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# A micro/macro approach for parallel computing of heterogeneous structures

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## Abstract

A new micro/macro computational strategy is proposed for the analysis of structures which are described up to the “micro” level, such as composite structures. This strategy is intended to overcome standard homogenization techniques coupled with a local re-analysis, at least for several domains of interest.

The description of micro and macro quantities is performed on the interfaces arising from the decomposition of the structure into an assembly of sub-structures and interfaces. Two examples of such a description are detailed; both are built from a kinematic point of view. Lastly, the performance of this multi-scale computational strategy is shown on an example of a 3D, strongly-heterogeneous structure.

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**Keywords:** domain decomposition, homogenization, LATIN method

## 1 Introduction

Structural analysis has become more heavily concerned with material models which are described up to a scale smaller than the macroscopic structural level. This is the case when dealing with composite materials, for instance.

For linear analysis, the treatment of such two-level problems is currently performed with techniques that take into account homogenization, coupled with a local re-analysis. The most mastered technique is probably the one initiated by Sanchez-Palencia for periodic media [14]. Further developments for associated computational approaches can be found in [7, 13, 12]. For other techniques, the reader can refer to [1]. Of course, one “constraint” in the use of this method lies in the fact that the ratio between the small-scale length and the large-scale length has to be small. Moreover, these techniques are not really suited to non-linear problems of evolution, in which they are applied to linear problems arising from successive linearizations related to the computational strategy.

The objectives of the micro/macro approach proposed herein are to avoid several of the limitations in classical homogenization techniques, and to be suited to the most powerful computing resources in use today, i.e. parallel-architecture computers. This iterative strategy has a strong mechanical base; it is built upon characteristic properties which are satisfied by structural models described up to the micro scale.

The first step is the decomposition of the structure into an assembly of simple constituents: sub-structures and interfaces. For instance, a sub-structure may gather one or several cells of composite structures. Each of these components possesses its own variables and equations. An interface transfers both a distribution of displacement and a distribution of forces.

The novelty, with respect to [8], is the splitting of the unknowns (displacements, forces, stress, strain) into the form:

$$\mathbf{s} = \mathbf{s}^M + \mathbf{s}^m$$

where  $\mathbf{s}^M$  is the set of the macroscopic quantities and  $\mathbf{s}^m$  is the additive “micro” complement. Several descriptions are conceivable. Two of them are described in this paper. The first is based on use of a hierarchical element or super-element for each sub-structure, i.e. with two discretization scales. The second

is related to a “continuum-medium” point of view and features a general method for homogenization and local re-analysis.

The second step of the micro/macro strategy is the use of the so-called LATIN method on the problem to be solved, expressed as an assembly of sub-structures and interfaces. The LATIN method is a non-incremental iterative computational strategy. It deals with the entire studied time interval, [8]. The resultant micro/macro strategy displays convergence for stable materials under standard assumptions. In order to focus on the main concepts, this method is described herein only for linear elasticity.

For linear problems, the strategy involves numerical parameters that can be interpreted as interface stiffnesses. At each iteration, one has to solve a “macro” problem, defined on the entire structure, along with a family of linear problems, independent of each sub-structure and interface. These are the “micro” problems, whereas the “macro” problem is related to the homogenized structure.

This overall scheme is well suited to parallel-architecture computers; a numerical example illustrates the possibilities of this method. It can be considered as a mixed domain decomposition method. An initial version, suited to slightly heterogeneous structures has been reported in [3]. This version is a priori less efficient than the current micro/macro computational strategy, yet is nevertheless comparable to the FETI domain decomposition method, which is today the reference within the field of parallelism [6]. The use of two scales or two grids pertains to other methods as well, such as multigrid methods, in which the basis is essentially numerical and far away from the “homogenization” background of the mechanician.

## 2 Reference problem and its re-formulation

The reference problem is related to the quasi-static behavior of a structure denoted by  $\Omega$ , for small perturbations and isothermal evolution. The loadings are:

- a prescribed displacement  $\underline{U}_d$  on a first part of the boundary  $\partial_1\Omega$ ,
- a prescribed traction force  $\underline{F}_d$  on the complementary part of the boundary  $\partial_2\Omega$ ,
- a prescribed body force  $\underline{f}_d$  on  $\Omega$ .

For the sake of simplicity, only the case of linear elasticity will be described herein. Therefore, only the final configuration is of interest, and time is no longer taken into account. The non-linear case is described in [9].

The current state of the structure is given by the stress field  $\boldsymbol{\sigma}$  at each point  $\underline{M}$  of  $\Omega$  and the displacement field  $\underline{U}$ .  $\boldsymbol{\sigma}$  is searched in the corresponding space  $\mathcal{S}$ , while  $\underline{U}$  is searched in  $\mathcal{U}$ .

The problem to be solved then is to find  $\mathbf{s} = (\underline{U}, \boldsymbol{\sigma})$  in  $\mathcal{U} \times \mathcal{S}$ , which satisfies:

- kinematic equations:  $\underline{U} \in \mathcal{U}$ ,  $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\underline{U})$ ,  $\underline{U}|_{\partial_1\Omega} = \underline{U}_d$
- equilibrium equations:  $\boldsymbol{\sigma} \in \mathcal{S}$ ,

$$\forall \underline{U}^* \in \mathcal{U}_0 \quad \int_{\Omega} \text{Tr}[\boldsymbol{\sigma} \boldsymbol{\varepsilon}(\underline{U}^*)] d\Omega = \int_{\Omega} \underline{f}_d \cdot \underline{U}^* d\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{U}^* dS$$

- the constitutive relation:  $\boldsymbol{\sigma} = \mathbf{K}\boldsymbol{\varepsilon}$ , where  $\mathbf{K}(\underline{M})$  is Hooke’s tensor, characteristic of the local material behavior.

The first step of the micro/macro strategy is the re-formulation of the problem in terms of a decomposition of the structure into an assembly of simple constituents: sub-structures and interfaces [8] (see Figure 1). Each of these components possesses its own variables and equations.

A sub-structure  $\Omega_E$ ,  $E \in \mathbf{E}$ , is submitted to the action of its environment (its neighboring interfaces): a force field  $\underline{F}_E$  and a displacement field  $\underline{W}_E$  on its boundary  $\partial\Omega_E$ .

An interface  $\Gamma_{EE'}$  between sub-structures  $E$  and  $E'$  transfers both the displacement field and force field on each side:  $\underline{W}_E$ ,  $\underline{W}_{E'}$  and  $\underline{F}_E$ ,  $\underline{F}_{E'}$ . The corresponding spaces then are  $\mathcal{W}_{EE'}$  and  $\mathcal{F}_{EE'}$ . Extended to all of the interfaces, they become  $\mathcal{W}$  and  $\mathcal{F}$ .

Since both the displacements and forces on the interfaces are the unknowns, the resulting approach is a “mixed” sub-structuring technique, as opposed to the primal sub-structuring [11, 10], or dual approach [6].

