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Markov operators in q -Fourier analysis

Lazhar Dhaouadi*

Abstract

In this paper there are many important results of q -Fourier analysis proved by new ways and new sets of markov operators acting on the $\mathcal{L}_{q,2,v}$ space of square q -integrable function which were defined.

1 Introduction

In the mathematical literature one finds many articles which deal with the theory of q -Fourier analysis associated with the q -Hankel transform. This theory finds its origins in an important article of H.T. Koornwinder and R.F. Swarttouw [5]. The two authors proved that the q -Bessel functions of Hahn-Exton satisfy an orthogonality relation, which gives a rigorous proof. They noticed that the q -Hankel transform would be an involutive isometric operator of the $\mathcal{L}_{q,2,v}$ space. Our goal is to give a detailed proof of this important result. It should be noticed that in [1] we provided the mains results of q -Fourier analysis in particular that the q -Hankel transform is prolonged on the $\mathcal{L}_{q,2,v}$ space like an isometric involutive operator. But we used the positivity of the q -Bessel translation operator in order to show these results. These property is not ensured for any q in the interval $]0, 1[$. Thus, we will prove some main results of q -Fourier analysis without the positivity of the q -Bessel translation operator, which are the following

- Inversion Formula in the $\mathcal{L}_{q,p,v}$ spaces with $p \geq 1$.
- Plancherel Formula in the $\mathcal{L}_{q,p,v} \cap \mathcal{L}_{q,1,v}$ spaces with $p > 2$.
- Plancherel Formula in the $\mathcal{L}_{q,2,v}$ spaces.
- Product formula for the q -Hankel transform .

Note that in a recent paper [2] we have proved that the positivity of the q -Bessel translation operator is ensured in all points of the interval $]0, 1[$

*Ecole Préparatoire d'Ingénieur, Bizerte, Tunisia. E-mail lazhardhaouadi@yahoo.fr

when $v \geq 0$. In this article we will try to show in a clear way the part in which the positivity of the q -Bessel translation operator plays a role in q -Fourier analysis. In particular, when we try to introduce the concept of Markov operators of the $\mathcal{L}_{q,2,v}$ space. Many interesting examples of such operators are the q -Bessel translation operator and the q -heat semigroup.

2 Preliminaries

In the following we will always assume $0 < q < 1$ and $v > -1$. We denote by

$$\mathbb{R}_q = \{\pm q^n, n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n, n \in \mathbb{Z}\}.$$

For more information on the q -series theory the reader can see the reference [3,4,6] and the reference [1,5] about the q -bessel Fourier analysis. Also for details of the proof of the following result in this section can be found in [1,5].

Definition 1 *The q -Bessel operator is defined as follows*

$$\Delta_{q,v}f(x) = \frac{1}{x^2} [f(q^{-1}x) - (1 + q^{2v})f(x) + q^{2v}f(qx)].$$

Definition 2 *The eigenfunction of $\Delta_{q,v}$ associated with the eigenvalue $-\lambda^2$ is the function $x \mapsto j_v(\lambda x, q^2)$, where $j_v(\cdot, q^2)$ is the normalized q -Bessel function defined by*

$$j_v(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2v+2}, q^2)_n (q^2, q^2)_n} x^{2n}.$$

Definition 3 *The q -Jackson integral of a function f defined on \mathbb{R}_q is*

$$\int_0^{\infty} f(t) d_q t = (1 - q) \sum_{n \in \mathbb{Z}} q^n f(q^n).$$

Definition 4 *We denote by $\mathcal{L}_{q,p,v}$ the space of even functions f defined on \mathbb{R}_q such that*

$$\|f\|_{q,p,v} = \left[\int_0^{\infty} |f(x)|^p x^{2v+1} d_q x \right]^{1/p} < \infty.$$

Definition 5 We denote by $\mathcal{C}_{q,0}$ the space of even functions defined on \mathbb{R}_q tending to 0 as $x \rightarrow \pm\infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $\mathcal{C}_{q,0}$ is complete with respect to the norm

$$\|f\|_{q,\infty} = \sup_{x \in \mathbb{R}_q} |f(x)|.$$

Proposition 1 The normalized q -Bessel function $j_v(\cdot, q^2)$ satisfies the orthogonality relation

$$c_{q,v}^2 \int_0^\infty j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t = \delta_q(x, y), \quad \forall x, y \in \mathbb{R}_q.$$

where

$$\delta_q(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ \frac{1}{(1-q)x^{2(v+1)}} & \text{if } x = y \end{cases}$$

and

$$c_{q,v} = \frac{1}{1-q} \frac{(q^{2v+2}, q^2)_\infty}{(q^2, q^2)_\infty}.$$

Remark 1 Let f be an even function defined on \mathbb{R}_q then

$$\int_0^\infty f(y) \delta_q(x, y) y^{2v+1} d_q y = f(x).$$

Proposition 2 The normalized q -Bessel function $j_v(\cdot, q^2)$ satisfies

$$|j_v(q^n, q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2v+2}; q^2)_\infty}{(q^{2v+2}; q^2)_\infty} \begin{cases} 1 & \text{if } n \geq 0 \\ q^{n^2 - (2v+1)n} & \text{if } n < 0 \end{cases}.$$

Definition 6 The q -Bessel Fourier transform $\mathcal{F}_{q,v}$ is defined by

$$\mathcal{F}_{q,v} f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t, \quad \forall x \in \mathbb{R}_q.$$

Proposition 3 Let $f \in \mathcal{L}_{q,1,v}$ then $\mathcal{F}_{q,v} f$ exists and $\mathcal{F}_{q,v} f \in \mathcal{C}_{q,0}$.

