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Pierre Andrieoletti, Roland Diel. Limit law of the local time for Brox's diffusion. 2009. hal-00321069v3

**HAL Id: hal-00321069**

**<https://hal.science/hal-00321069v3>**

Preprint submitted on 24 Apr 2010 (v3), last revised 13 Sep 2010 (v4)

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# Limit law of the local time for Brox's diffusion

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April 24, 2010

## Abstract

We consider Brox's model: a one-dimensional diffusion in a Brownian potential. We show that the normalized local time process  $(L(x + m_{\log t}, t)/t, x \in I \subset \mathbb{R})$ , centered at the coordinate of the bottom of the deepest valley  $m_{\log t}$  reached by the process before time  $t$ , converges in probability. As a consequence, we get the weak convergence of the local time to a functional of two independent 3-dimensional Bessel-square processes. We apply that result to get the limit law of the supremum of the normalized local time. These results are discussed and compared to the discrete time and space analogous model whose same questions have been solved recently by N. Gantert, Y. Peres and Z. Shi [6].

## 1 Introduction

### 1.1 The model

Let  $(W(x), x \in \mathbb{R})$  be a càdlàg real-valued stochastic process with  $W(0) = 0$ . A *diffusion process in the environment*  $W$  is a process  $(X(t), t \in \mathbb{R}^+)$  whose conditional generator, given  $W$ , is

$$\frac{1}{2}e^{W(x)}\frac{d}{dx}\left(e^{-W(x)}\frac{d}{dx}\right).$$

Notice that for almost surely differentiable  $W$ ,  $(X(t), t \in \mathbb{R}^+)$  is the solution of the following stochastic differential equation

$$\begin{cases} dX(t) = d\beta(t) - \frac{1}{2}W'(X(t))dt, \\ X(0) = 0. \end{cases}$$

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MSC 2000 60J25; 60J55.

*Key words* : *Diffusion process in Brownian potential, Local time*

where  $\beta$  is a standard one-dimensional Brownian motion independent of  $W$ . Of course when  $W$  is not differentiable, the previous equation has no rigorous sense.

The study of such a process starts with a choice for  $W$ , a classic one, originally introduced by S. Schumacher [17] and T. Brox [2], is to take for  $W$  a Lévy process. In fact only a few papers deal with the discontinuous case, see for example E. Carmona [3] or A. Singh [21], and most of the results concerns continuous  $W$ , *i.e.*  $(W(x) := B_x - \kappa x/2, x \in \mathbb{R})$ , with  $\kappa \in \mathbb{R}^+$  and  $B$  a two sided Brownian motion independent of  $\beta$ .

The case  $\kappa > 0$  was first studied by K. Kawazu and H. Tanaka [13], then by H. Tanaka [25] and Y. Hu, Z. Shi and M. Yor [11], more recently by for example M. Taleb [23], and A. Devulder [5, 4]. The universal characteristic of this  $X$  is the transience, however a wide range of limit behavior appears, depending on the value of  $\kappa$  see [11].

In this paper we choose  $\kappa = 0$ ,  $X$  is then recurrent and [2] shows that it is sub-diffusive with asymptotic behaviour in  $(\log t)^2$ , moreover  $X$  has the property, for a given instant  $t$ , to be localized in the neighborhood of a random point  $m_{\log t}$  depending only on  $t$  and  $W$ . The limit law of  $m_{\log t}/(\log t)^2$  and therefore of  $X_t/(\log t)^2$  were made explicit independently by H. Kesten [14] and A. O. Golosov [8].

In fact, the aim of H. Kesten and A. O. Golosov was to determine the limit law of the discrete time and space analogous of Brox's model introduced by F. Solomon [22] and then studied by Ya. G. Sinai [20]. This random walk in random environment, usually called Sinai's walk,  $(S_n, n \in \mathbb{N})$  has actually the same limit distribution as Brox's one.

Turning back to Brox's diffusion, notice that H. Tanaka [25, 24] obtained a deeper localization and later Y. Hu and Z. Shi [9] get the almost sure rates of convergence. It appears that these rates of convergence are exactly the same as the rate of convergence for Sinai's walk. The question of an invariance principle, that could exist between these two processes rises and remains open (see Z. Shi [19] for a survey). In fact a first attempt to link these two processes appears for the first time in the articles of S. Schumacher [17] and K. Kawazu, Y. Tamura and H. Tanaka [12].

This work is devoted to the limit distribution of the local time of  $X$ . Indeed to the diffusion  $X$  corresponds a local time process  $(L_X(t, x))_{t \geq 0, x \in \mathbb{R}}$  defined by the occupation time formula :  $L_X$  is the unique  $\mathbb{P}$ -a.s. jointly continuous process such that for any bounded Borel function  $f$  and for any  $t \geq 0$ ,

$$\int_0^t f(X_s) ds = \int_{\mathbb{R}} f(x) L_X(t, x) dx.$$

The first results on the behavior of  $L_X$  can be found in [18] and [10]. In particular in [10] it is proven that, for any  $x \in \mathbb{R}$

$$\frac{\log(L_X(t, x))}{\log t} \xrightarrow{\mathcal{L}} U \wedge \hat{U}, \quad t \rightarrow +\infty \quad (1)$$

where  $U$  and  $\hat{U}$  are independent variables uniformly distributed in  $(0, 1)$ ,  $\xrightarrow{\mathcal{L}}$  is the convergence in law and  $x \wedge y = \min(x, y)$ . Notice that in the same paper Y. Hu and Z. Shi also prove that this behavior is the same for Sinai's walk: if we denote by  $L_S(n, x) := \sum_{i=1}^n \mathbb{1}_{S_i=x}$  the local time of  $S$  in  $x \in \mathbb{Z}$  at time  $n$  then

$$\frac{\log(L_S(n, x))}{\log n} \xrightarrow{\mathcal{L}} U \wedge \hat{U}, \quad n \rightarrow +\infty.$$

For previous works on the local time for Sinai's diffusion we refer to the book of P. Révész [16].

In this article we show that the normalized local time process  $(L(t, x + m_{\log t})/t, x \in [-K, K])$  with  $K > 0$  converges in probability to a well defined process, which depends only on  $t$ ,  $m_{\log t}$  and  $W$ . We also make explicit the limit law of this process when  $t$  goes to infinity, it involves some 3-dimensional Bessel-square processes. The supremum of the local time process of  $X$  is given by

$$\forall t \geq 0, \quad L_X^*(t) := \sup_{x \in \mathbb{R}} L_X(t, x),$$

as a consequence of our results we show that  $L_X^*(t)/t$  converges weakly and determine its limit law. We also find interesting to compare the discrete Theorems of [6] with ours, pointing out the analogies and the differences.

## 1.2 Preliminary definitions and results

First let us describe the probability space where  $X$  is defined. It is composed of two Wiener's spaces, one for the environment and the other one for the diffusion itself:

Let  $\mathcal{W}$  be the space of continuous functions  $W : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $W(0) = 0$  and  $\mathcal{A}$  the  $\sigma$ -field generated by the topology of uniform convergence on compact sets on  $\mathcal{W}$ . We equip  $(\mathcal{W}, \mathcal{A})$  with Wiener measure  $\mathcal{P}$  i.e the coordinate process is a "two-sided" Brownian motion. We call *environment* an element of  $\mathcal{W}$ .

We also define the set  $\Omega := C([0; +\infty[, \mathbb{R})$ , the  $\sigma$ -field  $\mathcal{F}$  on  $\Omega$  generated by the topology of uniform convergence on compact sets and the probability

measure  $P$  such that the coordinate process on  $\Omega$  is a standard Brownian motion.

We denote by  $\mathbb{P}$  the probability product  $\mathcal{P} \otimes P$  on  $\mathcal{W} \times \Omega$ . We indifferently denote by  $W$  an undetermined element of  $\mathcal{W}$  and the first coordinate process on  $\mathcal{W} \times \Omega$  (i.e a “two-sided” Brownian motion under  $\mathbb{P}$  or  $P$ ) similarly  $B$  is indifferently an element of  $\Omega$  and the second coordinate process on  $\mathcal{W} \times \Omega$ . Finally  $\stackrel{\mathcal{L}_W}{=}$  means ”equality in law” under  $P$  that is under a fixed environment  $W$ , and  $\stackrel{\mathcal{L}}{=}$  (respectively  $\stackrel{\mathcal{L}}{\rightarrow}$ ) for an equality (resp. a convergence) in law under  $\mathbb{P}$ . We can now state our first result:

**Theorem 1.1** *We have*

$$\frac{L_X^*(t)}{t} \stackrel{\mathcal{L}}{\rightarrow} \frac{1}{\int_{-\infty}^{\infty} e^{-R(y)} dy} \quad (2)$$

where for any  $x \in \mathbb{R}$ ,  $R(x) := R_1(x)\mathbf{1}_{\{x \geq 0\}} + R_2(-x)\mathbf{1}_{\{x < 0\}}$ ,  $R_1$  and  $R_2$  are two independent 3-dimensional Bessel-square processes starting at 0.