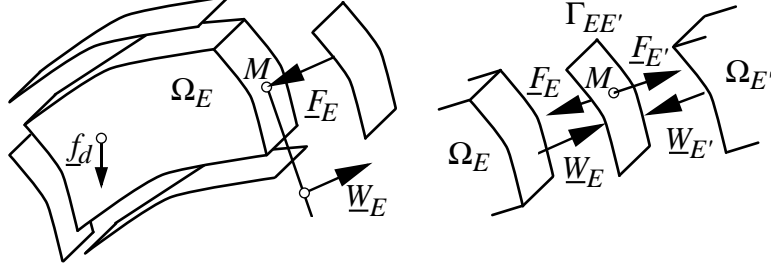


Figure 1: Sub-structures and interfaces

The solution to the reference problem,

$$\mathbf{s} = \bigcup_{E \in \mathbf{E}} \mathbf{s}_E \quad \text{with} \quad \mathbf{s}_E = (\underline{U}_E, \underline{W}_E, \boldsymbol{\sigma}_E, \underline{F}_E)$$

with the corresponding space being  $\mathbf{S}$ , must satisfy an initial set of equations,  $\mathbf{A}_d$ , in order to be admissible on each sub-structure:

- kinematic equations:  $\underline{U}_E \in \mathcal{U}_E$ ,  $\boldsymbol{\varepsilon}_E = \boldsymbol{\varepsilon}(\underline{U}_E)$ ,  $\underline{U}_E|_{\partial\Omega_E} = \underline{W}_E$
- equilibrium equations:  $\boldsymbol{\sigma}_E \in \mathcal{S}_E$ ,

$$\forall \underline{U}^* \in \mathcal{U}_E \quad \int_{\Omega_E} \text{Tr}[\boldsymbol{\sigma}_E \boldsymbol{\varepsilon}(\underline{U}^*)] d\Omega = \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^* d\Omega + \int_{\partial\Omega_E} \underline{F}_E \cdot \underline{U}^* dS$$

In addition,  $\mathbf{s}$  must also satisfy a second set of equations,  $\mathbf{\Gamma}$ , in order to verify the material and interface behaviors:

- constitutive relation:  $\boldsymbol{\sigma}_E = \mathbf{K} \boldsymbol{\varepsilon}_E$ ,
- interface behavior: for instance, with a perfect interface, the transmission conditions are  $\underline{W}_E = \underline{W}_{E'}$ ,  $\underline{F}_E + \underline{F}_{E'} = 0$  and the boundary conditions on  $\partial_1\Omega$  and  $\partial_2\Omega$ .

The regularity required for displacement field  $\underline{U}_E$  and stress field  $\boldsymbol{\sigma}_E$  is the classical one; for instance, with a tridimensional analysis,  $\mathcal{U}_E = [H^1(\Omega_E)]^3$  and  $\mathcal{S}_E = [L^2(\Omega_E)]^6$ .

Such a sub-structuring technique is well suited to the case of periodic structures [4], but this approach has not been used herein: boundary areas and interior areas are treated in the same way.

### 3 Description on the micro and macro scales

#### 3.1 General description

The state of the structure is expected to possess two parts, related to the micro scale, denoted by  $m$ , and to the macro scale  $M$ , each with a different length of variation [3].

The first step is the description of forces and displacements on the interfaces for both scales. For an interface  $\Gamma_{EE'}$  from the sub-structure  $E$ , the force  $\underline{F}_E|_{\Gamma_{EE'}}$  and the displacement  $\underline{W}_E|_{\Gamma_{EE'}}$  are split into:

$$\underline{F}_E = \underline{F}_E^M + \underline{F}_E^m \quad \underline{W}_E = \underline{W}_E^M + \underline{W}_E^m$$

A first description consists of defining a projection operator  $\boldsymbol{\pi}_{\Gamma_{EE'}}$  such that  $\underline{W}_E^M|_{\Gamma_{EE'}} = \boldsymbol{\pi}_{\Gamma_{EE'}} \underline{W}_E|_{\Gamma_{EE'}}$ . The micro and macro spaces for the displacement on  $\Gamma_{EE'}$  are  $\mathcal{W}_{EE'}^m$  and  $\mathcal{W}_{EE'}^M$ , respectively. The corresponding forces arise from the contribution of work on the interface  $\Gamma_{EE'}$  of sub-structure  $E$ :

$$(\underline{F}_E, \underline{W}_E)_{\Gamma_{EE'}} \equiv \int_{\Gamma_{EE'}} \underline{F}_E \cdot \underline{W}_E dS = (\underline{F}_E^m, \underline{W}_E^m)_{\Gamma_{EE'}} + (\underline{F}_E^M, \underline{W}_E^M)_{\Gamma_{EE'}} \quad (1)$$

A second description consists of defining a projection  $\boldsymbol{\pi}_{\Gamma_{EE'}}$  which operates on forces:  $\underline{F}_E^M|_{\Gamma_{EE'}} = \boldsymbol{\pi}_{\Gamma_{EE'}} \underline{F}_E|_{\Gamma_{EE'}}$ , with the micro and macro space for the forces on  $\Gamma_{EE'}$  denoted by  $\mathcal{F}_{EE'}^m$  and  $\mathcal{F}_{EE'}^M$ , respectively. For this description, the associated displacements are derived thanks to duality (1).

Extended to all of the interfaces of sub-structures, the previous splittings leads to:  $\mathcal{W} = \mathcal{W}^M + \mathcal{W}^m$  and  $\mathcal{F} = \mathcal{F}^M + \mathcal{F}^m$ . Other major choices must also be stated: the “micro” displacements and forces related to sub-structures  $E$  and  $E'$  with a common interface  $\Gamma_{EE'}$  do not have to satisfy the transmission conditions. On the contrary, the “macro” quantities satisfy these conditions in a weak sense:  $(\underline{W}^M, \underline{F}^M)$  must belong to  $\mathcal{W}_{\text{ad}}^M \times \mathcal{F}_{\text{ad}}^M$ ;  $\mathcal{W}_{\text{ad}}^M$  and  $\mathcal{F}_{\text{ad}}^M$  will be specified further.

Briefly, the state of the structure  $\mathbf{s}$ , is given by micro and macro quantities related to  $(\underline{W}^M, \underline{F}^M) \in \mathcal{W}_{\text{ad}}^M \times \mathcal{F}_{\text{ad}}^M$  and  $(\underline{W}^m, \underline{F}^m) \in \mathcal{W}^m \times \mathcal{F}^m$ . In the following sections, we will discuss two examples of descriptions associated with displacement-oriented projectors.

### 3.2 A first example of micro/macro description: hierarchical element or super-element

For the first example proposed herein, an approximation is added to the description of micro and macro quantities, because the micro/macro description is built on a finite element discretization of the problem. Let us consider the case where different meshes are used for each scale and, for purposes of simplification, when the discretization spaces are embedded, as shown in Figure 2-a.

The representation of the displacements on the two scales is performed on the hierarchical basis for macro and micro variables [15]. We will now denote with a subscript  $h$  the different spaces already mentioned, since they are strongly related to the hierarchical basis arising from the discretization.

Let us consider a common boundary  $\Gamma_{EE'}$  to sub-structures  $E$  and  $E'$ . The displacement  $\underline{W}_E$  on  $\Gamma_{EE'}$  is:

$$\underline{W}_E^M |_{\Gamma_{EE'}} = \sum_{i=1}^m \underline{W}_E^M(\underline{X}_i) \varphi_i |_{\Gamma_{EE'}} \quad \underline{W}_E^m |_{\Gamma_{EE'}} = \sum_{j=m+1}^n \underline{W}_E^m(\underline{X}_j) \varphi_j |_{\Gamma_{EE'}}$$

with  $\underline{W}_E^m(\underline{X}_i) = 0$  for  $i \in 1, \dots, m$ , and  $\varphi$  the hierarchical basis functions, see Figure 2-a. These relations serve to define  $\mathcal{W}_h^M$  and  $\mathcal{W}_h^m$ . The corresponding projector  $\pi_{\Gamma_{EE'}}$  is not orthogonal with respect to the standard scalar product. With the duality properties in (1), one obtains  $\mathcal{F}_h^M$  and  $\mathcal{F}_h^m$ . In particular,  $\underline{F}_E^M |_{\Gamma_{EE'}} = \pi_{\Gamma_{EE'}}^T \underline{F}_E |_{\Gamma_{EE'}}$  and  $\underline{F}_E^m |_{\Gamma_{EE'}} = (\mathbf{1} - \pi_{\Gamma_{EE'}}^T) \underline{F}_E |_{\Gamma_{EE'}}$ . The superscript  $T$  denotes the transposition associated to the symmetric form (1). Let us also denote the extension of  $\pi_{\Gamma_{EE'}}$  to a set of interfaces by  $\pi$ .

In order to define  $\mathcal{W}_{\text{ad},h}^M$ , we have elected to enforce the exact transmission and boundary conditions for the macro displacements  $\underline{W}^M$  only:

$$\mathcal{W}_{\text{ad},h}^M = \{ \underline{W} \in \mathcal{W}_h^M \mid \underline{W}_E^M |_{\Gamma_{EE'}} = \underline{W}_{E'}^M |_{\Gamma_{EE'}}, \quad \underline{W}_E^M |_{\partial_1 \Omega} = \pi \underline{U}_d \}$$

In this case,  $\mathcal{F}_{\text{ad},h}^M = \mathcal{F}_h^M$ .

