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Lazhar Dhaouadi

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On the q -Bessel Fourier transform

Lazhar Dhaouadi *

Abstract

In this work, we are interested by the q -Bessel Fourier transform with a new approach. Many important results of this q -integral transform are proved with a new constructive demonstrations and we establish in particular the associated q -Fourier-Neumen expansion which involves the q -little Jacobi polynomials.

1 Introduction

In the recent mathematical literature one finds many articles which deal with the theory of q -Fourier analysis associated with the q -Hankel transform. This theory was elaborated first by Koornwinder and R.F. Swarttouw [12] and then by Fitouhi and Al [5, 8].

It should be noticed that in [5] we provided the mains results of q -Fourier analysis in particular that the q -Hankel transform is extended to the $\mathcal{L}_{q,2,\nu}$ space like an isometric operator. Often we use the crucial properties namely the positivity of the q -Bessel translation operator to prove some results but these last property is not ensured for any q in the interval $]0, 1[$. Thus, we will prove some main results of q -Fourier analysis without the positivity argument especially the following statments:

- Inversion Formula in the $\mathcal{L}_{q,p,\nu}$ spaces with $p \geq 1$.
- Plancherel Formula in the $\mathcal{L}_{q,p,\nu} \cap \mathcal{L}_{q,1,\nu}$ spaces with $p > 2$.
- Plancherel Formula in the $\mathcal{L}_{q,2,\nu}$ spaces.

Note that in the paper [7] we have proved that the positivity of the q -Bessel translation operator is ensured in all points of the interval $]0, 1[$ when $\nu \geq 0$. In this article we will try to show in a clear way the part in which the positivity of the q -Bessel translation operator plays a role in q -Bessel Fourier analysis. In particular, when we try to prove a q -version of the Young's inequality for the associated convolution.

Many interesting result about the uncertainty principle for the q -Bessel transform was proved in the last years. We cite for examples [2, 3, 4, 9]. There are some differences of the results cited above and our result:

In this paper the Heisenberg uncertainty inequality is established for functions in $\mathcal{L}_{q,2,\nu}$ space. The Hardy's inequality discuss here is a quantitative uncertainty principles which give an information about how a function and its q -Bessel Fourier transform are linked.

In the end of this paper we use the remarkable work in [1] to establish a new result about the q -Fourier-Neumen expansion involving the q -little Jacobi polynomials.

*IPEIB, 7021 Zarzouna, Bizerte, Tunisia. E-mail: lazhardhaouadi@yahoo.fr

2 The q -Bessel transform

The reader can see the references [10, 11, 16] about q -series theory. The references [5, 8, 12] are devoted to the q -Bessel Fourier analysis. Throughout this paper, we consider $0 < q < 1$ and $\nu > -1$. We denote by

$$\mathbb{R}_q^+ = \{q^n, n \in \mathbb{Z}\}.$$

The q -Bessel operator is defined as follows [5]

$$\Delta_{q,\nu} f(x) = \frac{1}{x^2} [f(q^{-1}x) - (1 + q^{2\nu})f(x) + q^{2\nu}f(qx)].$$

The eigenfunction of $\Delta_{q,\nu}$ associated with the eigenvalue $-\lambda^2$ is the function $x \mapsto j_\nu(\lambda x, q^2)$, where $j_\nu(\cdot, q^2)$ is the normalized q -Bessel function defined by [5, 8, 10, 14, 16]

$$j_\nu(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\nu+2}, q^2)_n (q^2, q^2)_n} x^{2n}.$$

The q -Jackson integral of a function f defined on \mathbb{R}^+ is

$$\int_0^\infty f(t) d_q t = (1 - q) \sum_{n \in \mathbb{Z}} q^n f(q^n).$$

We denote by $\mathcal{L}_{q,p,\nu}$ the space of functions f defined on \mathbb{R}_q^+ such that

$$\|f\|_{q,p,\nu} = \left(\int_0^\infty |f(x)|^p x^{2\nu+1} d_q x \right)^{1/p} \text{ exist.}$$

We denote by $\mathcal{C}_{q,0}$ the space of functions defined on \mathbb{R}_q^+ tending to 0 as $x \rightarrow \infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $\mathcal{C}_{q,0}$ is complete with respect to the norm

$$\|f\|_{q,\infty} = \sup_{x \in \mathbb{R}_q^+} |f(x)|.$$

The normalized q -Bessel function $j_\nu(\cdot, q^2)$ satisfies the orthogonality relation

$$c_{q,\nu}^2 \int_0^\infty j_\nu(xt, q^2) j_\nu(yt, q^2) t^{2\nu+1} d_q t = \delta_q(x, y), \quad \forall x, y \in \mathbb{R}_q^+ \quad (1)$$

where

$$\delta_q(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ \frac{1}{(1-q)x^{2(\nu+1)}} & \text{if } x = y \end{cases}$$

and

$$c_{q,\nu} = \frac{1}{1-q} \frac{(q^{2\nu+2}, q^2)_\infty}{(q^2, q^2)_\infty}.$$

Let f be a function defined on \mathbb{R}_q^+ then

$$\int_0^\infty f(y) \delta_q(x, y) y^{2\nu+1} d_q y = f(x).$$

The normalized q -Bessel function $j_\nu(\cdot, q^2)$ satisfies

$$|j_\nu(q^n, q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2\nu+2}; q^2)_\infty}{(q^{2\nu+2}; q^2)_\infty} \begin{cases} 1 & \text{if } n \geq 0 \\ q^{n^2 - (2\nu+1)n} & \text{if } n < 0 \end{cases}.$$

The q -Bessel Fourier transform $\mathcal{F}_{q,\nu}$ is defined by [5, 8, 12]

$$\mathcal{F}_{q,\nu}f(x) = c_{q,\nu} \int_0^\infty f(t)j_\nu(xt, q^2)t^{2\nu+1}d_qt, \quad \forall x \in \mathbb{R}_q^+.$$

Proposition 1 Let $f \in \mathcal{L}_{q,1,\nu}$ then $\mathcal{F}_{q,\nu}f \in \mathcal{C}_{q,0}$ and we have

$$\|\mathcal{F}_{q,\nu}(f)\|_{q,\infty} \leq B_{q,\nu}\|f\|_{q,1,\nu}$$

where

$$B_{q,\nu} = \frac{1}{1-q} \frac{(-q^2; q^2)_\infty (-q^{2\nu+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

Theorem 1 Let f be a function in the $\mathcal{L}_{q,p,\nu}$ space where $p \geq 1$ then

$$\mathcal{F}_{q,\nu}^2 f = f. \tag{2}$$

Proof. If $f \in \mathcal{L}_{q,p,\nu}$ then $\mathcal{F}_{q,\nu}f$ exist, and we have

$$\begin{aligned} \mathcal{F}_{q,\nu}^2 f(x) &= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu}f(t)j_\nu(xt, q^2)t^{2\nu+1}d_qt \\ &= \int_0^\infty f(y) \left[c_{q,\nu}^2 \int_0^\infty j_\nu(xt, q^2)j_\nu(yt, q^2)t^{2\nu+1}d_qt \right] y^{2\nu+1}d_qy \\ &= \int_0^\infty f(y)\delta_q(x, y)y^{2\nu+1}d_qy \\ &= f(x). \end{aligned}$$

The computations are justified by the Fubini's theorem: If $p > 1$ then we use the Hölder's inequality

$$\begin{aligned} &\int_0^\infty |f(y)| \left[\int_0^\infty |j_\nu(xt, q^2)j_\nu(yt, q^2)|t^{2\nu+1}d_qt \right] y^{2\nu+1}d_qy \\ &\leq \left[\int_0^\infty |f(y)|^p y^{2\nu+1}d_qy \right]^{1/p} \times \left[\int_0^\infty \sigma(y)^{\bar{p}} y^{2\nu+1}d_qy \right]^{1/\bar{p}}. \end{aligned}$$