First notice that  $\int_{-\infty}^{\infty} e^{-R(y)} dy < +\infty$  a.s., moreover we can prove that

$$\frac{1}{\int_{-\infty}^{\infty} e^{-R(y)} dy} \stackrel{\mathcal{L}}{=} \frac{1}{4(\tau(1) + \tilde{\tau}(1))} \quad (3)$$

where  $\tau(1)$  and  $\tilde{\tau}(1)$  are the hitting time of 1 of two independent squared Bessel processes of dimension 2 starting from 0. Now we would like to state the equivalent of this theorem for Sinai’s walk recently found by [6],

$$\frac{L_S^*(n)}{n} \stackrel{\mathcal{L}}{\rightarrow} \sup_{x \in \mathbb{Z}} \pi(x) \quad (4)$$

where

$$\pi(x) = \frac{\exp(-Z_x) + \exp(-Z_{x-1})}{2 \sum_{y \in \mathbb{Z}} \exp(-Z_y)}, \quad x \in \mathbb{Z}, \quad (5)$$

and  $Z$  is a sum of i.i.d random variables (with mean zero, strictly positive variance and bounded) null at zero and conditionned to stay positive (see below Theorem 1.1 and Section 4 of [6] for the exact definition).

The analogy between the local time for  $X$  and the local time for  $S$  takes place in the fact that both  $R$  and  $S$  can be obtained from classical diffusion conditioned to stay positive,  $R_1$  and  $R_2$  are Brownian motions conditioned

to stay positive (see [27]) and  $Z$  a simple symmetric random walk conditioned to stay positive (see [7] and [1]). Note also that A. O. Golosov also proved that  $\sum_{y \in \mathbb{Z}} \exp(-Z_y) < +\infty$ .

However  $Z$  and  $R$  have not the same nature, one is discrete the other one continuous, notice also that the increments of  $Z$  are supposed to be bounded (see hypothesis 1.2 in [6]), and it is not the case for  $R$ . Finally the numerator of  $\pi(x)$  is not as simple as the numerator of our result, but this comes essentially from the difference of nature of the two processes discrete for one and continuous for the other.

Theorem 1.1 is an easy consequence of an interesting intermediate result (Theorem 1.2 below). Before introducing that result we need some extra definitions on the environment, these basic notions have been introduced by [2] see also [15]. Let  $h > 0$ , we say that  $W \in \mathcal{W}$  admits a  *$h$ -minimum* at  $x_0$  if there exists  $\xi$  and  $\zeta$  such that  $\xi < x_0 < \zeta$  and for any  $x \in [\xi, \zeta]$ ,

- $W(x) \geq W(x_0)$ ,
- $W(\xi) \geq W(x_0) + h$ ,
- $W(\zeta) \geq W(x_0) + h$ .

Similarly we say that  $W$  admits a  *$h$ -maximum* at  $x_0$  if  $-W$  admits a  *$h$ -minimum* at  $x_0$ . We denote by  $M_h(W)$  the set of  *$h$ -extrema* of  $W$ . It is easy to establish that  $\mathcal{P}$ -a.s.  $M_h$  has no accumulation point and that the points of  *$h$ -maximum* and of  *$h$ -minimum* alternate. Hence there exists exactly one triple  $\Delta_h = (p_h, m_h, q_h)$  of successive elements in  $M_h$  such that

- $m_h$  and 0 lay in  $[p_h, q_h]$ ,
- $p_h$  and  $q_h$  are  *$h$ -maxima*,
- $m_h$  is a  *$h$ -minimum*.

We call this triple the standard  *$h$ -valley* of  $W$  (see Figure 1). We can now state our second result:

**Theorem 1.2**

$$\left( \frac{L_X(t, m_{\log t} + x)}{t} \right)_{x \in \mathbb{R}} \xrightarrow{\mathcal{L}} (\mathfrak{R}(x))_{x \in \mathbb{R}}, \text{ with}$$

$$\mathfrak{R}(x) := \frac{e^{-R(x)}}{\int_{-\infty}^{\infty} e^{-R(y)} dy}.$$

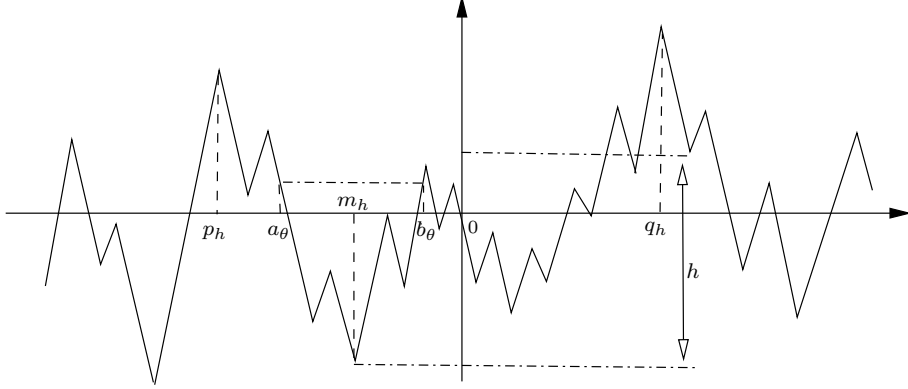


Figure 1: Example of standard valley

and  $R$  is the same as in Theorem 1.1.

This result is the analogue of Theorem 1.2 of [6], we recall their result:

$$\left( \frac{L_S(n, b_n + x)}{n}, x \in \mathbb{Z} \right) \xrightarrow{\mathcal{L}} (\pi(x))_{x \in \mathbb{Z}}, \quad (6)$$

where  $\pi(x)$  is given by (5) and  $b_n$  plays the same role for  $S$  as  $m_{\log t}$  plays for  $X$ . Notice that Theorem 1.1 can be deduced directly from Theorem 1.2, in the same way that (4) can be deduced from (6).

There is a second remark we can make here, H. Tanaka [26] shows that the limit when  $t$  goes to infinity of the distribution of the process  $(X_t - m_{\log t})$  converges to a distribution with a density (with respect to Lebesgue measure) given by  $\mathfrak{R}$ . There is a natural explanation to understand, at least intuitively, why  $\mathfrak{R}$  appears in both limits. For that we need first to present another result, indeed Theorems 1.1 and 1.2 are consequences of a result, in probability, on the asymptotic behavior of the local time in a neighborhood of  $m_{\log t}$  (Theorem 1.3 below) together with results on the random environment (see Sections 3.2 and 3.3). Before stating our third result, we need a new notation, let  $(W_x, x \in \mathbb{R})$  be the *shifted difference of potential*,

$$\forall x \in \mathbb{R}, W_x(\cdot) := W(x + \cdot) - W(x). \quad (7)$$

**Theorem 1.3**

Let  $K > 0$ ,  $r \in (0, 1)$ , then for all  $\delta > 0$ ,

$$\lim_{\alpha \rightarrow +\infty} \mathbb{P} \left( \sup_{-K \leq x \leq K} \left| \frac{L_X(e^\alpha, m_\alpha + x)}{e^\alpha} \frac{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)} dy}{e^{-W_{m_\alpha}(x)}} - 1 \right| \leq \delta \right) = 1$$

where for any  $\theta > 0$ ,

$$a_\theta = a_\theta(W_{m_\alpha}) := \sup \{x \leq 0 / W_{m_\alpha}(x) \geq \theta\} \quad \text{and} \\ b_\theta = b_\theta(W_{m_\alpha}) := \inf \{x \geq 0 / W_{m_\alpha}(x) \geq \theta\},$$

see also Figure 1.

We are ready to give an idea of the reason why  $\mathfrak{R}$  appears in the paper of H. Tanaka [26] and the present one. The important term in the last result, which appears when we study the inverse of the local time in  $m_\alpha$  (see section 2.2), is the following

$$\bar{\mathfrak{R}}(\alpha, x) := \frac{e^{-W_{m_\alpha}(x)}}{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)} dy}.$$

First, we will show in Section 3.2 that the process  $(\bar{\mathfrak{R}}(\alpha, x), x \in \mathbb{R})$  converges weakly when  $\alpha$  goes to infinity to  $(\mathfrak{R}(x), x \in \mathbb{R})$ , so Theorem 1.3 leads to Theorem 1.2. We therefore focus on  $\bar{\mathfrak{R}}(\alpha, x)$ .

Note that  $(\bar{\mathfrak{R}}(\alpha, x), x \in (a_{\alpha r}, b_{\alpha r}))$  can be seen as a local (in time) invariant probability measure for the process  $X$ . Indeed until the instant  $e^\alpha$ ,  $(X_s \equiv X_s(W), s \leq e^\alpha)$  spends, with a high probability, most of its time between the two points  $(a_{\alpha r}, b_{\alpha r})$ . So  $X$  can be approached, in some sense, by a simpler process  $(\tilde{X}_s \equiv \tilde{X}_s(\alpha W), s \leq e^\alpha)$  with the same generator as  $X$  but reflected at fixed barriers,  $\tilde{a}$  and  $\tilde{b}$  (see [2] or [26] page 159). This new process obviously possesses an invariant probability measure given by  $\tilde{\mu}_\alpha(dx) := \bar{\mathfrak{R}}(\alpha, x)dx := \frac{e^{-\alpha W_{m_1}(x)}}{\int_{\tilde{a}}^{\tilde{b}} e^{-\alpha W_{m_1}(y)} dy} dx$ . And  $\bar{\mathfrak{R}}(\alpha, x)$  is naturally involved in the limit behavior of the normalized local time  $L_{\tilde{X}}(e^\alpha, m_1 + \cdot)/e^\alpha$  and also in the limit distribution of  $\tilde{X}_{e^\alpha} - m_1$ . To turn back to  $\bar{\mathfrak{R}}$  a self similarity argument of the environment can be used.