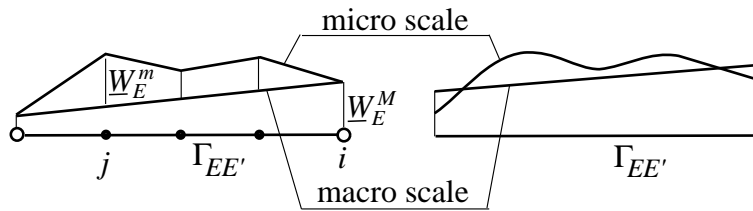


Figure 2: Description of macro and micro displacements on an interface: a) hierarchical description, b) continuum description

### 3.3 A second example of micro/macro description: a continuum mechanics point of view

In this section, we describe a second example of a micro/macro description without introducing any a priori discretization of the problem. The subscript  $h$  will no longer be used.

Once again, let us consider again a common boundary  $\Gamma_{EE'}$  to sub-structures  $E$  and  $E'$ . Using an orthogonal projector  $\pi_{\Gamma_{EE'}}$ , the macro displacement is  $\underline{W}_E^M |_{\Gamma_{EE'}} = \pi_{\Gamma_{EE'}} \underline{W}_E |_{\Gamma_{EE'}}$ . For instance, one can choose the linear part of the field  $\underline{W}_E |_{\Gamma_{EE'}}$  according to the classical scalar product, see Figure 2-b. This leads to  $\underline{W}_E^m |_{\Gamma_{EE'}} = (\mathbf{1} - \pi_{\Gamma_{EE'}}) \underline{W}_E |_{\Gamma_{EE'}}$ ,  $\underline{F}_E^M |_{\Gamma_{EE'}} = \pi_{\Gamma_{EE'}} \underline{F}_E |_{\Gamma_{EE'}}$  and  $\underline{F}_E^m |_{\Gamma_{EE'}} =$

$(\mathbf{1} - \pi_{\Gamma_{EE'}})\underline{F}_E|_{\Gamma_{EE'}}$ . Here, for the sake of simplicity, we assume that the resultant and the moment of  $\pi_{\Gamma_{EE'}}\underline{W}_E|_{\Gamma_{EE'}}$  and  $\underline{W}_E|_{\Gamma_{EE'}}$  are the same on  $\Gamma_{EE'}$ .

Once again,  $\mathcal{W}_{ad}^M$  is chosen to be the set of displacements in  $\mathcal{W}^M$  that satisfy the displacement transmission conditions on interfaces and the displacement boundary conditions on  $\partial_1\Omega$ . And once again,  $\mathcal{F}_{ad}^M = \mathcal{F}^M$ .

## 4 Computational micro/macro strategy: basis aspects

In order to solve the problem related to the assembly of sub-structures and interfaces, a strategy is developed with the LATIN method [8]. For the linear elasticity case, the corresponding duality changes: it is now a work-based duality and no longer a dissipative one.

The LATIN method is a non-incremental iterative strategy [8]. It successively builds an element  $\mathbf{s}$  of the space of admissible fields,  $\mathbf{A}_d$  (kinematic and equilibrium equations on each sub-structure), and an element of the second set  $\mathbf{\Gamma}$  (constitutive relation and interface behavior) within each iteration. At iteration  $n$ , the element  $\hat{\mathbf{s}}_{n+1/2}$  of  $\mathbf{\Gamma}$  is defined at the local stage from a previous element  $\mathbf{s}_n$  of  $\mathbf{A}_d$ , using the search direction  $\mathbf{E}^+$ . Then, the next element  $\mathbf{s}_{n+1}$  of  $\mathbf{A}_d$  is built using a second search direction  $\mathbf{E}^-$ , see Figure 3. These search directions are the parameters of the method.

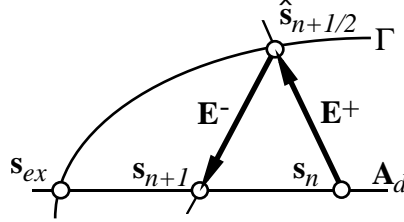


Figure 3: One iteration of the LATIN method

The two examples of micro/macro descriptions fall within the scope of this general framework. For the hierarchical element-based description, the spaces must be replaced by the same ones, with the subscript  $h$ .

### 4.1 Local stage at iteration $n$

At this stage, the material behavior, as well as the interface behavior, are satisfied. The problem consists of finding  $\hat{\mathbf{s}}_{n+1/2} \in \mathbf{\Gamma}$ , given  $\mathbf{s}_n \in \mathbf{A}_d$ . Moreover,  $\hat{\mathbf{s}}_{n+1/2} - \mathbf{s}_n$  must belong to the search direction  $\mathbf{E}^+$ . This last one is, for all sub-structure  $E$ :

$$(\hat{\boldsymbol{\sigma}}_{En+1/2} - \boldsymbol{\sigma}_{En}) + \mathbf{K}(\hat{\boldsymbol{\varepsilon}}_{En+1/2} - \boldsymbol{\varepsilon}_{En}) = 0$$

and for each interface:

$$\forall \underline{W}^{m\star} \in \mathcal{W}_{EE'}^m$$

$$(\hat{\underline{F}}_{En+1/2}^m - \underline{F}_{En}^m, \underline{W}^{m\star})_{\Gamma_{EE'}} - (k^m(\hat{\underline{W}}_{En+1/2}^m - \underline{W}_{En}^m), \underline{W}^{m\star})_{\Gamma_{EE'}} = 0$$

with a similar equation for the macro scale (with superscript  $M$  rather than  $m$ ).

$\mathbf{K}$  is the Hooke's tensor.  $k^m$  and  $k^M$  are two parameters of the method which are null or positive scalars.  $k^m$  is only related to micro quantities and to interface characteristics, while  $k^M$  is related to structural characteristics, [3].

For a perfect interface,  $\mathbf{\Gamma}$  contains the transmission conditions:

$$\underline{W}_E^m = \underline{W}_{E'}^m \quad \underline{W}_E^M = \underline{W}_{E'}^M$$

and  $\forall \underline{W}^{m\star} \in \mathcal{W}_{EE'}^m, \forall \underline{W}^{M\star} \in \mathcal{W}_{EE'}^M$

$$(\underline{F}_E^m + \underline{F}_{E'}^m, \underline{W}^{m\star})_{\Gamma_{EE'}} = (\underline{F}_E^M + \underline{F}_{E'}^M, \underline{W}^{M\star})_{\Gamma_{EE'}} = 0$$

as well as the boundary conditions on  $\partial_1\Omega$  and  $\partial_2\Omega$ .

## 4.2 Linear stage at iteration $n$

The problem now is to find  $\mathbf{s}_{n+1} \in \mathbf{A}_d$ , given  $\hat{\mathbf{s}}_{n+1/2} \in \mathbf{\Gamma}$ . For each sub-structure  $E$ , the stress field must balance the forces on interfaces:

$$\begin{aligned} \boldsymbol{\sigma}_E \in \mathcal{S}_E, \quad \underline{\mathbf{F}}_E^M \in \mathcal{F}_{\text{ad}}^M = \mathcal{F}^M, \quad \underline{\mathbf{F}}_E^m \in \mathcal{F}^m, \\ \forall \underline{\mathbf{U}}^* \in \mathcal{U}_E \int_{\Omega_E} \text{Tr}[\boldsymbol{\sigma}_E \boldsymbol{\varepsilon}(\underline{\mathbf{U}}^*)] d\Omega = \int_{\Omega_E} \underline{\mathbf{f}}_d \cdot \underline{\mathbf{U}}^* d\Omega + \int_{\partial\Omega_E} (\underline{\mathbf{F}}_E^M \cdot \underline{\mathbf{U}}^* + \underline{\mathbf{F}}_E^m \cdot \underline{\mathbf{U}}^*) dS \end{aligned} \quad (2)$$

The displacement field must be compatible with the displacements on interfaces:

$$\underline{\mathbf{U}}_E \in \mathcal{U}_E, \quad \underline{\mathbf{W}}_E^M \in \mathcal{W}_{\text{ad}}^M, \quad \underline{\mathbf{W}}_E^m \in \mathcal{W}^m, \quad \underline{\mathbf{U}}_E|_{\partial\Omega_E} = (\underline{\mathbf{W}}_E^M + \underline{\mathbf{W}}_E^m)|_{\partial\Omega_E}$$

In the previous conditions, note that  $(\underline{\mathbf{W}}_E^M, \underline{\mathbf{F}}_E^M)$  has been imposed to belong to  $\mathcal{W}_{\text{ad}}^M \times \mathcal{F}_{\text{ad}}^M$ .

