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Limit law of the local time for Brox's diffusion

Pierre Andreoletti, Roland Diel *

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Abstract

We consider Brox's model: a one-dimensional diffusion in a Brownian environment. We show the weak convergence of the normalized local time process $(L(x+m_{\log t},t)/t, x \in I \subset \mathbb{R})$, centered at the coordinate of the bottom of the deepest valley $m_{\log t}$ reached by the process before time t to a functional of two independent 3-dimensional Bessel processes. We apply that result to get the limit law of the supremum of the normalized local time. These results are discussed and compared to the discrete time and space analogous model whose same questions have been solved recently by N. Ganter, Y. Peres and Z. Shi [1].

1 Introduction

1.1 The model

Let $(W(x), x \in \mathbb{R})$ be a càdlàg real-valued stochastic process with $W(0) = 0$. A *diffusion process in the environment* W is a process $(X(t), t \in \mathbb{R}^+)$ whose conditional generator, given W, is

$$\frac{1}{2}e^{W(x)}\frac{d}{dx}\left(e^{-W(x)}\frac{d}{dx}\right).$$

Notice that for almost surely differentiable W, $(X(t), t \in \mathbb{R}^+)$ is the solution of the following stochastic differential equation

$$\begin{cases} \mathrm{d}X(t) = \mathrm{d}\beta(t) - \frac{1}{2}W'(X(t))\mathrm{d}t, \\ X(0) = 0. \end{cases}$$

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where β is a standard one-dimensional Brownian motion independent of W. Of course when W is not differentiable, the previous equation has no rigorous sense.
The study of such a process starts with a choice for W, a classic one, originally introduced by S. Schumacher [2] and T. Brox [3], is to take for W a Lévy process. In fact only a few papers deal with the discontinuous case, see for example E. Carmona [4] or A. Singh [5], and most of the results concerns continuous W, *i.e.* $(W(x) := B_x - \kappa/2x, x \in \mathbb{R})$, with $\kappa \in \mathbb{R}^+$ and B a two sided Brownian motion independent of β.
The case $\kappa > 0$ was first studied by K. Kawazu and H. Tanaka [6], then by H. Tanaka [7] and Y. Hu, Z. Shi and M. Yor [8], more recently by for example M. Taleb [9], and A. Devulder [10, 11]. The universal caracteristic of this X is the transience, however a wide range of limit behavior appears, depending on the value of κ, see [8].
In this paper we choose $\kappa = 0$, X is then recurrent and [3] shows that it is sub-diffusive with asymptotic behaviour in $(\log t)^2$, moreover X has the property, for a given instant t, to be localized in the neighborhood of a random point $m_{\log t}$ depending only on t and W. The limit law of $m_{\log t}/(\log t)^2$ and therefore of $X_t/(\log t)^2$ were made explicit independently by H. Kesten [12] and A. O. Golosov [13].
In fact, the aim of H. Kesten and A. O. Golosov was to determine the limit law of the discrete time and space analogous of Brox's model introduced by F. Solomon [14] and then studied by Ya. G. Sinai [15]. This random walk in random environment, usually called Sinai's walk, $(S_n, n \in \mathbb{N})$ has actually the same limit distribution as Brox's one.
Turning back to Brox's diffusion, notice that H. Tanaka [7, 16] obtained a deeper localization and later Y. Hu and Z. Shi [17] get the almost sure rates of convergence. It appears that these rates of convergence are exactly the same as the rate of convergence for Sinai's walk. The question of an invariance principle, that could exist between these two processes rises and remains open (see Z. Shi [18] for a survey). In fact a first attempt to link this two processes appears for the first time in the articles of S. Schumacher [2] and K. Kawazu, Y. Tamura and H. Tanaka [19].

This work is devoted to the limit distribution of the local time of X. Indeed to the diffusion X corresponds a local time process $(L_X(t,x))_{t\geq 0, x\in\mathbb{R}}$ defined by the occupation time formula : L_X is the unique $\mathbb{P}$-a.s. jointly continuous process such that for any bounded Borel function f and for any $t \geq 0$,

$$\int_0^t f(X_s)\mathrm{d}s = \int_{\mathbb{R}} f(x) L_X(t,x)\mathrm{d}x.$$

The first results on the behavior of L_X can be found in [20] and [21]. In particular in [21] it is proven that, for any $x \in \mathbb{R}$

$$\frac{\log(L_X(t,x))}{\log t} \xrightarrow{\mathcal{L}} U \wedge \hat{U}, \ t \to +\infty \tag{1}$$

where U and $\hat{U}$ are independent variables uniformely distributed in $(0,1)$ and $\xrightarrow{\mathcal{L}}$ is the convergence in law. Notice that in the same paper Y. Hu and Z. Shi also prove that this behavior is the same for Sinai's walk: if we denote by $L_S(n,x) := \sum_{i=1}^{n} \mathbb{1}_{S_i=x}$ the local time of S in $x \in \mathbb{Z}$ at time n then

$$\frac{\log(L_S(n,x))}{\log n} \xrightarrow{\mathcal{L}} U \wedge \hat{U}, \ n \to +\infty.$$

For original works on the local time for Sinai's diffusion we refer to the book of P. Révèsz [22].