3 Inversion formula in the $\mathcal{L}_{q,p,v}$ space

Theorem 1 Let f be a function in the $\mathcal{L}_{q,p,v}$ space where $p \geq 1$ then

$$\mathcal{F}_{q,v}^2 f = f.$$

Proof. If $f \in \mathcal{L}_{q,p,v}$ then $\mathcal{F}_{q,v}f$ exist, and we write

$$\begin{aligned}
\mathcal{F}_{q,v}^2 f(x) &= c_{q,v} \int_0^\infty \mathcal{F}_{q,v}f(t) j_v(xt, q^2) t^{2v+1} d_q t \\
&= \int_0^\infty f(y) \left[c_{q,v}^2 \int_0^\infty j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t \right] y^{2v+1} d_q y \\
&= \int_0^\infty f(y) \delta_q(x, y) y^{2v+1} d_q y \\
&= f(x).
\end{aligned}$$

We can permute the two integrals: in fact if $p > 1$ then we use the Hölder's inequality

$$\begin{aligned}
&\int_0^\infty |f(y)| \left[\int_0^\infty |j_v(xt, q^2) j_v(yt, q^2)| t^{2v+1} d_q t \right] y^{2v+1} d_q y \\
&\leq \left(\int_0^\infty |f(y)|^p y^{2v+1} d_q y \right)^{1/p} \times \left(\int_0^\infty \sigma(y)^{\bar{p}} y^{2v+1} d_q y \right)^{1/\bar{p}}.
\end{aligned}$$

The numbers p and \bar{p} above are conjugated and

$$\sigma(y) = \int_0^\infty |j_v(xt, q^2) j_v(yt, q^2)| t^{2v+1} d_q t,$$

then

$$\begin{aligned}
&\int_0^\infty \sigma(y)^{\bar{p}} y^{2v+1} d_q y \\
&= \int_0^1 \sigma(y)^{\bar{p}} y^{2v+1} d_q y + \int_1^\infty \sigma(y)^{\bar{p}} y^{2v+1} d_q y.
\end{aligned}$$

Not that

$$\begin{aligned}
\int_0^1 \sigma(y)^{\bar{p}} y^{2v+1} d_q y &\leq \|j_v(\cdot, q^2)\|_{q,\infty}^{\bar{p}} \int_0^1 \left\{ \int_0^\infty |j_v(xt, q^2)| t^{2v+1} d_q t \right\}^{\bar{p}} y^{2v+1} d_q y \\
&\leq \|j_v(\cdot, q^2)\|_{q,\infty}^{\bar{p}} \|j_v(\cdot, q^2)\|_{q,1,v}^{\bar{p}} x^{-2(v+1)\bar{p}} \left[\int_0^1 y^{2v+1} d_q y \right] < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\int_1^\infty \sigma(y)^{\bar{p}} y^{2v+1} d_q y \\
&\leq \|j_v(\cdot, q^2)\|_{q,\infty}^{\bar{p}} \|j_v(\cdot, q^2)\|_{q,1,v}^{\bar{p}} \int_1^\infty \frac{y^{2v+1}}{y^{2(v+1)\bar{p}}} d_q y \\
&\leq \|j_v(\cdot, q^2)\|_{q,\infty}^{\bar{p}} \|j_v(\cdot, q^2)\|_{q,1,v}^{\bar{p}} \int_1^\infty \frac{1}{y^{2(v+1)(\bar{p}-1)+1}} d_q y < \infty.
\end{aligned}$$

If $p = 1$ then

$$\begin{aligned} & \int_0^\infty |f(y)| \left\{ \int_0^\infty |j_v(xt, q^2) j_v(yt, q^2)| t^{2v+1} d_q t \right\} y^{2v+1} d_q y \\ & \leq \|f\|_{q,1,v} \|j_v(\cdot, q^2)\|_{q,\infty} \|j_v(\cdot, q^2)\|_{q,1,v} \times \frac{1}{x^{2(v+1)}}, \end{aligned}$$

this finishes the proof. ■

4 Plancherel Formula in the $\mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,p,v}$ space

Theorem 2 *Let f be a function in the $\mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,p,v}$ space, where $p > 2$ then*

$$\|\mathcal{F}_{q,v} f\|_{q,2,v} = \|f\|_{q,2,v}$$

Proof. Let $f \in \mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,p,v}$ then by Theorem 1 we see that

$$\mathcal{F}_{q,v}^2 f = f,$$

this implies

$$\begin{aligned} \int_0^\infty \mathcal{F}_{q,v} f(x)^2 x^{2v+1} d_q x &= \int_0^\infty \mathcal{F}_{q,v} f(x) \left\{ c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t \right\} x^{2v+1} d_q x \\ &= \int_0^\infty f(t) \left\{ c_{q,v} \int_0^\infty \mathcal{F}_{q,v} f(x) j_v(xt, q^2) x^{2v+1} d_q x \right\} t^{2v+1} d_q t \\ &= \int_0^\infty f(t)^2 t^{2v+1} d_q t. \end{aligned}$$