To finish with this discussion, it is interesting to notice that for the discrete time model the result in law plays an important role to get the almost sure asymptotic of the limit sup of  $L_S^*$ . Indeed (4) (together with Lemmata 3.1 and 3.2 in [6]) leads to

$$\limsup_n \frac{L_S^*(n)}{n} = \text{const} \in ]0, +\infty[, \mathbb{P}.a.s.$$



and the const is known explicitly. For Brox's model,  $L_X^*(t)$  is possibly larger than  $t$  and, in fact, if we use a similar argument than [6], then the limit in law (Theorem 1.2) only implies

$$\limsup_t \frac{L_X^*(t)}{t} = +\infty, \mathbb{P}.a.s.$$

So the limit in law implies a weaker result for the limsup than the already known result of Z. Shi ([18]) who proved  $\limsup_t \frac{L_X^*(t)}{t \log \log \log t} \geq \text{const}, \mathbb{P}.a.s..$

### 1.3 Basic facts for diffusion with potential

In this section we recall basic definitions and tools traditionally used to study diffusion in random environment. For all  $W \in \mathcal{W}$ , define

$$\forall x \in \mathbb{R}, S_W(x) := \int_0^x e^{W(y)} dy \quad (8)$$

and

$$\forall t \geq 0, T_W(t) := \int_0^t e^{-2W(S_W^{-1}(B(s)))} ds. \quad (9)$$

As Brox points out in [2], the standard diffusion theory implies that the process

$$\begin{aligned} X : \mathcal{W} \times \Omega &\longrightarrow \Omega \\ (W, B) &\longmapsto S_W^{-1} \circ B \circ T_W^{-1} \end{aligned} \quad (10)$$

is under  $\mathbb{P}$  a diffusion in Brownian environment. To simplify notations, we write when there is no possible mistake  $S$  and  $T$  for respectively  $S_W$  and  $T_W$ .

Using Formula (10), we easily obtain that for any  $x \in \mathbb{R}$  and  $t \geq 0$ ,

$$L_X(t, x) = e^{-W(x)} L_B(T^{-1}(t), S(x)) \quad (11)$$

where  $L_B$  is the local time process of the Brownian motion  $B$ .

T. Brox ([2]) noticed also that it is more convenient to study the asymptotic behavior of the process  $X_\alpha$  defined below instead of the one of  $X$ ,

$$X_\alpha(W, \cdot) := X(\alpha W, \cdot).$$

For all  $x \in \mathbb{R}$ , let us denote

$$W^\alpha(x) := \frac{1}{\alpha} W(\alpha^2 x).$$

For each  $\alpha > 0$ , we have a link between  $X_\alpha$  and  $X$  given by:

**Lemma 1.4** *For each  $W \in \mathcal{W}$  and  $\alpha > 0$ ,*

$$\begin{aligned} (X_\alpha(W^\alpha, t))_{t \geq 0} &:= (X(\alpha W^\alpha, t))_{t \geq 0} \stackrel{\mathcal{L}_W}{=} \left( \frac{1}{\alpha^2} X(W, \alpha^4 t) \right)_{t \geq 0}, \\ (L_{X_\alpha(W^\alpha, \cdot)}(t, x))_{t \geq 0, x \in \mathbb{R}} &\stackrel{\mathcal{L}_W}{=} \left( \frac{1}{\alpha^2} L_{X(W, \cdot)}(\alpha^4 t, \alpha^2 x) \right)_{t \geq 0, x \in \mathbb{R}}. \end{aligned}$$

We do not give any detail of the proof of this Lemma, the first relation can be found in Brox (see [2], Lemma 1.3) and the second is a straightforward consequence of the first one.

Formulas (8), (9) and (10) for  $X_\alpha$  are given by

$$\forall t \geq 0, X_\alpha(t) = S_\alpha^{-1}(B(T_\alpha^{-1}(t))) \quad (12)$$

where

$$\forall x \in \mathbb{R}, S_\alpha(x) := S_{\alpha W}(x) = \int_0^x e^{\alpha W(y)} dy \quad (13)$$

and

$$\forall t \geq 0, T_\alpha(t) := T_{\alpha W}(t) = \int_0^t e^{-2\alpha W(S_\alpha^{-1}(B(s)))} ds, \quad (14)$$

also for the local time we have,

$$\forall t \geq 0, \forall x \in \mathbb{R}, L_{X_\alpha}(t, x) = e^{-\alpha W(x)} L_B(T_\alpha^{-1}(t), S_\alpha(x)). \quad (15)$$

The rest of the paper is organized as follows: in the first part of Section 2 we get the asymptotic of the local time within a random amount of time, which is the inverse of the local time at  $m_{\log t}$ , in Section 2.2 the asymptotic of the inverse of the local time itself is studied. Note that Sections 2.1 and 2.2 can be read independently.

Propositions 2.1 and 2.5 of Sections 2 are the key results to get Theorem 1.3 proved at the beginning of Section 3. In the second and third subsection of Section 3, Theorems 1.1 and 1.2 are proved. Note that the Theorems come from Theorem 1.3 together with the study of a functional of the random environment involving 3-dimensional Bessel-square processes.

## 2 Asymptotics for the local time $L_{X_\alpha}$ and its inverse $\sigma_{X_\alpha}$

We begin with some definitions that will be used all along the paper. For any process  $M$  we define the following stopping times with the usual convention

$$\inf \emptyset = +\infty,$$

$$\forall x \in \mathbb{R}, \tau_M(x) := \inf\{t \geq 0 / M(t) = x\}, \quad (16)$$

$$\forall x \in \mathbb{R}, \forall r \geq 0, \sigma_M(r, x) := \inf\{t \geq 0 / L_M(t, x) \geq r\}. \quad (17)$$

We define for any  $W \in \mathcal{W}$  and for all  $x, y \in \mathbb{R}$ ,

$$\overline{W}(x, y) := \begin{cases} \sup_{[x, y]} W & \text{if } y \geq x, \\ \sup_{[y, x]} W & \text{if } y < x \end{cases}$$

and

$$\underline{W}(x, y) := \begin{cases} \inf_{[x, y]} W & \text{if } y \geq x, \\ \inf_{[y, x]} W & \text{if } y < x, \end{cases}$$

they represent respectively the maximum and the minimum of  $W$  between  $x$  and  $y$ . Finally we introduce the process starting in  $x \in \mathbb{R}$ ,

$$(X_\alpha^x(W, t))_{t \geq 0} := (x + X_\alpha(W_x, t))_{t \geq 0} = (x + X(\alpha W_x, t))_{t \geq 0}$$

where  $W_x$  is the *shifted difference of potential* (see (7)). Notice that we have the equivalent of (12):

$$\forall t \geq 0, X_\alpha^x(t) = x + (S_\alpha^x)^{-1}(B((T_\alpha^x)^{-1}(t))) \quad (18)$$

where  $S_\alpha^x := S_{\alpha W_x}$  and  $T_\alpha^x := T_{\alpha W_x}$  and it is easy to establish that for a fixed  $W \in \mathcal{W}$ ,  $X_\alpha^x$  is a strong Markov process.

## 2.1 Asymptotic behaviour of $L_{X_\alpha}$ at time $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)$

In this first sub-section we study the asymptotic behaviour of the local time at the inverse of the local time in  $m := m_1$ , recall that  $m_1$  is the coordinate of the bottom of the basic valley defined page 5.

### Proposition 2.1

Let  $K > 0$ ,  $W \in \mathcal{W}$  and let  $h$  be a function such that  $\lim_{\alpha \rightarrow +\infty} h(\alpha) = 1$ . Then, for all  $\delta > 0$ ,

$$\lim_{\alpha \rightarrow +\infty} P \left( \sup_{-K \leq x \leq K} \left| \frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m), m + \alpha^{-2}x)}{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}} - 1 \right| \leq \delta \right) = 1.$$

**Proof :**

For simplicity, we assume without loss of generality that  $m = m_1(W) \geq 0$  and to lighten notations, we denote for all  $x \in [-K, K]$ ,  $x_\alpha := m + \alpha^{-2}x$ .

