Parallel and multi-level strategies for structural analysis

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Abstract

In order to perform structural analysis on parallel computers, with shared or distributed memory, an iterative approach related to domain decomposition methods has been designed. The studied structure is described as an assembly defined from two mechanical entities: sub-structures and interfaces. The problem is here formulated in the case of linear elasticity, for which we are only interested in the final configuration.

After a description of the principles of this method, a multi-level approach is introduced in order to improve performances. Effectively, when decomposing into sub-structures, the presence of two scales can be noticed (one related to the dimension of the structure, and one related to the dimension of the sub-structures). Adding a large scale problem on the whole structure allows to quickly broadcast information among sub-structures because it takes into account the large variation length effects.

The goal is to illustrate the feasibility and the performances of such an approach, especially for a large number of sub-structures. A close attention is devoted to the influence of the discretisation choice for the unknowns of the problem, which are both displacement and efforts on interfaces.

1 INTRODUCTION

The most powerful computers are now based on a parallel architecture [17]. For using such architectures, designing suited algorithms is not obvious, and numerous researches are under development to build "parallel" algorithms [5], [2], [18].

The proposed parallel approach is related to domain decomposition methods [7], [21], [6]. The studied structure is described as an assembly defined by two mechanical entities: sub-structures and interfaces. The LArge Time INcrement method (LATIN) is then used to solve the implicit structural problem concurrently [13]. Such a method is a "mixed" approach as both displacements and efforts are the unknowns.

A general drawback of domain decomposition methods consists in a decrease of convergence when increasing the number of sub-structures [1]. In order to improve the former algorithm, a multi-level approach is used to rapidly propagate information among sub-structures. The use of a global problem is now implemented on several domain decomposition-like algorithms, such as the FETI method into which the rigid movement of sub-structures has to be introduced [8].

2 FORMULATION OF THE PROBLEM

In order to perform a parallel treatment of the problem, the elastic structure is decomposed into sub-structures $\Omega^E (E \in \mathbf{E})$ and interfaces $L^{EE'}$. Each is a mechanical entity with its own behaviour [4].

2.1 Sub-structure behaviour

Each sub-structure Ω^E is submitted to the action of its environment (the neighbouring interfaces): an effort density $\underline{F}^E$ and a displacement field $\underline{W}^E$ on its boundary $\partial\Omega^E$. Eventually, $\underline{f}_d$ is an additional density of body forces which the sub-structure is submitted to (Figure 1).

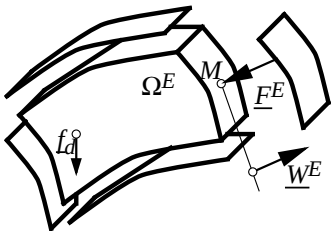

Figure 1: Sub-structure

$(\underline{W}^E; \underline{F}^E)$ has to satisfy for each $E \in \mathbf{E}$:

- kinematic equations:
  $\exists \underline{U}^E \in \mathcal{U}^E$
  $$\varepsilon^E = \varepsilon(\underline{U}^E) \quad \text{and} \quad \underline{U}^E_{|\partial\Omega^E} = \underline{W}^E \tag{1}$$
  where $\mathcal{U}^E$ is the set of displacement fields defined onto Ω^E which posses a finite energy.
- equilibrium equations:
  $\exists \sigma^E$ that equilibrates $\underline{f}_d$ and $\underline{F}^E$, i.e.
  $$\begin{aligned} &\forall \underline{U}^\star \in \mathcal{U}^E \\ \int_\Omega \mathrm{Tr}\left[\sigma^E \varepsilon(\underline{U}^\star)\right] d\Omega &= \int_{\Omega^E} \underline{f}_d \cdot \underline{U}^\star dS + \\ &+ \int_{\partial\Omega^E} \underline{F}^E \cdot \underline{U}^\star_{|\partial\Omega^E} dS \end{aligned} \tag{2}$$
- constitutive relation:
  $$\sigma^E = \mathbf{K}\varepsilon^E \tag{3}$$

where $\mathbf{K}$ is Hooke's tensor, which characterise the material. σ^E and ε^E are the stress and strain fields, while $\underline{U}^E$ is the displacement field within the sub-structure Ω^E.

