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On the vertices of the k -additive core

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Abstract

The core of a game v on N , which is the set of additive games ϕ dominating v such that $\phi(N) = v(N)$, is a central notion in cooperative game theory, decision making and in combinatorics, where it is related to submodular functions, matroids and the greedy algorithm. In many cases however, the core is empty, and alternative solutions have to be found. We define the k -additive core by replacing additive games by k -additive games in the definition of the core, where k -additive games are those games whose Möbius transform vanishes for subsets of more than k elements. For a sufficiently high value of k , the k -additive core is nonempty, and is a convex closed polyhedron. Our aim is to establish results similar to the classical results of Shapley and Ichiishi on the core of convex games (corresponds to Edmonds’ theorem for the greedy algorithm), which characterize the vertices of the core.

Key words: cooperative games; core; k -additive games; vertices

1 Introduction

Given a finite set N of n elements, and a set function $v : 2^N \rightarrow \mathbb{R}$ vanishing on the empty set (called hereafter a *game*), its *core* $\mathcal{C}(v)$ is the set of additive set

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4 functions ϕ on N such that $\phi(S) \geq v(S)$ for every $S \subseteq N$, and $\phi(N) = v(N)$.

5 Whenever nonempty, the core is a convex closed bounded polyhedron.

6 In many fields, the core is a central notion which has deserved a lot of studies.

7 In cooperative game theory, it is the set of imputations for players so that no

8 subcoalition has interest to form [18]. In decision making under uncertainty,

9 where games are replaced by *capacities* (monotonic games), it is the set of

10 probability measures which are coherent with the given representation of un-

11 certainty [19]. More on a combinatorial point of view, cores of convex games

12 are also known as base polytopes associated to supermodular functions [13,9],

13 for which the greedy algorithm is known to be a fundamental optimization

14 technique. Many studies have been done along this line, e.g., by Faigle and

15 Kern for the matching games [8], and cost games [7]. In game theory, which

16 will be our main framework here, related notions are the selectope [3], and the

17 Shapley value with many of its variations on combinatorial structures (see,

18 e.g., [1]).

19 It is a well known fact that the core is nonempty if and only if the game

20 is balanced [4]. In the case of emptiness, an alternative solution has to be

21 found. One possibility is to search for games more general than additive ones,

22 for example k -additive games and capacities proposed by Grabisch [10]. In

23 short, k -additive games have their Möbius transform vanishing for subsets

24 of more than k elements, so that 1-additive games are just usual additive

25 games. Since any game is a k -additive game for some k (possibly $k = n$), the

26 k -additive core, i.e., the set of dominating k -additive games, is never empty

27 provided k is high enough. The authors have justified this definition in the

28 framework of cooperative game theory [15]. Briefly speaking, an element of

29 the k -additive core implicitly defines by its Möbius transform an imputation

30 (possibly negative), which is now defined on groups of at most k players, and

31 no more on individuals. By definition of the k -additive core, the total worth

32 assigned to a coalition will be always greater or equal to the worth the coalition

33 can achieve by itself; however, the precise sharing among players has still to
34 be decided (e.g., by some bargaining process) among each group of at most k
35 players.

36 In game theory, elements of the core are imputations for players, and thus
37 it is natural that they fulfill monotonicity. We call monotonic core the core
38 restricted to monotonic games (capacities). We will see in the sequel that the
39 core is usually unbounded, while the monotonic core is not.

40 The properties of the (classical) core are well known, most remarkable being
41 the result characterizing the core of convex games, where the set of vertices is
42 exactly the set of additive games induced by maximal chains (or equivalently
43 by permutations on N) in the Boolean lattice $(2^N, \subseteq)$. This has been shown
44 by Shapley [17], and later Ichiishi proved the converse implication [12]. This
45 result is also known in the field of matroids, since vertices of the base polytope
46 can be found by a greedy algorithm.

47 A natural question arises: is it possible to generalize the Shapley-Ichiishi the-
48 orem for k -additive (monotonic) cores? More precisely, can we find the set of
49 vertices for some special classes of games? Are they induced by some general-
50 ization of maximal chains? The paper shows that the answer is more complex
51 than expected. It is possible to define notions similar to permutations and
52 maximal chains, so as to generate vertices of the k -additive core of $(k + 1)$ -
53 monotone games, a result which is a true generalization of the Shapley-Ichiishi
54 theorem, but this does not permit to find all vertices of the core. A full ana-
55 lytical description of vertices seems to be difficult to find, but we completely
56 explicit the case $k = n - 1$.

57 After a preliminary section introducing necessary concepts, Section 3 presents
58 our basic ingredients, that is, orders on subsets of at most k elements, and
59 achievable families, which play the role of maximal chains in the classical case.
60 Then Section 4 presents the main result on the characterization of vertices for

61 $(k + 1)$ -monotone games induced by achievable families.

62 2 Preliminaries

63 Throughout the paper, $N := \{1, \dots, n\}$ denotes a set of n elements (players in
64 a game, nodes of a graph, etc.). We use indifferently 2^N or $\mathcal{P}(N)$ for denoting
65 the set of subsets of N , and the set of subsets of N containing at most k
66 elements is denoted by $\mathcal{P}^k(N)$, while $\mathcal{P}_*^k(N) := \mathcal{P}^k(N) \setminus \{\emptyset\}$. For convenience,
67 subsets like $\{i\}, \{i, j\}, \{2\}, \{2, 3\}, \dots$ are written in the compact form $i, ij, 2, 23$
68 and so on.

69 A *game* on N is a function $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. The set of games
70 on N is denoted by $\mathcal{G}(N)$. For any $A \in 2^N \setminus \{\emptyset\}$, the *unanimity game centered*
71 *on* A is defined by $u_A(B) := 1$ iff $B \supseteq A$, and 0 otherwise.

72 A game v on N is said to be:

- 73 (i) *additive* if $v(A \cup B) = v(A) + v(B)$ whenever $A \cap B = \emptyset$;
- 74 (ii) *convex* if $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$, for all $A, B \subseteq N$;
- 75 (iii) *monotone* if $v(A) \leq v(B)$ whenever $A \subseteq B$;
- (iv) *k -monotone* for $k \geq 2$ if for any family of k subsets A_1, \dots, A_k , it holds

$$v\left(\bigcup_{i=1}^k A_i\right) \geq \sum_{\substack{K \subseteq \{1, \dots, k\} \\ K \neq \emptyset}} (-1)^{|K|+1} v\left(\bigcap_{j \in K} A_j\right)$$

- 76 (v) *infinitely monotone* if it is k -monotone for all $k \geq 2$.

78 Convexity corresponds to 2-monotonicity. Note that k -monotonicity implies
79 k' -monotonicity for all $2 \leq k' \leq k$. Also, $(n - 2)$ -monotone games on N
80 are infinitely monotone [2]. The set of monotone games on N is denoted by
81 $\mathcal{MG}(N)$, while the set of infinitely monotone games is denoted by $\mathcal{G}_\infty(N)$.

Let v be a game on N . The *Möbius transform* of v [16] (also called *dividends* of v , see Harsanyi [11]) is a function $m : 2^N \rightarrow \mathbb{R}$ defined by:

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} v(B), \quad \forall A \subseteq N.$$

The Möbius transform is invertible since one can recover v from m by:

$$v(A) = \sum_{B \subseteq A} m(B), \quad \forall A \subseteq N.$$

82 If v is an additive game, then m is non null only for singletons, and $m(\{i\}) =$
 83 $v(\{i\})$. The Möbius transform of u_A is given by $m(A) = 1$ and m is 0 otherwise.

84 A game v is said to be *k-additive* [10] for some integer $k \in \{1, \dots, n\}$ if
 85 $m(A) = 0$ whenever $|A| > k$, and there exists some A such that $|A| = k$, and
 86 $m(A) \neq 0$.