$\mathbf{s}_{n+1}$  is also defined with the search direction  $\mathbf{E}^-$ : for each sub-structure  $E$ :

$$(\boldsymbol{\sigma}_{E_{n+1}} - \hat{\boldsymbol{\sigma}}_{E_{n+1/2}}) - \mathbf{K}(\boldsymbol{\varepsilon}_{E_{n+1}} - \hat{\boldsymbol{\varepsilon}}_{E_{n+1/2}}) = 0$$

and for each interface  $\Gamma_{EE'}$  and sub-structure  $E$ :

$$\begin{aligned} \forall \underline{\mathbf{W}}_E^{m*} \in \mathcal{W}_{EE'}^m \\ (\underline{\mathbf{F}}_E^m|_{\Gamma_{EE'}} - \hat{\underline{\mathbf{F}}}_E^m|_{\Gamma_{EE'}}, \underline{\mathbf{W}}_E^{m*})_{\Gamma_{EE'}} + (k^m(\underline{\mathbf{W}}_E^m|_{\Gamma_{EE'}} - \hat{\underline{\mathbf{W}}}_E^m|_{\Gamma_{EE'}}), \underline{\mathbf{W}}_E^{m*})_{\Gamma_{EE'}} = 0 \end{aligned} \quad (3)$$

Concerning the macro quantities, the search direction must be global:

$$\begin{aligned} \forall \underline{\mathbf{W}}_E^{M*} \in \mathcal{W}_{\text{ad},0}^M \sum_{\Gamma_{EE'}} \sum_E (\underline{\mathbf{F}}_E^M|_{\Gamma_{EE'}} - \hat{\underline{\mathbf{F}}}_E^M|_{\Gamma_{EE'}}, \underline{\mathbf{W}}_E^{M*})_{\Gamma_{EE'}} + \\ + (k^M(\underline{\mathbf{W}}_E^M|_{\Gamma_{EE'}} - \hat{\underline{\mathbf{W}}}_E^M|_{\Gamma_{EE'}}), \underline{\mathbf{W}}_E^{M*})_{\Gamma_{EE'}} = 0 \end{aligned} \quad (4)$$

The resulting problem is then split into two kinds of sub-problems: a global macro problem and a micro problem on each sub-structure. In the following discussions, subscripts  $n+1/2$  and  $n+1$  will be omitted.

## 5 A micro/macro strategy based on a hierarchical finite element description

Let us first consider the hierarchical finite element description. An example of meshes for a tridimensional beam is described in Figure 4. In order to easily solve the discretized problem, the prolongation of both the macro and micro parts of the displacement inside each sub-structure  $E$  is defined according to the hierarchical basis of the meshes. An additional approximation is then introduced, after that related to the discretization on interfaces, and a subscript  $h$  will be added in order to recall these additions. In particular, the interior micro and macro displacement fields will be the displacements of  $\mathcal{U}_{E,h}^m$  and  $\mathcal{U}_{E,h}^M$ , respectively. Therefore, the displacement  $\underline{\mathbf{U}}$  is split into:  $\underline{\mathbf{U}} = \underline{\mathbf{U}}^m + \underline{\mathbf{U}}^M$  with  $\underline{\mathbf{U}}_E \in \mathcal{U}_{E,h}$ ,  $\underline{\mathbf{U}}_E^m \in \mathcal{U}_{E,h}^m$  and  $\underline{\mathbf{U}}_E^M \in \mathcal{U}_{E,h}^M$ .

### 5.1 Micro-scale problem

This problem is defined on a sub-structure  $E$  and is related to quantities defined on the micro subspaces, with all macro quantities considered as fixed. Let us recall that  $\hat{\mathbf{s}}$  is given. Using the search direction on micro quantities, independent of each sub-structure, the problem is:

$$\begin{aligned} \underline{\mathbf{U}}_E^m \in \mathcal{U}_{E,h}^m, \quad \underline{\mathbf{W}}_E^m \in \mathcal{W}_{EE',h}^m, \quad \underline{\mathbf{U}}_E^m|_{\partial\Omega_E} = \underline{\mathbf{W}}_E^m|_{\partial\Omega_E}, \\ \forall \underline{\mathbf{U}}^{m*} \in \mathcal{U}_{E,h}^m \int_{\Omega_E} \text{Tr}[\boldsymbol{\varepsilon}(\underline{\mathbf{U}}_E^m) \mathbf{K} \boldsymbol{\varepsilon}(\underline{\mathbf{U}}^{m*})] d\Omega + \int_{\partial\Omega_E} \underline{\mathbf{U}}_E^m \cdot k^m \underline{\mathbf{U}}^{m*} dS = \\ = \int_{\Omega_E} \underline{\mathbf{f}}_d \cdot \underline{\mathbf{U}}^{m*} d\Omega + \int_{\partial\Omega_E} (\hat{\underline{\mathbf{F}}}_E^m + k^m \hat{\underline{\mathbf{W}}}_E^m) \cdot \underline{\mathbf{U}}^{m*} dS + \\ - \int_{\Omega_E} \text{Tr}[\boldsymbol{\varepsilon}(\underline{\mathbf{U}}^M)|_{\Omega_E} \mathbf{K} \boldsymbol{\varepsilon}(\underline{\mathbf{U}}^{m*})] d\Omega \end{aligned} \quad (5)$$

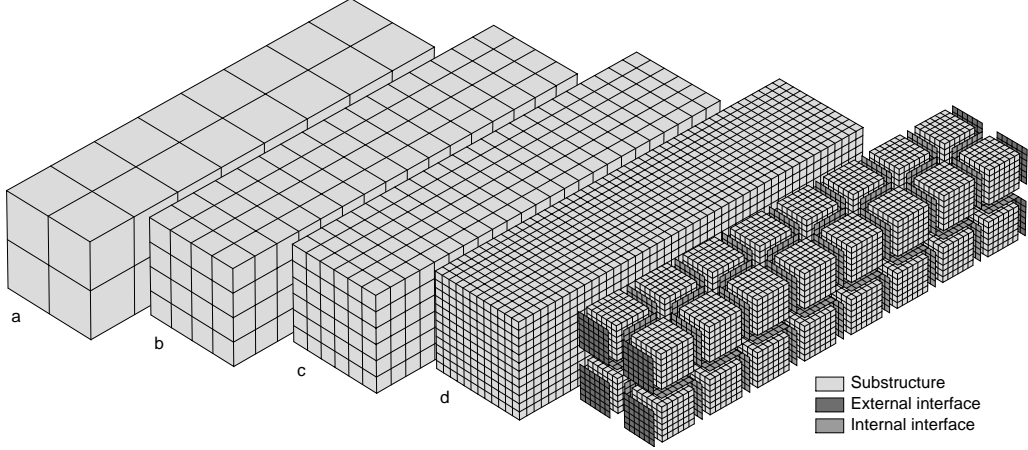


Figure 4: Meshes of both the large-scale problem (a,b,c,d) with 8-node elements and the sub-structured problem (assembly of 32 sub-structures and interfaces) with 20-node elements

Due to the definition of  $\pi$  used herein, this problem possesses a unique solution when  $k^m \geq k > 0$  ( $k$  is a constant).