The set of unknowns $(\underline{W}^E, \underline{F}^E, \underline{U}^E, \sigma^E)$ for $E \in \mathbf{E}$ is denoted by $\mathbf{s}$.

2.2 Interface behaviour

The state of the liaison $L^{EE'}$ between sub-structures Ω^E and $\Omega^{E'}$ is define by values on its surface $\Gamma^{EE'}$ of the displacements and efforts $(\underline{W}^E; \underline{F}^E)$ and $(\underline{W}^{E'}; \underline{F}^{E'})$ (Figure 2).

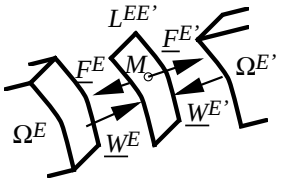

Figure 2: Interface

For instance, for a perfect liaison, they have to satisfy:

$$\underline{F}^E + \underline{F}^{E'} = 0 \quad \text{and} \quad \underline{W}^E = \underline{W}^{E'} \tag{4}$$

Of course, other kinds of liaison can be expressed, such as prescribed effort liaison, prescribed displacement liaison, unilateral contact with or without friction,... , as described in [12].

2.3 Description of the algorithm

The structure is now considered as an assembly of sub-structures and interfaces. Following the framework of the LArge Time INcrement method (LATIN) [14], [11], equations are splitted into two groups:

- $\mathbf{A}_d$ is the set of unknowns $\mathbf{s}$ satisfying each sub-structure behaviour (1), (2), (3);
- $\mathbf{\Gamma}$ is the set of unknowns $\mathbf{s}$ satisfying the interfaces behaviour (4).

The solution $\mathbf{s}_{ex}$ is the intersection of $\mathbf{A}_d$ and $\mathbf{\Gamma}$.

A two-stage algorithm builds successively an element of $\mathbf{A}_d$ and an element of $\mathbf{\Gamma}$ (Figure 3).

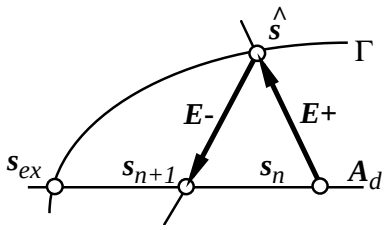

Figure 3: The LATIN 2-stage algorithm

Each stage involves a search direction; they are the parameters of the method:

- the local stage uses the search direction $E+$:

$$(\hat{\underline{F}} - \underline{F}) - k(\hat{\underline{W}} - \underline{W}) = 0 \tag{5}$$

Finding $\hat{\mathbf{s}} \in \mathbf{\Gamma}$ such that $\hat{\mathbf{s}} - \mathbf{s}_n$ belongs to the search direction $E+$ is a local problem on interfaces, which is then parallelisable.

- the global stage uses the search direction E-:

$$(\underline{F} - \hat{\underline{F}}) + k(\underline{W} - \hat{\underline{W}}) = 0 \tag{6}$$

The problem is then to find $\mathbf{s}_{n+1} \in \mathbf{A}_d$ such that $\mathbf{s}_{n+1} - \hat{\mathbf{s}}$ belongs to E-. It leads to a problem which is global only per sub-structure, though it can be solved in parallel once the sub-structures are distributed among the available processors as well as their related interfaces.

In the case of linear elasticity, this algorithm is similar to those proposed by [10], [15], [9].

3 DISCRETISATION CHOICE

For a continuum point of view, the previous algorithm converges toward the solution of the "reference problem" without decomposition. Precise convergence conditions are detailed in [12].

