Folding optimal polygons from squares

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What is the largest regular n-gon that fits in a unit square? Can it be folded from a square piece of paper using standard moves from origami? Answering the first question is relatively easy, using simple ideas from geometry. The second is more interesting; our answer illustrates the difference between origami and the standard compass-and-straightedge constructions of the Greeks, where, for instance, the 7-gon cannot be constructed. Not only can we fold a 7-gon, but we can fold the largest one possible from a given square piece of paper.

Origami (from japanese *oru*, to fold and *kami*, paper), is the ancient art of paperfolding. When we fold a paper in half, we create a line segment and bisect a length. These simple moves can be combined to reproduce *any* compass-and-straightedge construction [1, 14]. Thus, by origami, as with an unmarked straightedge and compass, we can construct roots of any second-order polynomial from a given unit length.

However, many constructions known to be impossible under the standard Greek rules, such as trisecting a given angle, become possible with origami. For instance, using a construction technique due to Lill and first used for origami by M. P. Beloch [2], we can construct roots of cubic polynomials by folding [1, 5, 7, 12]. Origami also simplifies certain constructions that are possible, but cumbersome, with compass and straightedge.

Since origami often begins with a square piece of paper, we propose not only to fold a regular n-gon, but to fold the one with the largest area that fits in the square. Such polygons will be called *optimal* polygons. For instance, the side of the largest equilateral triangle that fits in a unit square is known to have length $\sqrt{6} - \sqrt{2} \approx 1.035$ (shown in Figure 4). Wetzel [18] takes this as the starting point for his article "Fits and Covers," which gives many answers to similar problems, but does not address our question.

Our first step is to determine the proper orientation of an optimal polygon with respect to the square. We do this in complete generality and then consider how to construct them by folding. We show how to fold the optimal hexagon and pentagon, which can also be constructed with the standard compass and straightedge. Moving into the realm of techniques that break the Greek rules, we trisect an angle and show how to fold the optimal 7-gon and 9-gon, neither of which can be constructed with straightedge and compass alone. It turns out that in each case we fold a stellated polygon as an intermediate step.

Are you eager to fold the optimal 11-gon? If so, you will have to invent a folding technique that permits you to construct roots of a quintic polynomial!

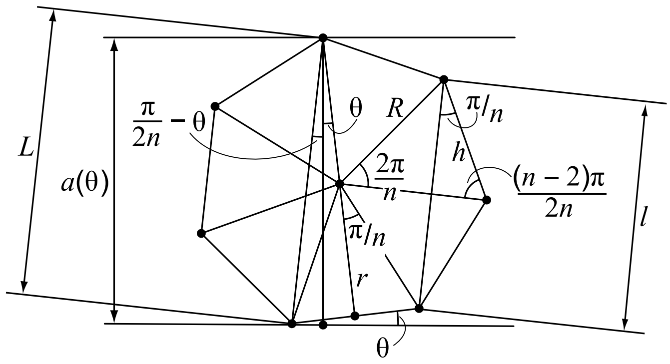

Figure 1: The width of a strip that contains a tilted polygon

Facts about optimal polygons

The goal is to find the largest regular n-gon that can be folded from a square piece of paper, for $n \geq 3$. Of course, the case $n = 4$ is trivial, with no folding required. For the general case, let us review some facts about the regular n-gon.

Let R be the radius of the circumscribed circle and r the radius of the inscribed circle. (Another name for r is the *apothem* of the polygon.) The reader may wish to confirm that $r = R\cos(\pi/n)$. The *diameter* of the n-gon, denoted L, is the maximum distance between any two of its points. A contrasting quantity is the *altitude*, meaning the shortest perpendicular distance from a vertex to an opposite side. When n is even, these quantities are simple, the diameter is just $2R$ and the altitude is $2r$. When n is odd, the altitude is rather easily seen to be $R+r$, which is $2R\cos^2(\pi/(2n))$. The diameter can then be found to be $2R\cos(\pi/(2n))$. (Figure 1 will help with this.)

Another useful quantity is the side of a stellated polygon, l. Figure 1 shows it to be $2R\sin(2\pi/n)$. The side of the polygon is $h = 2R\sin(\pi/n)$.