The numbers p and \bar{p} above are conjugates and

$$\sigma(y) = \int_0^\infty |j_\nu(xt, q^2)j_\nu(yt, q^2)|t^{2\nu+1}d_qt,$$

then

$$\begin{aligned} &\int_0^\infty \sigma(y)^{\bar{p}} y^{2\nu+1}d_qy \\ &= \int_0^1 \sigma(y)^{\bar{p}} y^{2\nu+1}d_qy + \int_1^\infty \sigma y^{\bar{p}} y^{2\nu+1}d_qy. \end{aligned}$$

Note that

$$\begin{aligned} &\int_0^1 \sigma(y)^{\bar{p}} y^{2\nu+1}d_qy \\ &\leq \|j_\nu(\cdot, q^2)\|_{q,\infty}^{\bar{p}} \int_0^1 \left[\int_0^\infty |j_\nu(xt, q^2)|t^{2\nu+1}d_qt \right]^{\bar{p}} y^{2\nu+1}d_qy \\ &\leq \|j_\nu(\cdot, q^2)\|_{q,\infty}^{\bar{p}} \|j_\nu(\cdot, q^2)\|_{q,1,\nu}^{\bar{p}} x^{-2(\nu+1)\bar{p}} \left[\int_0^1 y^{2\nu+1}d_qy \right] < \infty, \end{aligned}$$

and

$$\begin{aligned}
& \int_1^\infty \sigma(y)^{\bar{p}} y^{2\nu+1} d_q y \\
& \leq \|j_\nu(\cdot, q^2)\|_{q, \infty}^{\bar{p}} \|j_\nu(\cdot, q^2)\|_{q, 1, \nu} \int_1^\infty \frac{y^{2\nu+1}}{y^{2(\nu+1)\bar{p}}} d_q y \\
& \leq \|j_\nu(\cdot, q^2)\|_{q, \infty}^{\bar{p}} \|j_\nu(\cdot, q^2)\|_{q, 1, \nu}^{\bar{p}} \int_1^\infty \frac{1}{y^{2(\nu+1)(\bar{p}-1)+1}} d_q y < \infty.
\end{aligned}$$

If $p = 1$ then

$$\begin{aligned}
& \int_0^\infty \|f(y)\| \left[\int_0^\infty |j_\nu(xt, q^2) j_\nu(yt, q^2)| t^{2\nu+1} d_q t \right] y^{2\nu+1} d_q y \\
& \leq \|f\|_{q, 1, \nu} \|j_\nu(\cdot, q^2)\|_{q, \infty} \|j_\nu(\cdot, q^2)\|_{q, 1, \nu} \times \frac{1}{x^{2(\nu+1)}}.
\end{aligned}$$

■

Theorem 2 Let f be a function in the $\mathcal{L}_{q, 1, \nu} \cap \mathcal{L}_{q, p, \nu}$ space, where $p > 2$ then

$$\|\mathcal{F}_{q, \nu} f\|_{q, 2, \nu} = \|f\|_{q, 2, \nu}.$$

Proof. Let $f \in \mathcal{L}_{q, 1, \nu} \cap \mathcal{L}_{q, p, \nu}$ then by Theorem 1 we see that

$$\mathcal{F}_{q, \nu}^2 f = f.$$

This implies

$$\begin{aligned}
\int_0^\infty \mathcal{F}_{q, \nu} f(x)^2 x^{2\nu+1} d_q x &= \int_0^\infty \mathcal{F}_{q, \nu} f(x) \left[c_{q, \nu} \int_0^\infty f(t) j_\nu(xt, q^2) t^{2\nu+1} d_q t \right] x^{2\nu+1} d_q x \\
&= \int_0^\infty f(t) \left[c_{q, \nu} \int_0^\infty \mathcal{F}_{q, \nu} f(x) j_\nu(xt, q^2) x^{2\nu+1} d_q x \right] t^{2\nu+1} d_q t \\
&= \int_0^\infty f(t)^2 t^{2\nu+1} d_q t.
\end{aligned}$$

The computations are justified by the Fubini's theorem

$$\begin{aligned}
& \int_0^\infty |f(t)| \left[c_{q, \nu} \int_0^\infty |\mathcal{F}_{q, \nu} f(x)| |j_\nu(xt, q^2)| x^{2\nu+1} d_q x \right] t^{2\nu+1} d_q t \\
& \leq \left[\int_0^\infty |f(t)|^p t^{2\nu+1} d_q t \right]^{1/p} \times \left[\int_0^\infty |\phi(t)|^{\bar{p}} t^{2\nu+1} d_q t \right]^{1/\bar{p}},
\end{aligned}$$

where

$$\phi(t) = c_{q, \nu} \int_0^\infty |\mathcal{F}_{q, \nu} f(x)| |j_\nu(xt, q^2)| x^{2\nu+1} d_q x,$$

then

$$\begin{aligned}
\|\mathcal{F}_{q, \nu} f(x)\| &\leq c_{q, \nu} \int_0^\infty |f(y)| |j_\nu(xy, q^2)| y^{2\nu+1} d_q y \\
&\leq c_{q, \nu} \left[\int_0^\infty |f(y)|^p y^{2\nu+1} d_q y \right]^{1/p} \times \left[\int_0^\infty |j_\nu(xy, q^2)|^{\bar{p}} y^{2\nu+1} d_q y \right]^{1/\bar{p}} \\
&\leq c_{q, \nu} \left[\int_0^\infty |f(y)|^p y^{2\nu+1} d_q y \right]^{1/p} \times \left[\int_0^\infty |j_\nu(y, q^2)|^{\bar{p}} y^{2\nu+1} d_q y \right]^{1/\bar{p}} x^{-2(\nu+1)/\bar{p}} \\
&\leq c_{q, \nu} \|f\|_{q, p, \nu} \|j_\nu(\cdot, q^2)\|_{q, \bar{p}, \nu} x^{-2(\nu+1)/\bar{p}}.
\end{aligned}$$

This gives

$$\begin{aligned}
\phi(t) &\leq c_{q,\nu}^2 \|f\|_{q,p,\nu} \|j_\nu(\cdot, q^2)\|_{q,\bar{p},\nu} \int_0^\infty |j_\nu(xt, q^2)| x^{(2\nu+1)-2(\nu+1)/\bar{p}} d_q x \\
&\leq c_{q,\nu}^2 \|f\|_{q,p,\nu} \|j_\nu(\cdot, q^2)\|_{q,\bar{p},\nu} \left[\int_0^\infty |j_\nu(x, q^2)| x^{2(\nu+1)/p-1} d_q x \right] t^{-2(\nu+1)/p} \\
&\leq C_1 t^{-2(\nu+1)/p},
\end{aligned}$$

and

$$\begin{aligned}
\phi(t) &= c_{q,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(x)| |j_\nu(xt, q^2)| x^{2\nu+1} d_q x \\
&= \left[c_{q,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(x/t)| |j_\nu(x, q^2)| x^{2\nu+1} d_q x \right] t^{-2(\nu+1)} \\
&\leq c_{q,\nu} \|\mathcal{F}_{q,\nu} f\|_{q,\infty} \times \|j_\nu(\cdot, q^2)\|_{q,1,\nu} \times t^{-2(\nu+1)} \\
&\leq C_2 t^{-2(\nu+1)}.
\end{aligned}$$

Note that

$$\begin{cases} -1 < -2(\nu+1)\frac{\bar{p}}{p} + 2\nu + 1 \\ -2(\nu+1)\bar{p} + 2\nu + 1 < -1 \end{cases} \Leftrightarrow \begin{cases} 0 < -2(\nu+1)(\bar{p}-2) \\ -2(\nu+1)(\bar{p}-1) < 0 \end{cases} \Leftrightarrow 1 < \bar{p} < 2 \Leftrightarrow p > 2.$$