The proof is based on the decomposition of the local time into two terms, the first one is the contribution of the local time in  $x_\alpha$  before  $\tau_{X_\alpha}(m)$  (the first time  $X_\alpha$  hits  $m$ ) and the second one is the contribution of the local time between  $\tau_{X_\alpha}(m)$  and  $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)$  (the inverse of the local time in  $m$ ):

$$\begin{aligned} L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m), x_\alpha) &= L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha) \\ &+ \left( L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m), x_\alpha) - L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha) \right). \end{aligned}$$

We treat these two terms in the Lemmata 2.2 and 2.3 below. Lemma 2.2 states that, asymptotically, the local time in a point  $x_\alpha$  until the process reaches  $m$  is negligible compared to  $e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}$ . Thanks to the strong Markov property for  $X_\alpha$ , it remains to study the asymptotic behaviour of

$$\left( L_{X_\alpha^m}(\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m), m + \alpha^{-2}x) \right)_{-K \leq x \leq K}$$

when  $\alpha$  goes to infinity, this is what is done in Lemma 2.3 which says that the local time in  $x_\alpha$  of  $X_\alpha^m$  within the interval of time  $[0, \sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)]$  is of the order of  $e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}$ .  $\square$

Let us state and prove

**Lemma 2.2** *For any  $\delta > 0$ ,*

$$\lim_{\alpha \rightarrow +\infty} P \left( \sup_{-K \leq x \leq K} \frac{L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha)}{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}} \leq \delta \right) = 1.$$

**Proof**

First, as we have assumed that  $m \geq 0$ , for all  $x > 0$   $L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha) = 0$  so we only have to consider non positive  $x$ . Notice also that for all  $x \in [-K, 0]$ ,  $L_{X_\alpha}(\tau_{X_\alpha}(m - \alpha^{-2}K), x_\alpha) = 0$ , therefore

$$L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha) = L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha) - L_{X_\alpha}(\tau_{X_\alpha}(m - \alpha^{-2}K), x_\alpha). \quad (19)$$

Let  $\kappa_\alpha = m - \alpha^{-2}K$  thanks to (19) and the strong Markov property for  $X_\alpha$ , we only need to prove that

$$\lim_{\alpha \rightarrow +\infty} P \left( \sup_{-K \leq x \leq 0} \frac{L_{X_\alpha^{\kappa_\alpha}}(\tau_{X_\alpha^{\kappa_\alpha}}(m), x_\alpha)}{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}} \leq \delta \right) = 1. \quad (20)$$

It follows from (18) with  $x = \kappa_\alpha$  that

$$\tau_{X_\alpha^{\kappa_\alpha}}(m) = \tau_{X_\alpha(W_{\kappa_\alpha}, \cdot)}(\alpha^{-2}K) = T_\alpha^{\kappa_\alpha}(\tau_B(S_\alpha^{\kappa_\alpha}(\alpha^{-2}K))),$$

so according to (15), we have for all  $x \in \mathbb{R}$ ,

$$\begin{aligned} L_{X_\alpha^{\kappa_\alpha}}(\tau_{X_\alpha^{\kappa_\alpha}}(m), x_\alpha) &= \\ e^{-\alpha W_{\kappa_\alpha}(\alpha^{-2}(x+K))} L_B(\tau_B(S_\alpha^{\kappa_\alpha}(\alpha^{-2}K)), S_\alpha^{\kappa_\alpha}(\alpha^{-2}(x+K))). \end{aligned}$$

The classic scaling property of the local time of the Brownian motion given by:

$$\forall \lambda > 0, \forall y_1 > 0, \quad (\lambda L_B(\tau_B(y_1), y))_{y \in \mathbb{R}} \stackrel{\mathcal{L}_W}{=} (L_B(\tau_B(\lambda y_1), \lambda y))_{y \in \mathbb{R}}, \quad (21)$$

yields that the processes  $\left( L_{X_\alpha^{\kappa_\alpha}}(\tau_{X_\alpha^{\kappa_\alpha}}(m), x_\alpha) \right)_{x \in \mathbb{R}}$  and

$$\left( S_\alpha^{\kappa_\alpha}(\alpha^{-2}K) e^{-\alpha W_{\kappa_\alpha}(\alpha^{-2}(x+K))} L_B(\tau_B(1), s_\alpha(\alpha^{-2}(x+K))) \right)_{x \in \mathbb{R}}$$

are equal in law, where  $s_\alpha(z) := S_\alpha^{\kappa_\alpha}(z)/S_\alpha^{\kappa_\alpha}(\alpha^{-2}K)$ .

We claim that for all  $x \in [-K, 0]$

$$\begin{aligned} & S_\alpha^{\kappa_\alpha}(\alpha^{-2}K) e^{-\alpha W_{\kappa_\alpha}(\alpha^{-2}(x+K))} L_B(\tau_B(1), s_\alpha(\alpha^{-2}(x+K))) \\ & \leq \alpha^{-2}K e^{\alpha(\overline{W}_m(-\alpha^{-2}K, 0) - W_m(\alpha^{-2}x))} \sup_{y \leq 1} L_B(\tau_B(1), y). \end{aligned}$$

Indeed

$$S_\alpha^{\kappa_\alpha}(\alpha^{-2}K) = \int_0^{\frac{K}{\alpha^2}} e^{\alpha W_{\kappa_\alpha}(y)} dy \leq \frac{K}{\alpha^2} e^{\alpha \overline{W}_{\kappa_\alpha}(0, \alpha^{-2}K)},$$

for all  $x \in [-K, 0]$

$$\overline{W}_{\kappa_\alpha}(0, \alpha^{-2}K) - W_{\kappa_\alpha}(\alpha^{-2}(x+K)) = \overline{W}_m(-\alpha^{-2}K, 0) - W_m(\alpha^{-2}x)$$

and  $s_\alpha(\alpha^{-2}(x+K)) \leq 1$ .

Assembling the above estimates, we get for any  $\delta > 0$ ,

$$\begin{aligned} & P \left( \sup_{-K \leq x \leq 0} \frac{L_{X_\alpha^{\kappa_\alpha}}(\tau_{X_\alpha^{\kappa_\alpha}}(m), x_\alpha)}{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}} \leq \delta \right) \\ & \geq P \left( \alpha^{-2}K e^{\alpha \overline{W}_m(-\alpha^{-2}K, 0) - \alpha h(\alpha)} \sup_{y \leq 1} L_B(\tau_B(1), y) \leq \delta \right). \end{aligned}$$

As  $\lim_{\alpha \rightarrow +\infty} \overline{W}_m(-\alpha^{-2}K, 0) = 0$ ,  $\lim_{\alpha \rightarrow +\infty} h(\alpha) = 1$  and  $\sup_{y \leq 1} L_B(\tau_B(1), y) < \infty$   $P$ -a.s., the right hand side of the last inequality tends to 1 as  $\alpha$  goes to infinity. (20) is proved together with the Lemma.  $\square$

We move to the proof of the second Lemma

**Lemma 2.3** *For any  $\delta > 0$ ,*

$$\lim_{\alpha \rightarrow +\infty} P \left( \sup_{-K \leq x \leq K} \left| \frac{L_{X_\alpha^m}(\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m), m + \alpha^{-2}x)}{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}} - 1 \right| \leq \delta \right) = 1.$$

**Proof**

For simplicity we denote for any process  $M$ ,  $\sigma_M(r) := \sigma_M(r, 0)$ , we will also assume without loss of generality that  $m = 0$ , Lemma 2.3 can therefore be rewritten in the following way:

$$\lim_{\alpha \rightarrow +\infty} P \left( \sup_{-K \leq x \leq K} \left| \frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}), \alpha^{-2}x)}{e^{\alpha h(\alpha) - \alpha W(\alpha^{-2}x)}} - 1 \right| \leq \delta \right) = 1. \quad (22)$$

Like for  $\tau$  in Lemma 2.2, we easily get that  $\sigma_{X_\alpha}(t) = T_\alpha(\sigma_B(t))$ , thus formula (15), together with the scale invariance for the local time of the Brownian motion

$$\forall r > 0, \forall \lambda > 0, (\lambda L_B(\sigma_B(r), y))_{y \in \mathbb{R}} \stackrel{\mathcal{L}_W}{=} (L_B(\sigma_B(\lambda r), \lambda y))_{y \in \mathbb{R}} \quad (23)$$

yields

$$\left( L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}), \alpha^{-2}x) \right)_{x \in \mathbb{R}} \stackrel{\mathcal{L}_W}{=} \left( e^{\alpha h(\alpha) - \alpha W(\alpha^{-2}x)} L_B(\sigma_B(1), \tilde{s}_\alpha(\alpha^{-2}x)) \right)_{x \in \mathbb{R}},$$

where  $\tilde{s}_\alpha(\alpha^{-2}x) := S_\alpha(\alpha^{-2}x)e^{-\alpha h(\alpha)}$ . Let  $K_\alpha := \alpha^{-2}K e^{\alpha \overline{W}(-\alpha^{-2}K, \alpha^{-2}K) - \alpha h(\alpha)}$ , we have for all  $x \in [-K, K]$ ,  $-K_\alpha \leq \tilde{s}_\alpha(\alpha^{-2}x) \leq K_\alpha$ . Collecting what we did above we get for any  $\delta > 0$ ,

$$\begin{aligned} & P \left( \sup_{-K \leq x \leq K} \left| \frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}), \alpha^{-2}x)}{e^{\alpha h(\alpha) - \alpha W(\alpha^{-2}x)}} - 1 \right| \leq \delta \right) \\ &= P \left( \sup_{-K \leq x \leq K} |L_B(\sigma_B(1), \tilde{s}_\alpha(\alpha^{-2}x)) - 1| \leq \delta \right), \\ &\geq P \left( \sup_{-K_\alpha \leq y \leq K_\alpha} |L_B(\sigma_B(1), y) - 1| \leq \delta \right). \end{aligned}$$

Moreover  $\lim_{\alpha \rightarrow +\infty} K_\alpha = 0$  and  $y \rightarrow L_B(\sigma_B(1), y)$  is continuous at 0, so (22) and the Lemma are proved.  $\square$

## 2.2 Asymptotic behaviour of $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)$

This section is devoted to the study of the asymptotic behavior of  $\sigma_{X_\alpha}(e^{\alpha h(\alpha)})$ , the main result is Proposition 2.5 below.