When implementing the algorithm in a finite element code, we need to discretise the unknowns. A first approach is to treat the problem with a displacement-oriented formulation. In this case, the displacement $\underline{U}^E$ is discretised according to the mesh of the sub-structure Ω^E, while $\underline{W}^E$ is the trace of $\underline{U}^E$ on the boundary $\partial\Omega^E$. Let $\mathcal{U}_h^E$ be the corresponding space of discretisation of $\underline{U}^E$. The displacement formulation of the global stage is then:

$$\min_{\underline{U}^E \in \mathcal{U}_h^E} J(\underline{U}^E) \tag{7}$$

where

$$\begin{aligned} J(\underline{U}^E) = \frac{1}{2}\int_{\Omega^E} \mathrm{Tr}\left[\varepsilon(\underline{U}^E)\mathbf{K}\varepsilon(\underline{U}^E)\right] d\Omega + \\ - \int_{\Omega^E} \underline{f}_d \cdot \underline{U}^E d\Omega - \int_{\partial\Omega^E} (\hat{\underline{F}}^E + k\hat{\underline{W}}^E) \cdot \underline{U}^E dS \end{aligned} \tag{8}$$

We can notice here that the problem to solve is global on the sub-structure and is an elasticity-like one. After discretisation, it can be written:

$$([K^E] + [k^E])[q^E] = [f^E] \tag{9}$$

where $[K^E]$ is a classical rigidity matrix and $[k^E]$ an additional rigidity on the boundary, traducing the search direction. Both are constant all along the iterations. $\underline{W}^E$ is then the trace of $\underline{U}^E$ on the boundary and $\underline{F}^E$ is back-substituted thanks to the search direction:

$$\underline{F}^E = \hat{\underline{F}}^E + k\hat{\underline{W}}^E - k\underline{W}^E \tag{10}$$

In fact, it conducts to have a discretisation of efforts $\underline{F}$ similar to the one of $\underline{W}$.

In a second time, one can remark that at the continuum level, displacement and efforts are treated symmetrically. For instance, all over the iterations, nor displacement neither efforts of $\mathbf{s}_n$ are continuous through interfaces but both gaps are improved; some other approaches privilege

a continuous displacement, as primal Schur method [19] does, or a continuous normal stress, as dual Schur method [7] does. We can take advantage of this symmetrical treatment by choosing the discretisation of efforts on the boundary. Lets say, for instance, that efforts $\underline{F}^E$ and $\underline{\hat{F}}^E$ are naturally interpolated one degree below the displacements, and note $\mathcal{F}_h^E$ the corresponding space of discretisation. $\underline{W}^E$ and $\underline{\hat{W}}^E$ can be discretised on the same fashion, as they are the "deformation" associated to the efforts; $\mathcal{W}_h^E$ is their space of discretisation. In such a case and without a major change in the algorithm, the solution now tends to the one of a mixed formulation problem:

Find the saddle point of:

$$\begin{aligned} &\sum_E \frac{1}{2}\int_{\Omega^E} \mathrm{Tr}\left[\varepsilon(\underline{U}^E)\mathbf{K}\varepsilon(\underline{U}^E)\right] d\Omega + \\ &-\sum_{\Gamma^{EE'}}\int_{\Gamma^{EE'}} \underline{\hat{F}}^E \cdot (\underline{W}^E - \underline{W}^{E'})\, dS + \\ &-\sum_E \int_{\Omega^E} \underline{f}_d \cdot \underline{U}^E d\Omega - \int_{\partial_2\Omega^E} \underline{F}_d \cdot \underline{U}^E dS \end{aligned} \tag{11}$$

For $\underline{U}^E \in \mathcal{U}_h^E$, $\underline{W}^E \in \mathcal{W}_h^E$ kinematically admissible, i.e. :

$$\begin{aligned} &\forall \quad \underline{W}^\star \in \mathcal{W}_h, \\ &\int_{\partial\Omega^E} (\underline{U}^E{}_{|\partial\Omega^E} - \underline{W}^E) \cdot k\underline{W}^{E\star} dS = 0 \end{aligned} \tag{12}$$

and for $\underline{\hat{F}}^E + \underline{\hat{F}}^{E'} = 0 \in \mathcal{F}_h^E$.