In any case, the micro-scale problem (5) is linear. Its solution depends separately on:

- $f_d |_{\Omega_E}$  and  $(\hat{F}_E^m + k^m \hat{W}_E^m) |_{\partial\Omega_E}$ ,
- $\varepsilon(\underline{U}^M) |_{\Omega_E}$

Therefore, this solution can be written as:  $\underline{U}_E^m = \hat{\underline{U}}_{E,d}^m + \tilde{\underline{U}}_E^m$  where  $\tilde{\underline{U}}_E^m$  is related to  $\varepsilon(\underline{U}^M) |_{\Omega_E}$  and where  $\hat{\underline{U}}_{E,d}^m$  involves only known quantities at this stage. More precisely, we have:

$$\mathbf{K}\varepsilon(\tilde{\underline{U}}_E^m) = -\mathbf{H}_E\varepsilon(\underline{U}^M) |_{\Omega_E}$$

where  $\mathbf{H}_E$  is a linear operator. Moreover,  $\mathbf{H}_E$  verifies:

**Property 1** *The operator  $\mathbf{H}_E$  is linear, symmetric and positive definite.*

**Proof 1** *Let us write the problem that defines  $\mathbf{H}_E$ : find  $\tilde{\underline{U}}_E^m \in \mathcal{U}_{E,h}^m$  such that*

$$\begin{aligned} \forall \underline{U}^{m*} \in \mathcal{U}_{E,h}^m \quad \int_{\Omega_E} \text{Tr}[\varepsilon(\tilde{\underline{U}}_E^m)\mathbf{K}\varepsilon(\underline{U}^{m*})]d\Omega + \int_{\partial\Omega_E} \tilde{\underline{U}}_E^m \cdot k^m \underline{U}^{m*} dS = \\ = - \int_{\Omega_E} \text{Tr}[\varepsilon(\underline{U}^M) |_{\Omega_E} \mathbf{K}\varepsilon(\underline{U}^{m*})]d\Omega \quad (6) \end{aligned}$$

Let us now consider two macro strains  $\varepsilon_1^M$  and  $\varepsilon_2^M$  and their corresponding solutions  $\tilde{\underline{U}}_{E,1}^m$  and  $\tilde{\underline{U}}_{E,2}^m$ . From (6) and the symmetry of  $\mathbf{K}$ , one obtains:

$$\begin{aligned} \int_{\Omega_E} \text{Tr}[\varepsilon(\tilde{\underline{U}}_{E,1}^m)\mathbf{K}\varepsilon(\tilde{\underline{U}}_{E,2}^m)]d\Omega + \int_{\partial\Omega_E} \tilde{\underline{U}}_{E,1}^m \cdot k^m \tilde{\underline{U}}_{E,2}^m dS = \\ = \int_{\Omega_E} \text{Tr}[\varepsilon_1^M \mathbf{H}_E \varepsilon_2^M]d\Omega = \int_{\Omega_E} \text{Tr}[\varepsilon_2^M \mathbf{H}_E \varepsilon_1^M]d\Omega \quad (7) \end{aligned}$$

This relation holds for any  $\varepsilon_1^M$  and  $\varepsilon_2^M$  and hence  $\mathbf{H}_E$  is a symmetric operator. From (7), we also obtain:

$$\int_{\Omega_E} \text{Tr}[\varepsilon(\tilde{\underline{U}}_E^m)\mathbf{K}\varepsilon(\tilde{\underline{U}}_E^m)]d\Omega + \int_{\partial\Omega_E} \tilde{\underline{U}}_E^m \cdot k^m \tilde{\underline{U}}_E^m dS = \int_{\Omega_E} \text{Tr}[\varepsilon^M \mathbf{H}_E \varepsilon^M]d\Omega$$

Thus, since  $\mathbf{K}$  is a positive definite operator and  $k^m > 0$ ,  $\mathbf{H}_E$  is also positive definite.



## 5.2 Macro-scale problem

The macro-scale problem is related to macro quantities defined on the entire structure and arises from the macro-search direction:

$$\forall \underline{W}^{M\star} \in \mathcal{W}_{\text{ad},0,h}^M \quad \sum_{E \in \mathbf{E}} \int_{\partial\Omega_E} (\underline{F}_E^M - \hat{\underline{F}}_E^M) \cdot \underline{W}^{M\star} dS + \int_{\partial\Omega_E} (\underline{W}_E^M - \hat{\underline{W}}_E^M) \cdot k^M \underline{W}^{M\star} dS = 0 \quad (8)$$

and  $\underline{W}^M \in \mathcal{W}_{\text{ad},h}^M$ ,  $\underline{W}_E^M|_{\partial\Omega_E} = \underline{U}_E^M|_{\partial\Omega_E}$ ,  $\underline{U}^M \in \mathcal{U}_h^M$ .

Moreover, for each sub-structure:

$$\begin{aligned} \forall \underline{U}^{M\star} \in \mathcal{U}_{E,h}^M \quad \int_{\Omega_E} \text{Tr}[\varepsilon(\underline{U}_E^M) \mathbf{K} \varepsilon(\underline{U}^{M\star})] d\Omega = \\ = \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^{M\star} d\Omega + \int_{\partial\Omega_E} \underline{F}_E^M \cdot \underline{U}^{M\star} dS - \int_{\Omega_E} \text{Tr}[\varepsilon(\underline{U}_E^m) \mathbf{K} \varepsilon(\underline{U}^{M\star})] d\Omega \end{aligned} \quad (9)$$

Using (8) and (9), the displacement-oriented formulation of the macro-scale problem then becomes: find  $\underline{U}^M \in \mathcal{U}_h^M$  such that

$$\begin{aligned} \forall \underline{U}^{M\star} \in \mathcal{U}_{0,h}^M \quad \sum_{E \in \mathbf{E}} \int_{\Omega_E} \text{Tr}[\varepsilon(\underline{U}_E^M) \mathbf{K} \varepsilon(\underline{U}^{M\star})] d\Omega + \\ + \int_{\partial\Omega_E} \underline{U}_E^M \cdot k^M \underline{U}^{M\star} dS = \sum_{E \in \mathbf{E}} \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^{M\star} d\Omega + \int_{\partial\Omega_E} (\hat{\underline{F}}_E^M + k^M \hat{\underline{W}}_E^M) \cdot \underline{U}^{M\star} dS + \\ - \int_{\Omega_E} \text{Tr}[\varepsilon(\underline{U}_E^m) \mathbf{K} \varepsilon(\underline{U}^{M\star})] d\Omega \end{aligned} \quad (10)$$

Using the operator  $\mathbf{H}_E$ , this becomes: find  $\underline{U}^M \in \mathcal{U}_h^M$  such that

$$\begin{aligned} \forall \underline{U}^{M\star} \in \mathcal{U}_{0,h}^M \quad \sum_{E \in \mathbf{E}} \int_{\Omega_E} \text{Tr}[\varepsilon(\underline{U}_E^M) (\mathbf{K} - \mathbf{H}_E) \varepsilon(\underline{U}^{M\star})] d\Omega + \\ + \int_{\partial\Omega_E} \underline{U}_E^M \cdot k^M \underline{U}^{M\star} dS = \sum_{E \in \mathbf{E}} \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^{M\star} d\Omega + \int_{\partial\Omega_E} (\hat{\underline{F}}_E^M + k^M \hat{\underline{W}}_E^M) \cdot \underline{U}^{M\star} dS + \\ - \int_{\Omega_E} \text{Tr}[\varepsilon(\hat{\underline{U}}_{E,d}^m) \mathbf{K} \varepsilon(\underline{U}^{M\star})] d\Omega \end{aligned} \quad (11)$$

The operator  $(\mathbf{K} - \mathbf{H}_E)$  is the homogenized Hooke's operator. It depends both upon the material characteristics of the sub-structure  $E$  and the choice of the micro/macro description.