We can permute the two integrals because

$$\begin{aligned} & \int_0^\infty |f(t)| \left\{ c_{q,v} \int_0^\infty |\mathcal{F}_{q,v} f(x)| |j_v(xt, q^2)| x^{2v+1} d_q x \right\} t^{2v+1} d_q t \\ & \leq \left(\int_0^\infty |f(t)|^p t^{2v+1} d_q t \right)^{1/p} \times \left(\int_0^\infty |\phi(t)|^{\bar{p}} t^{2v+1} d_q t \right)^{1/\bar{p}}, \end{aligned}$$

where

$$\phi(t) = c_{q,v} \int_0^\infty |\mathcal{F}_{q,v} f(x)| |j_v(xt, q^2)| x^{2v+1} d_q x,$$

then

$$\begin{aligned}
|\mathcal{F}_{q,v}f(x)| &\leq c_{q,v} \int_0^\infty |f(y)| |j_v(xy, q^2)| y^{2v+1} d_q y \\
&\leq c_{q,v} \left(\int_0^\infty |f(y)|^p y^{2v+1} d_q y \right)^{1/p} \times \left(\int_0^\infty |j_v(xy, q^2)|^{\bar{p}} y^{2v+1} d_q y \right)^{1/\bar{p}} \\
&\leq c_{q,v} \left(\int_0^\infty |f(y)|^p y^{2v+1} d_q y \right)^{1/p} \times \left(\int_0^\infty |j_v(y, q^2)|^{\bar{p}} y^{2v+1} d_q y \right)^{1/\bar{p}} x^{-2(v+1)/\bar{p}} \\
&\leq c_{q,v} \|f\|_{q,p,v} \|j_v(\cdot, q^2)\|_{q,\bar{p},v} x^{-2(v+1)/\bar{p}}.
\end{aligned}$$

This gives

$$\begin{aligned}
\phi(t) &\leq c_{q,v}^2 \|f\|_{q,p,v} \|j_v(\cdot, q^2)\|_{q,\bar{p},v} \int_0^\infty |j_v(xt, q^2)| x^{(2v+1)-2(v+1)/\bar{p}} d_q x \\
&\leq c_{q,v}^2 \|f\|_{q,p,v} \|j_v(\cdot, q^2)\|_{q,\bar{p},v} \left[\int_0^\infty |j_v(x, q^2)| x^{2(v+1)/p-1} d_q x \right] t^{-2(v+1)/p} \\
&\leq C_1 t^{-2(v+1)/p},
\end{aligned}$$

and

$$\begin{aligned}
\phi(t) &= c_{q,v} \int_0^\infty |\mathcal{F}_{q,v}f(x)| |j_v(xt, q^2)| x^{2v+1} d_q x \\
&= \left[c_{q,v} \int_0^\infty |\mathcal{F}_{q,v}f(x/t)| |j_v(x, q^2)| x^{2v+1} d_q x \right] t^{-2(v+1)} \\
&\leq c_{q,v} \|\mathcal{F}_{q,v}f\|_{q,\infty} \times \|j_v(\cdot, q^2)\|_{q,1,v} \times t^{-2(v+1)} \\
&\leq C_2 t^{-2(v+1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^\infty |\phi(t)|^{\bar{p}} t^{2v+1} d_q t &= \int_0^1 |\phi(t)|^{\bar{p}} t^{2v+1} d_q t + \int_1^\infty |\phi(t)|^{\bar{p}} t^{2v+1} d_q t \\
&\leq C_1 \int_0^1 t^{-2(v+1)\bar{p}/p} t^{2v+1} d_q t + C_2 \int_1^\infty t^{-2(v+1)\bar{p}} t^{2v+1} d_q t < \infty.
\end{aligned}$$

In fact

$$\begin{cases} -1 < -2(v+1)\frac{\bar{p}}{p} + 2v+1 \\ -2(v+1)\bar{p} + 2v+1 < -1 \end{cases} \Leftrightarrow \begin{cases} 0 < -2(v+1)(\bar{p}-2) \\ -2(v+1)(\bar{p}-1) < 0 \end{cases} \Leftrightarrow 1 < \bar{p} < 2 \Leftrightarrow p > 2,$$

this finishes the proof. ■

5 Plancherel Formula in the $\mathcal{L}_{q,2,v}$ space

Theorem 3 *Let f be a function in the $\mathcal{L}_{q,2,v}$ space then*

$$\|\mathcal{F}_{q,v}f\|_{q,2,v} = \|f\|_{q,2,v}.$$

Proof. We introduce the function ψ_x as follows

$$\psi_x(t) = c_{q,v} j_v(tx, q^2)$$

The inner product \langle, \rangle in the Hilbert space $\mathcal{L}_{q,2,v}$ is defined by

$$f, g \in \mathcal{L}_{q,2,v} \Rightarrow \langle f, g \rangle = \int_0^\infty f(t)g(t)t^{2v+1}d_qt.$$

By Proposition 1 we write

$$x \neq y \Rightarrow \langle \psi_x, \psi_y \rangle = 0$$

$$\|\psi_x\|_{q,2,v}^2 = \frac{1}{1-q} x^{-2(v+1)}.$$

We have

$$\mathcal{F}_{q,v}f(x) = \langle f, \psi_x \rangle,$$

and by Theorem 1

$$f \in \mathcal{L}_{q,2,v} \Rightarrow \mathcal{F}_{q,v}^2 f = f,$$

then

$$\langle f, \psi_x \rangle = 0, \forall x \in \mathbb{R}_q^+ \Rightarrow \mathcal{F}_{q,v}f = 0 \Rightarrow f = 0.$$

Hence, $\{\psi_x, x \in \mathbb{R}_q^+\}$ form an orthogonal basis of the Hilbert space $\mathcal{L}_{q,2,v}$ and we have

$$\overline{\{\psi_x, x \in \mathbb{R}_q^+\}} = \mathcal{L}_{q,2,v} \quad (1)$$

Now

$$f \in \mathcal{L}_{q,2,v} \Rightarrow f = \sum_{x \in \mathbb{R}_q^+} \frac{1}{\|\psi_x\|_{q,2,v}^2} \langle f, \psi_x \rangle \psi_x,$$

and then

$$\|f\|_{q,2,v}^2 = \sum_{x \in \mathbb{R}_q^+} \frac{1}{\|\psi_x\|_{q,2,v}^2} \langle f, \psi_x \rangle^2 = (1-q) \sum_{x \in \mathbb{R}_q^+} x^{2(v+1)} \mathcal{F}_{q,v}f(x)^2 = \|\mathcal{F}_{q,v}f\|_{q,2,v}^2,$$

this finishes the proof. ■

6 q -translation operator and q -convolution product

Definition 7 The q -translation operator is given as follows (see [1])

$$T_{q,x}^v f(y) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v} f(t) j_v(yt, q^2) j_v(xt, q^2) t^{2v+1} d_q t.$$