Hence

$$\begin{aligned}
\int_0^\infty |\phi(t)|^{\bar{p}} t^{2\nu+1} d_q t &= \int_0^1 |\phi(t)|^{\bar{p}} t^{2\nu+1} d_q t + \int_1^\infty |\phi(t)|^{\bar{p}} t^{2\nu+1} d_q t \\
&\leq C_1 \int_0^1 t^{-2(\nu+1)\bar{p}/p} t^{2\nu+1} d_q t + C_2 \int_1^\infty t^{-2(\nu+1)\bar{p}} t^{2\nu+1} d_q t < \infty,
\end{aligned}$$

which prove the result. ■

Theorem 3 *Let f be a function in the $\mathcal{L}_{q,2,\nu}$ space then*

$$\|\mathcal{F}_{q,\nu} f\|_{q,2,\nu} = \|f\|_{q,2,\nu}.$$

Proof. We introduce the function ψ_x as follows

$$\psi_x(t) = c_{q,\nu} j_\nu(tx, q^2).$$

The inner product $\langle \cdot, \cdot \rangle$ in the Hilbert space $\mathcal{L}_{q,2,\nu}$ is defined by

$$f, g \in \mathcal{L}_{q,2,\nu} \Rightarrow \langle f, g \rangle = \int_0^\infty f(t)g(t)t^{2\nu+1} d_q t. \quad (3)$$

Using (1) we write

$$\begin{aligned}
x \neq y &\Rightarrow \langle \psi_x, \psi_y \rangle = 0 \\
\|\psi_x\|_{q,2,\nu}^2 &= \frac{1}{1-q} x^{-2(\nu+1)}.
\end{aligned}$$

We have

$$\mathcal{F}_{q,\nu} f(x) = \langle f, \psi_x \rangle,$$

and by Theorem 1

$$f \in \mathcal{L}_{q,2,\nu} \Rightarrow \mathcal{F}_{q,\nu}^2 f = f,$$

then

$$\langle f, \psi_x \rangle = 0, \forall x \in \mathbb{R}_q^+ \Rightarrow \mathcal{F}_{q,\nu} f(x) = 0, \forall x \in \mathbb{R}_q^+ \Rightarrow f = 0.$$

Hence, $\{\psi_x, x \in \mathbb{R}_q^+\}$ form an orthogonal basis of the Hilbert space $\mathcal{L}_{q,2,\nu}$ and we have

$$\overline{\{\psi_x, \forall x \in \mathbb{R}_q^+\}} = \mathcal{L}_{q,2,\nu}.$$

Now

$$f \in \mathcal{L}_{q,2,\nu} \Rightarrow f = \sum_{x \in \mathbb{R}_q^+} \frac{1}{\|\psi_x\|_{q,2,\nu}^2} \langle f, \psi_x \rangle \psi_x,$$

and then

$$\|f\|_{q,2,\nu}^2 = \sum_{x \in \mathbb{R}_q^+} \frac{1}{\|\psi_x\|_{q,2,\nu}^2} \langle f, \psi_x \rangle^2 = (1-q) \sum_{x \in \mathbb{R}_q^+} x^{2(\nu+1)} \mathcal{F}_{q,\nu} f(x)^2 = \|\mathcal{F}_{q,\nu} f\|_{q,2,\nu}^2,$$

which achieve the proof. ■

Proposition 2 *Let $f \in \mathcal{L}_{q,p,\nu}$ where $p \geq 1$ then $\mathcal{F}_{q,\nu} f \in \mathcal{L}_{q,\bar{p},\nu}$. If $1 \leq p \leq 2$ then*

$$\|\mathcal{F}_{q,\nu} f\|_{q,\bar{p},\nu} \leq B_{q,\nu}^{\frac{2}{p}-1} \|f\|_{q,p,\nu}. \quad (4)$$

Proof. This is an immediate consequence of Proposition 1, Theorem 3, the Riesz-Thorin theorem and the inversion formula (2). ■

The q -translation operator is given as follow

$$T_{q,x}^\nu f(y) = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(t) j_\nu(yt, q^2) j_\nu(xt, q^2) t^{2\nu+1} d_q t.$$

Let us now introduce

$$Q_\nu = \{q \in]0, 1[, \quad T_{q,x}^\nu \text{ is positive for all } x \in \mathbb{R}_q^+\}$$

the set of the positivity of $T_{q,x}^\nu$. We recall that $T_{q,x}^\nu$ is called positive if $T_{q,x}^\nu f \geq 0$ for $f \geq 0$. In a recent paper [6] it was proved that if $-1 < \nu < \nu'$ then $Q_\nu \subset Q_{\nu'}$. As a consequence :

- If $0 \leq \nu$ then $Q_\nu =]0, 1[$.
- If $-\frac{1}{2} \leq \nu < 0$ then $]0, q_0] \subset Q_{-\frac{1}{2}} \subset Q_\nu \subsetneq]0, 1[, \quad q_0 \simeq 0.43$.
- If $-1 < \nu \leq -\frac{1}{2}$ then $Q_\nu \subset Q_{-\frac{1}{2}}$.

Theorem 4 Let $f \in \mathcal{L}_{q,p,\nu}$ then $T_{q,x}^\nu f$ exists and we have

$$\int_0^\infty T_{q,x}^\nu f(y) y^{2\nu+1} d_q y = \int_0^\infty f(y) y^{2\nu+1} d_q y.$$

and

$$T_{q,x}^\nu f(y) = \int_0^\infty f(z) D_\nu(x, y, z) z^{2\nu+1} d_q z,$$

where

$$D_\nu(x, y, z) = c_{q,\nu}^2 \int_0^\infty j_\nu(xs, q^2) j_\nu(ys, q^2) j_\nu(zs, q^2) s^{2\nu+1} d_q s.$$

If we suppose that $T_{q,x}^\nu$ is a positive operator then for all $p \geq 1$ we have

$$\|T_{q,x}^\nu f\|_{q,p,\nu} \leq \|f\|_{q,p,\nu}. \quad (5)$$

Proof. We write the operator $T_{q,x}^\nu$ in the following form

$$\begin{aligned} T_{q,x}^\nu f(y) &= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(z) j_\nu(xz, q^2) j_\nu(yz, q^2) z^{2\nu+1} d_q z \\ &= \mathcal{F}_{q,\nu} [\mathcal{F}_{q,\nu} f(z) j_\nu(xz, q^2)] (y). \end{aligned}$$

So we have

$$\begin{aligned} \int_0^\infty T_{q,x}^\nu f(y) y^{2\nu+1} d_q y &= \int_0^\infty \mathcal{F}_{q,\nu} [\mathcal{F}_{q,\nu} f(z) j_\nu(xz, q^2)] (y) y^{2\nu+1} d_q y \\ &= \frac{1}{c_{q,\nu}} c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} [\mathcal{F}_{q,\nu} f(z) j_\nu(xz, q^2)] (y) j_\nu(0, q^2) y^{2\nu+1} d_q y \\ &= \frac{1}{c_{q,\nu}} \mathcal{F}_{q,\nu}^2 [\mathcal{F}_{q,\nu} f(z) j_\nu(xz, q^2)] (0) \\ &= \frac{1}{c_{q,\nu}} \mathcal{F}_{q,\nu} f(0) \\ &= \int_0^\infty f(y) y^{2\nu+1} d_q y. \end{aligned}$$

On the other hand

$$\begin{aligned} T_{q,x}^\nu f(y) &= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(z) j_\nu(xz, q^2) j_\nu(yz, q^2) z^{2\nu+1} d_q z \\ &= c_{q,\nu} \int_0^\infty \left[c_{q,\nu} \int_0^\infty f(t) j_\nu(tz, q^2) t^{2\nu+1} d_q t \right] j_\nu(xz, q^2) j_\nu(yz, q^2) z^{2\nu+1} d_q z \\ &= \int_0^\infty \left[c_{q,\nu}^2 \int_0^\infty j_\nu(xz, q^2) j_\nu(yz, q^2) j_\nu(tz, q^2) z^{2\nu+1} d_q z \right] f(t) t^{2\nu+1} d_q t \\ &= \int_0^\infty D_{q,\nu}(x, y, t) f(t) t^{2\nu+1} d_q t. \end{aligned}$$