**Property 2**  $(\mathbf{K} - \mathbf{H}_E)$  is still a symmetric positive definite operator.

**Proof 2** Using the same notations as in the previous proof and from the equality:

$$\begin{aligned} \int_{\Omega_E} \text{Tr}[\varepsilon_1^M (\mathbf{K} - \mathbf{H}_E) \varepsilon_2^M] d\Omega = \\ = \int_{\Omega_E} \text{Tr}[(\varepsilon(\tilde{\underline{U}}_{E,1}^m) + \varepsilon_1^M) \mathbf{K} (\varepsilon(\tilde{\underline{U}}_{E,2}^m) + \varepsilon_2^M)] d\Omega + \int_{\partial\Omega_E} \tilde{\underline{U}}_{E,1}^m \cdot k^m \tilde{\underline{U}}_{E,2}^m dS \end{aligned}$$

we can conclude both symmetry and positive definiteness.

The problem in (10) is thereby a standard finite-element discretized problem with  $k^M = 0$  for which a unique solution exist. For  $k^M > 0$ , rigidities associated with the interfaces are added and the problem has a unique solution again.

Once  $\underline{U}^M$  has been determined, one obtains  $\underline{W}_E^M = \underline{U}_E^M|_{\partial\Omega_E}$  and  $\underline{F}_E^M$  with the search direction

$$\forall \underline{W}^{M\star} \in \mathcal{W}_{EE'}^M \quad (\underline{F}_E^M, \underline{W}^{M\star})_{\Gamma_{EE'}} = (\hat{\underline{F}}_E^M + k^M \hat{\underline{W}}_E^M - k^M \underline{W}_E^M, \underline{W}^{M\star})_{\Gamma_{EE'}}$$

$\underline{U}^M$  is then needed to compute the generalized loading on the micro scale, and thus, the micro-scale corrections  $\underline{U}_E^m$ .

Both problems (5) and (11) involve linear operators which can be factorized once in the case of linear elasticity problems, while the right hand sides are iteration-dependent.

## 6 A continuum mechanics point-of-view description

In this section, we are concerned with the continuum mechanics description that uses an orthogonal projector.

### 6.1 Micro-scale problem

Let us consider a sub-structure  $E$ . The displacement field  $\underline{U}_E$  has been split on the boundary  $\partial\Omega_E$  in the micro and macro parts. In particular,  $(\mathbf{1} - \boldsymbol{\pi})\underline{U}_E|_{\partial\Omega_E} = \underline{W}_E^m|_{\partial\Omega_E}$ . The micro-search direction (3) then becomes:

$$\forall \underline{W}^{m*} \in \mathcal{W}_{EE'}^m$$

$$(\underline{F}_E^m, \underline{W}^{m*})_{\Gamma_{EE'}} = (\hat{\underline{F}}_E^m + k^m \hat{\underline{W}}_E^m - k^m (\mathbf{1} - \boldsymbol{\pi}_{\Gamma_{EE'}})\underline{U}_E|_{\Gamma_{EE'}}, \underline{W}^{m*})_{\Gamma_{EE'}}$$

The admissibility, satisfied in the linear stage (2), along with the previous micro-search direction give the formulation: find  $\underline{U}_E \in \mathcal{U}_E$  such that

$$\forall \underline{U}^* \in \mathcal{U}_E$$

$$\int_{\Omega_E} \text{Tr}[\boldsymbol{\varepsilon}(\underline{U}_E) \mathbf{K} \boldsymbol{\varepsilon}(\underline{U}^*)] d\Omega + \int_{\partial\Omega_E} (\mathbf{1} - \boldsymbol{\pi})\underline{U}_E \cdot k^m (\mathbf{1} - \boldsymbol{\pi})\underline{U}^* dS =$$

$$= \int_{\partial\Omega_E} (\hat{\underline{F}}_E^m + k^m \hat{\underline{W}}_E^m) \cdot \underline{U}^* dS + \int_{\partial\Omega_E} \underline{F}_E^M \cdot \underline{U}^* dS + \int_{\Omega_E} \underline{f}_d \cdot \underline{U}^* d\Omega \quad (12)$$

The uniqueness of the solution to this problem depends on the choice of projection operator. For instance, when resultant and moment are preserved, the micro-scale problem has a solution with an undefined additive rigid body displacement, as soon as  $\underline{F}_E^M|_{\partial\Omega_E}$  balances  $\underline{f}_d|_{\Omega_E}$  (it can be noticed that  $\hat{\underline{F}}_E^m + k^m \hat{\underline{W}}_E^m$  is always orthogonal to the kernel of the left-hand side of (12)).

**Proof 3** Let us consider a force field  $\underline{F}_E^M|_{\partial\Omega_E}$  with a priori two solutions  $\underline{U}_{E,1}$  and  $\underline{U}_{E,2}$  on the sub-structure  $E$ . We then have:

$$\int_{\Omega_E} \text{Tr}[\boldsymbol{\varepsilon}(\underline{U}_{E,2} - \underline{U}_{E,1}) \mathbf{K} \boldsymbol{\varepsilon}(\underline{U}_{E,2} - \underline{U}_{E,1})] d\Omega +$$

$$+ \int_{\partial\Omega_E} (\mathbf{1} - \boldsymbol{\pi})(\underline{U}_{E,2} - \underline{U}_{E,1}) \cdot k^m (\mathbf{1} - \boldsymbol{\pi})(\underline{U}_{E,2} - \underline{U}_{E,1}) dS = 0$$

Due to the positiveness of  $\mathbf{K}$  and  $k^m$ , we can conclude that  $(\underline{U}_{E,2} - \underline{U}_{E,1})$  is a rigid body displacement. In the case where  $\boldsymbol{\pi}_{\Gamma_{EE'}}$  preserves both resultant and moment,  $(\mathbf{1} - \boldsymbol{\pi}_{\Gamma_{EE'}})(\underline{U}_{E,2} - \underline{U}_{E,1})|_{\Gamma_{EE'}} = 0$  anyway.

Since this problem is linear, its solution depends separately on:

- $\underline{f}_d|_{\Omega_E}$  and  $(\hat{\underline{F}}_E^m + k^m \hat{\underline{W}}_E^m)|_{\partial\Omega_E}$ ,
- $\underline{F}_E^M|_{\partial\Omega_E}$

Therefore, this solution can be written on the boundary as:

$$\boldsymbol{\pi}\underline{U}_E|_{\partial\Omega_E} = \hat{\underline{W}}_{E,d}^M + \mathbf{L}_E(\underline{F}_E^M|_{\partial\Omega_E} - \hat{\underline{F}}_{E,d}^M)$$

where  $\hat{\underline{F}}_{E,d}^M \in \mathcal{F}^M$  is a specific macro force field on  $\partial\Omega_E$  that balances the resultant and moment of  $\underline{f}_d|_{\Omega_E}$ .  $(\underline{F}_E^M|_{\partial\Omega_E} - \hat{\underline{F}}_{E,d}^M)$  is thus the first part of the right-hand side of (12), with a null resultant and moment on  $\partial\Omega_E$ .  $\mathbf{L}_E$  is an operator defined on  $\mathcal{F}_E^M$ , whose values are in  $\mathcal{W}_E^M$ . These spaces have the same dimension.

The remaining part, which is constant for the linear stage, leads to the solution part  $\hat{\underline{W}}_{E,d}^M$  on  $\partial\Omega_E$ .

**Property 3**  $\mathbf{L}_E$  is a symmetric positive linear operator from  $\mathcal{F}_E^M$  to  $\mathcal{W}_E^M$ , with a finite dimension. Its restriction to fields with a null resultant and moment on  $\partial\Omega_E$  is regular. It depends on both material characteristics of the sub-structure  $E$  and the choice of the micro/macro description.