Proposition 4 If $f \in \mathcal{L}_{q,1,v}$ then $T_{q,x}^v f$ exists and

$$T_{q,x}^v f(y) = \int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z,$$

where

$$D_v(x, y, z) = c_{q,v}^2 \int_0^\infty j_v(xs, q^2) j_v(ys, q^2) j_v(zs, q^2) s^{2v+1} d_q s.$$

Proof. In fact for all $x \in \mathbb{R}_q$

$$\begin{aligned} |T_{q,x}^v f(y)| &\leq \int_0^\infty |\mathcal{F}_{q,v} f(t)| |j_v(yt, q^2)| |j_v(xt, q^2)| t^{2v+1} d_q t \\ &\leq \|\mathcal{F}_{q,v} f\|_{q,\infty} \|j_v(\cdot, q^2)\|_{q,\infty} \|j_v(\cdot, q^2)\|_{q,1,v} \frac{1}{x^{2(v+1)}} < \infty. \end{aligned}$$

On the other hand

$$\begin{aligned} T_{q,x}^v f(y) &= c_{q,v} \int_0^\infty \mathcal{F}_{q,v} f(t) j_v(yt, q^2) j_v(xt, q^2) t^{2v+1} d_q t \\ &= c_{q,v} \int_0^\infty \left[c_{q,v} \int_0^\infty f(z) j_v(zt, q^2) z^{2v+1} d_q z \right] j_v(yt, q^2) j_v(xt, q^2) t^{2v+1} d_q t \\ &= \int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z. \end{aligned}$$

We can permute the two integrals because

$$\begin{aligned} &\int_0^\infty \left[\int_0^\infty |f(z)| |j_v(zt, q^2)| z^{2v+1} d_q z \right] |j_v(yt, q^2)| |j_v(xt, q^2)| t^{2v+1} d_q t \\ &\leq \|f\|_{q,1,v} \|j_v(\cdot, q^2)\|_{q,\infty}^2 \|j_v(\cdot, q^2)\|_{q,1,v} \times \frac{1}{x^{2(v+1)}} < \infty, \end{aligned}$$

which finishes the proof. ■

Definition 8 The q -convolution product is defined by

$$f *_q g(x) = c_{q,v} \int_0^\infty T_{q,x}^v f(y) g(y) y^{2v+1} d_q y.$$

Proposition 5 Let $f, g \in \mathcal{L}_{q,1,v}$ then $f *_q g$ exists and we have

$$f *_q g = g *_q f.$$

Proof. Let $f, g \in \mathcal{L}_{q,1,v}$ then

$$\begin{aligned} f *_q g(x) &= c_{q,v} \int_0^\infty T_{q,x}^v f(y) g(y) y^{2v+1} d_q y \\ &= c_{q,v} \int_0^\infty \left[c_{q,v} \int_0^\infty \mathcal{F}_{q,v} f(t) j_v(yt, q^2) j_v(xt, q^2) t^{2v+1} d_q t \right] g(y) y^{2v+1} d_q y \\ &= c_{q,v} \int_0^\infty \mathcal{F}_{q,v} f(t) \left[c_{q,v} \int_0^\infty g(y) j_v(yt, q^2) y^{2v+1} d_q y \right] j_v(xt, q^2) t^{2v+1} d_q t \\ &= c_{q,v} \int_0^\infty \mathcal{F}_{q,v} f(t) \mathcal{F}_{q,v} g(t) j_v(xt, q^2) t^{2v+1} d_q t \\ &= g *_q f(x), \end{aligned}$$

The exchange of the integral signs has a valid sense because

$$\begin{aligned} &\int_0^\infty \int_0^\infty |\mathcal{F}_{q,v} f(t)| \times |j_v(yt, q^2)| \times |j_v(xt, q^2)| \times |g(y)| y^{2v+1} t^{2v+1} d_q t d_q y \\ &\leq \|\mathcal{F}_{q,v} f\|_{q,\infty} \|j_v(\cdot, q^2)\|_{q,\infty} \|g\|_{q,1,v} \|j_v(\cdot, q^2)\|_{q,1,v} \times x^{-2(v+1)}, \end{aligned}$$

which prove the result. ■

Proposition 6 Let $f \in \mathcal{L}_{q,1,v}$ and $g \in \mathcal{L}_{q,2,v}$ then

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \times \mathcal{F}_{q,v}(g).$$

Proof. In fact we have

$$\mathcal{F}_{q,v} : \mathcal{L}_{q,2,v} \rightarrow \mathcal{L}_{q,2,v}$$

and

$$\mathcal{F}_{q,v} : \mathcal{L}_{q,1,v} \rightarrow \mathcal{C}_{q,0}.$$

If $f \in \mathcal{L}_{q,1,v}$ and $g \in \mathcal{L}_{q,2,v}$ then

$$\omega = \mathcal{F}_{q,v} f \times \mathcal{F}_{q,v} g \in \mathcal{L}_{q,2,v}$$

On the other hand

$$\mathcal{F}_{q,v}\omega(x) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}f(t)\mathcal{F}_{q,v}g(t)j_v(xt, q^2)t^{2v+1}d_qt.$$

From the inversion formula in Theorem 1 we have

$$\mathcal{F}_{q,v}(\mathcal{F}_{q,v}\omega) = \mathcal{F}_{q,v}f \times \mathcal{F}_{q,v}g.$$