The computations are justified by the Fubini's theorem

$$\begin{aligned} &\int_0^\infty \left[\int_0^\infty |f(t)| |j_\nu(tz, q^2)| t^{2\nu+1} d_q t \right] |j_\nu(xz, q^2)| |j_\nu(yz, q^2)| z^{2\nu+1} d_q z \\ &\leq \|f\|_{q,p,\nu} \int_0^\infty \left[\int_0^\infty |j_\nu(tz, q^2)|^{\bar{p}} t^{2\nu+1} d_q t \right]^{\frac{1}{\bar{p}}} |j_\nu(xz, q^2)| |j_\nu(yz, q^2)| z^{2\nu+1} d_q z \\ &\leq \|f\|_{q,p,\nu} \|j_\nu(\cdot, q^2)\|_{q,\bar{p},\nu} \int_0^\infty |j_\nu(xz, q^2)| |j_\nu(yz, q^2)| z^{2(\nu+1)(1-\frac{1}{\bar{p}})-1} d_q z. \end{aligned}$$

Now suppose that $T_{q,x}^\nu$ is positive. Given a function $f \in \mathcal{C}_{q,0}$ we obtains

$$\begin{aligned} |T_{q,x}^\nu f(y)| &= \left| \int_0^\infty D_{q,\nu}(x,y,t) f(t) t^{2\nu+1} d_q t \right| \\ &\leq \int_0^\infty |D_{q,\nu}(x,y,t)| |f(t)| t^{2\nu+1} d_q t \\ &\leq \left[\int_0^\infty D_{q,\nu}(x,y,t) t^{2\nu+1} d_q t \right] \|f\|_{q,\infty} = \|f\|_{q,\infty} \end{aligned}$$

which implies

$$\|T_{q,x}^\nu f\|_{q,\infty} \leq \|f\|_{q,\infty}.$$

If the function $f \in \mathcal{L}_{q,1,\nu}$ then we obtains

$$\begin{aligned} \|T_{q,x}^\nu f\|_{q,1,\nu} &= \int_0^\infty |T_{q,x}^\nu f(y)| y^{2\nu+1} d_q y \\ &\leq \int_0^\infty \left[\int_0^\infty |D_{q,\nu}(x,y,t)| |f(t)| t^{2\nu+1} d_q t \right] y^{2\nu+1} d_q y \\ &\leq \int_0^\infty \left[\int_0^\infty D_{q,\nu}(x,y,t) y^{2\nu+1} d_q y \right] |f(t)| t^{2\nu+1} d_q t \\ &\leq \int_0^\infty |f(t)| t^{2\nu+1} d_q t = \|f\|_{q,1,\nu}. \end{aligned}$$

The result is a consequence of the Riesz-Thorin theorem.

Notice that the kernel $D_{q,\nu}(x,y,t)$ can be written as follows

$$\begin{aligned} D_{q,\nu}(x,y,t) &= c_{q,\nu}^2 \int_0^\infty j_\nu(xz, q^2) j_\nu(yz, q^2) j_\nu(tz, q^2) z^{2\nu+1} d_q z \\ &= c_{q,\nu} \mathcal{F}_{q,\nu} [j_\nu(xz, q^2) j_\nu(yz, q^2)] (t), \end{aligned}$$

which implies

$$\begin{aligned} \int_0^\infty D_{q,\nu}(x,y,t) t^{2\nu+1} d_q t &= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} [j_\nu(xz, q^2) j_\nu(yz, q^2)] (t) t^{2\nu+1} d_q t \\ &= \mathcal{F}_{q,\nu}^2 [j_\nu(xz, q^2) j_\nu(yz, q^2)] (0) = 1. \end{aligned}$$

■

The q -convolution product is defined by

$$f *_q g = \mathcal{F}_{q,\nu} [\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g].$$

Theorem 5 *Let $1 \leq p, r, s$ such that*

$$\frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{s}$$

*Given two functions $f \in \mathcal{L}_{q,p,\nu}$ and $g \in \mathcal{L}_{q,r,\nu}$ then $f *_q g$ exists and we have*

$$f *_q g(x) = c_{q,\nu} \int_0^\infty T_{q,x}^\nu f(y) g(y) y^{2\nu+1} d_q y.$$

and

$$\begin{aligned} f *_q g &\in \mathcal{L}_{q,s,\nu}. \\ \mathcal{F}_{q,\nu}(f *_q g) &= \mathcal{F}_{q,\nu}(f) \times \mathcal{F}_{q,\nu}(g). \end{aligned}$$

If $s \geq 2$ then

$$\|f *_q g\|_{q,s,\nu} \leq B_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu}. \quad (6)$$

If we suppose that $T_{q,x}^\nu$ is a positive operator then

$$\|f *_q g\|_{q,s,\nu} \leq c_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu}. \quad (7)$$

Proof. We have

$$\begin{aligned} f *_q g(x) &= \mathcal{F}_{q,\nu} [\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g](x) \\ &= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(y) \times \mathcal{F}_{q,\nu} g(y) j_\nu(xy, q^2) y^{2\nu+1} d_q y \\ &= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(y) \times \left[c_{q,\nu} \int_0^\infty g(z) j_\nu(zy, q^2) z^{2\nu+1} d_q z \right] j_\nu(xy, q^2) y^{2\nu+1} d_q y \\ &= c_{q,\nu} \int_0^\infty \left[c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(y) j_\nu(zy, q^2) j_\nu(xy, q^2) y^{2\nu+1} d_q y \right] g(z) z^{2\nu+1} d_q z \\ &= c_{q,\nu} \int_0^\infty T_{q,x}^\nu f(z) g(z) z^{2\nu+1} d_q z. \end{aligned}$$

The computations are justified by the Fubini's theorem

$$\begin{aligned} &\int_0^\infty |F_{q,\nu} f(y)| \times \left[\int_0^\infty |g(z)| \times |j_\nu(zy, q^2)| z^{2\nu+1} d_q z \right] |j_\nu(xy, q^2)| y^{2\nu+1} d_q y \\ &\leq \|g\|_{q,r,\nu} \int_0^\infty |F_{q,\nu} f(y)| \times \left[\int_0^\infty |j_\nu(zy, q^2)|^{\bar{r}} z^{2\nu+1} d_q z \right]^{\frac{1}{\bar{r}}} |j_\nu(xy, q^2)| y^{2\nu+1} d_q y \\ &\leq \|g\|_{q,r,\nu} \|j_\nu(\cdot, q^2)\|_{q,\bar{r},\nu} \int_0^\infty |F_{q,\nu} f(y)| \times \left[|j_\nu(xy, q^2)| y^{-\frac{2\nu+2}{\bar{r}}} \right] y^{2\nu+1} d_q y \\ &\leq \|g\|_{q,r,\nu} \|j_\nu(\cdot, q^2)\|_{q,\bar{r},\nu} \|F_{q,\nu} f\|_{q,\bar{p},\nu} \left(\int_0^\infty \left[|j_\nu(xy, q^2)| y^{-\frac{2\nu+2}{\bar{r}}} \right]^p y^{2\nu+1} d_q y \right)^{\frac{1}{p}} \\ &\leq \|g\|_{q,r,\nu} \|j_\nu(\cdot, q^2)\|_{q,\bar{r},\nu} \|F_{q,\nu} f\|_{q,\bar{p},\nu} \left(\int_0^\infty |j_\nu(xy, q^2)|^p y^{2(\nu+1)(1-\frac{p}{\bar{r}})-1} d_q y \right)^{\frac{1}{p}}. \end{aligned}$$