**Proof 4** Let us consider two force fields with a null resultant and moment,  $\underline{F}_1^M$  and  $\underline{F}_2^M$ , which correspond to two displacement fields,  $\underline{U}_1$  and  $\underline{U}_2$ , with an undefined rigid body displacement on the sub-structure  $E$ . Then, on this sub-structure:

$$\begin{aligned} \int_{\Omega_E} \text{Tr}[\varepsilon(\underline{U}_1)\mathbf{K}\varepsilon(\underline{U}_2)]d\Omega + \int_{\partial\Omega_E} (\mathbf{1} - \boldsymbol{\pi})\underline{U}_1 \cdot k^m(\mathbf{1} - \boldsymbol{\pi})\underline{U}_2 dS = \\ = \int_{\partial\Omega_E} \underline{F}_2^M \cdot \underline{U}_1 dS = \int_{\partial\Omega_E} \underline{F}_2^M \cdot \mathbf{L}_E \underline{F}_1^M dS \end{aligned}$$

$\mathbf{L}_E$  is clearly symmetric and positive, with  $k^m \geq 0$ . For the uniqueness feature, with  $\underline{F}_1^M = \underline{F}_2^M$ , we obtain:

$$\begin{aligned} \int_{\Omega_E} \text{Tr}[\varepsilon(\underline{U}_1 - \underline{U}_2)\mathbf{K}\varepsilon(\underline{U}_1 - \underline{U}_2)]d\Omega + \\ + \int_{\partial\Omega_E} (\mathbf{1} - \boldsymbol{\pi})(\underline{U}_1 - \underline{U}_2) \cdot k^m(\mathbf{1} - \boldsymbol{\pi})(\underline{U}_1 - \underline{U}_2) dS = 0 \end{aligned}$$

From the choice of projector  $\boldsymbol{\pi}_{\Gamma_{EE'}}$  we made, we are able to conclude that  $\underline{U}_1 - \underline{U}_2$  is a rigid body displacement. If subjected to a null resultant and moment, this leads to  $\underline{U}_1 = \underline{U}_2$ .  $\mathbf{L}_E$  is then injective. Let  $\mathcal{W}_E^{M0}$  (respectively  $\mathcal{F}_E^{M0}$ ) be the space of displacements belonging to  $\mathcal{W}_E^M$  (respectively forces belonging to  $\mathcal{F}_E^M$ ) with a null resultant and moment. They have the same dimension; hence, the restriction of  $\mathbf{L}_E$  to  $\mathcal{F}_E^{M0} \mapsto \mathcal{W}_E^{M0}$  is regular.

This property allows us to define the Moore-Penrose generalized inverse of  $\mathbf{L}_E$ :  $\mathbf{L}_E^+$ . As a consequence:

$$\underline{F}_E^M |_{\partial\Omega_E} - \hat{\underline{F}}_{E,d}^M = \mathbf{L}_E^+(\boldsymbol{\pi}\underline{U}_E |_{\partial\Omega_E} - \hat{\underline{W}}_{E,d}^M) \quad (13)$$

and  $(\underline{F}_E^M |_{\partial\Omega_E} - \hat{\underline{F}}_{E,d}^M)$  has a null resultant and moment; then,  $(\underline{F}_E^M |_{\partial\Omega_E}, \underline{f}_d |_{\Omega_E})$  are globally balanced (a required condition for a well-posed micro-scale problem).

## 6.2 Macro-scale problem

The macro search direction has been written in its weak form in (4). Using the form in (13) with  $\underline{W}_E^M |_{\Gamma_{EE'}} = \boldsymbol{\pi}_{\Gamma_{EE'}} \underline{U}_E |_{\Gamma_{EE'}}$ , it can be expressed as: find  $\underline{W}^M \in \mathcal{W}_{\text{ad}}^M$  such that

$$\begin{aligned} \forall \underline{W}^{M*} \in \mathcal{W}_{\text{ad},0}^M \\ \sum_{\Gamma_{EE'}} \sum_E (k^M \underline{W}_E^M + \mathbf{L}_E^+(\underline{W}_E^M - \hat{\underline{W}}_d^M), \underline{W}^{M*})_{\Gamma_{EE'}} - (\hat{\underline{F}}_E^M + k^M \hat{\underline{W}}_E^M, \underline{W}^{M*})_{\Gamma_{EE'}} = 0 \end{aligned}$$

**Property 4** This problem possesses a unique solution as soon as  $k^M \geq 0$ .

**Proof 5** This problem has a finite dimension. With two solutions  $\underline{W}_1^M$  and  $\underline{W}_2^M$ , we obtain:

$$\begin{aligned} \forall \underline{W}^{M*} \in \mathcal{W}_{\text{ad},0}^M \\ \sum_{\Gamma_{EE'}} \sum_E (k^M (\underline{W}_1^M - \underline{W}_2^M) + \mathbf{L}_E^+(\underline{W}_1^M - \underline{W}_2^M), \underline{W}^{M*})_{\Gamma_{EE'}} = 0 \end{aligned}$$

with  $(\underline{W}_1^M - \underline{W}_2^M) \in \mathcal{W}_{\text{ad},0}^M$ . Due to the previous positiveness and regularity of  $\mathbf{L}_E^+$ , this leads to  $\underline{W}_1^M = \underline{W}_2^M$ .

The macro-scale problem thus has a unique solution, and its finite dimension leads to the existence of this solution, even when  $k^M = 0$ .

## 7 Convergence

Following the convergence proof of the mono-level strategy given in [8], with standard assumptions for elasticity, convergence is reached if the search directions are such that:  $\infty > k_2 \geq k^m |_{\Gamma_{EE'}} \geq k_1 > 0$

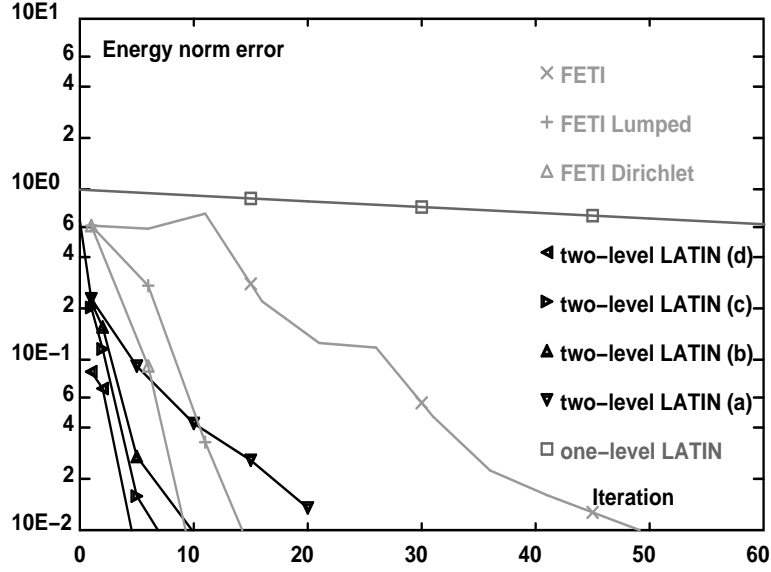


Figure 5: Comparison of methods – error versus iterations

and  $\infty > k_2 \geq k^M$   $|\Gamma_{EE'}| \geq 0$  (where  $k_1$  and  $k_2$  are constants). In particular, if  $\mathbf{s}_{ex}$  denotes the solution to the reference problem (which belongs to both  $\mathbf{A}_d$  and  $\Gamma$ , see Figure 3),  $\lim_{n \rightarrow \infty} \|\mathbf{s}_n - \mathbf{s}_{ex}\| = 0$  and  $\lim_{n \rightarrow \infty} \|\hat{\mathbf{s}}_{n+1/2} - \mathbf{s}_{ex}\| = 0$ , where (in the case  $k^M > 0$ ):

$$\begin{aligned} \|\mathbf{s}\|^2 = & \sum_{E \in \mathbf{E}} \int_{\Omega_E} (\text{Tr}[\boldsymbol{\sigma}_E \mathbf{K}^{-1} \boldsymbol{\sigma}_E] + \text{Tr}[\boldsymbol{\varepsilon}_E \mathbf{K} \boldsymbol{\varepsilon}_E]) d\Omega + \\ & + \int_{\partial\Omega_E} (\underline{F}_E^m \cdot k^{m-1} \underline{F}_E^m + \underline{W}_E^m \cdot k^m \underline{W}_E^m + \underline{F}_E^M \cdot k^{M-1} \underline{F}_E^M + \underline{W}_E^M \cdot k^M \underline{W}_E^M) dS \end{aligned}$$

## 8 Comparison with the FETI method

This first example uses a previous version of the micro/macro approach suitable for slightly heterogeneous structures, with a hierarchical description of scales [3]. It is a priori less efficient than the full micro/macro approach described above.

