of characteristic zero. Let \mathcal{D} be a graded set, satisfying the conditions of lemma 3. There exists a graded set \mathcal{D}' , such that $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$ is isomorphic, as a graded Hopf algebra, to $\mathcal{H}_R^{\mathcal{D}'}$.*

Proof. $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$ is isomorphic, as a graded Hopf algebra, to $\mathcal{U}(\mathcal{B}r(\mathcal{D}))^*$. For a good choice of \mathcal{D}' , $\mathcal{B}r(\mathcal{D})$ is isomorphic to $\mathcal{P}\mathcal{L}(\mathcal{D}')$ as a prelie algebra, so also as a Lie algebra. So $\mathcal{U}(\mathcal{B}r(\mathcal{D}))$ is isomorphic to $\mathcal{U}(\mathcal{P}\mathcal{L}(\mathcal{D}'))$. Dually, $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$ is isomorphic to $\mathcal{H}_R^{\mathcal{D}'}$. \square

Corollary 13 *Let us assume that K is a field of characteristic zero. Let \mathcal{D} be graded set, satisfying the conditions of lemma 3. Then $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$ is a cofree coalgebra. Moreover, $\mathcal{B}r(\mathcal{D})$ is free as a Lie algebra.*

Proof. It is proved in [3] that $(\mathcal{H}_R^{\mathcal{D}'})^*$ is a free algebra, so $\text{Prim}((\mathcal{H}_R^{\mathcal{D}'})^*) = \mathcal{P}\mathcal{L}(\mathcal{D}')$ is a free Lie algebra and $\mathcal{H}_R^{\mathcal{D}'}$ is a cofree coalgebra. So $\text{Prim}((\mathcal{H}_{PR}^{\mathcal{D}})^*) = \mathcal{B}r(\mathcal{D})$ is a free Lie algebra and $\mathcal{H}_{PR}^{\mathcal{D}}$ is a cofree coalgebra. \square

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