From Proposition 2 we deduce that

$$\mathcal{F}_{q,\nu} f \in \mathcal{L}_{q,\bar{p},\nu} \text{ and } \mathcal{F}_{q,\nu} g \in \mathcal{L}_{q,\bar{r},\nu}.$$

Then, using the Hölder inequality and the fact that

$$\frac{1}{\bar{p}} + \frac{1}{\bar{r}} = \frac{1}{\bar{s}}$$

to conclude that

$$\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g \in \mathcal{L}_{q,\bar{s},\nu}.$$

Which implies that

$$f *_q g = \mathcal{F}_{q,\nu} [\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g] \in \mathcal{L}_{q,s,\nu}$$

and by the inversion formula (2) we obtain

$$\mathcal{F}_{q,\nu}(f *_q g) = \mathcal{F}_{q,\nu}f \times \mathcal{F}_{q,\nu}g.$$

Suppose that $s \geq 2$, so $1 \leq \bar{s} \leq 2$ and we can write

$$\begin{aligned} \|f *_q g\|_{q,s,\nu} &= \|\mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu}f \times \mathcal{F}_{q,\nu}g]\|_{q,s,\nu} \\ &\leq B_{q,\nu}^{\frac{2}{s}-1} \|\mathcal{F}_{q,\nu}f\|_{q,\bar{p},\nu} \|\mathcal{F}_{q,\nu}g\|_{q,\bar{r},\nu} \\ &\leq B_{q,\nu}^{\frac{2}{s}-1} B_{q,\nu}^{\frac{2}{p}-1} B_{q,\nu}^{\frac{2}{r}-1} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu} \\ &\leq B_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu}. \end{aligned}$$

Now suppose that $T_{q,x}^\nu$ is a positive operator.

We introduce the operator K_f as follows

$$K_f g(x) = c_{q,\nu} \int_0^\infty T_{q,x}^\nu f(z) g(z) z^{2\nu+1} d_q z.$$

By the Hölder inequality and (5) we get

$$\|K_f g\|_{q,\infty} \leq c_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,\bar{p},\nu}.$$

The Minkowski inequality leads to

$$\|K_f g\|_{q,p,\nu} \leq c_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,1,\nu}.$$

Hence we have

$$K_f : \mathcal{L}_{q,\bar{p},\nu} \rightarrow \mathcal{C}_{q,0}, \quad K_f : \mathcal{L}_{q,1,\nu} \rightarrow \mathcal{L}_{q,p,\nu}.$$

Then the operator K_f satisfies

$$K_f : \mathcal{L}_{q,r,\nu} \rightarrow \mathcal{L}_{q,s,\nu}$$

and

$$\|f *_q g\|_{q,s,\nu} = \|K_f g\|_{q,s,\nu} \leq c_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu}.$$

■

Remark 1 We discuss here the sharp results for the Hausdorff-Young inequality provided above. An inequality already sharper than (6) is given in formula (7). In fact we have $c_{q,\nu} < B_{q,\nu}$. To obtain (7) without the positivity argument, we can do by using which is a q -Riemann-Liouville fractional integral generalizing the q -Mehler integral representation for the q -Bessel function $j_\nu(\cdot, q^2)$ which can be proved in a straightforward way [8]

$$j_\nu(\lambda, q^2) = [2\nu]_q \int_0^1 \frac{(q^2 t^2, q^2)_\infty}{(q^{2\nu} t^2, q^2)_\infty} j_0(\lambda t, q^2) t d_q t$$

together with the inequalities for the q -Bessel function which is given as formula (24) in the paper [4]

$$|j_0(x; q^2)| \leq 1, \quad \forall x \in \mathbb{R}_q^+.$$

Combine this formulas we arrive at

$$|j_\nu(x; q^2)| \leq 1, \quad \forall x \in \mathbb{R}_q^+, \quad \nu \geq 0.$$

Then the inequalities (4) can be written as follows

$$\|\mathcal{F}_{q,\nu} f\|_{q,\bar{p},\nu} \leq c_{q,\nu}^{\frac{2}{p}-1} \|f\|_{q,p,\nu}.$$

This should give the sharpest version of (6) in the cases $\nu \geq 0$. Unfortunately the positivity of the operator $T_{q,x}^\nu$ is satisfied in this case.

In fact we can prove that if we are in the positivity cases then

$$\|j_\nu(\cdot, q^2)\|_{q,\infty} \leq 1.$$

To prove this recalling that

$$T_{q,x}^\nu j_\nu(y, q^2) = j_\nu(x, q^2) j_\nu(y, q^2).$$

So we have

$$\int_0^\infty D_\nu(x, y, t) j_\nu(t, q^2) t^{2\nu+1} d_q t = j_\nu(x, q^2) j_\nu(y, q^2).$$

We obtains for all $x, y \in \mathbb{R}_q^+$

$$\begin{aligned} |j_\nu(x, q^2)| \times |j_\nu(y, q^2)| &\leq \int_0^\infty D_\nu(x, y, t) |j_\nu(t, q^2)| t^{2\nu+1} d_q t \\ &\leq \left[\int_0^\infty D_\nu(x, y, t) t^{2\nu+1} d_q t \right] \|j_\nu(\cdot, q^2)\|_{q,\infty}. \end{aligned}$$

The fact that

$$\int_0^\infty D_\nu(x, y, t) t^{2\nu+1} d_q t = 1$$

implies

$$\|j_\nu(\cdot, q^2)\|_{q,\infty}^2 \leq \|j_\nu(\cdot, q^2)\|_{q,\infty}$$

which gives the result.

3 Uncertainty principle

We introduce two q -difference operators

$$\partial_q f(x) = \frac{f(q^{-1}x) - f(x)}{x}$$

and

$$\partial_q^* f(x) = \frac{f(x) - q^{2\nu+1} f(qx)}{x}.$$

Then we have

$$\partial_q \partial_q^* f(x) = \partial_q^* \partial_q f(x) = \Delta_{q,\nu} f(x).$$

Proposition 3 *If $\langle \partial_q f, g \rangle$ exist and $\lim_{a \rightarrow \infty} |a^{2\nu+1} f(q^{-1}a)g(a)| = 0$ then*

$$\langle \partial_q f, g \rangle = - \langle f, \partial_q^* g \rangle.$$

Proof. The following computation

$$\begin{aligned}
& \int_0^a \partial_q f(x) g(x) x^{2\nu+1} d_q x \\
&= \int_0^a \frac{f(q^{-1}x) - f(x)}{x} g(x) x^{2\nu+1} d_q x \\
&= \int_0^a \frac{f(q^{-1}x)}{x} g(x) x^{2\nu+1} d_q x - \int_0^a \frac{f(x)}{x} g(x) x^{2\nu+1} d_q x \\
&= q^{2\nu+1} \int_0^{q^{-1}a} \frac{f(x)}{x} g(qx) x^{2\nu+1} d_q x - \int_0^a \frac{f(x)}{x} \partial_q g(x) x^{2\nu+1} d_q x \\
&= q^{2\nu+1} \int_0^a \frac{f(x)}{x} \partial_q g(qx) x^{2\nu+1} d_q x - \int_0^a \frac{f(x)}{x} g(x) x^{2\nu+1} d_q x + a^{2\nu+1} f(q^{-1}a) g(a) \\
&= - \int_0^a f(x) \frac{g(x) - q^{2\nu+1} g(qx)}{x} x^{2\nu+1} d_q x + a^{2\nu+1} f(q^{-1}a) g(a) \\
&= - \int_0^a f(x) \partial_q^* g(x) x^{2\nu+1} d_q x + a^{2\nu+1} f(q^{-1}a) g(a)
\end{aligned}$$