Before fitting our n-gon into a square, we first fit it into a strip. It turns out to be simplest to consider the n-gon to have a fixed radius R and find out how wide the strip must be to contain it. Depending on the orientation of the n-gon, the necessary width will fall somewhere between the diameter and the altitude. Since the altitude is smaller, we might decide that it is best to orient the polygon with its altitude along one dimension of the square. Unfortunately, the n-gon is always fatter in the perpendicular direction.

Therefore, let us find the narrowest strip of paper that can contain a given polygon when it is tilted with respect to the strip at an angle θ. For a *fixed* rotation angle θ, the minimum strip width is denoted $a(\theta)$. The odd case is shown in Figure 1.

Aided by Figure 1, which shows the odd case, the reader can verify the following formula for $a(\theta)$, given in terms of the number n of edges (or vertices) of the polygon, and the radius R of the circumscribed circle:

- if n is odd, $a(\theta)$ is (π/n)-periodic, and $a(\theta) = L\cos(\theta - \pi/(2n))$ for $\theta \in [0, \pi/n]$. Here, $L = 2R\cos(\pi/(2n))$ is the diameter of the polygon;
- if n is even, $a(\theta)$ is $(2\pi/n)$-periodic, and $a(\theta) = L\cos(\theta - \pi/n)$ for $\theta \in$

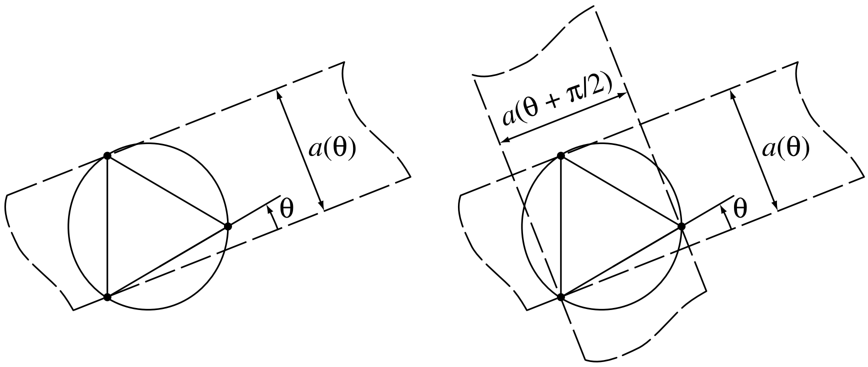

Figure 2: Two perpendicular strips

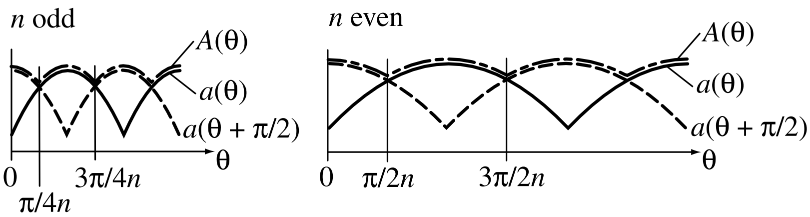

Figure 3: Strip problem solving procedure

$[0, 2\pi/n]$. In this case, $L = 2R$.

Now that we know the narrowest strip that contains a tilted regular polygon, we add a second strip of paper, orthogonal to the first one, and require that it too must contain the polygon. Observe from Figure 2 that the width of this second strip will be $a(\theta + \pi/2)$. If the polygon is to fit into a square, each strip must have width $A(\theta) = \max\{a(\theta), a(\theta + \pi/2)\}$ as shown in Figure 3. Thus, if we minimize $A(\theta)$, we find the smallest square, which is equivalent to find the largest regular polygon within a given square. From the previous expressions for $a(\theta)$ (depicted in Figure 3), we derive the following values for the side of the smallest square:

- $\theta_{\text{opt}} = \pi/(4n)$ (modulo $\pi/(2n)$) if n is odd,
- $\theta_{\text{opt}} = \pi/(2n)$ (modulo π/n) if n is even.

Note that in each case, $a(\theta_{\text{opt}}) = a(\theta_{\text{opt}} + \pi/2)$ and so one can conclude that each side of the square touches at least one vertex of the optimal polygon. Moreover, using the formulas for these angles θ_{opt}, the reader may prove that each optimal polygon has at least one diagonal of the square as an axis of symmetry. Figure 4 shows the optimal polygon placements up to the octagon ($n = 8$ edges).

We remark that this pattern also gives us the optimal polygons that fit any rectangular piece of paper, and not only square ones.