Now we write

$$\begin{aligned} \mathcal{F}_{q,v}\omega(x) &= c_{q,v} \int_0^\infty \mathcal{F}_{q,v}f(t)\mathcal{F}_{q,v}g(t)j_v(xt, q^2)t^{2v+1}d_qt \\ &= c_{q,v} \int_0^\infty \mathcal{F}_{q,v}f(t) \left[c_{q,v} \int_0^\infty g(y)j_v(ty, q^2)y^{2v+1}d_qy \right] j_v(xt, q^2)t^{2v+1}d_qt \\ &= c_{q,v} \int_0^\infty g(y) \left[c_{q,v} \int_0^\infty \mathcal{F}_{q,v}f(t)j_v(ty, q^2)j_v(xt, q^2)t^{2v+1}d_qt \right] y^{2v+1}d_qy \\ &= c_{q,v} \int_0^\infty g(y)T_{q,x}^v f(y)y^{2v+1}d_qy \\ &= f *_q g(x). \end{aligned}$$

This implies that

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}f \times \mathcal{F}_{q,v}g.$$

The exchange of the integral signs can be justified, indeed let

$$\begin{aligned} \phi(y) &= \int_0^\infty |\mathcal{F}_{q,v}f(t)| \times |j_v(xt, q^2)| \times |j_v(yt, q^2)| t^{2v+1}d_qt \\ &= \int_0^\infty \varrho(t) \times |j_v(yt, q^2)| t^{2v+1}d_qt \end{aligned}$$

where

$$\varrho(t) = |\mathcal{F}_{q,v}f(t)| \times |j_v(xt, q^2)| \Rightarrow \varrho \in \mathcal{L}_{q,1,v}$$

then

$$\lim_{y \downarrow 0} \phi(y) < \infty,$$

and

$$\begin{aligned} \phi(y) &= \int_0^\infty \varrho(t) \times |j_v(yt, q^2)| t^{2v+1}d_qt \\ &= \left[\int_0^\infty \varrho\left(\frac{t}{y}\right) \times |j_v(t, q^2)| t^{2v+1}d_qt \right] y^{-2(v+1)} \\ &\leq \left[\|\mathcal{F}_{q,v}f\|_{q,\infty} \|j_v(\cdot, q^2)\|_{q,\infty} \|j_v(\cdot, q^2)\|_{q,1,\infty} \right] y^{-2(v+1)}, \end{aligned}$$

which prove that $\phi \in \mathcal{L}_{q,2,v}$. This leads to the result. ■

7 Positivity of the q -translation operator

Definition 9 The operator $T_{q,x}^v$ is said positive if $T_{q,x}^v f \geq 0$ when $f \geq 0$ for all $x \in \mathbb{R}_q$. We denote by Q_v the domain of positivity of $T_{q,x}^v$

$$Q_v = \{q \in]0, 1[, \quad T_{q,x}^v \text{ is positive}\}.$$

In a recent paper [2] the authors prove that if $v \geq 0$ then $Q_v =]0, 1[$. In the following we assume that $q \in Q_v$.

Proposition 7 The operator $T_{q,x}^v$ is positive if and only if

$$D_v(x, y, z) \geq 0, \quad \forall x, y, z \in \mathbb{R}_q.$$

Proof. If we take $f(z) = \delta_a(z)$ then

$$T_{q,x}^v f(y) = (1 - q)a^{2(v+1)} D_v(x, y, a),$$

which prove the result. ■

Proposition 8 If $f \in \mathcal{L}_{q,1,v}$ then

$$\int_0^\infty T_{q,x}^v f(y) y^{2v+1} d_q y = \int_0^\infty f(y) y^{2v+1} d_q y.$$

Proof. In fact

$$\begin{aligned} \int_0^\infty T_{q,x}^v f(y) y^{2v+1} d_q y &= \int_0^\infty \left[\int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z \right] y^{2v+1} d_q y \\ &= \int_0^\infty \left[\int_0^\infty D_v(x, y, z) y^{2v+1} d_q y \right] f(z) z^{2v+1} d_q z. \end{aligned}$$

On the other hand

$$D_v(x, y, z) = \mathcal{F}_{q,v} \phi(y)$$

where

$$\phi(t) = c_{q,v} j_v(tx, q^2) j_v(tz, q^2),$$

then

$$\begin{aligned} \int_0^\infty D_v(x, y, z) y^{2v+1} d_q y &= \int_0^\infty \mathcal{F}_{q,v} \phi(y) y^{2v+1} d_q y \\ &= \frac{1}{c_{q,v}} \mathcal{F}_{q,v}^2 \phi(0) = \frac{1}{c_{q,v}} \phi(0) = 1. \end{aligned}$$

This leads to the result. ■

Proposition 9 Let $f, g \in \mathcal{L}_{q,1,v}$ then

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \times \mathcal{F}_{q,v}(g).$$

Proof. In fact

$$\begin{aligned} & \mathcal{F}_{q,v}(f *_q g)(t) \\ &= c_{q,v} \int_0^\infty f *_q g(x) j_v(xt, q^2) x^{2v+1} d_q x \\ &= c_{q,v} \int_0^\infty \left[\int_0^\infty T_{q,x}^v f(y) g(y) y^{2v+1} d_q y \right] j_v(xt, q^2) x^{2v+1} d_q x \\ &= c_{q,v} \int_0^\infty \left(\int_0^\infty g(y) \left[\int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z \right] y^{2v+1} d_q y \right) j_v(xt, q^2) x^{2v+1} d_q x \\ &= c_{q,v} \int_0^\infty f(z) \left(\int_0^\infty g(y) \left[\int_0^\infty D_v(x, y, z) j_v(xt, q^2) x^{2v+1} d_q x \right] y^{2v+1} d_q y \right) z^{2v+1} d_q z \\ &= c_{q,v}^2 \int_0^\infty f(z) \left(\int_0^\infty g(y) j_v(yt, q^2) j_v(xt, q^2) y^{2v+1} d_q y \right) z^{2v+1} d_q z \\ &= \mathcal{F}_{q,v}(f)(t) \times \mathcal{F}_{q,v}(g)(t). \end{aligned}$$