87 Clearly, 1-additive games are additive. The set of games on N being at most
 88 k -additive (resp. infinitely monotone games at most k -additive) is denoted by
 89 $\mathcal{G}^k(N)$ (resp. $\mathcal{G}_\infty^k(N)$). As above, replace \mathcal{G} by \mathcal{MG} if monotone games are
 90 considered instead.

91 We recall the fundamental following result.

92 **Proposition 1** [5] *Let v be a game on N . For any $A, B \subseteq N$, with $A \subseteq B$,*
 93 *we denote $[A, B] := \{L \subseteq N \mid A \subseteq L \subseteq B\}$.*

(i) *Monotonicity is equivalent to*

$$\sum_{L \in [i, B]} m(L) \geq 0, \quad \forall B \subseteq N, \quad \forall i \in B.$$

94 (ii) *For $2 \leq k \leq n$, k -monotonicity is equivalent to*

$$\sum_{L \in [A, B]} m(L) \geq 0, \quad \forall A, B \subseteq N, A \subseteq B, \quad 2 \leq |A| \leq k.$$

95

96 Clearly, a monotone and infinitely monotone game has a nonnegative Möbius
 97 transform.

The *core* of a game v is defined by:

$$\mathcal{C}(v) := \{\phi \in \mathcal{G}^1(N) \mid \phi(A) \geq v(A), \quad \forall A \subseteq N, \text{ and } \phi(N) = v(N)\}.$$

98

99 A *maximal chain* in 2^N is a sequence of subsets $A_0 := \emptyset, A_1, \dots, A_{n-1}, A_n := N$
 100 such that $A_i \subset A_{i+1}$, $i = 0, \dots, n-1$. The set of maximal chains of 2^N is
 101 denoted by $\mathcal{M}(2^N)$.

To each maximal chain $C := \{\emptyset, A_1, \dots, A_n = N\}$ in $\mathcal{M}(2^N)$ corresponds
 a unique permutation σ on N such that $A_1 = \sigma(1)$, $A_2 \setminus A_1 = \sigma(2)$, \dots ,
 $A_n \setminus A_{n-1} = \sigma(n)$. The set of all permutations over N is denoted by $\mathfrak{S}(N)$.
 Let v be a game. Each permutation σ (or maximal chain C) induces an additive
 game ϕ^σ (or ϕ^C) on N defined by:

$$\phi^\sigma(\sigma(i)) := v(\{\sigma(1), \dots, \sigma(i)\}) - v(\{\sigma(1), \dots, \sigma(i-1)\})$$

or

$$\phi^C(\sigma(i)) := v(A_i) - v(A_{i-1}), \quad \forall i \in N.$$

102 with the above notation. The following is immediate.

Proposition 2 *Let v be a game on N , and C a maximal chain of 2^N . Then*

$$\phi^C(A) = v(A), \quad \forall A \in C.$$

103

104 **Theorem 1** *The following propositions are equivalent.*

105 (i) v is a convex game.

106 (ii) All additive games ϕ^σ , $\sigma \in \mathfrak{S}(N)$, belong to the core of v .

107 (iii) $\mathcal{C}(v) = \text{co}(\{\phi^\sigma\}_{\sigma \in \mathfrak{S}(N)})$.

108 (iv) $\text{ext}(\mathcal{C}(v)) = \{\phi^\sigma\}_{\sigma \in \mathfrak{S}(N)}$,

109 where $\text{co}(\cdot)$ and $\text{ext}(\cdot)$ denote respectively the convex hull of some set, and the
 110 extreme points of some convex set.

111 (i) \Rightarrow (ii) and (i) \Rightarrow (iv) are due to Shapley [17], while (ii) \Rightarrow (i) was proved
 112 by Ichiishi [12].

A natural extension of the definition of the core is the following. For some integer $1 \leq k \leq n$, the k -additive core of a game v is defined by:

$$\mathcal{C}^k(v) := \{\phi \in \mathcal{G}^k(N) \mid \phi(A) \geq v(A), \quad \forall A \subseteq N, \phi(N) = v(N)\}.$$

In a context of game theory where elements of the core are imputations, it is natural to consider that monotonicity must hold, i.e., the imputation allocated to some coalition $A \in \mathcal{P}_*^k(N)$ is larger than for any subset of A . We call it the *monotone k -additive core*:

$$\mathcal{MC}^k(v) := \{\phi \in \mathcal{MG}^k(N) \mid \phi(A) \geq v(A), \quad \forall A \subseteq N, \phi(N) = v(N)\}.$$

We introduce also the *core of k -additive infinitely monotone games*:

$$\mathcal{C}_\infty^k(v) := \{\phi \in \mathcal{G}_\infty^k(N) \mid \phi(A) \geq v(A), \quad \forall A \subseteq N, \text{ and } \phi(N) = v(N)\}.$$

113 The latter is introduced just for mathematical convenience, and has no clear
 114 application. Note that $\mathcal{C}(v) = \mathcal{C}^1(v) = \mathcal{C}_\infty^1(v)$.

115 **3 Orders on $\mathcal{P}_*^k(N)$ and achievable families**

116 As our aim is to give a generalization of the Shapley-Ichiishi results, we need
 117 counterparts of permutations and maximal chains. These are given in this sec-
 118 tion. Exact connections between our material and permutations and maximal
 119 chains will be explicated at the end of this section. First, we introduce total
 120 orders on subsets of at most k elements as a generalization of permutations.

121 We denote by \prec a total (strict) order on $\mathcal{P}_*^k(N)$, \preceq denoting the corresponding
 122 weak order.

- 123 (i) \prec is said to be *compatible* if for all $A, B \in \mathcal{P}_*^k(N)$, $A \prec B$ if and only
124 if $A \cup C \prec B \cup C$ for all $C \subseteq N$ such that $A \cup C, B \cup C \in \mathcal{P}_*^k(N)$,
125 $A \cap C = B \cap C = \emptyset$.
- 126 (ii) \prec is said to be \subseteq -*compatible* if $A \subset B$ implies $A \prec B$.
- 127 (iii) \prec is said to be *strongly compatible* if it is compatible and \subseteq -compatible.

We introduce the *binary order* \prec^2 on 2^N as follows. To any subset $A \subseteq N$ we associate an integer $\eta(A)$, whose binary code is the indicator function of A , i.e., the i th bit of $\eta(A)$ is 1 if $i \in A$, and 0 otherwise. For example, with $n = 5$, $\{1, 3\}$ and $\{4\}$ have binary codes 00101 and 01000 respectively, hence $\eta(\{1, 3\}) = 5$ and $\eta(\{4\}) = 8$. Then $A \prec^2 B$ if $\eta(A) < \eta(B)$. This gives

$$1 \prec^2 2 \prec^2 12 \prec^2 3 \prec^2 13 \prec^2 23 \prec^2 123 \prec^2 4 \prec^2 14 \prec^2 24 \prec^2 \\ 124 \prec^2 34 \prec^2 134 \prec^2 234 \prec^2 1234 \prec^2 5 \prec^2 \dots \quad (1)$$

- 128 Note the recursive nature of \prec^2 . Obviously, \prec^2 is a strongly compatible order,
129 as well as all its restrictions to $\mathcal{P}_*^k(N)$, $k = 1, \dots, n - 1$.

We introduce now a generalization of maximal chains associated to permutations. Let \prec be a total order on $\mathcal{P}_*^k(N)$. For any $B \in \mathcal{P}_*^k(N)$, we define

$$\mathcal{A}(B) := \{A \subseteq N \mid [A \supseteq B] \text{ and } [\forall K \subseteq A \text{ s.t. } K \in \mathcal{P}_*^k(N), \text{ it holds } K \preceq B]\}$$

- 130 the *achievable family* of B .