leads to the result. ■

Corollary 1 *If $f \in \mathcal{L}_{q,2,\nu}$ such that $x\mathcal{F}_{q,\nu}f \in \mathcal{L}_{q,2,\nu}$ then*

$$\|\partial_q f\|_2 = \|x\mathcal{F}_{q,\nu}f\|_2.$$

Proof. In fact we have

$$\begin{aligned}
\|\partial_q f\|_2^2 &= \langle \partial_q f, \partial_q f \rangle = - \langle f, \partial_q^* \partial_q f \rangle \\
&= - \langle f, \Delta_{q,\nu} f \rangle \\
&= - \langle \mathcal{F}_{q,\nu} f, \mathcal{F}_{q,\nu} \Delta_{q,\nu} f \rangle \\
&= \langle \mathcal{F}_{q,\nu} f, x^2 \mathcal{F}_{q,\nu} f \rangle \\
&= \|x\mathcal{F}_{q,\nu}f\|_2^2,
\end{aligned}$$

which prove the result. ■

Theorem 6 *Assume that f belongs to the space $\mathcal{L}_{q,2,\nu}$. Then the q -Bessel transform satisfies the following uncertainty principal*

$$\|f\|_2^2 \leq k_{q,\nu} \|xf\|_2 \|x\mathcal{F}_{q,\nu}f\|_2$$

where

$$k_{q,\nu} = \frac{[1 + \sqrt{q} \times q^{\nu+1}]}{1 - q^{2(\nu+1)}}.$$

Proof. In fact

$$\begin{aligned}
\partial_q^* x f &= f(x) - q^{2\nu+2} f(qx) \\
x \partial_q f &= f(q^{-1}x) - f(x).
\end{aligned}$$

We introduce the following operator

$$\Lambda_q f(x) = f(qx),$$

then

$$\langle \Lambda_q f, g \rangle = q^{-2(\nu+1)} \langle f, \Lambda_q^{-1} g \rangle.$$

So

$$\frac{1}{1 - q^{2(\nu+1)}} [\partial_q^* x f(x) - q^{2\nu+2} \Lambda_q x \partial_q f(x)] = f(x)$$

Assume that xf and $x\mathcal{F}_{q,\nu}f$ belongs to the space $\mathcal{L}_{q,2,\nu}$. Then we have

$$\langle f, f \rangle = -\frac{1}{1 - q^{2(\nu+1)}} \langle xf, \partial_q f \rangle - \frac{1}{1 - q^{2(\nu+1)}} \langle \partial_q f, x\Lambda_q^{-1} f \rangle.$$

By Cauchy-Schwartz inequality we get

$$\langle f, f \rangle \leq \frac{1}{1 - q^{2(\nu+1)}} \|xf\|_2 \|\partial_q f\|_2 + \frac{1}{1 - q^{2(\nu+1)}} \|\partial_q f\|_2 \|x\Lambda_q^{-1} f\|_2.$$

On the other hand

$$\|x\Lambda_q^{-1} f\|_2 = \sqrt{q} \times q^{\nu+1} \|xf\|_2,$$

Corollary 1 leads to the result. ■

4 Hardy's theorem

The following Lemma from complex analysis is crucial for the proof of our main theorem.

Lemma 1 *For every $p \in \mathbb{N}$, there exist $\sigma_p > 0$ for which*

$$|z|^{2p} |j_\nu(z, q^2)| < \sigma_p e^{|z|}, \quad \forall z \in \mathbb{C}.$$

Proof. In fact

$$\begin{aligned} |z|^{2p} |j_\nu(z, q^2)| &\leq \frac{1}{(q^2, q^2)_\infty (q^{2\nu+2}, q^2)_\infty} \sum_{n=0}^{\infty} q^{n(n-1)} |z|^{2n+2p} \\ &\leq \frac{q^{p(p+1)}}{(q^2, q^2)_\infty (q^{2\nu+2}, q^2)_\infty} \sum_{n=p}^{\infty} q^{n(n-2p-1)} |z|^{2n}. \end{aligned}$$

Now using the Stirling's formula

$$n! \sim \sqrt{2\pi n} \frac{n^n}{e^n},$$

we see that there exist an entire $n_0 \geq p$ such that

$$q^{n(n-2p-1)} < \frac{1}{(2n)!}, \quad \forall n \geq n_0,$$

which implies

$$\sum_{n=n_0}^{\infty} q^{n(n-2p-1)} |z|^{2n} < \sum_{n=n_0}^{\infty} \frac{1}{(2n)!} |z|^{2n} < e^{|z|}.$$

Finally there exist $\sigma_p > 0$ such that

$$\frac{|z|^{2p}|j_\nu(z, q^2)|}{e^{|z|}} < \sigma_p, \quad \forall z \in \mathbb{C}$$

This complete the proof. ■

Lemma 2 *Let h be an entire function on \mathbb{C} such that*

$$|h(z)| \leq Ce^{a|z|^2}, \quad z \in \mathbb{C},$$

$$|h(x)| \leq Ce^{-ax^2}, \quad x \in \mathbb{R},$$

for some positive constants a and C . Then there exist $C^* \in \mathbb{R}$ such

$$h(x) = C^*e^{-ax^2}.$$

The reader can see the reference [17] for the proof.

Now we are in a position to state and prove the q -analogue of the Hardy's theorem

Theorem 7 *Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following estimates*

$$|f(x)| \leq Ce^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R}_q^+, \quad (8)$$

$$|\mathcal{F}_{q,\nu}f(x)| \leq Ce^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R},$$

where C is a positive constant. Then there exist $A \in \mathbb{R}$ such that

$$f(z) = Ac_{q,\nu}\mathcal{F}_{q,\nu}\left(e^{-\frac{1}{2}x^2}\right)(z), \quad \forall z \in \mathbb{C}.$$

Proof. We claim that $\mathcal{F}_{q,\nu}f$ is an analytic function and there exist $C' > 0$ such that

$$|\mathcal{F}_{q,\nu}f(z)| \leq C'e^{\frac{1}{2}|z|^2}, \quad \forall z \in \mathbb{C}.$$

We have

$$|\mathcal{F}_{q,\nu}f(z)| \leq c_{q,\nu} \int_0^\infty |f(x)||j_\nu(zx, q^2)|x^{2\nu+1}d_qx.$$

From the Lemma 1, if $|z| > 1$ then there exist $\sigma_1 > 0$ such that

$$x^{2\nu+1}|j_\nu(zx, q^2)| = \frac{1}{|z|^{2\nu+1}}(|z|x)^{2\nu+1}|j_\nu(zx, q^2)| < \frac{\sigma_1}{1+|z|^2}e^{x|z|}, \quad \forall x \in \mathbb{R}_q^+.$$

Then we obtain

$$|\mathcal{F}_{q,\nu}f(z)| \leq C\sigma_1c_{q,\nu} \left[\int_0^\infty \frac{e^{-\frac{1}{2}(x-|z|)^2}}{1+|z|^2}d_qx \right] e^{\frac{1}{2}|z|^2} < C\sigma_1c_{q,\nu} \left[\int_0^\infty \frac{1}{1+x^2}d_qx \right] e^{\frac{1}{2}|z|^2}.$$

Now, if $|z| \leq 1$ then there exist $\sigma_2 > 0$ such that

$$x^{2\nu+1}|j_\nu(zx, q^2)| \leq \sigma_2e^x, \quad \forall x \in \mathbb{R}_q^+.$$

Therefore

$$|\mathcal{F}_{q,\nu}f(z)| \leq C\sigma_2c_{q,\nu} \left[\int_0^\infty e^{-\frac{1}{2}x^2+x} d_q x \right] \leq C\sigma_2c_{q,\nu} \left[\int_0^\infty e^{-\frac{1}{2}x^2+x} d_q x \right] e^{\frac{1}{2}|z|^2},$$

which leads to the estimate (8). Using Lemma 2, we obtain

$$\mathcal{F}_{q,\nu}f(z) = \text{const.}e^{-\frac{1}{2}z^2}, \quad \forall z \in \mathbb{C},$$

and by Theorem 1, we conclude that

$$f(z) = \text{const.}\mathcal{F}_{q,\nu} \left(e^{-\frac{1}{2}t^2} \right) (z), \quad \forall z \in \mathbb{C}.$$