One can see that

$$\int_0^\infty D_v(x, y, z) j_v(xt, q^2) x^{2v+1} d_q x = c_{q,v} j_v(zt, q^2) j_v(yt, q^2),$$

and

$$\int_0^\infty D_v(x, y, z) x^{2v+1} d_q x = 1.$$

We can permute the two integral signs because

$$\begin{aligned} & \int_0^\infty \left(\int_0^\infty |g(y)| \left[\int_0^\infty |f(z)| |D_v(x, y, z)| z^{2v+1} d_q z \right] y^{2v+1} d_q y \right) |j_v(xt, q^2)| x^{2v+1} d_q x \\ & \leq \|j_v(\cdot, q^2)\|_{q,\infty} \int_0^\infty |f(z)| \left(\int_0^\infty |g(y)| \left[\int_0^\infty |D_v(x, y, z)| x^{2v+1} d_q x \right] y^{2v+1} d_q y \right) z^{2v+1} d_q z \\ & \leq \|j_v(\cdot, q^2)\|_{q,\infty} \int_0^\infty |f(z)| \left(\int_0^\infty |g(y)| \left[\int_0^\infty D_v(x, y, z) x^{2v+1} d_q x \right] y^{2v+1} d_q y \right) z^{2v+1} d_q z \\ & \leq \|j_v(\cdot, q^2)\|_{q,\infty} \|f\|_{q,1,v} \|g\|_{q,1,v} < \infty. \end{aligned}$$

Note that if $q \in Q_v$ then $|D_v(x, y, z)| = D_v(x, y, z)$. ■

8 Markov operators on the $\mathcal{L}_{q,2,v}$ space

Definition 10 A linear bounded mapping $T : \mathcal{L}_{q,2,v} \rightarrow \mathcal{L}_{q,2,v}$ is called a Markov operator if

- T is positive, that is $f \geq 0$ implies $Tf \geq 0$.
- The constant function $f = 1$ is a fixed point of T .
- For all function $f, g \in \mathcal{L}_{q,2,v}$ we have

$$\langle Tf, g \rangle = \langle f, Tg \rangle.$$

Lemma 1 Let μ be a probability measure on \mathbb{R}_q^+ . If f is a positive function in $\mathcal{L}_{q,v,1}$ and g is convex on \mathbb{R}^+ then

$$g\left(\int_0^\infty f d\mu(x)\right) \leq \int_0^\infty g \circ f(x) d\mu(x).$$

In particular

$$\left[\int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z\right]^2 \leq \int_0^\infty f(z)^2 D_v(x, y, z) z^{2v+1} d_q z.$$

Proof. There exists $c \in \mathbb{R}$ such that for all x, y in a convex set of \mathbb{R}^+ it holds

$$g(x) \geq g(y) + c(x - y).$$

Let $t = \int_0^\infty f(x) d\mu(x)$. Then $t \in \mathbb{R}^+$. Now let $x \in \mathbb{R}_q^+$ then

$$g(f(x)) \geq g(t) + c(f(x) - t).$$

Integrating both sides and using the special value of t gives

$$\int_0^\infty g(f(x)) d\mu(x) \geq \int_0^\infty [g(t) + c(f(x) - t)] d\mu(x) = g(t).$$

In particular for $g(x) = x^2$ and $d\mu(z) = D_v(x, y, z) z^{2v+1} d_q z$ which is a probability measure on \mathbb{R}_q^+ the result follows immediately. ■

Proposition 10 $T_{q,x}^v$ is a Markov operator of $\mathcal{L}_{q,2,v}$.

Proof. We will prove that $T_{q,x}^v$ sends $\mathcal{L}_{q,2,v}$ to $\mathcal{L}_{q,2,v}$. Let $f \in \mathcal{L}_{q,2,v}$ then

$$\begin{aligned} [T_{q,x}^v f(y)]^2 &= \left[\int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z \right]^2 \\ &\leq \int_0^\infty f(z)^2 D_v(x, y, z) z^{2v+1} d_q z = T_{q,x}^v f^2(y). \end{aligned}$$

$$\int_0^\infty [T_{q,x}^v f(y)]^2 y^{2v+1} d_q y \leq \int_0^\infty T_{q,x}^v f^2(y) y^{2v+1} d_q y.$$

$$f \in \mathcal{L}_{q,2,v} \Rightarrow f^2 \in \mathcal{L}_{q,1,v}$$

Hence

$$\int_0^\infty T_{q,x}^v f^2(y) y^{2v+1} d_q y = \int_0^\infty f^2(y) y^{2v+1} d_q y$$

which gives

$$\|T_{q,x}^v f\|_{q,2,v} \leq \|f\|_{q,2,v}.$$

The operator $T_{q,x}^v$ is said to be a contraction of the $\mathcal{L}_{q,2,v}$ space. Now we will prove that $T_{q,x}^v$ is symmetric. Let $f, g \in \mathcal{L}_{q,2,v}$ then

$$|\langle T_{q,x}^v f, g \rangle| = \int_0^\infty |T_{q,x}^v f(y)| |g(y)| y^{2v+1} d_q y \leq \|T_{q,x}^v f\|_{q,2,v} \|g\|_{q,2,v} \leq \|f\|_{q,2,v} \|g\|_{q,2,v}$$

$$\begin{aligned} \langle T_{q,x}^v f, g \rangle &= \int_0^\infty T_{q,x}^v f(y) g(y) y^{2v+1} d_q y = \int_0^\infty \left[\int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z \right] g(y) y^{2v+1} d_q y \\ &= \int_0^\infty \left[\int_0^\infty g(z) D_v(x, y, z) z^{2v+1} d_q z \right] f(y) y^{2v+1} d_q y \\ &= \langle f, T_{q,x}^v g \rangle. \end{aligned}$$