EXAMPLE 1: Consider $n = 3$, $k = 2$, and the following order: $1 \prec 2 \prec 12 \prec$
 $13 \prec 23 \prec 3$. Then

$$\mathcal{A}(1) = \{1\}, \quad \mathcal{A}(2) = \{2\}, \quad \mathcal{A}(12) = \{12\}, \quad \mathcal{A}(13) = \mathcal{A}(23) = \emptyset, \\ \mathcal{A}(3) = \{3, 13, 23, 123\}.$$

131

- 132 **Proposition 3** $\{\mathcal{A}(B)\}_{B \in \mathcal{P}_*^k(N)}$ is a partition of $\mathcal{P}(N) \setminus \{\emptyset\}$.

133 **Proof:** Let $\emptyset \neq C \in \mathcal{P}(N)$. It suffices to show that there is a unique $B \in$
134 $\mathcal{P}_*^k(N)$ such that $C \in \mathcal{A}(B)$. Let K_1, K_2, \dots, K_p be the nonempty collection
135 of subsets of C in $\mathcal{P}^k(N)$, assuming $K_1 \prec K_2 \prec \dots \prec K_p$. Then $C \in \mathcal{A}(K_p)$
136 is the unique possibility, since any B outside the collection will fail to satisfy
137 the condition $B \subseteq C$, and any $B \neq K_p$ inside the collection will fail to satisfy
138 the condition $K_p \preceq B$. ■

139

140 **Proposition 4** For any $B \in \mathcal{P}_*^k(N)$ such that $\mathcal{A}(B) \neq \emptyset$, $(\mathcal{A}(B), \subseteq)$ is an
141 inf-semilattice, with bottom element B .

142 **Proof:** If $\mathcal{A}(B) \neq \emptyset$, any $C \in \mathcal{A}(B)$ contains B , hence every $K \subseteq B \subseteq C$,
143 $K \in \mathcal{P}_*^k(N)$, is such that $K \preceq B$. Hence $B \in \mathcal{A}(B)$, and it is the smallest
144 element.

145 Let $K, K' \in \mathcal{A}(B)$, assuming $\mathcal{A}(B)$ contains at least 2 elements (otherwise,
146 we are done). $K \in \mathcal{A}(B)$ is equivalent to $K \supseteq B$ and any $L \subseteq K$, $L \in \mathcal{P}_*^k(N)$
147 is such that $L \preceq B$. The same holds for K' . Therefore, $K \cap K' \supseteq B$, and if
148 $L \subseteq K \cap K'$, $L \in \mathcal{P}_*^k(N)$, then $L \subseteq K$ and $L \subseteq K'$, which entails $L \preceq B$.
149 Hence $K \cap K' \in \mathcal{A}(B)$. ■

150

151 From the above proposition we deduce:

152 **Corollary 1** Let $B \in \mathcal{P}_*^k(N)$ and \prec be some total order on $\mathcal{P}_*^k(N)$. Then
153 $\mathcal{A}(B) \neq \emptyset$ if and only if for all $C \in \mathcal{P}_*^k(N)$, $C \subseteq B$ implies $C \preceq B$. Conse-
154 quently, if $|B| = 1$ then $\mathcal{A}(B) \neq \emptyset$.

155 **Corollary 2** $\mathcal{A}(B) \neq \emptyset$ for all $B \in \mathcal{P}_*^k(N)$ if and only if \prec is \subseteq -compatible.

156 It is easy to build examples where achievable families are not lattices.

157 EXAMPLE 2: Consider $n = 4, k = 2$ and the following order: 2, 3, 24, 12,
 158 4, 13, 34, 1, 23, 14. Then $\mathcal{A}(23) = \{23, 123, 234\}$, and $1234 \notin \mathcal{A}(23)$ since
 159 $14 \succ 23$.

160 Assuming $\mathcal{A}(B)$ is a lattice, we denote by \check{B} its top element.

161 **Proposition 5** *Let \prec be a total order on $\mathcal{P}_*^k(N)$. Consider $B \in \mathcal{P}_*^k(N)$ such
 162 that $\mathcal{A}(B)$ is a lattice. Then it is a Boolean lattice isomorphic to $(\mathcal{P}(\check{B} \setminus B), \subseteq)$.*

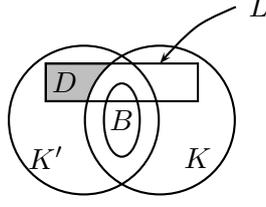
163 **Proof:** It suffices to show that $\mathcal{A}(B) = \{B \cup K \mid K \subseteq \check{B} \setminus B\}$. Taking
 164 $\check{K} := \check{B} \setminus B$, we have $B \cup \check{K} \in \mathcal{A}(B)$. Hence, any $L \subseteq B \cup \check{K}$, $L \in \mathcal{P}_*^k(N)$, is
 165 such that $L \preceq B$. This is a fortiori true for $L \subseteq B \cup K$, $L \in \mathcal{P}_*^k(N)$, $\forall K \subseteq \check{K}$.
 166 Hence $B \cup K$ belongs to $\mathcal{A}(B)$, for all $K \subseteq \check{K}$. ■

167

168 **Proposition 6** *Assume \prec is compatible. For any $B \in \mathcal{P}_*^k(N)$ such that
 169 $\mathcal{A}(B) \neq \emptyset$, $\mathcal{A}(B)$ is the Boolean lattice $[B, \check{B}]$.*

170 **Proof:** If $\mathcal{A}(B)$ is a lattice, we know by Prop. 5 that it is a Boolean lattice
 171 with bottom element B . Since we know that $\mathcal{A}(B)$ is an inf-semilattice by
 172 Prop. 4, it remains to show that $K, K' \in \mathcal{A}(B)$ implies $K \cup K' \in \mathcal{A}(B)$.
 173 Assume $K \cup K' \notin \mathcal{A}(B)$. Then there exists $L \subseteq K \cup K'$, $L \in \mathcal{P}_*^k(N)$ such that
 174 $L \succ B$. Necessarily, $L \setminus K \neq \emptyset$, otherwise $L \subseteq K$ and $K \in \mathcal{A}(B)$ imply $L \prec B$,
 175 a contradiction. Similarly, $L \setminus K' \neq \emptyset$. Moreover, $L \not\subseteq B$ since $\mathcal{A}(B) \neq \emptyset$ (see
 176 Cor. 1).

177 We consider $D := L \setminus K$, not empty by definition of L . Since $L \setminus D \subseteq K$ and
 178 $L \setminus D \in \mathcal{P}_*^k(N)$, we have $L \setminus D \preceq B$, with strict inequality since $L \setminus D$ has
 179 elements outside $K \cap K'$, hence outside B (see Figure below).



180

181 Suppose first that $|B| < k$, and let $D := \{i, j, \dots\}$. We have $B \cup l \in \mathcal{P}_*^k(N)$
 182 and $B \cup l \subseteq K'$ for any $l \in D$, which implies $B \cup l \prec B$. Taking $l = i$, by
 183 compatibility, $L \setminus D \prec B$ implies $(L \setminus D) \cup i \prec B \cup i \prec B$. By compatibility
 184 again, $(L \setminus D) \cup i \prec B$ implies $(L \setminus D) \cup i \cup j \prec B \cup j \prec B$. Continuing the
 185 process till all elements of D have been taken, we finally end with $L \prec B$, a
 186 contradiction.

187 Secondly, assume that $|B| = k$. Take $K'' \subset B$ such that $K'' \supseteq L \cap B$ and
 188 $|K'' \cup D| = k$, which is always possible by assumption. Since $K'' \subset B \subseteq K$
 189 and $K'' \in \mathcal{P}_*^k(N)$, we have $K'' \prec B$. Then

- 190 (i) Either $L \setminus D \prec K'' \prec B$. By compatibility, $L \setminus D \prec K''$ implies $L \prec K'' \cup D$.
 191 Since $K'' \cup D \in \mathcal{P}_*^k(N)$ and $K'' \cup D \subseteq K'$, we deduce that $K'' \cup D \prec B$,
 192 hence $L \prec B$, a contradiction.
- 193 (ii) Or $K'' \prec L \setminus D \prec B$. Since $(L \setminus D) \cap (B \setminus K'') = \emptyset$, from compatibility
 194 $K'' \prec L \setminus D$ implies $B = K'' \cup (B \setminus K'') \prec (L \setminus D) \cup (B \setminus K'')$. We have by
 195 assumption $|(L \setminus D) \cup (B \setminus K'')| = |L| \leq k$ and $(L \setminus D) \cup (B \setminus K'') \subseteq K$,
 196 from which we deduce $(L \setminus D) \cup (B \setminus K'') \prec B$. Hence we get $B \prec B$, a
 197 contradiction.