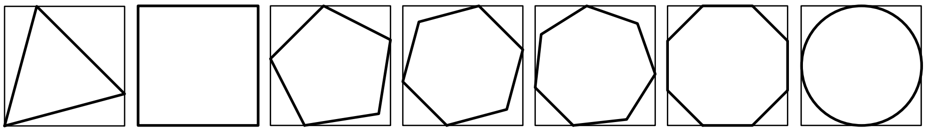

Figure 4: Placement of optimal polygons

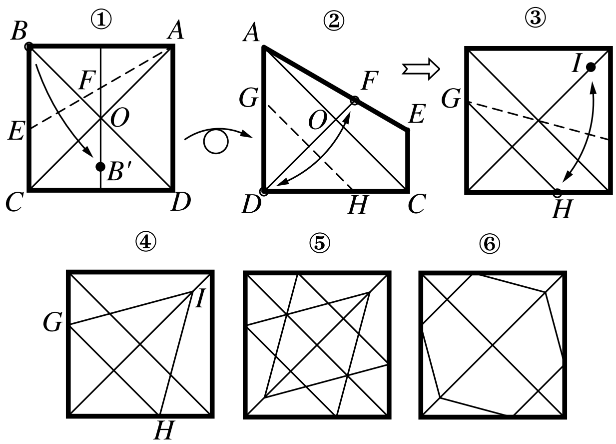

Figure 5: Folding sequence of the optimal hexagon

Folding the optimal hexagon and pentagon

The case of the optimal hexagon ($n = 6$ edges) involves the angle $2\pi/6$, and is very easy to construct. Given a square of paper with side 1, we use the previous formula for θ_{opt} to deduce that $R = 1/(2cos(\pi/12))$. This gives us the edge length, $h = R = \sqrt{2}(\sqrt{3}-1)/2$. The side of the stellated hexagon is $l = 2h\cos(\pi/6) = \sqrt{2}(3-\sqrt{3})/2$. Our construction, which first produces the stellated hexagon, is shown in Figure 5.

- Step 1: Fold corner B onto the central vertical line to create line AE, which meets diagonal BD at F. Since $\sin(\angle B'AD) = 1/2$, this is a simple technique to get an angle $\angle B'AD = \pi/6$. Then, $\angle BAE = \angle EAB' = \angle BAB'/2 = (\pi/2 - \angle B'AD)/2 = \pi/6$ again. (We have trisected the angle $\angle BAD = \pi/2$; this is easy for this particular angle, but more difficult for general case, as we will see for the nonagon. Of course, if have only straightedge and compass we cannot do trisect a general angle at all.) Notice that $\angle EAC = \pi/4 - \angle BAE = \pi/12$, so, $DF = DO + OF = OA + OA\tan(\angle EAC) = \sqrt{2}(1+\tan(\pi/12))/2 = \sqrt{2}(3-\sqrt{3})/2$. This is the length l of the desired polygon side. The following steps are needed to move the crease of length l in a correct position to obtain an edge of the optimal hexagon.

- Step 2: Turn over the model. Split DF in two by folding, and create the fold GH. Its length is $GH = DF$ and due to symmetry with respect to

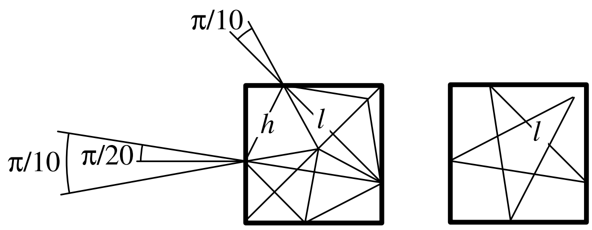

Figure 6: Optimal pentagon and its stellated version (or pentagram)