■

Corollary 2 Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following estimates

$$|f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,$$

$$|\mathcal{F}_{q,\nu}f(x)| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R},$$

where C, p, σ are a positive constant and $p\sigma = \frac{1}{4}$. We suppose that there exist $a \in \mathbb{R}_q^+$ such that $a^2p = \frac{1}{2}$. Then there exist $A \in \mathbb{R}$ such that

$$f(z) = Ac_{q,\nu}\mathcal{F}_{q,\nu} \left(e^{-\sigma t^2} \right) (z), \quad \forall z \in \mathbb{C}.$$

Proof. Let $a \in \mathbb{R}_q^+$, and put

$$f_a(x) = f(ax),$$

then

$$\mathcal{F}_{q,\nu}f_a(x) = \frac{1}{a^{2\nu+2}}\mathcal{F}_{q,\nu}f(x/a).$$

In the end, applying Theorem 7 to the function f_a .

■

Corollary 3 Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following estimates

$$|f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,$$

$$|\mathcal{F}_{q,\nu}f(x)| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R}, \tag{9}$$

where C, p, σ are a positive constant and $p\sigma > \frac{1}{4}$. We suppose that there exist $a \in \mathbb{R}_q^+$ such that $a^2p = \frac{1}{2}$. Then $f \equiv 0$.

Proof. In fact there exists $\sigma' < \sigma$ such that $p\sigma' = \frac{1}{4}$. Then the function f satisfying the estimates of Corollary 2, if we replacing σ by σ' . Which implies

$$\mathcal{F}_{q,\nu}f(x) = \text{const.}e^{-\sigma'x^2}, \quad \forall x \in \mathbb{R}.$$

On the other hand, f satisfying the estimates (9), then

$$\left| \text{const.}e^{-\sigma'x^2} \right| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R}.$$

This implies $\mathcal{F}_{q,\nu}f \equiv 0$, and by Theorem 1 we conclude that $f \equiv 0$.

■

5 The q -Fourier-Neumann Expansions

The little q -Jacobi polynomials are defined for $\nu, \beta > -1$ by [15]

$$p_n(x; q^\nu, q^\beta; q) = {}_2\phi_1 \left(\begin{matrix} q^{n+\nu+\beta+1}, q^{-n} \\ q^{\nu+1} \end{matrix} \middle| q; qx \right).$$

We define the functions

$$P_{\nu,n}(x; q^2) = \sigma_{q,\nu}(n) q^{-n(\nu+1)} \frac{(q^{2+2n}, q^{2\nu+2}; q^2)_\infty}{(q^{2+2n+2\nu}, q^2; q^2)_\infty} p_n(x^2; q^{2\nu}, 1; q^2)$$

and

$$\mathcal{J}_{\nu,n}(x; q^2) = \sigma_{q,\nu}(n) \frac{J_{\nu+2n+1}(q^n x; q^2)}{x^{\nu+1}},$$

where

$$\sigma_{q,\nu}(n) = \sqrt{\frac{1 - q^{2\nu+4n+2}}{1 - q}}.$$

Consider $\mathcal{L}_{q,2}^\nu$ as an Hilbert space with the inner product

$$\langle f|g \rangle = \int_0^1 f(x)g(x)x^{2\nu+1}d_qx.$$

The q -Paley-Wiener space is defined by

$$PW_q^\nu = \left\{ f \in \mathcal{L}_{q,2,\nu} : f(x) = c_{q,\nu} \int_0^1 u(t)j_\nu(xt, q^2)t^{2\nu+1}d_qt, \quad u \in \mathcal{L}_{q,2}^\nu \right\}.$$

Proposition 4 PW_q^ν is a closed subspace of $\mathcal{L}_{q,2,\nu}$ and with the inner product given in (3) is an Hilbert space.

Proof. In fact, given $f \in \mathcal{L}_{q,2,\nu}$ and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of element of PW_q^ν which converge to f in L^2 -norm. For $n \in \mathbb{N}$, there exist $u_n \in \mathcal{L}_{q,2}^\nu$ such that

$$f_n(x) = c_{q,\nu} \int_0^1 u_n(t)j_\nu(xt, q^2)t^{2\nu+1}d_qt.$$

Moreover

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{q,2,\nu} = 0.$$

This give

$$\lim_{n \rightarrow \infty} \|\mathcal{F}_{q,\nu}f_n - \mathcal{F}_{q,\nu}f\|_{q,2,\nu} = 0,$$

and then

$$\lim_{n \rightarrow \infty} \left[\int_0^1 |\mathcal{F}_{q,\nu}f_n(x) - \mathcal{F}_{q,\nu}f(x)|^2 x^{2\nu+1} d_qx + \int_1^\infty |\mathcal{F}_{q,\nu}f(x)|^2 x^{2\nu+1} d_qx \right] = 0,$$

which implies

$$\int_1^\infty |\mathcal{F}_{q,\nu}f(x)|^2 x^{2\nu+1} d_qx = 0 \Rightarrow \mathcal{F}_{q,\nu}f(x) = 0, \quad \forall x \in \mathbb{R}_q^+ \cap]1, +\infty[.$$

Then $f \in PW_q^\nu$. ■

Proposition 5 *We have*

$$\mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,n})(x) = P_{\nu,n}(x; q^2)\chi_{[0,1]}(x), \quad \forall x \in \mathbb{R}_q^+.$$

As a consequence

$$\int_0^1 P_{\nu,n}(x; q^2)P_{\nu,m}(x; q^2)x^{2\nu+1} d_q x = \delta_{n,m}.$$

Proof. The following proof is identical to the proof of Lemma 1 in [1]. Using an identity established in [12, 13]

$$\begin{aligned} & \int_0^\infty t^{-\lambda} J_\mu(q^m t; q^2) J_\theta(q^n t; q^2) d_q t \\ &= (1-q)q^{n(\lambda-1)+(m-n)\mu} \frac{(q^{1+\lambda+\theta-\mu}, q^{2\mu+2}; q^2)_\infty}{(q^{1-\lambda+\theta+\mu}, q^2; q^2)_\infty} \\ & \quad \times {}_2\phi_1 \left(\begin{matrix} q^{1-\lambda+\mu+\theta}, q^{1-\lambda+\mu-\theta} \\ q^{2\mu+2} \end{matrix} \middle| q^2; q^{2m-2n+1+\lambda+\theta-\mu} \right), \end{aligned} \quad (10)$$

where $n, m \in \mathbb{Z}$ and $\theta, \mu, \lambda \in \mathbb{C}$ such that $\operatorname{Re}(1 - \lambda + \theta + \mu) > 0$, θ, μ are not equal to a negative integer and

$$(\lambda + \theta + 1 - \mu)/2, \quad m - n + (\lambda + \theta + 1 - \mu)/2$$

are not a non-positive integer [13].