Such a formulation is intend to exhibits solutions more accurate than those produced by pure displacement formulations.

The algorithm has been implemented within the industrial-like finite element code CASTEM 2000 [20]. In order to compare the solution for both kinds of discretisation, let us consider the 2D plane stress problem of the holed plate in traction for which a quarter of the geometry is presented on Figure 4.

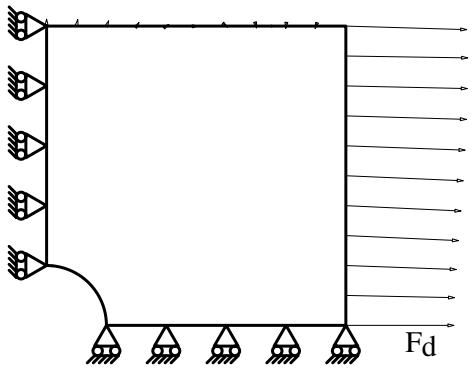

Figure 4: Holed plate problem

The loading is chosen such that the continuum level solution is the one of the infinite holed plate; this solution will be the reference one σ_{ref}. Figure 5 presents the decomposition into 42 sub-structures. The computation has been performed on 42 processors among the 64 ones of an nCUBE 2 MIMD computer, with 4 Mbytes RAM each.

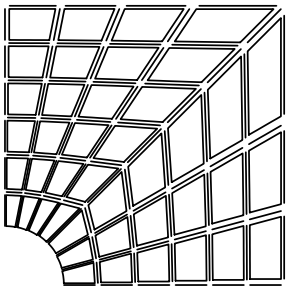

Figure 5: Decomposition into 42 sub-structures and interfaces

When convergence is reached, the solutions are compared with the relative error:

$$e = \frac{\|\sigma - \sigma_{ref}\|}{\|\sigma_{ref}\|} \tag{13}$$

where $\|\sigma\| = \sup \frac{1}{2}\sigma \mathbf{K}^{-1}\sigma$ is evaluated on the integration points of the meshes. Figure 6 then reports this error for several numbers of elements (3-node triangles).

One can notice that when the number of elements increases (while keeping the number of sub-structures constant), the two solutions become equivalent. Effectively, the ratio between the number of elements located on the boundary of sub-structures, and the number of internal elements decreases.

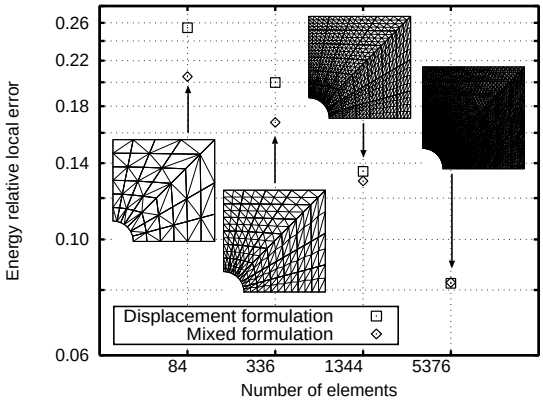

Figure 6: Decomposition into 42 sub-structures and interfaces

4 A MULTI-LEVEL APPROACH

A general and well-known behaviour of domain decomposition approaches is to have a degrading convergence rate when increasing the number of sub-structures [1]. For instance, on a model example consisting in a two dimensional beam with a flexion loading (Figure 7), error in energy with respect to the iteration number is represented on Figure 8 for several numbers of sub-structures.