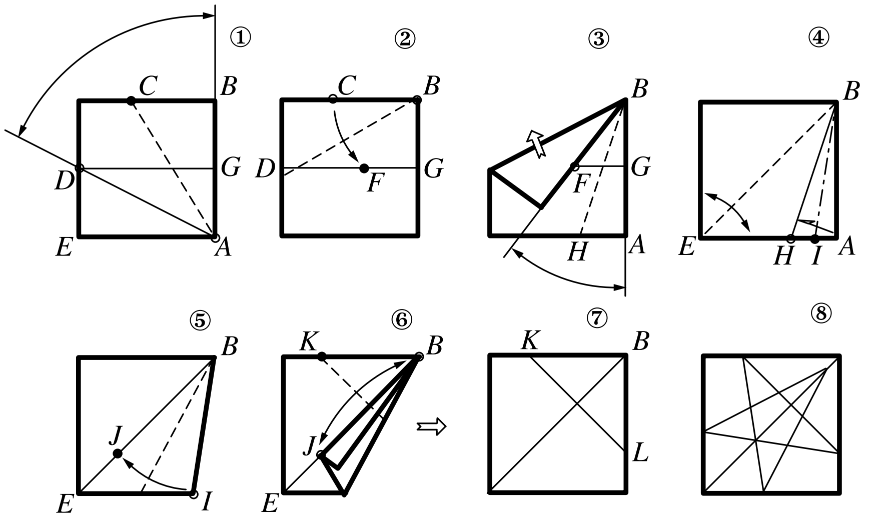

Figure 7: Folding sequence of the optimal pentagon

the diagonal, GH is an edge of the optimal stellated hexagon.

- Step 3: Unfold. Bring H onto the main diagonal at I, with G as an end point of the crease.
- Step 4: Fold GI and HI.
- Step 5: Complete the stellated polygon, using symmetries.

Folding a pentagon ($n = 5$ edges) requires the angle $2\pi/5$, and is more difficult than folding the hexagon. As early as 1989, Roberto Morassi [13] designed an origami construction of the optimal pentagon. The technique shown in Figure 7 has been developed independently and seems much simpler. As before, the stellated version of the polygon is used. With an initial square of unitary edge length, we get $l = 1/\cos(\pi/20)$ in Figure 6.

- Step 1: With D as the middle of the edge, bisect the angle $\angle BAD$. The crease is AC, $\tan(\angle DAE) = 1/2$ and $\angle BAC = \angle CAD = (1/2)\angle BAD = (1/2)(\pi/2 - \angle DAE)$. We can compute $\tan(\angle BAC) = (\sqrt{5} - 1)/2 = BC$; this is the so-called "golden ratio."

- Step 2: Fold C on the central horizontal line DG at F, with B as an end point of the crease. Since $\cos(\pi/5) = 1/(\sqrt{5}-1)$ and $BG = 1/2$, we get $BF = BC = (\sqrt{5}-1)/2 = 1/(2\cos(\pi/5)) = BG/\cos(\pi/5)$. This allows us to conclude that $\cos(\angle FBG) = BG/BF = \cos(\pi/5)$ and $\angle FBG = \pi/5$.

- Step 3: Bisect $\angle FBG$ to get $\angle HBA = \pi/10$. Unfold.

- Step 4: Bisect $\angle ABH$ (fold behind) to get $\angle ABI = \pi/20$ and $BI = 1/\cos(\pi/20)$. This is the length l of the optimal pentagon edge. As before, the following steps are needed to move the crease in a correct position.

- Step 5: Bring I on BE at J.

- Step 6: Fold in half BJ to get K. Unfold.

- Step 7: $KL = BJ = BI = l$ is the correct edge.

- Step 8: Complete the polygon.

Folding other optimal polygons

If the optimal square, triangle, and octagon, are the easiest regular polygons to design (the reader may begin to try to fold them, except for the square...), the optimal hexagon and pentagon are the next easiest. As soon as a regular polygon can be constructed by folding, the corresponding optimal polygons can be also folded with a technique similar to the one used in this paper [4].

In a publication from 1837, P. L. Wantzel [16] demonstrated which regular polygons are constructible with straight-edge and compass (see also [3]). A necessary condition concerns the number of edges of these polygons: they must have $n = 2^p f_1 f_2 ... f_s$ edges, with p as an integer, and where the f_i are all different *primes* of the form $2^m + 1$, where m is also an integer. This result can be further simplified because a necessary (but not sufficient) condition for these f_i to be primes, is to be Fermat numbers, that is, numbers of the form $2^{2^m} + 1$, m still being an integer. Up to the author knowledge, the only known Fermat primes to day are 3, 5, 17, 257 and 65537 [17, 9]. Therefore, one can expect that all the previously folded polygons ($n = 3$, $n = 2^2 = 4$, $n = 5$, $n = 2^1 \times 3 = 6$, $n = 2^3 = 8$) can also be built with straight-edge and compass. This won't be the case for heptagon ($n = 7$ edges) and nonagon ($n = 9$ edges), for instance. Using the technique mentioned in the introduction to solve third-order equations by folding, the previous set of constructible polygons can be extended to the set of polygons with $n = 2^m 3^q g_1 g_2 ... g_s$ edges, where the g_i are all different primes, of the form $2^p 3^r + 1$, m, p, q, and r being integers (see for instance [15, 12, 1]). Such a construction will be necessary to fold the optimal heptagon and nonagon.