Before stating that Proposition we need a preliminary result on the random environment which gives precisions on the standard  $h$ -valley  $\Delta_h(W) = (p_h, m_h, q_h)$  defined Section 1.2. We denote

$$W^\#(x, y) := \max_{[x \wedge y, x \vee y]} (W(z) - \underline{W}(x, z))$$

where  $x \vee y = \max(x, y)$ . Notice that the function  $W^\#$  represents the largest barrier of potential we have to cross in the path from  $x$  to  $y$ . We call *depth of the valley*  $\Delta_h(W)$  the quantity

$$D(\Delta_h(W)) := (W(p_h) - W(m_h)) \wedge (W(q_h) - W(m_h))$$

and *inner directed ascent* the quantity

$$A(\Delta_h(W)) := W^\#(p_h, m_h) \vee W^\#(q_h, m_h).$$

Note that the above notions have already been introduced by Sinai [20], Brox [2], and Tanaka [25]. According to Brox, we have the following

**Lemma 2.4** *There exists a subset  $\widetilde{\mathcal{W}}$  of  $\mathcal{W}$  of  $\mathcal{P}$ -measure 1 such that for any  $W \in \widetilde{\mathcal{W}}$ , the standard 1-valley  $\Delta_1(W) := (p_1, m_1, q_1)$  satisfies  $A(\Delta_1(W)) < 1 < D(\Delta_1(W))$ .*

Throughout this Section we write  $p, m, q, D$  and  $A$  for respectively  $m_1(W)$ ,  $p_1(W)$ ,  $q_1(W)$ ,  $D(\Delta_1(W))$  and  $A(\Delta_1(W))$ .

We can now state the main result of this section:

### Proposition 2.5

*Let  $W \in \widetilde{\mathcal{W}}$ ,  $r \in (0, 1)$  and  $h$  be a function such that  $\lim_{\alpha \rightarrow +\infty} h(\alpha) = 1$ , then for all  $\delta > 0$ ,*

$$\lim_{\alpha \rightarrow +\infty} P \left( \left| \frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)}{e^{\alpha h(\alpha)} \int_{a_r}^{b_r} e^{-\alpha W_m(x)} dx} - 1 \right| \leq \delta \right) = 1$$

where  $a_r$  and  $b_r$  are defined in Theorem 1.3 page 6.

To lighten notations, in the rest of the paper we denote,

$$g(\alpha) := \int_{a_r}^{b_r} e^{-\alpha W_m(y)} dy, \quad \alpha > 0.$$

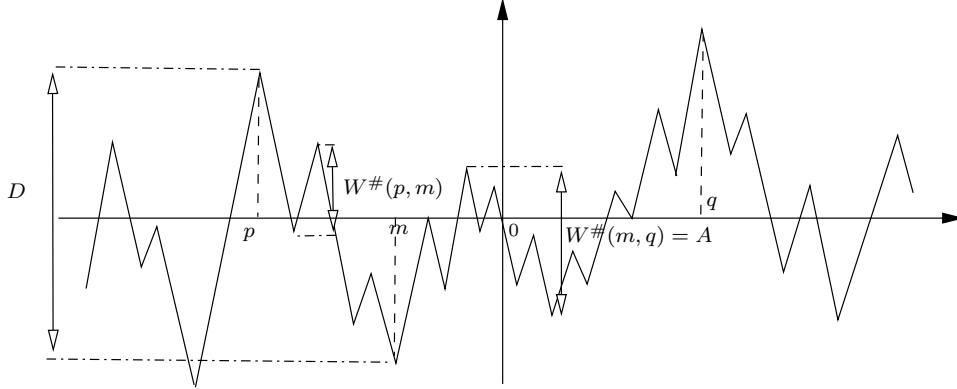


Figure 2: Example of 1-standard valley with its depth and its inner directed ascent.

**Proof :**

We assume that  $m > 0$ , we get the other case by reflection, note that we work at fixed  $W$  which belongs to  $\widetilde{\mathcal{W}}$ . We follow the same steps of the proof of Proposition 2.1: we decompose  $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)$  into two terms,

$$\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m) = \tau_{X_\alpha}(m) + \left( \sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m) - \tau_{X_\alpha}(m) \right).$$

The first one  $\tau_{X_\alpha}(m)$  is treated in Lemma 2.6, we show that its contribution is negligible comparing to  $g(\alpha)e^{\alpha h(\alpha)}$ . Then thanks to the strong Markov property, it is enough to prove that  $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)/g(\alpha)e^{\alpha h(\alpha)}$  converge to 1 in  $P$  probability, this is what Lemma 2.7 tells.  $\square$

Let us state and prove the first Lemma

**Lemma 2.6** *For any  $\delta > 0$ ,*

$$\lim_{\alpha \rightarrow +\infty} P \left( \frac{\tau_{X_\alpha}(m)}{g(\alpha)e^{\alpha h(\alpha)}} \leq \delta \right) = 1.$$

**Proof**

This proof has the same outline of the proof of point (i) of Lemma 3.1 in [2], however because of some slight differences and for completeness we give some details.



By definition of the local time together with (15) and (21), the hitting time of  $m$  can be written

$$\begin{aligned}\tau_{X_\alpha}(m) &= \int_{-\infty}^m L_{X_\alpha}(\tau_{X_\alpha}(m), z) dz, \\ &= \int_{-\infty}^m e^{-\alpha W(z)} L_B(\tau_B(S_\alpha(m)), S_\alpha(z)) dz, \\ &\stackrel{\mathcal{L}_W}{=} S_\alpha(m) \int_{-\infty}^m e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z)) dz\end{aligned}\quad (24)$$

where  $\hat{s}_\alpha(z) := S_\alpha(z)/S_\alpha(m)$ . Let  $n := \operatorname{argmax}_{[0, m]} W$  we denote

$$\begin{aligned}I_{\alpha,1} &:= S_\alpha(m) \int_{-\infty}^n e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z)) dz, \\ I_{\alpha,2} &:= S_\alpha(m) \int_n^m e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z)) dz,\end{aligned}$$

and formula (24) can be rewritten

$$\tau_{X_\alpha}(m) \stackrel{\mathcal{L}_W}{=} I_{\alpha,1} + I_{\alpha,2}. \quad (25)$$

The rest of the proof consists essentially in finding an upper bound for  $I_{\alpha,1}$  and  $I_{\alpha,2}$ .

*We begin with  $I_{\alpha,1}$ ,* first we prove that, with a probability which tends to 1 when  $\alpha$  goes to infinity, the process  $X_\alpha$  does not visit coordinates smaller than  $p$ , where  $p$  is the left vertex of the standard 1-valley defined page 5. Thanks to this, the lower bound in the integral of  $I_{\alpha,1}$  will be  $p$  and not  $-\infty$ , the upper bound for  $I_{\alpha,1}$  follows almost immediatly.

Let us define

$$l := \inf \{x \leq 0 / L_B(\tau_B(1), x) > 0\}, \quad (26)$$

we claim that,

$$P.a.s, \exists \alpha_0 \text{ such that } \forall \alpha > \alpha_0, \hat{s}_\alpha^{-1}(l) > p. \quad (27)$$

Indeed

$$\hat{s}_\alpha^{-1}(l) \geq p \iff l \geq \hat{s}_\alpha(p) = -\frac{\int_p^0 e^{\alpha W(x)} dx}{\int_0^m e^{\alpha W(x)} dx},$$

moreover Laplace's method gives

$$\begin{aligned}\lim_{\alpha \rightarrow +\infty} \frac{1}{\alpha} \log \int_p^0 e^{\alpha W(x)} dx &= \overline{W}(p, 0) = W(p), \text{ and} \\ \lim_{\alpha \rightarrow +\infty} \frac{1}{\alpha} \log \int_0^m e^{\alpha W(x)} dx &= \overline{W}(0, m) = W(n),\end{aligned}$$

so

$$\hat{s}_\alpha(p) = -\exp(\alpha(W(p) - W(n)) + o(\alpha)),$$

finally according to the definition of the standard valley  $W(p) > W(n)$ , therefore

$$\lim_{\alpha \rightarrow +\infty} \hat{s}_\alpha(p) = -\infty$$

and (27) is true.

On the event  $\{\hat{s}_\alpha^{-1}(l) > p\}$ , we have

$$\begin{aligned}I_{\alpha,1} &= S_\alpha(m) \int_p^n e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z)) dz, \\ &\leq S_\alpha(m)(n-p) e^{-\alpha \underline{W}(p,n)} \max_{x \leq 1} L_B(\tau_B(1), x),\end{aligned}$$

moreover

$$S_\alpha(m) \leq m e^{\alpha \overline{W}(0,m)} \leq (q-p) e^{\alpha W(n)},$$

and we get the upper bound

$$I_{\alpha,1} \leq (q-p)^2 e^{\alpha A} \max_{x \leq 1} L_B(\tau_B(1), x) \quad (28)$$

where  $A$  is the inner direct ascent of the valley defined at the beginning of this section.