To evaluate $\mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,n})(x)$ when $x = q^m \leq 1$, we take in (10)

$$q^m = x, \mu = \nu, \theta = \nu + 2n + 1, \lambda = 0$$

then

$$\begin{aligned} \mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,n})(x) &= \sigma_{q,\nu}(n) \frac{x^{-\nu}}{1-q} \int_0^\infty J_\nu(xt; q^2) J_{\nu+2n+1}(q^n t; q^2) d_q t \\ &= \sigma_{q,\nu}(n) q^{-n(\nu+1)} \frac{(q^{2+2n}, q^{2\nu+2}; q^2)_\infty}{(q^{2+2n+2\nu}, q^2; q^2)_\infty} {}_2\phi_1 \left(\begin{matrix} q^{2+2\nu+2n}, q^{-2n} \\ q^{2\nu+2} \end{matrix} \middle| q^2; q^2 x^2 \right) \\ &= P_{\nu,n}(x; q^2). \end{aligned}$$

To evaluate $\mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,m})(x)$ when $x = q^n > 1$, we consider in (10)

$$q^n = x, \mu = \nu + 2m + 1, \theta = \nu, \lambda = 0$$

In this way, $1 + \lambda + \theta - \mu = -2m$. This gives for $m \in \mathbb{N}$ a factor

$$(q^{-2m}; q^2)_\infty = 0$$

on the numerator and then

$$\mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,m})(x) = 0, \quad x > 1$$

By setting $\lambda = 1$, $\theta = \nu + 2n + 1$, and $\mu = \nu + 2m + 1$ in , it is clear that, for $n, m = 0, 1, 2, \dots$,

$$\int_0^\infty J_{\nu+2n+1}(q^n x; q^2) J_{\nu+2m+1}(q^m x; q^2) \frac{d_q x}{x} = \frac{1}{\sigma_{q,\nu}(n)^2} \delta_{n,m}$$

and then

$$\int_0^\infty \mathcal{J}_{\nu,n}(x; q^2) \mathcal{J}_{\nu,m}(x; q^2) x^{2\nu+1} d_q x = \delta_{n,m}.$$

Now we use the arguments of q -Bessel Fourier analysis provided in this paper to show that

$$\langle P_{\nu,n} \chi_{[0,1]}, P_{\nu,m} \chi_{[0,1]} \rangle = \langle \mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,n}), \mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,m}) \rangle = \langle \mathcal{J}_{\nu,n}, \mathcal{J}_{\nu,m} \rangle = \delta_{n,m}. \quad (11)$$

Another proof of the orthogonality of the little q -Jacobi polynomials can be found in [15] ■

Proposition 6 *The systems*

$$\{\mathcal{J}_{\nu,n}\}_{n=0}^\infty, \quad \{P_{\nu,n}\}_{n=0}^\infty$$

forme two orthonormal basis respectively of the Hilbert spaces PW_q^ν and $\mathcal{L}_{q,2}^\nu$.

Proof. From (11) we derive the orthonormality. To prove that the system $\{\mathcal{J}_{\nu,n}\}_{n=0}^\infty$ is complet in PW_q^ν , given a function $f \in PW_q^\nu$ such that

$$\langle f, \mathcal{J}_{\nu,n} \rangle = 0, \quad \forall n \in \mathbb{N}.$$

Then

$$\langle \mathcal{F}_{q,\nu}(f), \mathcal{F}_{q,\nu}(\mathcal{J}_{\nu,n}) \rangle = 0, \quad \forall n \in \mathbb{N},$$

which implies

$$\langle \mathcal{F}_{q,\nu}(f), P_{\nu,n} \chi_{[0,1]} \rangle = \langle \mathcal{F}_{q,\nu}(f) \chi_{[0,1]}, P_{\nu,n} \rangle = \langle \mathcal{F}_{q,\nu}(f), P_{\nu,n} \rangle = 0, \quad \forall n \in \mathbb{N}.$$

From the definition of the polynomial $P_{\nu,n}$ we conclude that

$$\langle \mathcal{F}_{q,\nu}(f), t^{2n} \rangle = 0, \quad \forall n \in \mathbb{N}.$$

Then

$$c_{q,\nu} \sum_{n=0}^\infty (-1)^n \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2\nu+2}, q^2)_n} \langle \mathcal{F}_{q,\nu}(f), t^{2n} \rangle x^{2n} = 0, \quad \forall x \in \mathbb{R}_q^+,$$

which can be written as

$$\mathcal{F}_{q,\nu}^2(f)(x) = 0, \quad \forall x \in \mathbb{R}_q^+.$$

By the inversion formula (2) we conclude that $f = 0$. From (11) we derive the orthonormality. To prove that the system $\{P_{\nu,n}\}_{n=0}^\infty$ is complet in $\mathcal{L}_{q,2}^\nu$, given a function $f \in \mathcal{L}_{q,2}^\nu$ such that

$$\langle f | P_{\nu,n} \rangle = 0, \quad \forall n \in \mathbb{N}$$

Then

$$\langle f | t^{2n} \rangle = 0, \quad \forall n \in \mathbb{N}.$$

Which leads to the result. ■

Proposition 7 Let $\lambda \in \mathbb{R}_q^+$ then

$$c_{q,\nu} j_\nu(\lambda x; q^2) = \sum_{n=0}^{\infty} \mathcal{J}_{n,\nu}(\lambda; q^2) P_{n,\nu}(x), \quad \forall x \in [0, 1] \cap \mathbb{R}_q^+.$$

As a consequence we have

$$\sum_{n=0}^{\infty} [P_{n,\nu}(x; q^2)]^2 = \frac{x^{-2(\nu+1)}}{1-q}, \quad \forall x \in [0, 1] \cap \mathbb{R}_q^+$$

and for all $\lambda \in \mathbb{R}_q^+$

$$\begin{aligned} \sum_{n=0}^{\infty} [\mathcal{J}_{n,\nu}(\lambda; q^2)]^2 &= -\frac{q^\nu}{2(1-q)\lambda^{1+2\nu}} \\ &\times \left[\frac{\lambda}{q} J_{\nu+1}(\lambda; q^2) J'_\nu(\lambda/q; q^2) - J_{\nu+1}(\lambda; q^2) J_\nu(\lambda/q; q^2) - J'_{\nu+1}(\lambda; q^2) J_\nu(\lambda/q; q^2) \right]. \end{aligned}$$

Proof. Let $\lambda \in \mathbb{R}$ and consider the function

$$\psi_\lambda : [0, 1] \cap \mathbb{R}_q^+ \rightarrow \mathbb{R}, \quad x \mapsto c_{q,\nu} j_\nu(\lambda x; q^2).$$

Then $\psi_\lambda \in \mathcal{L}_{q,2}^\nu$ and we can write

$$\psi_\lambda(x) = \sum_{n=0}^{\infty} \langle \psi_\lambda | P_{n,\nu} \rangle P_{n,\nu}(x), \quad \forall x \in [0, 1] \cap \mathbb{R}_q^+. \quad (12)$$

Note that

$$\langle \psi_\lambda | P_{n,\nu} \rangle = \langle \psi_\lambda, P_{n,\nu} \chi_{[0,1]} \rangle = \langle \psi_\lambda, \mathcal{F}_{q,\nu}(\mathcal{J}_{n,\nu}) \rangle = \mathcal{F}_{q,\nu}^2(\mathcal{J}_{n,\nu})(\lambda) = \mathcal{J}_{n,\nu}(\lambda; q^2).$$

Then we deduce the result. Using the Parseval's theorem and (12) we obtain

$$\sum_{n=0}^{\infty} [P_{n,\nu}(x; q^2)]^2 = \|\psi_\lambda\|_{q,2,\nu}^2 = \frac{x^{-2(\nu+1)}}{1-q}.$$

The second identity is deduced also from the Parseval's theorem

$$\sum_{n=0}^{\infty} [\mathcal{J}_{n,\nu}(\lambda; q^2)]^2 = N_{q,\nu,2}^2(\psi_\lambda),$$

and the following relation proved in [14]

$$\begin{aligned} \int_0^1 [J_\nu(aqt; q^2)]^2 t d_q t &= -\frac{(1-q)q^{\nu-1}}{2a} \\ &\times \left[a J_{\nu+1}(aq; q^2) J'_\nu(a; q^2) - J_{\nu+1}(aq; q^2) J_\nu(a; q^2) - J'_{\nu+1}(aq; q^2) J_\nu(a; q^2) \right]. \end{aligned}$$

■

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