The exchange of the signs integrals is valid sense

$$\begin{aligned} &\int_0^\infty \left[\int_0^\infty |g(z)| D_v(x, y, z) z^{2v+1} d_q z \right] |f(y)| y^{2v+1} d_q y \\ &\leq \left(\int_0^\infty \left[\int_0^\infty |g(z)| D_v(x, y, z) z^{2v+1} d_q z \right]^2 y^{2v+1} d_q y \right)^{1/2} \left(\int_0^\infty |f(y)|^2 y^{2v+1} d_q y \right)^{1/2} \\ &\leq \left(\int_0^\infty \left[\int_0^\infty |g(z)|^2 D_v(x, y, z) z^{2v+1} d_q z \right] y^{2v+1} d_q y \right)^{1/2} \|f\|_{q,2,v} = \|g\|_{q,2,v} \|f\|_{q,2,v}. \end{aligned}$$

On the other hand

$$T_{q,x}^v 1 = \int_0^\infty D_v(x, y, z) z^{2v+1} d_q z = 1.$$

Also

$$f \geq 0 \Rightarrow T_{q,x}^v f \geq 0.$$

Not that if f is bounded then

$$\begin{aligned} |T_{q,x}^v f(x)| &\leq \int_0^\infty |f(z)| |D_v(x, y, z)| z^{2v+1} d_q z \\ &\leq \|f\|_{q,\infty} \left[\int_0^\infty D_v(x, y, z) z^{2v+1} d_q z \right] = \|f\|_{q,\infty}. \end{aligned}$$

which finishes the proof. ■

Theorem 4 Let $c_{q,v} \varrho(t) t^{2v+1} d_q t$ be a probability measure on \mathbb{R}_q^+ . Then

$$K : \mathcal{L}_{q,2,v} \rightarrow \mathcal{L}_{q,2,v}, \quad f \mapsto f *_q \varrho$$

is a Markov operator.

Proof. From Proposition 6 we see that K send $\mathcal{L}_{q,2,v}$ to $\mathcal{L}_{q,2,v}$. On the other hand

$$Kf(x) = f *_q \varrho(x) = c_{q,v} \int_0^\infty T_{q,x} f(y) \varrho(y) y^{2v+1} d_q y,$$

which implies that $K1 = 1$ and if $f \geq 0$ then $Kf \geq 0$. Given two functions $f, g \in \mathcal{L}_{q,2,v}$ we write

$$\begin{aligned} \langle Kf, g \rangle &= \int_0^\infty Kf(x) g(x) x^{2v+1} d_q x \\ &= \int_0^\infty \left[c_{q,v} \int_0^\infty T_{q,x} f(y) \varrho(y) y^{2v+1} d_q y \right] g(x) x^{2v+1} d_q x \\ &= \int_0^\infty \left[c_{q,v} \int_0^\infty T_{q,y} f(x) g(x) x^{2v+1} d_q x \right] \varrho(y) y^{2v+1} d_q y \\ &= \int_0^\infty \left[c_{q,v} \int_0^\infty f(x) T_{q,y} g(x) x^{2v+1} d_q x \right] \varrho(y) y^{2v+1} d_q y \\ &= \int_0^\infty \left[c_{q,v} \int_0^\infty T_{q,x} g(y) \varrho(y) y^{2v+1} d_q y \right] f(x) x^{2v+1} d_q x \\ &= \langle f, Kg \rangle. \end{aligned}$$

The exchange of the integral signs has a valid sense

$$\begin{aligned} &\int_0^\infty \left[\int_0^\infty |T_{q,y} f(x)| |g(x)| x^{2v+1} d_q x \right] \varrho(y) y^{2v+1} d_q y \\ &\leq \|f\|_{q,2,v} \|g\|_{q,2,v} \|\varrho\|_{q,1,v}. \end{aligned}$$

Not that if f is bonded then

$$|Kf(x)| \leq \left[c_{q,v} \int_0^\infty \varrho(y) y^{2v+1} d_q y \right] \|f\|_{q,\infty} = \|f\|_{q,\infty}$$

which finishes the proof. ■

Proposition 11 *The eigenfunctions of the operator $T_{q,x}^v$ are in the following form*

$$f_n(t) = \frac{\psi_{q^n}(t)}{\|\psi_{q^n}\|_{q,2,v}}, \quad \forall n \in \mathbb{Z}$$

associated to the eigenvalues

$$\lambda_n = \frac{f_n(x)}{f_n(0)}, \quad \forall n \in \mathbb{Z}.$$

Proof. From (1), the sequence $\{f_n\}_{n \in \mathbb{Z}}$ forme a normal orthogonal basis of the Hilbert space $\mathcal{L}_{q,2,v}$. On the other hand

$$T_{q,x}^v j_v(q^n y, q^2) = j_v(q^n x, q^2) j_v(q^n y, q^2),$$

then

$$T_{q,x}^v f_n(y) = \frac{f_n(x)}{f_n(0)} f_n(y),$$

which leads to the result. ■

Remark 2 *Note that the kernel of $T_{q,x}^v$ can be written as follows*

$$D_v(x, y, z) = \sum_{n \in \mathbb{Z}} \frac{f_n(x) f_n(y) f_n(z)}{f_n(0)} \geq 0,$$

then the sequence $\{f_n\}$ satisfies the hypergroup property at the point 0 with measure $x^{2v+1} d_q x$.