We continue with  $I_{\alpha,2}$ , the main ingredient to get an upper bound in this case is to use the first Ray-Knight which leads to the study of an integral involving a two-dimensional Bessel process: first we rewrite  $I_{\alpha,2}$  in the following way

$$I_{\alpha,2} = S_\alpha(m) \int_n^m e^{-\alpha W(z)} L(\tau_B(1), 1 - \bar{s}_\alpha(z)) dz \quad (29)$$

where

$$\bar{s}_\alpha := 1 - \hat{s}_\alpha(z) = \frac{1}{S_\alpha(m)} \int_z^m e^{\alpha W(x)} dx.$$

Let  $R$  be a two-dimensional Bessel squared process starting from the origin, according to the First Ray-Knight theorem

$$(L_B(\tau_B(1), 1 - \bar{s}_\alpha(z)))_{z \in [0, m]} \stackrel{\mathcal{L}_W}{=} (R(\bar{s}_\alpha(z)))_{z \in [0, m]},$$

together with the scale invariance  $(t^2 R_{\frac{1}{t}})_{t \in \mathbb{R}_+} \stackrel{\mathcal{L}_W}{=} (R_t)_{t \in \mathbb{R}_+}$  we get

$$\int_n^m e^{-\alpha W(z)} R(\bar{s}_\alpha(z)) dz \stackrel{\mathcal{L}_W}{=} \int_n^m \left\{ e^{-\alpha W(z)} \bar{s}_\alpha(z) \right\} \bar{s}_\alpha(z) R\left(\frac{1}{\bar{s}_\alpha(z)}\right) dz. \quad (30)$$

We are now able to get a preliminary upper bound for  $I_{\alpha,2}$ :

$$\begin{aligned} & S_\alpha(m) \int_n^m \left\{ e^{-\alpha W(z)} \bar{s}_\alpha(z) \right\} \bar{s}_\alpha(z) R\left(\frac{1}{\bar{s}_\alpha(z)}\right) dz, \\ & \leq \max_{n \leq z \leq m} \left[ e^{-\alpha W(z)} \int_z^m e^{\alpha W(x)} dx \right] \int_n^m \bar{s}_\alpha(z) R\left(\frac{1}{\bar{s}_\alpha(z)}\right) dz, \\ & \leq (q-p) \exp \left\{ \alpha \max_{n \leq z \leq m} [-W(z) + \overline{W}(z, m)] \right\} \int_n^m \bar{s}_\alpha(z) R\left(\frac{1}{\bar{s}_\alpha(z)}\right) dz, \\ & \leq (q-p)^2 e^{\alpha A} \underbrace{\frac{1}{m-n} \int_n^m \bar{s}_\alpha(z) R\left(\frac{1}{\bar{s}_\alpha(z)}\right) dz}_{J_\alpha} \end{aligned} \quad (31)$$

and the last inequality comes from the relation  $\max_{n \leq z \leq m} [-W(z) + \overline{W}(z, m)] = W^\#(n, m) \leq A$ .

According to Jensen's inequality and Fubini's theorem the expectation of  $(J_\alpha)^2$  satisfies

$$\begin{aligned} E[(J_\alpha)^2] & \leq \frac{1}{m-n} \int_n^m E[\bar{s}_\alpha^2(z) (R(\frac{1}{\bar{s}_\alpha(z)}))^2] dz, \\ & = \frac{1}{m-n} \int_n^m \int_0^{+\infty} \frac{\bar{s}_\alpha(z)^3 y^2}{2} e^{-\frac{\bar{s}_\alpha(z)y}{2}} dy dz = 8. \end{aligned} \quad (32)$$

*End of the proof of the Lemma:* using (25), we obtain for all  $\alpha > 0$

$$\begin{aligned} & P\left(\frac{\tau_{X_\alpha}(m)}{g(\alpha)e^{\alpha h(\alpha)}} \leq \delta\right) = P\left(\frac{S_\alpha(m)(I_{\alpha,1} + I_{\alpha,2})}{g(\alpha)e^{\alpha h(\alpha)}} \leq \delta\right), \\ & \geq P\left(\frac{S_\alpha(m)I_{\alpha,1}}{g(\alpha)e^{\alpha h(\alpha)}} \leq \frac{\delta}{2}; \hat{s}_\alpha^{-1}(l) \geq p\right) + P\left(\frac{S_\alpha(m)I_{\alpha,2}}{g(\alpha)e^{\alpha h(\alpha)}} \leq \frac{\delta}{2}\right) - 1. \end{aligned}$$

For the first term in the above expression, (28) yields

$$\begin{aligned} & P\left(\frac{S_\alpha(m)I_{\alpha,1}}{g(\alpha)e^{\alpha h(\alpha)}} \leq \frac{\delta}{2}; \hat{s}_\alpha^{-1}(l) \geq p\right) \geq \\ & P\left(\max_{x \leq 1} L_B(\tau_B(1), x) \leq G(\alpha); \hat{s}_\alpha^{-1}(l) \geq p\right) \end{aligned} \quad (33)$$

where

$$G(\alpha) := \frac{\delta g(\alpha)}{2(q-p)^2} e^{\alpha(h(\alpha)-A)}.$$

By Laplace's method we know that  $\lim_{\alpha \rightarrow +\infty} \frac{\log g(\alpha)}{\alpha} = 0$ , so  $g(\alpha) = e^{o(\alpha)}$ , therefore as  $\lim_{\alpha \rightarrow +\infty} h(\alpha) = 1 > A$ ,  $G(\alpha)$  tends to infinity when  $\alpha$  does. Using that  $y \rightarrow L_B(\tau_B(1), y)$  is  $P$ -a.s. finite and (27) we get that (33) tends to 1 when  $\alpha$  goes to infinity.

For the second term we collect (29), (30) and (31), we get

$$P\left(\frac{S_\alpha(m)I_{\alpha,2}}{g(\alpha)e^{\alpha h(\alpha)}} > \frac{\delta}{2}\right) \leq P(J_\alpha > G(\alpha)),$$

then by Tchebychev's inequality and (32)

$$P(J_\alpha > G(\alpha)) \leq \frac{E[(J_\alpha)^2]}{(G(\alpha))^2} \leq \frac{8}{(G(\alpha))^2},$$

by using once again that  $G(\alpha)$  tends to infinity we get the Lemma.  $\square$

Next step is to prove

**Lemma 2.7** *For any  $\delta > 0$ ,*

$$\lim_{\alpha \rightarrow +\infty} P\left(\left|\frac{\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m)}{g(\alpha)e^{\alpha h(\alpha)}} - 1\right| \leq \delta\right) = 1.$$

**Proof:**

Just like for the proof of Lemma 2.3 we assume without loss of generality that  $m = 0$ , as a consequence  $\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m) = \sigma_{X_\alpha}(e^{\alpha h(\alpha)})$  and we simply have to establish that :

$$\lim_{\alpha \rightarrow +\infty} P\left(\left|\frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)})}{g(\alpha)e^{\alpha h(\alpha)}} - 1\right| \leq \delta\right) = 1. \quad (34)$$

In the same way we get (24), one can prove that

$$\sigma_{X_\alpha}(e^{\alpha h(\alpha)}) \stackrel{\mathcal{L}_W}{=} e^{\alpha h(\alpha)} \int_{-\infty}^{+\infty} e^{-\alpha W(x)} L_B(\sigma_B(1), \tilde{s}_\alpha(x)) dx, \quad (35)$$

recall that  $\tilde{s}_\alpha(y) = S_\alpha(y)e^{-\alpha h(\alpha)}$ . The rest of the proof is devoted to estimate the integral and the main difficulty is to get the upper bound.

We begin with the lower bound, we easily get that

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-\alpha W(x)} L_B(\sigma_B(1), \tilde{s}_\alpha(x)) dx &\geq \int_{a_r}^{b_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \tilde{s}_\alpha(x)) dx, \\ &\geq \inf_{y \in [\tilde{s}_\alpha(a_r), \tilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) g(\alpha), \end{aligned}$$

where  $a_r$  and  $b_r$  are defined at the end of Theorem 1.3, therefore

$$\begin{aligned} P \left( \frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)})}{g(\alpha) e^{\alpha h(\alpha)}} \geq 1 - \delta \right) &\geq \\ P \left( \inf_{y \in [\tilde{s}_\alpha(a_r), \tilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) \geq 1 - \delta \right). \end{aligned} \quad (36)$$

Also we recall that  $r \in (0, 1)$  therefore we can prove easily by using the Laplace transform that  $\lim_{\alpha \rightarrow +\infty} \tilde{s}_\alpha(a_r) = \lim_{\alpha \rightarrow +\infty} \tilde{s}_\alpha(b_r) = 0$ . We conclude by noticing that  $P - a.s.$ ,  $\lim_{\alpha \rightarrow +\infty} \inf_{y \in [\tilde{s}_\alpha(a_r), \tilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) = 1$ , thanks to the continuity of the function  $y \rightarrow L_B(\sigma_B(1), y)$  at 0.