198



199

200 The following example shows that compatibility is not a necessary condition.

EXAMPLE 3: Consider $n = 4$, $k = 2$, and the following order: 1, 3, 2, 12,
 23, 13, 4, 14, 24, 34. This order is not compatible since $3 \prec 2$ and $12 \prec 13$.

We obtain:

$$\begin{aligned} \mathcal{A}(1) = 1, \quad \mathcal{A}(3) = 3, \quad \mathcal{A}(2) = 2, \quad \mathcal{A}(12) = 12, \quad \mathcal{A}(23) = 23, \quad \mathcal{A}(13) = \{13, 123\}, \\ \mathcal{A}(4) = 4, \quad \mathcal{A}(14) = 14, \quad \mathcal{A}(24) = \{24, 124\}, \quad \mathcal{A}(34) = \{34, 134, 234, 1234\}. \end{aligned}$$

201 All families are lattices.

202 In the above example, \prec was \subseteq -compatible. However, this is not enough to
203 ensure that achievable families are lattices, as shown by the following example.

EXAMPLE 4: Let us consider the following \subseteq -compatible order with $n = 4$
and $k = 2$:

$$3 \prec 4 \prec 34 \prec 2 \prec 24 \prec 1 \prec 13 \prec 12 \prec 23 \prec 14.$$

204 Then $\mathcal{A}(23) = \{23, 123, 234\}$.

205 We give some fundamental properties of achievable families when they are
206 lattices, in particular of their top elements.

207 **Proposition 7** *Assume \prec is compatible, and consider a nonempty achiev-*
208 *able family $\mathcal{A}(B)$, with top element \check{B} . Then $\{\mathcal{A}(B_i) \mid B_i \in \mathcal{P}_*^k(N), B_i \subseteq$*
209 *$\check{B}, \mathcal{A}(B_i) \neq \emptyset\}$ is a partition of $\mathcal{P}(\check{B}) \setminus \{\emptyset\}$.*

210 **Proof:** We know by Prop. 3 that all $\mathcal{A}(B_i)$'s are disjoint. It remains to show
211 that (1) any $K \subseteq \check{B}$ is in some $\mathcal{A}(B_i)$, $B_i \subseteq \check{B}$, and (2) conversely that any
212 K in such $\mathcal{A}(B_i)$ is a subset of \check{B} .

213 (1) Assume $K \in \mathcal{A}(B_i)$, $B_i \not\subseteq \check{B}$. Then $B_i \subseteq K \subseteq \check{B}$, a contradiction.

214 (2) Assume $K \in \mathcal{A}(B_i)$, $B_i \subseteq \check{B}$, and $K \not\subseteq \check{B}$. Then there exists $l \in K$ such
215 that $l \notin \check{B}$ (and hence not in B_i). Note that this implies $B_i \cup l \prec B_i$, provided
216 $|B_i| < k$. First we show that $l \prec j$ for any $j \in B_i$. Since $K \supseteq B_i \cup \{l\}$, we
217 deduce that for any $j \in B_i$, $\{j, l\} \prec B_i$ and $l \prec B_i$. If $B_i = \{j\}$, we can
218 further deduce that $l \prec j$. Otherwise, if $B_i = \{j, j'\}$, from $\{j, l\} \prec \{j, j'\}$ and

219 $\{j', l\} \prec \{j, j'\}$, by compatibility $l \prec j$ and $l \prec j'$. Generalizing the above, we
 220 conclude that $l \prec j$ for all $j \in B_i$.

221 Next, if $l \notin \check{B}$, it means that for some $B' \subseteq \check{B}$ such that $B' \cup l \in \mathcal{P}_*^k(N)$, we
 222 have $B' \cup l \succ B$ (otherwise l should belong to \check{B}). We prove that $B' \not\subseteq B_i$. The
 223 case $|B_i| = k$ is obvious, let us consider $|B_i| < k$. Suppose on the contrary that
 224 $B' = B_i \cup L$, $L \subseteq N \setminus B_i$. Then $B_i \cup l \prec B_i$ implies that $B' \cup l = B_i \cup l \cup L \prec$
 225 $B_i \cup L \prec B$, the last inequality coming from $B_i \cup L \subseteq \check{B}$, $B_i \cup L \in \mathcal{P}_*^k(N)$.
 226 But $B' \cup l \succ B$, a contradiction.

227 Choose any $j \in B_i \setminus B'$. Since $j \succ l$, we deduce $B' \cup j \succ B' \cup l \succ B$, but since
 228 $B' \cup j \subseteq \check{B}$ and $B' \cup j \in \mathcal{P}_*^k(N)$, it follows that $B' \cup j \prec B$, a contradiction.

229 ■

230

231 **Proposition 8** *Let \prec be a compatible order on $\mathcal{P}_*^k(N)$. For any $B \in \mathcal{P}_*^k(N)$
 232 such that $\mathcal{A}(B)$ is nonempty, putting $\check{B} := \{i_1, \dots, i_l\}$ with $i_1 \prec \dots \prec i_l$, then
 233 necessarily there exists $j \in \{1, \dots, l\}$ such that $B = \{i_j, \dots, i_l\}$.*

234 **Proof:** Assume $\check{B} \neq B$, otherwise we have simply $j = 1$. Consider i_j the
 235 element in B with the lowest index in the list $\{1, \dots, l\}$. Let us prove that all
 236 successors i_{j+1}, \dots, i_l are also in B . Assume $j < l$ (otherwise we are done),
 237 and suppose that $i_{j'} \notin B$ for some $j < j' \leq l$. Then by compatibility, $B =$
 238 $(B \setminus i_j) \cup i_j \prec (B \setminus i_j) \cup i_{j'}$. Since $(B \setminus i_j) \cup i_{j'} \subseteq \check{B}$ and $(B \setminus i_j) \cup i_{j'} \in \mathcal{P}_*^k(N)$,
 239 the converse inequality should hold. ■

240

241 **Proposition 9** *Assume that \prec is strongly compatible. Then for all $B \subseteq N$,
 242 $|B| < k$, $\check{B} = B$.*

243 **Proof:** By Prop. 6 and Cor. 2, we know that $\mathcal{A}(B)$ is a Boolean lattice with
 244 top element denoted by \check{B} . Suppose that $\check{B} \neq B$. Then there exists $i \in \check{B} \setminus B$,
 245 and $B \cup i \in \mathcal{A}(B)$. Remark that $|B \cup i| \leq k$ and $\mathcal{A}(B \cup i) \ni B \cup i$ by Prop. 6
 246 and Cor. 2 again. This contradicts the fact that the achievable families form
 247 a partition of $\mathcal{P}_*^k(N)$ (Prop. 3). ■

248

Proposition 10 *Let \prec be a strongly compatible order on $\mathcal{P}_*^k(N)$, and assume
 w.l.o.g. that $1 \prec 2 \prec \dots \prec n$. Then the collection $\check{\mathcal{B}}$ of \check{B} 's is given by:*

$$\check{\mathcal{B}} = \left\{ \{1, 2, \dots, l\} \cup \{j_1, \dots, j_{k-1}\} \mid l = 1, \dots, n - k + 1 \right. \\ \left. \text{and } \{j_1, \dots, j_{k-1}\} \subseteq \{l + 1, \dots, n\} \right\} \cup \left\{ A \subseteq N \mid |A| < k \right\}.$$