To introduce this new construction, let us recall one of its applications: the trisection of an arbitrary angle θ [6], known to be impossible with Euclidean constructions (for a nice discussion on this subject, the reader may refer to the web page `http://www.jimloy.com/geometry/trisect.htm`). To trisect an arbitrary angle, consider the construction of Figure 8: an isoceles triangle $AA'B$ has γ as half main angle. Build the perpendicular AD to AB to get $\angle A'AD = \gamma$. The main idea is then to look at the axis of symmetry Δ that

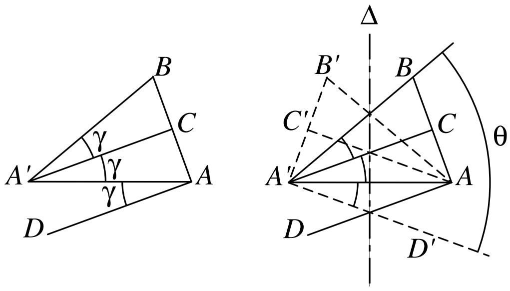

Figure 8: Illustration of trisection problem

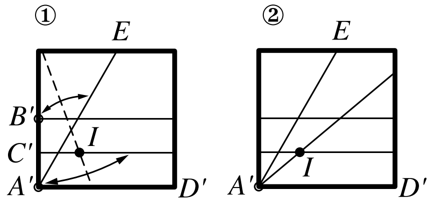

Figure 9: Solving the trisection problem

reflects A onto A', B onto B', C onto C', D onto D', and to consider the angle $\theta = \angle D'A'B = 3\gamma$.

The whole construction can now be reversed to perform the trisection of an angle θ on Figure 9:

- Step 1: Begin with a fold $A'E$ and the angle $\angle D'A'E = \theta$, and two horizontal folds that only have to be equally distant ($A'C' = C'B'$). Fold the paper to get simultaneously B' on $A'E$ and A' on the first horizontal fold (that intersects the new fold Δ at I). Unfold.
- Step 2: You get $\angle IA'E = \angle D'A'E/3$.

The trisection of an angle requires to solve a third-order polynomial equation, while intersecting circles and lines (basically, the Euclidean constructions) requires only second-order polynomial equations. Equipped with the operation of Figure 9, the paperfolder is now able to construct additional polygons.

Since folding a nonagon ($n = 9$ edges) is a little easier than folding the heptagon, let us begin with it. Figure 10 describes shortly the corresponding sequence. Folding it requires precision (and a large square of paper), and proving that an exact optimal nonagon is obtained is a not so easy task [4]. Both are left as challenges to the reader. Here are some guidelines:

- Step 1: Precrease diagonal and central lines. Fold $\pi/3 = \angle BAE$; D should lie on the central horizontal line. Unfold.
- Step 2: With the trisection method, fold $2\pi/9 = \angle BAF$ (then, $\angle FAC = \pi/4 - 2\pi/9 = \pi/36$). The point G is the intersection of AF and the central vertical line. Since $AH = 1/2$, $AG = 1/(2\cos(2\pi/9))$.