We continue with the upper bound. First we use the same idea of the proof of Lemma 2.6 when we had to deal with  $I_{\alpha,1}$ : we establish that with a probability which tends to 1 as  $\alpha$  goes to infinity,  $X_\alpha$  does not exit from the standard valley  $(p, m, q)$ . Define

$$L := \inf \{x \leq 0 / L_B(\sigma_B(1), x) > 0\}, U := \sup \{x \geq 0 / L_B(\sigma_B(1), x) > 0\},$$

we claim that,

$$P - a.s. \exists \alpha_0, \forall \alpha > \alpha_0, p < \tilde{s}_\alpha^{-1}(L) < 0 < \tilde{s}_\alpha^{-1}(U) < q. \quad (37)$$

Indeed, we have

$$\tilde{s}_\alpha^{-1}(L) \geq p \iff L \geq \tilde{s}_\alpha(p) = -e^{-\alpha h(\alpha)} \int_p^0 e^{\alpha W(x)} dx,$$

and by Laplace's method we get  $\tilde{s}_\alpha(p) = -e^{\alpha(W(p) - h(\alpha)) + o(\alpha)}$ . It follows from the fact that  $W \in \widetilde{\mathcal{W}}$  and  $\lim_{\alpha \rightarrow +\infty} h(\alpha) = 1 < D \leq W(p)$  that  $\lim_{\alpha \rightarrow +\infty} \tilde{s}_\alpha(p) = -\infty$ ,  $P$ -a.s.. In a similar way we obtain  $\lim_{\alpha \rightarrow +\infty} \tilde{s}_\alpha(q) = +\infty$ ,  $P$ -a.s. and (37) is satisfied.

On the event  $\{p < \tilde{s}_\alpha^{-1}(L) < 0 < \tilde{s}_\alpha^{-1}(U) < q\}$ , we can write

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-\alpha W(x)} L_B(\sigma_B(1), \tilde{s}_\alpha(x)) dx &= \int_p^{a_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \tilde{s}_\alpha(x)) dx \\ &+ \int_{a_r}^{b_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \tilde{s}_\alpha(x)) dx \\ &+ \int_{b_r}^q e^{-\alpha W(x)} L_B(\sigma_B(1), \tilde{s}_\alpha(x)) dx. \end{aligned}$$

We only have to found an upper bound for these integrals, first we have

$$\begin{aligned} &\int_p^{a_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \tilde{s}_\alpha(x)) dx + \int_{b_r}^q e^{-\alpha W(x)} L_B(\sigma_B(1), \tilde{s}_\alpha(x)) dx \\ &\leq (q - p) \exp(-\alpha \min_{x \in I_r} W(x)) \sup_{y \in \mathbb{R}} L_B(\sigma_B(1), y) \end{aligned}$$

where  $I_r := [p, a_r] \cup [b_r, q]$ , and moreover

$$\begin{aligned} \int_{a_r}^{b_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \tilde{s}_\alpha(x)) dx &\leq \sup_{y \in [\tilde{s}_\alpha(a_r), \tilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) \int_{a_r}^{b_r} e^{-\alpha W(x)} dx, \\ &= g(\alpha) \sup_{y \in [\tilde{s}_\alpha(a_r), \tilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y). \end{aligned}$$

Therefore, assembling the last two inequalities and the equality in law (35)

$$\begin{aligned} P \left( \frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)})}{g(\alpha)e^{\alpha h(\alpha)}} + 1 \leq \delta \right) &\geq \\ P \left( \sup_{y \in [\tilde{s}_\alpha(a_r), \tilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) - 1 + \frac{(q - p)}{g(\alpha)} e^{-\alpha \min_{I_r} W} \sup_{y \in \mathbb{R}} L_B(\sigma_B(1), y) \leq \delta ; \right. \\ &\left. p < \tilde{s}_\alpha^{-1}(L) < 0 < \tilde{s}_\alpha^{-1}(U) < q \right). \end{aligned} \quad (38)$$

By hypothesis  $r < 1$ , so  $\lim_{\alpha \rightarrow +\infty} \tilde{s}_\alpha(a_r) = \lim_{\alpha \rightarrow +\infty} \tilde{s}_\alpha(b_r) = 0$ , moreover  $y \rightarrow L_B(\sigma_B(1), y)$  is  $P$ -a.s. continuous at 0, it follows

$$\lim_{\alpha \rightarrow +\infty} \sup_{y \in [\tilde{s}_\alpha(a_r), \tilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) = 1 \quad P\text{-a.s..}$$

We also know that  $\lim_{\alpha \rightarrow +\infty} \frac{\log g(\alpha)}{\alpha} = 0$ , moreover according to the definition of  $\widetilde{W}$ ,  $\min_{x \in I_r} W(x) > 0$ , and finally  $\sup_{y \in \mathbb{R}} L_B(\sigma_B(1), y)$  is  $P$ -a.s. finite, so

$$\lim_{\alpha \rightarrow +\infty} \frac{(q - p)}{g(\alpha)} e^{-\alpha \min_{I_r} W} \sup_{y \in \mathbb{R}} L_B(\sigma_B(1), y) = 0 \quad P\text{-a.s..}$$

Putting the last two assertions together with (38) we get the upper bound and finally the Lemma.  $\square$

### 3 Proof of the main results

One of the key result of this paper is Theorem 1.3, the other results can be deduced from that theorem together with estimates on the random environment, so we naturally start with the

#### 3.1 Proof of Theorem 1.3

We begin with a Proposition which resume Propositions 2.1 and 2.5, we get the asymptotic behaviour of  $L_{X_\alpha}$  within a deterministic interval of time :

**Proposition 3.1**

Let  $K > 0$ ,  $r \in (0, 1)$ ,  $W \in \widetilde{\mathcal{W}}$  and  $h$  a real function such that  $\lim_{\alpha \rightarrow \infty} h(\alpha) = 1$ . For all  $\delta > 0$ , we have

$$\lim_{\alpha \rightarrow +\infty} P \left( \sup_{-K \leq x \leq K} \left| \frac{L_{X_\alpha}(e^{\alpha h(\alpha)}, m + \alpha^{-2}x)}{e^{\alpha h(\alpha)}} \frac{\int_{a_r}^{b_r} e^{-\alpha W_m(y)} dy}{e^{-\alpha W_m(\alpha^{-2}x)}} - 1 \right| \leq \delta \right) = 1.$$

**Proof :**

Let  $\delta > 0$ , and  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\lim_{\alpha \rightarrow +\infty} f(\alpha) = 1$ , define

$$A_{\alpha, f} := \left\{ \sup_{-K \leq x \leq K} \left| \frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha f(\alpha)}, m), m + \alpha^{-2}x)}{e^{\alpha f(\alpha) - \alpha W_m(\alpha^{-2}x)}} - 1 \right| \leq \delta \right\}$$

and

$$B_{\alpha, f} := \left\{ \left| \frac{\sigma_{X_\alpha}(e^{\alpha f(\alpha)}, m)}{g(\alpha) e^{\alpha f(\alpha)}} - 1 \right| \leq \delta \right\}$$

where, as in the previous section,  $g(\alpha) = \int_{a_r}^{b_r} e^{-\alpha W_m(y)} dy$ . We also define two functions

$$\begin{aligned} h^+(\alpha) &:= h(\alpha) - \alpha^{-1} \log(g(\alpha)(1 - \delta)), \\ h^-(\alpha) &:= h(\alpha) - \alpha^{-1} \log(g(\alpha)(1 + \delta)). \end{aligned}$$

On  $B_{\alpha, h^+}$  the following inequality holds:

$$\begin{aligned} \sigma_{X_\alpha}(e^{\alpha h^+(\alpha)}, m) &\geq (1 - \delta) g(\alpha) e^{\alpha h^+(\alpha)}, \\ &\geq e^{\alpha h(\alpha)}, \end{aligned}$$

moreover in its first coordinate the local time is an increasing function, therefore on  $A_{\alpha,h^+} \cap B_{\alpha,h^+}$ ,  $\forall x \in [-K, K]$ ,

$$\begin{aligned} L_{X_\alpha}(e^{\alpha h(\alpha)}, m + \alpha^{-2}x) &\leq L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h^+(\alpha)}, m), m + \alpha^{-2}x), \\ &\leq e^{\alpha h^+(\alpha) - \alpha W_m(\alpha^{-2}x)}(1 + \delta), \\ &\leq \frac{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}}{g(\alpha)} \frac{1 + \delta}{1 - \delta}. \end{aligned}$$

In the same way, on  $A_{\alpha,h^-} \cap B_{\alpha,h^-}$  we obtain

$$L_{X_\alpha}(e^{\alpha h(\alpha)}, m + \alpha^{-2}x) \geq \frac{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}}{g(\alpha)} \frac{1 - \delta}{1 + \delta}.$$

By Laplace's method,  $\lim_{\alpha \rightarrow +\infty} \frac{\log g(\alpha)}{\alpha} = 0$ , so  $h^+$  and  $h^-$  tend to 1 when  $\alpha$  goes to infinity and we can apply Propositions 2.1 and 2.5, finally

$$\lim_{\alpha \rightarrow +\infty} P(A_{\alpha,h^+} \cap B_{\alpha,h^+} \cap A_{\alpha,h^-} \cap B_{\alpha,h^-}) = 1$$

and the Proposition is proved.  $\square$

We turn back to the proof of the Theorem, notice that the difference between Theorem 1.3 and Proposition 3.1 above is the process itself: one deals with  $X$  whereas the other deals with  $X_\alpha$ . To finish the proof we need to show that thanks to Lemma 1.4 we can get the theorem from the proposition.