249 *If \prec is compatible, then $\check{\mathcal{B}}$ is a subcollection of the above, where some subsets
 250 of at most $k - 1$ elements may be absent.*

251 **Proof:** From Prop. 9, we know that $\check{\mathcal{B}}$ contains all subsets having less than
 252 k elements. This proves the right part of “ \cup ” in $\check{\mathcal{B}}$. By Prop. 9 again, the left
 253 part uniquely comes from those B 's of exactly k elements. Take such a B . From
 254 Prop. 8, we know that \check{B} cannot contain elements ranked after the last one
 255 of B in the sequence $1, 2, \dots, n$. In other words, letting $B := \{l, j_1, \dots, j_{k-1}\}$,
 256 with l the lowest ranked element, we know that $\check{B} = B' \cup \{l, j_1, \dots, j_{k-1}\}$,
 257 with all elements of B' ranked before l . It remains to show that necessarily
 258 B' contains all elements from 1 to l excluded. Assume $j \notin B'$, $1 \leq j < l$.
 259 Then it should exist $K \in \mathcal{P}_*^k(N)$, $j \in K \subseteq \check{B} \cup j$, such that $K \succ B$. Since
 260 $|B| = k$, it cannot be that $K \supseteq B$, so that say $j' \in B$ is not in K . Hence
 261 we have $j \prec j'$, and by compatibility, $K = (K \setminus j) \cup j \prec (K \setminus j) \cup j'$. Now,
 262 $(K \setminus j) \cup j' \in \mathcal{P}_*^k(N)$ and $(K \setminus j) \cup j' \subseteq \check{B}$, which entails $(K \setminus j) \cup j' \prec B$, so
 263 that $K \prec B$, a contradiction.

264 Finally, consider that \prec is only compatible. Then by Cor. 2, there exists $B \in$
265 $\mathcal{P}_*^k(N)$ such that $\mathcal{A}(B) = \emptyset$. This implies that there exist some proper subsets
266 of B in $\mathcal{P}_*^k(N)$ ranked after B , let us call K the last ranked such subset. Then
267 $|K| < k$, and $\mathcal{A}(K) \neq \{K\}$ since it contains at least B , because all subsets of
268 B are ranked before K by definition of K . Hence K does not belong to $\check{\mathcal{B}}$. ■

269

We finish this section by explaining why achievable families induced by orders on $\mathcal{P}_*^k(N)$ are generalizations of maximal chains induced by permutations. Taking $k = 1$, $\mathcal{P}_*^1(N) = N$, and total orders on singletons coincide with permutations on N . Trivially, any order on N is strongly compatible, so that all achievable families are nonempty lattices. Denoting by σ the permutation corresponding to \prec , i.e., $\sigma(1) \prec \sigma(2) \prec \dots \prec \sigma(n)$, then

$$\mathcal{A}(\{\check{\sigma}(j)\}) = [\{\sigma(j)\}, \{\sigma(1), \dots, \sigma(j)\}],$$

270 i.e., the top element $\{\check{\sigma}(j)\}$ is $\{\sigma(1), \dots, \sigma(j)\}$. Then the collection of all top
271 elements $\{\check{\sigma}(j)\}$ is exactly the maximal chain associated to σ .

272 4 Vertices of $\mathcal{C}^k(v)$ induced by achievable families

273 Let us consider a game v and its k -additive core $\mathcal{C}^k(v)$. We suppose hereafter
274 that $\mathcal{C}^k(v) \neq \emptyset$, which is always true for a sufficiently high k . Indeed, taking

275 at worst $k = n$, $v \in \mathcal{C}^n(v)$ always holds.

276 4.1 General facts

A k -additive game v^* with Möbius transform m^* belongs to $\mathcal{C}^k(v)$ if and only if it satisfies the system

$$\sum_{\substack{K \subseteq A \\ |K| \leq k}} m^*(K) \geq \sum_{K \subseteq A} m(K), \quad A \in 2^N \setminus \{\emptyset, N\} \quad (2)$$

$$\sum_{\substack{K \subseteq N \\ |K| \leq k}} m^*(K) = v(N). \quad (3)$$

277 The number of variables is $N(k) := \binom{n}{1} + \dots + \binom{n}{k}$, but due to (3), this
 278 gives rise to a $(N(k) - 1)$ -dim closed polyhedron. (2) is a system of $2^n - 2$
 279 inequalities. The polyhedron is convex since the convex combination of any
 280 two elements of the core is still in the core, but it is not bounded in general.
 281 To see this, consider the simple following example.

EXAMPLE 5: Consider $n = 3$, $k = 2$, and a game v defined by its Möbius transform m with $m(i) = 0.1$, $m(ij) = 0.2$ for all $i, j \in N$, and $m(N) = 0.1$. Then the system of inequalities defining the 2-additive core is:

$$m^*(1) \geq 0.1$$

$$m^*(2) \geq 0.1$$

$$m^*(3) \geq 0.1$$

$$m^*(1) + m^*(2) + m^*(12) \geq 0.4$$

$$m^*(1) + m^*(3) + m^*(13) \geq 0.4$$

$$m^*(2) + m^*(3) + m^*(23) \geq 0.4$$

$$m^*(1) + m^*(2) + m^*(3) + m^*(12) + m^*(13) + m^*(23) = 1.$$

282 Let us write for convenience $m^* := (m^*(1), m^*(2), m^*(3), m^*(12), m^*(13), m^*(23))$.

283 Clearly $m_0^* := (0.2, 0.1, 0.1, 0.2, 0.2, 0.2)$ is a solution, as well as

284 $m_0^* + t(1, 0, 0, -1, 0, 0)$ for any $t \geq 0$. Hence $(1, 0, 0, -1, 0, 0)$ is
 285 a ray, and the core is unbounded.

For the monotone core, from Prop. 1 (i) there is an additional system of $n2^{n-1}$ inequalities

$$\sum_{\substack{K \in [i, L] \\ |K| \leq k}} m^*(K) \geq 0, \quad \forall i \in N, \forall L \ni i. \quad (4)$$

For monotone games, Miranda and Grabisch [14] have proved that the Möbius transform is bounded as follows:

$$-\binom{|A| - 1}{l'_{|A|}} v(N) \leq m(A) \leq \binom{|A| - 1}{l_{|A|}} v(N), \quad \forall A \subseteq N,$$

286 where $l_{|A|}, l'_{|A|}$ are given by:

- 287 (i) $l_{|A|} = \frac{|A|}{2}$, and $l'_{|A|} = \frac{|A|}{2} - 1$ if $|A| \equiv 0 \pmod{4}$
 288 (ii) $l_{|A|} = \frac{|A| - 1}{2}$, and $l'_{|A|} = \frac{|A| - 3}{2}$ or $l'_{|A|} = \frac{|A| + 1}{2}$ if $|A| \equiv 1 \pmod{4}$
 289 (iii) $l_{|A|} = \frac{|A|}{2} - 1$, and $l'_{|A|} = \frac{|A|}{2}$ if $|A| \equiv 2 \pmod{4}$
 290 (iv) $l_{|A|} = \frac{|A| - 3}{2}$ or $l_{|A|} = \frac{|A| + 1}{2}$, and $l'_{|A|} = \frac{|A| - 1}{2}$ if $|A| \equiv 3 \pmod{4}$.

291 Since $v(N)$ is fixed and bounded, the monotone k -additive core is always
 292 bounded.

For $\mathcal{C}_\infty^k(v)$, using Prop. 1 (ii) system (4) is replaced by a system of $N(k) - n$ inequalities:

$$m^*(K) \geq 0, \quad K \in \mathcal{P}_*^k(N), |K| > 1. \quad (5)$$

293 Since in addition we have $m^*(\{i\}) \geq m(\{i\})$, $i \in N$ coming from (2), m^* is
 294 bounded from below. Then (3) forces m^* to be bounded from above, so that
 295 $\mathcal{C}_\infty^k(v)$ is bounded.