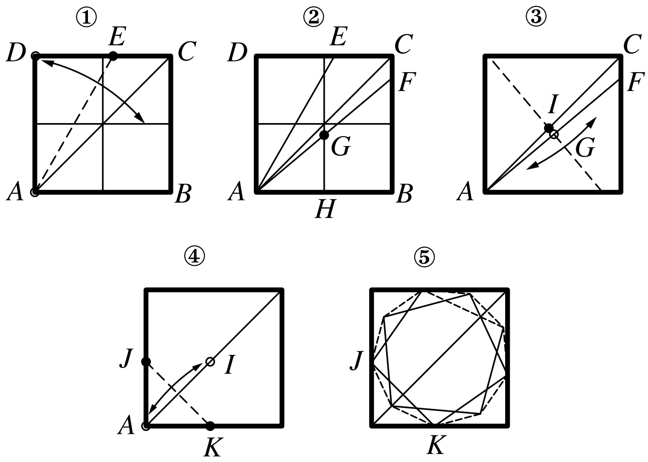

Figure 10: Folding sequence of the optimal nonagon

- Step 3: Fold the perpendicular to AF (and not to AC!) at G. It intersects diagonal AC at I (then, $AI = AG/\cos(\pi/36)$; this is the length of the edge of one stellated version of the nonagon. The nonagram is usually an other star polygon, called 9/3 since edges connect vertices every 3 ones. Here, we have a 9/2 regular star polygon).
- Step 4: Report this distance at JK.
- Step 5: Complete the nonagon (quite challenging, again).

An even more challenging construction is the one of the heptagon ($n = 7$ edges). This time, the angle $2\pi/7$ is involved, still anattainable with Euclidean constructions. Therefore, it requires also the previous technique, designed to solve any third-order polynomial root [12] (let us just mention that $2\cos(2\pi/7)$ is a root of $t^3 + t^2 - 2t - 1 = 0$, and the solving technique is the same as in [11, 8]). Figure 11 describes shortly the corresponding sequence:

- Step 1: Precrease diagonal and central lines. Fold in half and unfold.
- Step 2: Use previous technique, to get a fold HI of slope $2\cos(2\pi/7) = AI/AH$ (G is located at the middle of the previous two horizontal folds).
- Step 3: Fold in half and unfold.
- Step 4: Fold H onto horizontal previous line to obtain $\angle JAH' = 2\pi/7$ (because $AJ/AH' = AI/(2\,AH) = \cos(2\pi/7)$).
- Step 5: The intersection of the folded edge and the initial central line is K. Note that $\angle CAK = 2\pi/7 - \pi/4 = \pi/28$ and $AK = AE/\cos(2\pi/7) = 1/(2\cos(2\pi/7))$. Fold the perpendicular to AK (and not to AC!) at K: it intersects diagonal AC at L. Unfold (then $AL = AK/\cos(\pi/28)$; this is the length of the edge of one stellated version of the heptagon, the 7/2 regular star polygon, or heptagram).

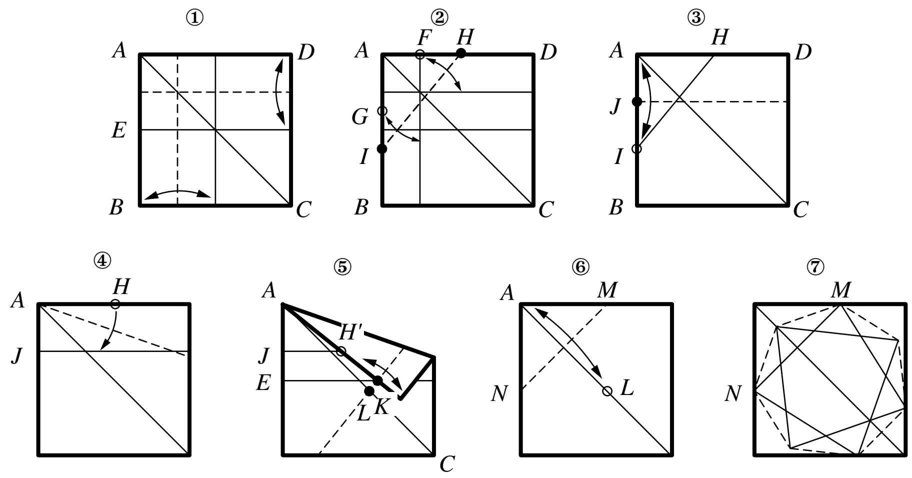

Figure 11: Folding sequence of the optimal heptagon

- Step 6: Report this distance at MN.
- Step 7: Complete the heptagon.

Prospects

Several regular polygons can be folded with Euclidean constructions. Their optimal versions can be folded also, though practically, they become less and less easy to obtain. With a special basic fold that is recalled in this paper, and that is now widely spread through the community of paperfolders, more can be done. But even with it, not all of the polygons can be folded: for instance, the first unreachable regular polygon is the hendecagon ($n = 11$ edges). A construction, simple enough to enter the standard repertoire, and allowing the construction of the regular hendecagon, would require the solution of a 5th-order polynomial equation (the trigonometric functions of $2\pi/11$ are roots of such a polynomial equation). It is still to come.

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