Let  $\alpha > 0$ , recall that  $W^\alpha(\cdot) := \alpha^{-1}W(\alpha^2 \cdot)$ . First, remark that for all  $W \in \mathcal{W}$ ,

$$\begin{aligned} m_1(W^\alpha) &= \alpha^{-2}m_\alpha(W), \\ a_r(W_{m_1(W^\alpha)}^\alpha) &= \alpha^{-2}a_{\alpha r}(W_{m_\alpha(W)}), \\ b_r(W_{m_1(W^\alpha)}^\alpha) &= \alpha^{-2}b_{\alpha r}(W_{m_\alpha(W)}), \end{aligned}$$

and for any  $x \in \mathbb{R}$ ,

$$W_{m_1(W^\alpha)}^\alpha(x) = \frac{1}{\alpha}W_{m_\alpha(W)}(\alpha^2 x).$$

Now replacing  $t$  by  $\alpha^{-4}e^\alpha$  in the second part of Lemma 1.4 page 9, we obtain for all  $W \in \mathcal{W}$ ,

$$(L_{X(\alpha W^\alpha, \cdot)}(\alpha^{-4}e^\alpha, m_1(W^\alpha) + \alpha^{-2}x))_{x \in \mathbb{R}} \stackrel{\mathcal{L}_W}{=} \left( \frac{1}{\alpha^2} L_{X(W, \cdot)}(e^\alpha, m_\alpha(W) + x) \right)_{x \in \mathbb{R}}.$$



Therefore, for any  $\alpha > 0$ ,  $\delta > 0$ ,  $K > 0$  and  $W \in \mathcal{W}$ ,

$$P \left( \sup_{-K \leq x \leq K} \left| \frac{L_{X(W, \cdot)}(e^\alpha, m_\alpha(W) + x)}{e^\alpha} \frac{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)} dy}{e^{-W_{m_\alpha}(x)}} - 1 \right| < \delta \right) = \\ P \left( \sup_{-K \leq x \leq K} \left| \frac{L_{X(\alpha W^\alpha, \cdot)}(\alpha^{-4} e^\alpha, m_1(W^\alpha) + \alpha^{-2} x)}{\alpha^{-2} e^\alpha} \frac{\int_{\alpha^2 a_r}^{\alpha^2 b_r} e^{-\alpha W_{m_1}^\alpha(\alpha^{-2} y)} dy}{e^{-\alpha W_{m_1}^\alpha(\alpha^{-2} x)}} - 1 \right| < \delta \right).$$

Moreover, for all  $\alpha > 0$ ,  $\mathcal{P}$  is invariant under the transformation  $W \mapsto W^\alpha$ , we get that

$$\mathbb{P} \left( \sup_{-K \leq x \leq K} \left| \frac{L_{X(W, \cdot)}(e^\alpha, m_\alpha(W) + x)}{e^\alpha} \frac{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)} dy}{e^{-W_{m_\alpha}(x)}} - 1 \right| < \delta \right) = \\ \mathbb{P} \left( \sup_{-K \leq x \leq K} \left| \frac{L_{X(\alpha W, \cdot)}(\alpha^{-4} e^\alpha, m_1(W) + \alpha^{-2} x)}{\alpha^{-4} e^\alpha} \frac{\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy}{e^{-\alpha W_{m_1}(\alpha^{-2} x)}} - 1 \right| < \delta \right)$$

and we recall that  $\mathbb{P} = \mathcal{P} \otimes P$ . To finish the proof we notice that  $\mathcal{P}(\widetilde{\mathcal{W}}) = 1$ ,  $\alpha^{-4} e^\alpha = e^{\alpha(1 - \frac{4}{\alpha} \log \alpha)}$  and  $\lim_{\alpha \rightarrow +\infty} (1 - \frac{4}{\alpha} \log \alpha) = 1$ , so applying Proposition 3.1 we get Theorem 1.3.  $\square$

As we said at the beginning of the section, once Theorem 1.3 is proved, we get the other results by studying in details some properties of the random environment. We continue with the

### 3.2 Proof of Theorems 1.1 and 1.2

We recall that Theorem 1.1 is a straight forward consequence of Theorem 1.2, so we are left to prove Theorem 1.2. The main ingredients to get this theorem are Theorem 1.3, Lemmata 3.2 and 3.3 below. We begin with the proof of the first Lemma:

**Lemma 3.2** *For all  $r \in (0, 1)$ ,  $\mathcal{P}$ -almost surely for all  $a_r \leq a < 0 < b \leq b_r$ , we get*

$$\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy \underset{\alpha \rightarrow +\infty}{\sim} \int_a^b e^{-\alpha W_{m_1}(y)} dy.$$

**Proof :**

For any  $W \in \mathcal{W}$ ,

$$\begin{aligned} \int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy - \int_a^b e^{-\alpha W_{m_1}(y)} dy &= \int_{a_r}^a e^{-\alpha W_{m_1}(y)} dy + \int_b^{b_r} e^{-\alpha W_{m_1}(y)} dy \\ &\leq (b_r - a_r) e^{-\alpha \min_I W_{m_1}} \end{aligned}$$

where  $I := [a_r, a] \cup [b, b_r]$ . Moreover, by Laplace's method,

$$\lim_{\alpha \rightarrow +\infty} \frac{1}{\alpha} \log \int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy = - \min_{[a_r, b_r]} (W_{m_1}).$$

As  $a_r \leq 0 \leq b_r$ , this minimum is equal to 0, thus  $\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy = e^{o(\alpha)}$  and so,

$$\frac{\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy - \int_a^b e^{-\alpha W_{m_1}(y)} dy}{\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy} \leq (b_r - a_r) e^{-\alpha \min_I W_{m_1} + o(\alpha)}.$$

And for any  $W \in \widetilde{\mathcal{W}}$ ,  $\min_I W_{m_1} > 0$ , then letting  $\alpha$  go to infinity we obtain the equivalence for any  $W \in \widetilde{\mathcal{W}}$ . As  $\mathcal{P}(\widetilde{\mathcal{W}}) = 1$ , this implies the result of the lemma.  $\square$

Now, the proof of the theorem will be finished once we will have shown

**Lemma 3.3** *For any positive constant  $K$  and for any bounded continuous functional  $F$  on  $C(\mathbb{R}, \mathbb{R})$  such that  $F(f)$  depends only on the values of the function  $f$  on  $[-K, K]$ ,*

$$\lim_{\alpha \rightarrow +\infty} \mathbb{E} \left[ F \left( \frac{e^{-W_{m_\alpha}}}{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)} dy} \right) \right] = \mathbb{E} \left[ F \left( \frac{e^{-R}}{\int_{-\infty}^{\infty} e^{-R(y)} dy} \right) \right]. \quad (39)$$

We recall that

$$\forall x \in \mathbb{R}, R(x) := R_1(x) \mathbf{1}_{\{x \geq 0\}} + R_2(-x) \mathbf{1}_{\{x < 0\}},$$

$R_1$  and  $R_2$  being two independent 3-dimensional Bessel processes.

**Proof :**

According to the scaling property of Brownian Motion,

$$\mathbb{E} \left[ F \left( \frac{e^{-W_{m_\alpha}(\cdot)}}{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)} dy} \right) \right] = \mathbb{E} \left[ F \left( \frac{e^{-\alpha W_{m_1}(\alpha^{-2} \cdot)}}{\alpha^2 \int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy} \right) \right].$$

So, thanks to Lemma 3.2, denoting

$$\tilde{a}_r := \begin{cases} a_r & \text{if } m_1 < 0 \\ a_r \vee 0 & \text{if } m_1 \geq 0 \end{cases} \text{ and } \tilde{b}_r := \begin{cases} b_r & \text{if } m_1 \geq 0 \\ b_r \wedge 0 & \text{if } m_1 < 0 \end{cases} ,$$

it is enough to prove that

$$\lim_{\alpha \rightarrow +\infty} \mathbb{E} \left[ F \left( \frac{e^{-\alpha W_{m_1}(\alpha^{-2} \cdot)}}{\alpha^2 \int_{\tilde{a}_r}^{\tilde{b}_r} e^{-\alpha W_{m_1}(y)} dy} \right) \right] = \mathbb{E} \left[ F \left( \frac{e^{-R}}{\int_{-\infty}^{\infty} e^{-R(y)} dy} \right) \right] .$$

This can be done using exactly the same arguments as in the proof of Lemma 3.2 in Tanaka [25].  $\square$

**Acknowledgment :** We would like to thank Romain Abraham for very helpful discussions, many remarks and comments on the different versions of this paper.

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