296 In summary, we have the following.

297 **Proposition 11** *For any game v , $\mathcal{C}^k(v)$, $\mathcal{MC}^k(v)$ and $\mathcal{C}_\infty^k(v)$ are closed convex*
 298 *$(N(k) - 1)$ -dimensional polyhedra. Only $\mathcal{MC}^k(v)$ and $\mathcal{C}_\infty^k(v)$ are always bounded.*

299 The following result about rays of $\mathcal{C}^k(v)$ is worthwhile to be noted.

300 **Proposition 12** *The components of rays of $\mathcal{C}^k(v)$ do not depend on v , but*
 301 *only on k and n .*

302 **Proof:** For any polyhedron defined by a system of m inequalities and n
 303 variables (including slack variables) $\mathbf{Ax} = \mathbf{b}$, it is well known that its conical
 304 part is given by $\mathbf{Ax} = \mathbf{0}$, and that rays (also called basic feasible directions)
 305 are particular solutions of the latter system with $n - m$ non basic components
 306 all equal to zero but one (see, e.g., [6]). Hence, components of rays do not
 307 depend on \mathbf{b} .

308 Applied to our case, this means that components of rays do not depend on v ,
 309 but only on k and n . ■

310

311 4.2 A Shapley-Ichiishi-like result

We turn now to the characterization of vertices induced by achievable families.
 Let v be a game on N , m its Möbius transform, and \prec be a total order on
 $\mathcal{P}_*^k(N)$. We define a k -additive game v_{\prec} by its Möbius transform as follows:

$$m_{\prec}(B) := \begin{cases} \sum_{A \in \mathcal{A}(B)} m(A), & \text{if } \mathcal{A}(B) \neq \emptyset \\ 0, & \text{else} \end{cases} \quad (6)$$

312 for all $B \in \mathcal{P}_*^k(N)$, and $m_{\prec}(B) := 0$ if $B \notin \mathcal{P}_*^k(N)$.

313 Due to Prop. 3, m_{\prec} satisfies $\sum_{B \subseteq N} m_{\prec}(B) = \sum_{B \subseteq N} m(B) = v(N)$, hence
 314 $v_{\prec}(N) = v(N)$.

315 This definition is a generalization of the definition of ϕ^{σ} or ϕ^C (see Sec. 2).

316 Indeed, denoting by σ the permutation on N corresponding to \prec , we get:

$$\begin{aligned}
m_{\prec}(\{\sigma(i)\}) &= \sum_{A \subseteq \{\sigma(1), \dots, \sigma(i-1)\}} m(A \cup \sigma(i)) \\
&= \sum_{A \subseteq \{\sigma(1), \dots, \sigma(i)\}} m(A) - \sum_{A \subseteq \{\sigma(1), \dots, \sigma(i-1)\}} m(A) \\
&= v(\{\sigma(1), \dots, \sigma(i)\}) - v(\{\sigma(1), \dots, \sigma(i-1)\}) = \phi^\sigma(\{\sigma(i)\}) = m^\sigma(\{\sigma(i)\}),
\end{aligned}$$

317 where m^σ is the Möbius transform of ϕ^σ (see Sec. 2).

318 **Proposition 13** *Assume that $\mathcal{A}(B)$ is a nonempty lattice. Then $v_{\prec}(\check{B}) =$
319 $v(\check{B})$ if and only if $\{\mathcal{A}(C) \mid C \in \mathcal{P}_*^k(N), C \subseteq \check{B}, \mathcal{A}(C) \neq \emptyset\}$ is a partition of
320 $\mathcal{P}(\check{B}) \setminus \{\emptyset\}$.*

Proof: We have by Eq. (6)

$$v_{\prec}(\check{B}) = \sum_{\substack{C \subseteq \check{B} \\ C \in \mathcal{P}_*^k(N) \\ \mathcal{A}(C) \neq \emptyset}} m_{\prec}(C) = \sum_{\substack{C \subseteq \check{B} \\ C \in \mathcal{P}_*^k(N) \\ \mathcal{A}(C) \neq \emptyset}} \sum_{K \in \mathcal{A}(C)} m(K). \quad (7)$$

321 On the other hand, $v(\check{B}) = \sum_{K \subseteq \check{B}} m(K)$. To ensure $v_{\prec}(\check{B}) = v(\check{B})$ for any
322 v , every $K \subseteq \check{B}$ must appear exactly once in the last sum of (7), which is
323 equivalent to the desired condition. ■

324

325 The following is immediate from Prop. 13 and 7.

326 **Corollary 3** *Assume \prec is compatible, and consider a nonempty achievable*
327 *family $\mathcal{A}(B)$. Then $v_{\prec}(\check{B}) = v(\check{B})$.*

328 **Proposition 14** *Let us suppose that all nonempty achievable families are lat-*
329 *tices. Then v k -monotone implies that v_{\prec} is infinitely monotone.*

Proof: It remains to show that $m_{\prec}(B) \geq 0$ for any B such that $1 < |B| \leq k$.
For all such B satisfying $\mathcal{A}(B) \neq \emptyset$,

$$m_{\prec}(B) = \sum_{A \in \mathcal{A}(B)} m(A) = \sum_{A \in [B, \check{B}]} m(A).$$

330 Since $1 < |B| \leq k$, by Prop. 1, it follows from k -monotonicity that $m_{\prec}(B) \geq 0$
 331 for all $B \in \mathcal{P}_*^k(N)$. ■

332

333 The next corollary follows from Prop. 6.

334 **Corollary 4** *Let us suppose that \prec is compatible. Then v k -monotone implies*
 335 *that v_{\prec} is infinitely monotone.*

336 **Theorem 2** *v is $(k+1)$ -monotone if and only if for all compatible orders \prec ,*
 337 *$v_{\prec}(A) \geq v(A)$, $\forall A \subseteq N$.*

Proof: For any compatible order \prec , and any $A \subseteq N$, $A \neq \emptyset$, by compatibility and Prop. 6, we can write

$$v_{\prec}(A) = \sum_{\substack{B \subseteq A \\ B \in \mathcal{P}_*^k(N) \\ \mathcal{A}(B) \neq \emptyset}} \sum_{C \in [B, \check{B}]} m(C). \quad (8)$$

Let $C \subseteq A$. Then by Prop. 3, $C \in \mathcal{A}(B)$ for some $B \subseteq A$. Indeed $B \subseteq C \subseteq A$. Hence (8) writes

$$v_{\prec}(A) = v(A) + \sum_{\substack{B \subseteq A \\ B \in \mathcal{P}_*^k(N) \\ \mathcal{A}(B) \neq \emptyset}} \sum_{\substack{C \in [B, \check{B}] \\ C \not\subseteq A}} m(C). \quad (9)$$

338

(\Rightarrow) Let us take any compatible order \prec . By (9), it suffices to show that

$$\sum_{\substack{C \in [B, \check{B}] \\ C \not\subseteq A}} m(C) \geq 0, \quad \forall B \subseteq A, B \in \mathcal{P}_*^k(N), \mathcal{A}(B) \neq \emptyset. \quad (10)$$

339 For simplicity define \mathcal{C} as the set of subsets C satisfying the condition in the
 340 summation in (10). If $\check{B} \subseteq A$, then $\mathcal{C} = \emptyset$, and so (10) holds for such B 's.
 341 Assume then that $\check{B} \setminus A \neq \emptyset$. Let us take $i \in \check{B} \setminus A$. Then $C_0 := B \cup i$ is
 342 a minimal element of \mathcal{C} , of cardinality $1 < |B| + 1 \leq k + 1$. Observe that

343 $[C_0, \check{B}] \subseteq \mathcal{C}$, and that it is a Boolean sublattice of $[B, \check{B}]$. Hence, $(k+1)$ -
 344 monotonicity implies that $\sum_{C \in [C_0, \check{B}]} m(C) \geq 0$ (see Prop. 1).