In this article we show that the normalized local time process $(L(t, x + m_{\log t})/t, x \in [-K, K])$ with $K > 0$ is equivalent in probability to a well defined process, which depends only on t, $m_{\log t}$ and W. We also make explicit the limit law of this process when t goes to infinity, it involves some 3-dimensional Bessel processes. The supremum of the local time process of X is given by

$$\forall t \geq 0, \quad L_X^*(t) := \sup_{x \in \mathbb{R}} L_X(t,x),$$

as a consequence of our results we show that $L_X^*(t)/t$ converges weakly and determine its limit law. We also find interesting to compare the discrete Theorems of [1] with ours, pointing out the analogies and the differences.

1.2 Preliminary definitions and results

First let us describe the probability space where X is defined. It is composed of two Wiener's spaces, one for the environment and the other one for the diffusion itself:

Let $\mathcal{W}$ be the space of continuous functions $W : \mathbb{R} \to \mathbb{R}$ satisfying $W(0) = 0$ and $\mathcal{A}$ the σ-field generated by the topology of uniform convergence on compact sets on $\mathcal{W}$. We equip $(\mathcal{W}, \mathcal{A})$ with Wiener measure $\mathcal{P}$ i.e the coordinate process is a "two-sided" Brownian motion. We call *environment* an element of $\mathcal{W}$.

We also define the set $\Omega := C([0; +\infty[, \mathbb{R})$, the σ-field $\mathcal{F}$ on Ω generated by the topology of uniform convergence on compact sets and the probability measure P such that the coordinate process on Ω is a standard Brownian

motion.
We denote by $\mathbb{P}$ the probability product $\mathcal{P} \otimes P$ on $\mathcal{W} \times \Omega$. We indifferently denote by W an undetermined element of $\mathcal{W}$ and the first coordinate process on $\mathcal{W} \times \Omega$ (i.e a "two-sided" Brownian motion under $\mathbb{P}$ or P) similarly B is indifferently an element of Ω and the second coordinate process on $\mathcal{W} \times \Omega$. Finally $\overset{\mathcal{L}_W}{=}$ means "equality in law" under P that is under a fixed environment W, and $\overset{\mathcal{L}}{=}$ (respectively $\xrightarrow{\mathcal{L}}$) for an equality (resp. a convergence) in law under $\mathbb{P}$. We can now state our first result:

Theorem 1.1 *We have*

$$\frac{L_X^*(t)}{t} \xrightarrow{\mathcal{L}} \frac{1}{\int_{-\infty}^{\infty} e^{-R(y)} \mathrm{d}y} \tag{2}$$

where for any $x \in \mathbb{R}$, $R(x) := R_1(x)\mathbf{1}_{\{x \geq 0\}} + R_2(-x)\mathbf{1}_{\{x<0\}}$, R_1 and R_2 are two independent 3-dimensional Bessel processes starting at 0.

First notice that $\int_{-\infty}^{\infty} e^{-R(y)} \mathrm{d}y < +\infty$ *a.s.* , then we would like to state the equivalent of this Theorem for Sinai's walk recently found by [1],

$$\frac{L_S^*(n)}{n} \xrightarrow{\mathcal{L}} \sup_{x \in \mathbb{Z}} \pi(x) \tag{3}$$

where

$$\pi(x) = \frac{\exp(-Z_x) + \exp(-Z_{x-1})}{2\sum_{y \in \mathbb{Z}} \exp(-Z_y)}, \ x \in \mathbb{Z}, \tag{4}$$

and Z is a sum of i.i.d random variables (with mean zero, strictly positive variance and bounded) null at zero and conditionned to stay positive (see below Theorem 1.1 and Section 4 of [1] for the exact definition).
The analogy between the local time for X and the local time for S takes place in the fact that both R and S can be obtained from classical diffusion conditioned to stay positive, R_1 and R_2 are Brownian motions conditioned to stay positive (see [23]) and Z a simple symmetric random walk conditionned to stay positive (see [24] and [25]). Note also that A. O. Golosov also proved that $\sum_{y \in \mathbb{Z}} \exp(-Z_y) < +\infty$.
However Z and R have not the same nature, one is discrete the other one continuous, notice also that the increments of Z are supposed to be bounded (see hypothesis 1.2 in [1]), and it is not the case for R. Finally the numerator of $\pi(x)$ is not as simple as the numerator of our result, but this comes

essentially from the difference of nature of the two processes discrete for one and continuous for the other.

Theorem 1.1 is an easy consequence of an interesting intermediate result (Theorem 1.2 below). Before introducing that result we need some extra definitions on the environment, these basic notions have been introduced by [3], see also [26]. Let $h > 0$, we say that $W \in \mathcal{W}$ admits a *h-minimum* at x_0 if there exists ξ and ζ such that $\xi < x_0 < \zeta$ and for any $x \in [\xi, \zeta]$,

- $W(x) \geq W(x_0)$,
- $W(\xi) \geq W(x_0) + h$,
- $W(\zeta) \geq W(x_0) + h$.