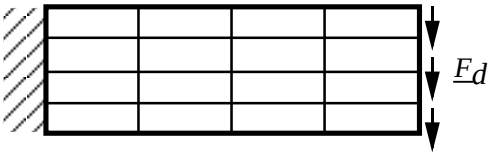

Figure 7: Model problem and example of decomposition in 16 sub-structures

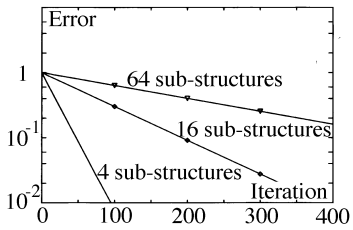

Figure 8: Convergence

The reference is here the direct finite element solution without decomposition, and is kept identical for each sub-structuring. The discretisation has then been chosen in order to converge toward this solution, i.e. it is the previously described displacement-oriented discretisation.

The form of the search direction parameter k has been chosen to be $k = k_0 I_d$ where I_d is the identity operator and $k_0 = \dfrac{E}{L_0}$, E is the Young modulus of the material and L_0 a length that characterises the search direction. Here, the "optimal" value of L_0 is related to the length of the structure (typically L_0 is twice the length for such a flexion problem) [4].

To remedy to such a drawback, a global exchange of information is generally efficient [7], [16]. In order to built such a mechanism, we choose to express the solution onto two different scales:

$$(\underline{U}^E; \sigma^E) = (\underline{U}_1^E; \sigma_1^E) + (\underline{U}_2^E; \sigma_2^E) \tag{14}$$

where 1 and 2 denote respectively effective unknowns on the large scale and corrections on the fine scale. Each scale can arise from a different modelling of the structure, for instance here, from two meshes with embedded elements (Figure 9). Let Ω_1 and Ω_2 denote them. Transfers between them is performed through a prolongation and a restriction operators, respectively $\mathbf{P}$ and $\mathbf{R} = \mathbf{P}^T$. Transposition is derived from the symmetric form:

$$\begin{aligned} &\sum_{E\in\mathbf{E}} \int_{\partial\Omega^E} \mathrm{Tr}[\varepsilon(\mathbf{P}^E \overline{\underline{U}})\sigma^E] d\Omega = \\ &= \int_{\partial\Omega_1} \mathrm{Tr}[\varepsilon(\overline{\underline{U}}) \sum_{E\in\mathbf{E}} \mathbf{R}^E \sigma^E] d\Omega \end{aligned} \tag{15}$$

Where $(\overline{\underline{U}}, \overline{\sigma})$ denotes the effective solution, i.e. the solution defined onto the mesh Ω_1 (then $\underline{U}_1^E = \mathbf{P}^E \overline{\underline{U}}$ and $\overline{\sigma} = \sum_{E\in\mathbf{E}} \mathbf{R}^E \sigma_1^E$).

When using two embedded grids, prolongation can be performed with a hierarchical finite element projection [23], [22].

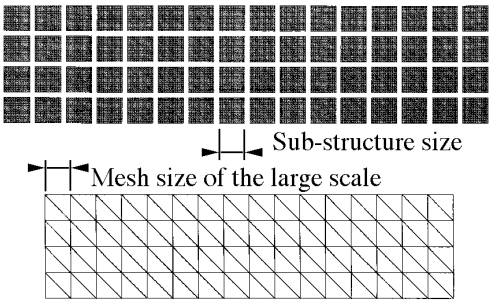

Figure 9: Meshes of the two scales

With such a splitting, the problem is now expressed separately on both scales: the equilibrium equations become

- on the fine scale 2, on each sub-structure Ω_2^E

$$\begin{aligned} &\in \mathcal{U}^E \forall \underline{U}^\star \in \mathcal{U}_2^E \\ &\int_{\Omega^E} \mathrm{Tr}[\sigma_2^E \varepsilon(\underline{U}^\star)] d\Omega = \int_{\Omega^E} \underline{f}_d \cdot \underline{U}^\star d\Omega + \\ &+ \int_{\partial\Omega^E} \underline{F}_2^E \cdot \underline{U}^\star dS - \int_{\Omega^E} \mathrm{Tr}[\varepsilon(\mathbf{P}^E \overline{\underline{U}}) \mathbf{K} \varepsilon(\underline{U}^\star)] d\Omega \end{aligned} \tag{16}$$

$\mathcal{U}_2^E$ is the part of $\mathcal{U}^E$ corresponding to the fine scale 2.