345 Consider $j \in \check{B} \setminus A$, $j \neq i$. If no such j exists, then $[C_0, \check{B}] = \mathcal{C}$, and we
 346 have shown (10) for such B 's. Otherwise, define $C_1 := B \cup j$ and the interval
 347 $[C_1, \check{B} \setminus i]$, which is disjoint from $[C_0, \check{B}]$. Applying again $(k+1)$ -monotonicity
 348 we deduce that $\sum_{D \in [C_1, \check{B}]} m(D) \geq 0$. Continuing this process until all elements
 349 of $\check{B} \setminus A$ have been taken, the set \mathcal{C} has been partitioned into intervals $[B \cup$
 350 $i, \check{B}]$, $[B \cup j, \check{B} \setminus i]$, $[B \cup k, \check{B} \setminus \{i, j\}]$, \dots , $[B \cup l, A \cup l]$ where the sum of $m(C)$
 351 over these intervals is non negative by $(k+1)$ -monotonicity. Hence (10) holds
 352 in any case and the sufficiency is proved.

353 (\Leftarrow) Consider $K, L \subseteq N$ such that $1 < |K| \leq k+1$ and $L \supseteq K$. We have to
 354 prove that $\sum_{C \in [K, L]} m(C) \geq 0$. Without loss of generality, let us assume for
 355 simplicity that $K := \{i, i+1, \dots, l\}$ and $L := \{1, \dots, l\}$, with $l-k \leq i < l \leq n$.
 356 Define $B := K \setminus i = \{i+1, \dots, l\}$ and $A := L \setminus i$. Take a total order on $\mathcal{P}_*^k(N)$
 357 as follows:

- 358 (i) put first all subsets in $\mathcal{P}_*^k(L)$, with increasing cardinality, except B which
 359 is put the last
- 360 (ii) then put remaining subsets in $\mathcal{P}_*^k(N)$ such that they form a compatible
 361 order (for example: consider the above fixed sequence in $\mathcal{P}_*^k(L)$ augmented
 362 with the empty set as first element of the sequence, then take any subset
 363 D in $N \setminus L$ belonging to $\mathcal{P}_*^k(N)$, and add it to any subset of the sequence,
 364 discarding subsets not in $\mathcal{P}_*^k(N)$. Do this for any subset D of $N \setminus L$).
- 365 (iii) subsets in $\mathcal{P}_*^k(L)$ with same cardinality are ordered according to the lex-
 366 icographic order, which means in particular $1 \prec 2 \prec \dots \prec l$.

¹ For example, with $n = 5, l = 4, i = 3, k = 3$:

$1 \prec 2 \prec 3 \prec 12 \prec 13 \prec 14 \prec 23 \prec 24 \prec 34 \prec 123 \prec 124 \prec 134 \prec 234 \prec 4 \prec 5 \prec 51 \prec 52 \dots$

367 One can check that such an order is compatible¹. By construction, we have
368 $\mathcal{A}(B) = [B, L]$. Indeed, for any $C \in \mathcal{A}(B)$, any subset of C in $\mathcal{P}_*^k(N)$ is ranked
369 before B . Moreover, $[K, L] = [B \cup i, L] = \{C \in \mathcal{A}(B) \mid C \not\subseteq A\}$. Now, take
370 any $B' \neq B$ in $\mathcal{P}_*^k(L)$ such that $B' \subseteq A$. Let us prove that any $C \in \mathcal{A}(B')$
371 is such that $i \notin C$, or equivalently $C \subseteq A$. Indeed, up to the fact that B is
372 ranked last, the sequence $\mathcal{P}_*^k(L)$ forms a strongly compatible order. Adapting
373 slightly Prop. 9, it is easy to see that if $|B'| < k$, then either $\check{B}' = B'$ or
374 $\mathcal{A}(B') = \emptyset$, the latter arising if $B' \supset B$. Then trivially any $C \in \mathcal{A}(B')$ satisfies
375 $C \subseteq A$. Assume now $|B'| = k$. If B' contains some $j \prec i$, then $B' \cup i$ cannot
376 belong to $\mathcal{A}(B')$ since by lexicographic ordering $B' \cup i \setminus j$ is ranked after B' ,
377 which implies that for any $C \in \mathcal{A}(B')$, $i \notin C$. Hence, the condition $i \in C$ can
378 be true for some $C \in \mathcal{A}(B')$ only if all elements of B' are ranked after i . But
379 since $B = \{i + 1, \dots, l\}$, this implies that either $B' = B$, a contradiction, or
380 B' does not exist (if $|B| < k$).

Let us apply the dominance condition for $v_{\prec}(A)$. Using (9), dominance is
equivalent to write:

$$\sum_{\substack{B \subseteq A \\ B \in \mathcal{P}_*^k(N) \\ \mathcal{A}(B) \neq \emptyset}} \sum_{\substack{C \in [B, \check{B}] \\ C \not\subseteq A}} m(C) \geq 0.$$

381 Using the above, this sum reduces to $\sum_{C \in [K, L]} m(C) \geq 0$. This finishes the
382 proof. ■

383

384 The following is an interesting property of the system $\{(2), (3)\}$.

385 **Proposition 15** *Let \prec be a compatible order. Then the linear system of equal-*
386 *ities $v_{\prec}(\check{B}) = v(\check{B})$, for all \check{B} 's induced by \prec , is triangular with no zero on*
387 *the diagonal, and hence has a unique solution.*

388 **Proof:** We consider w.l.o.g. that $1 \prec 2 \prec \dots \prec n$ and consider the binary
389 order \prec^2 for ordering variables $m^*(B)$.

390 Delete all variables such that $\mathcal{A}(B) = \emptyset$, and consider the list of subsets
391 in $\mathcal{P}_*^k(N)$ corresponding to non deleted variables. Take all B 's in the list,
392 and their corresponding \check{B} 's (always exist since by compatibility, $\mathcal{A}(B)$ is a
393 lattice). They are all different by Prop. 3, so we get a linear system of the
394 same number of equations (namely $v_{\prec}(\check{B}) = v(\check{B})$) and variables. Take one
395 particular equation corresponding to B . Then variables used in this equation
396 are necessarily $m^*(B)$ itself (because $\check{B} \supseteq B$), and some variables ranked
397 before B in the binary order. Indeed, if $\check{B} = B$, then all variables used in the
398 equation are ranked before B by \prec^2 . If $\check{B} \neq B$, supersets B' of B in $\mathcal{P}_*^k(N)$
399 are ranked after B by \prec^2 (because \prec^2 is \subseteq -compatible), and ranked before
400 B by \prec (otherwise $\mathcal{A}(B)$ would not contain \check{B}), but since they contain B ,
401 necessarily $\mathcal{A}(B') = \emptyset$, so corresponding variables are deleted.

402 Hence the system is triangular. ■

403

404 Note that the proof holds under the condition that all achievable families are
405 lattices, so compatibility is even not necessary.

406 **Theorem 3** *Let v be a $(k + 1)$ -monotone game. Then*

407 (i) *If \prec is strongly compatible, then v_{\prec} is a vertex of $\mathcal{C}^k(v)$.*

408 (ii) *If \prec is compatible, then v_{\prec} is a vertex of $\mathcal{C}_{\infty}^k(v)$.*

409 **Proof:** By standard results on polyhedra, it suffices to show that v_{\prec} is an
410 element of $\mathcal{C}^k(v)$ (resp. $\mathcal{C}_{\infty}^k(v)$) satisfying at least $N(k) - 1$ linearly independent
411 equalities among (2) (resp. among (2) and (5)). Assume \prec is compatible. Then
412 by Cor. 4, v_{\prec} is infinitely monotone, and it dominates v by $(k+1)$ -monotonicity
413 (Th. 2). Moreover, for any $B \in \mathcal{P}_*^k(N)$, $\mathcal{A}(B)$ is either empty or a lattice, hence
414 either $m_{\prec}(B) = 0$ or $v_{\prec}(\check{B}) = v(\check{B})$ by Cor. 3. Since if $|B| = 1$, $\mathcal{A}(B) \neq \emptyset$,
415 this gives $N(k)$ equalities in the system defining $\mathcal{C}_{\infty}^k(v)$, including (3), hence

416 we have the exact number of equalities required, which form a nonsingular
 417 system by Prop 15, and (ii) is proved. If the order is strongly compatible, then
 418 all achievable families are lattices, which proves the result for $\mathcal{C}^k(v)$, since
 419 again by Prop. 15, the system is nonsingular. ■

420

421 **REMARK 1:** Vertices induced by (strongly) compatible orders are also ver-
 422 tices of the monotone k -additive core. They are induced only by dominance
 423 constraints, not by monotonicity constraints.