The micro-scale problems (5) and the macro-scale problem (11) are not decoupled. The reference problem is solved herein by building a staggered solution method with only one iteration per LATIN iteration [3]. This approach has proved to be efficient for slightly heterogeneous cases, in which it performs similarly to the dual multi-level FETI method [6].

We report herein a comparison test, with the meshes in Figure 4, submitted to a terminal parabolic transverse loading. For the large scale, the influence of the discretization with 8-node cubic elements is studied: the different meshes correspond to cases (a), (b), (c) and (d). Figure 5 shows the error (with respect to the monolithic direct resolution) versus iterations, for the FETI method without preconditioning, then with lumped preconditioning, and finally with optimal Dirichlet preconditioning, [6]. These three computations have been performed by F.-X. Roux with a 32-processor PARAGON machine at ONERA-Châtillon, France. The hierarchical micro/macro approach has been implemented in the finite element code CASTEM 2000 developed at CEA-Saclay, and computations have been performed on a 32-processor SGI ORIGIN 2000.

In order to identify the major trends in computational costs, we set the CPU costs of initializations for the FETI approach at 1, in terms of CPU equivalent time (accumulated on the 32 processors). Afterwards, the FETI iteration and the 2-level LATIN iteration for case (a) were determined in terms of cost. Figure 6 shows the evolution of the error versus this CPU equivalent time.

The cost of a direct finite element approach is 18 in terms of CPU equivalent time. When using the multi-frontal scheme [2], [5], the costs are: 3 for local condensations and forward-backward substitutions (which can be performed concurrently) and 2.6 for the resolution of the condensed problem (performed sequentially). Total cost of the analysis therefore is 5.6 in CPU equivalent time. The cost of a local

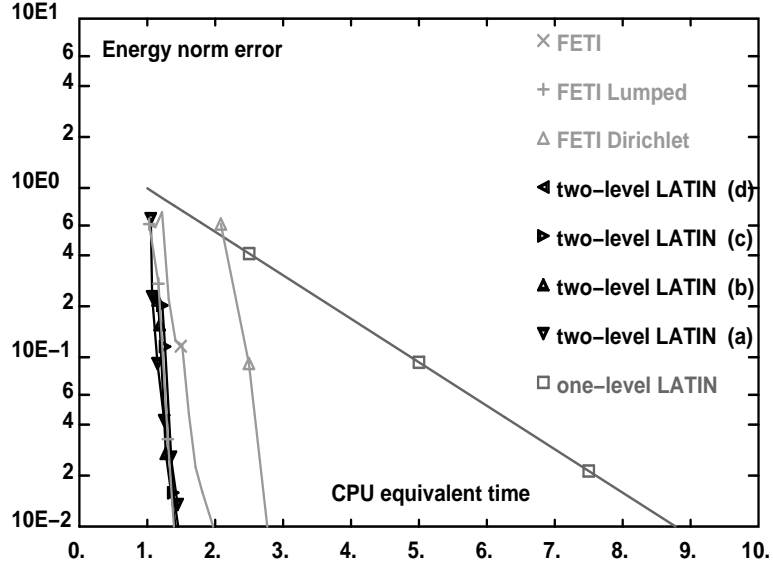


Figure 6: Comparison of costs

condensation is higher than a simple factorisation due to the higher fill-in of the local rigidity matrix (in order to treat the boundary d.o.f. at the end). One can initially note that when increasing the large-scale problem size of the 2-level LATIN algorithm, the error indicator starts out lower at the first iteration because the large-scale first solution is used to initiate the algorithm. Another effect is the increase in the convergence rate (Figure 5), but since iteration costs are also increasing, the two effects cancel each other for the proposed example, (Figure 6).

The search directions for this example are  $k^m = \frac{E}{l^4}$  and  $k^M = 0$ ;  $E$  is the Young's modulus, and  $l$  is the length of an interface edge.

## 9 Example of a heterogeneous structure

In order to illustrate the proposed micro/macro computational strategy, a test example is described in Figure 7. It concerns a tridimensional cantilever structure which is weakened with multi-perforations in the transverse direction and submitted to a terminal parabolic load.

The micro/macro description used herein corresponds to the hierarchical basis.

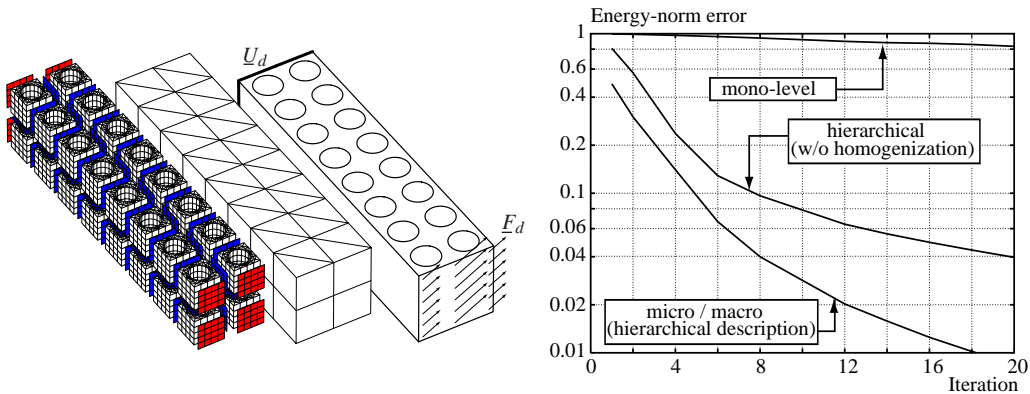


Figure 7: Discretizations of the micro and macro scales, the considered problem and the convergence curves

The Mises equivalent stress field of the solution is illustrated in Figure 8, where localized high gradient areas are shown.

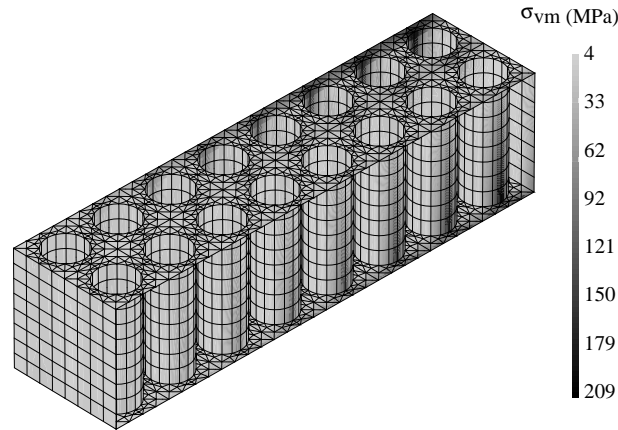


Figure 8: Equivalent stress field level of the solution

Figure 7 also displays the error with respect to the standard discretized reference problem (96 819 degrees of freedom). It is compared to the error obtained using the hierarchical approach without homogenization [3], as well as to the error obtained with the former mono-level approach without the macro scale [8]. In terms of the iteration count, the level of efficiency is quite high.

The search directions in this example are  $k^m = \frac{E}{0.6l}$  and  $k^M = 0$ ;  $E$  is the average Young's modulus according to the mixing law, and  $l$  is the length of a side of interface.

## 10 Conclusion

The technique proposed herein belongs to the set of structural analysis with homogenization. It uses a formulation on both the micro and macro scales within the LATIN method. A key point is the description of the micro and macro quantities on the interfaces between sub-structures. Several choices are possible and two of them, which arise from the partitioning of displacement have been detailed. Nevertheless, dual approaches can be built when focussing on interface forces.

Moreover, this approach leads to a parallel and mechanical approach which is related to domain decomposition methods, and well suited to parallel architecture computers: the underlying algorithm can be interpreted as a “mixed” and 2-level domain decomposition method.

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