Proposition 12 *A linear mapping*

$$K : \mathcal{L}_{q,2,v} \rightarrow \mathcal{L}_{q,2,v}, \quad f_n \mapsto c_n f_n$$

is a Markov operator if and only if

$$c_n = c_{q,v} \int_0^\infty \frac{f_n(y)}{f_n(0)} \varrho(y) y^{2v+1} d_q y$$

where $c_{q,v} \varrho(y) y^{2v+1} d_q y$ is a probability measure on \mathbb{R}_q^+ .

Proof. In fact

$$\begin{aligned} f_n *_q \varrho(x) &= c_{q,v} \int_0^\infty T_{q,x} f_n(y) \varrho(y) y^{2v+1} d_q y \\ &= \left[c_{q,v} \int_0^\infty \frac{f_n(y)}{f_n(0)} \varrho(y) y^{2v+1} d_q y \right] f_n(x) = c_n f_n(x), \end{aligned}$$

Then

$$K(f) = f *_q \varrho, \quad \forall f \in \mathcal{L}_{q,2,v}.$$

Theorem 4 leads to the result. ■

9 The q -heat semigroup

Definition 11 The q -exponential function is defined by

$$e(z, q) = \sum_{n=0}^{\infty} \frac{z^n}{(q, q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1.$$

Lemma 2 The unique solution of the following q -difference equation

$$\begin{cases} -x^2 \psi(t) = (1 - q^2) D_{q^2, t} \psi(t) \\ \psi(0) = c(x) \end{cases}$$

is in the following form

$$\psi(t) = c(x) e(-tx^2, q^2).$$

Lemma 3 Let $f \in \mathcal{L}_{q,2,v}$ such that $\Delta_{q,v} f \in \mathcal{L}_{q,2,v}$ then

$$\mathcal{F}_{q,v} [\Delta_{q,v} f](x) = -x^2 \mathcal{F}_{q,v} f(x).$$

Proof. Let $g = \mathcal{F}_{q,v} f$. From the inversion formula we obtain

$$f = \mathcal{F}_{q,v}^2 f = \mathcal{F}_{q,v} g.$$

Then

$$\Delta_{q,v} f(x) = \mathcal{F}_{q,v} \left[y \mapsto -y^2 g(y) \right](x)$$

Again, we use the inversion formula we obtain the result. ■

Definition 12 The q -heat semigroup is introduced in [1] as follows:

$$P_{q,t}^v f(x) = G^v(., t, q^2) *_q f(x), \quad t > 0$$

where $G^v(., t, q^2)$ is the q -Gauss kernel

$$G^v(x, t, q^2) = \mathcal{F}_{q,v} \left[y \mapsto e(-ty^2, q^2) \right] (x) = A(t) e \left(-\frac{q^{-2v}}{t} x^2, q^2 \right),$$

and

$$A(t) = \frac{(-q^{2v+2}t, -q^{-2v}/t; q^2)_\infty}{(-t, -q^2/t; q^2)_\infty}.$$

Note that $A(t^2) = t^{-2(v+1)} A(1)$.

Proposition 13 The unique solution of the q -heat equation

$$\begin{cases} \Delta_{q,v} u(x, t) = (1 - q^2) D_{q^2, t} u(x, t), & \forall x \in \mathbb{R}_q, \quad \forall t > 0 \\ u(x, 0) = f(x), & f \in \mathcal{L}_{q,2,v} \\ x \mapsto u(x, t) \in \mathcal{L}_{q,2,v}, & \forall t > 0. \end{cases}$$

is in the following form

$$u(x, t) = P_{q,t}^v f(x).$$

Proof. Let

$$\psi(t) = \mathcal{F}_{q,v} \left[u(., t) \right] (x).$$

From the following equation

$$\Delta_{q,v} u(x, t) = (1 - q^2) D_{q^2, t} u(x, t)$$

we see that

$$\Delta_{q,v} u(., t) \in \mathcal{L}_{q,2,v},$$

then from Lemma 3 we have

$$\mathcal{F}_{q,v} \left[\Delta_{q,v} u(., t) \right] (x) = -x^2 \mathcal{F}_{q,v} \left[u(., t) \right] (x)$$

and

$$\mathcal{F}_{q,v} \left[(1 - q^2) D_{q^2, t} u(., t) \right] (x) = (1 - q^2) D_{q^2, t} \mathcal{F}_{q,v} \left[u(., t) \right] (x),$$

which prove that ψ satisfies

$$(1 - q^2) D_{q^2, t} \psi(t) = -x^2 \psi(t), \quad \psi(0) = \mathcal{F}_{q,v} f(x).$$

From Lemma 1 and Proposition 5 we see that

$$\psi(t) = e(-tx^2, q^2) \mathcal{F}_{q,v} f(x) = \mathcal{F}_{q,v} \left[G^v(\cdot, t, q^2) \right] (x) \times \mathcal{F}_{q,v} f(x) = \mathcal{F}_{q,v} \left[G^v(\cdot, t, q^2) *_q f \right] (x).$$

From Theorem 1 we obtain

$$u(x, t) = G^v(\cdot, t, q^2) *_q f(x).$$

This finishes the proof. ■

Proposition 14 *The q -heat semigroup $P_{q,t}^v$ is a Markov operator of $\mathcal{L}_{q,2,v}$.*

Proof. In [1] it is proved that

$$\|G^v(\cdot, t, q^2)\|_{q,1,v} = \frac{1}{c_{q,v}},$$

then

$$y \mapsto c_{q,v} G^v(y, t, q^2) y^{2v+1} d_q y$$

is a probability measure on \mathbb{R}_q^+ . Theorem 4 leads to the result. ■

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