424 **REMARK 2:** Cor. 3 generalizes Prop. 2, while Theorems 2 and 3 generalize
 425 the Shapley-Ichiishi results summarized in Th. 1. Indeed, recall that con-
 426 vexity is 2-monotonicity. Then clearly Th. 2 is a generalization of (i) \Rightarrow (ii)
 427 of Th. 1, and Th. 3 (i) is a part of (iv) in Th. 1. But as it will become clear
 428 below, all vertices are not recovered by achievable families, mainly because
 429 they can induce only infinitely monotone games. In particular, $\mathcal{MC}^k(v)$ con-
 430 tains many more vertices.

431 Let us examine more precisely the number of vertices induced by strongly
 432 compatible orders. In fact, there are much fewer than expected, since many
 433 strongly compatible orders lead to the same v_{\prec} . The following is a consequence
 434 of Prop. 10.

435 **Corollary 5** *The number of vertices of $\mathcal{C}^k(v)$ given by strongly compatible*
 436 *orders is at most $\frac{n!}{k!}$.*

437 **Proof:** Given the order $1 \prec 2 \prec \dots \prec n$, a permutation over the last k
 438 singletons would not change the collection $\check{\mathcal{B}}$. ■

439

440 Note that when $k = 1$, we recover the fact that vertices are induced by all
 441 permutations, and that with $k = n$, we find only one vertex (which is in fact

442 the only vertex of $\mathcal{C}^n(v)$, which is v itself (use Prop. 10 and the definition of
 443 m_{\prec}).

444 4.3 Other vertices

445 In this last section we give some insights about other vertices. Even for the
 446 (non monotonic) k -additive core, in general for $k \neq 1, n$, not all vertices are
 447 induced by strongly compatible orders. However, for the case $k = n - 1$, it
 448 is possible to find all vertices of $\mathcal{C}^k(v)$. For $1 < k < n - 1$ and also for the
 449 monotonic core, the problem becomes highly combinatorial.

450 **Theorem 4** *Let v be any game in $\mathcal{G}(N)$, with Möbius transform m .*

(i) *If $m(N) > 0$, $\mathcal{C}^{n-1}(v)$ contains exactly 2^{n-1} (if n is even) or $2^{n-1} - 1$ (if
 n is odd) vertices, among which n vertices come from strongly compatible
 orders. They are given by their Möbius transform:*

$$m_{B_0}^*(K) = \begin{cases} m(K), & \text{if } K \not\supseteq B_0 \\ m(K) + (-1)^{|K \setminus B_0|} m(N), & \text{else} \end{cases}$$

451 *for all $B_0 \subset N$ such that $|N \setminus B_0|$ is odd.*

452 (ii) *If $m(N) = 0$, then there is only one vertex, which is v itself.*

(iii) *If $m(N) < 0$, $\mathcal{C}^{n-1}(v)$ contains exactly $2^{n-1} - 1$ (if n is odd) or $2^{n-1} - 2$
 (if n is even) vertices, of which none comes from a strongly compatible
 order. They are given by their Möbius transform:*

$$m_{B_0}^*(K) = \begin{cases} m(K), & \text{if } K \not\supseteq B_0 \\ m(K) - (-1)^{|K \setminus B_0|} m(N), & \text{else} \end{cases}$$

453 *for all $B_0 \subset N$ such that $|N \setminus B_0|$ is even.*

Proof: We assume $m(N) \geq 0$ (the proof is much the same for the case
 $m(N) \leq 0$). We consider the system of $2^n - 1$ inequalities $\{(2),(3)\}$, which has

$N(n-1) = 2^n - 2$ variables. We have to fix $2^n - 2$ equalities, among which (3), so we have to choose only one inequality in (2) to remain strict, say for $B_0 \subset N$, $B_0 \neq \emptyset$:

$$\sum_{K \subseteq B_0} m^*(K) > \sum_{K \subseteq B_0} m(K). \quad (11)$$

From the definition of the Möbius transform, we have

$$0 = m^*(N) = \sum_{K \subseteq N} (-1)^{|N \setminus K|} v^*(K).$$

454 Note that for any $\emptyset \neq K \subseteq N$, $v^*(K)$ is the left member of some inequality
 455 or equality of the system. Hence, by turning all inequalities into equalities,
 456 we get, by doing the above summation on the system, $0 = m(N)$. Hence, if
 457 $m(N) = 0$, there is only one vertex, which is v itself, otherwise taking equality
 458 everywhere gives a system with no solution. Since strict inequality holds only
 459 for $B_0 \subset N$, we get instead $0 > m(N)$ if $|N \setminus B_0|$ is even, and $0 < m(N)$ if
 460 $|N \setminus B_0|$ is odd. The first case is impossible by assumption on $m(N)$, so only
 461 the case where $|N \setminus B_0|$ odd can produce a vertex. Note that if $|B_0| = n - 1$,
 462 we recover all n vertices induced by strongly compatible orders. In total we
 463 get $\binom{n}{n-1} + \binom{n}{n-3} + \dots + \binom{n}{1} = 2^{n-1}$ potential different vertices when n is even,
 464 and $2^{n-1} - 1$ when n is odd. Clearly, there is no other possibility.

It remains to show that the corresponding system of equalities is non singular, and eventually to solve it. Assume $B_0 \subset N$ in (11) is chosen. From the linear system of equalities we easily deduce $m^*(K) = m(K)$ for all $K \not\supseteq B_0$. Substituting into all equations, the system reduces to

$$\begin{aligned} \sum_{K \subseteq B \setminus B_0} m^*(B_0 \cup K) &= \sum_{K \subseteq B \setminus B_0} m(B_0 \cup K), \quad \forall B \supset B_0, B \neq N \\ \sum_{K \subset N \setminus B_0} m^*(B_0 \cup K) &= \sum_{K \subset N \setminus B_0} m(B_0 \cup K) + m(N). \end{aligned}$$

B_0 being present everywhere, we may rename all variables after deleting B_0 , i.e., we set $N' = N \setminus B_0$, $m'(A) := m(A \cup B_0)$ and $m'^*(A) = m^*(A \cup B_0)$, for

all $A \subseteq N'$. The system becomes

$$\sum_{K \subseteq B} m'^*(K) = \sum_{K \subseteq B} m'(K), \quad \forall B \subset N'$$

$$\sum_{K \subset N'} m'^*(K) = \sum_{K \subset N'} m'(K) + m'(N').$$

465 Summing equations of the system as above, i.e., computing $\sum_{B \subseteq N'} (-1)^{|N' \setminus B|} \sum_{K \subseteq B} m'^*(K)$,
 466 we get $m'^*(\emptyset) = m'(\emptyset) + m'(N')$, or equivalently $m^*(B_0) = m(B_0) + m(N)$.
 467 Substituting in the above system, we get a system which is triangular (use, e.g.,
 468 Prop. 15 with $k = n = n'$). We get easily $m^*(K) = m(K) + (-1)^{|K \setminus B_0|} m(N)$,
 469 for all $K \supseteq B_0$. ■

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