- on the large scale 1

$$\forall\, \overline{\underline{U}}^\star \in \mathcal{U}_1$$
$$\int_{\Omega_1} \mathrm{Tr}[\overline{\sigma}\varepsilon(\overline{\underline{U}}^\star)]d\Omega = \int_{\Omega_1} \underline{f}_d \cdot \mathbf{P}\ \overline{\underline{U}}^\star d\Omega + \\ - \int_{\Omega_1} \mathrm{Tr}[\varepsilon(\overline{\underline{U}}^\star) \sum_{E\in\mathbf{E}} \mathbf{R}^E \mathbf{K}\varepsilon(\underline{U}_2^E)]d\Omega \tag{17}$$

where $\mathcal{U}_1$ is this time related to fields defined on the whole structure Ω satisfying homogeneous boundary condition.

One can notice that the two scales are not separated, due to coupling terms: on the large scale problem, the stress field $\overline{\sigma}$ has to equilibrate also $-\sum_{E\in\mathbf{E}} \mathbf{R}^E \mathbf{K}\varepsilon(\underline{U}_2^E)$; on the fine scale, the stress field σ_2^E has to equilibrate also $-\mathbf{K}\varepsilon(\mathbf{P}^E\overline{\underline{U}})$.

The large scale problem being chosen to remain global, we search successively the solution on the two scales at each iteration of the LATIN method on the sub-structured fine scale. On the previous example of a two dimensional beam, the convergence rate gain is illustrated on Figure 10 when each sub-structure is meshed with 512 3-node elements; the reference problem posses 33 400 d.o.f. A drawback is the necessity to solve a global problem on the whole structure due to the large scale. This problem is then to be kept small to avoid a decrease of degree of parallelism for the method; here it is meshed with a number of 3-node elements equal to twice the number of sub-structures.

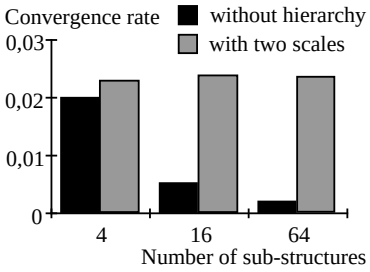

Figure 10: Convergence rate

On the other hand, the use of a large scale avoid the aforementioned degrading of convergence rate with the number of sub-structures. Figure 10 also illustrate the influence of using two scales when increasing the number of sub-structures (here, the sub-structure size remains equal to the element size of the large scale, while the reference problem is still the same).

One can notice that the optimal search direction for this example when using the large scale of Figure 9, is now related to the length of the substructures (L_0 was chosen to be 0.26 times the length of one sub-structure). It is then characterised by the sub-structures behaviour and no more by the global structure behaviour.

5 CONCLUSIONS

The originality of the coupling of the LArge Time INcrement method with a sub-structuration technique is the major role played by interfaces which are considered as structures on their own.

As mechanical unknowns are both displacements and efforts, the last ones could have their own discretisation. The consequence is the convergence toward the solution of a mixed formulation.

It conducts to a "pure parallel" algorithm that can be improved by introducing a two level scheme. The consequence is the arising of a global problem to solve onto the whole structure at each iteration.

Comparisons with other domain decomposition methods are under way and the final goal of the approach is the extension to non-linear structural analysis, following the LATIN design.

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