Similarly we say that W admits a *h-maximum* at x_0 if $-W$ admits a h-minimum at x_0. We denote by $M_h(W)$ the set of *h-extrema* of W. It is easy to establish that $\mathcal{P}$-a.s. M_h has no accumulation point and that the points of h-maximum and of h-minimum alternate. Hence there exists exactly one triple $\Delta_h = (p_h, m_h, q_h)$ of elements in M_h such that

- m_h and 0 lay in $[p_h, q_h]$,
- p_h and q_h are h-maxima,
- m_h is a h-minimum.

We call this triple the standard h-valley of W (see Figure 1). We can now state our second result:

Theorem 1.2

$$\left(\frac{L_X(t, m_{\log t} + x)}{t}\right)_{x\in\mathbb{R}} \xrightarrow{\mathcal{L}} \left(\frac{e^{-R(x)}}{\int_{-\infty}^{\infty} e^{-R(y)}\mathrm{d}y}\right)_{x\in\mathbb{R}}$$

where R is the same as in Theorem 1.1.

This result is the analog of Theorem 1.2 of [1], we recall their result:

$$\left(\frac{L_S(n, b_n + x)}{n}, x \in \mathbb{Z}\right) \xrightarrow{\mathcal{L}} (\pi(x))_{x\in\mathbb{Z}}, \tag{5}$$

where $\pi(x)$ is given by (4) and b_n plays the same roll for S as $m_{\log t}$ plays for X. Notice that Theorem 1.1 can be deduced directly from Theorem 1.2, in the same way that (3) can be deduced from (5).

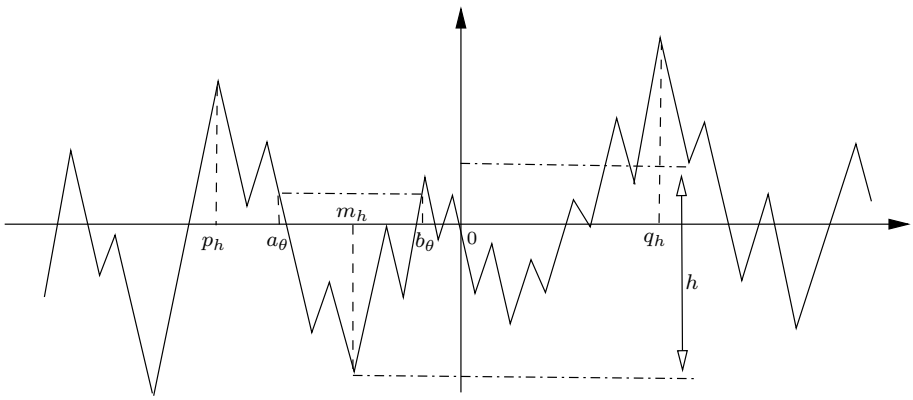

Figure 1: Example of standard valley

Theorems 1.1 and 1.2 are consequences of a result, in probability, on the asymptotic behavior of the local time in a neighborhood of $m_{\log t}$ (Theorem 1.3 below) together with results on the random environment (see Sections 3.2 and 3.3). Before stating our third result, we need a new notation, let $(W_x, x \in \mathbb{R})$ be the *shifted difference of potential*,

$$\forall x \in \mathbb{R}, W_x(\cdot) := W(x + \cdot) - W(x). \tag{6}$$

Theorem 1.3
Let $K > 0$, $r \in (0,1)$, then for all $\delta > 0$,

$$\lim_{\alpha \to +\infty} \mathbb{P}\left(\sup_{-K \le x \le K} \left| \frac{L_X(e^\alpha, m_\alpha + x)}{e^\alpha} \frac{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)} \mathrm{d}y}{e^{-W_{m_\alpha}(x)}} - 1 \right| \le \delta \right) = 1$$

where for any $\theta > 0$,

$$a_\theta = a_\theta(W_{m_\alpha}) := \sup\{x \le 0 / W_{m_\alpha}(x) \ge \theta\} \text{ and}$$
$$b_\theta = b_\theta(W_{m_\alpha}) := \inf\{x \ge 0 / W_{m_\alpha}(x) \ge \theta\},$$

see also Figure 1.

There is no real equivalent of this Theorem in the paper of [1]. An important term is the $\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)} \mathrm{d}y$, it appears naturally when we study the inverse of the local time in m_α (see section 2.2). The above Theorem leads, with a few more work, to the following Corollary where the convergence in probability of the local time process appears clearly.

Corollary 1.4 *Let $K > 0$, $c \geq 6$, and for all $\alpha > 1$ define $\theta_\alpha := (\log \alpha)^c$ then for all $\delta > 0$,*

$$\lim_{\alpha \to +\infty} \mathbb{P}\left(\sup_{-K \leq x \leq K} \left| \frac{L_X(e^\alpha, m_\alpha + x)}{e^\alpha} - \frac{e^{-W_{m_\alpha}(x)}}{\int_{-\theta_\alpha}^{\theta_\alpha} e^{-W_{m_\alpha}(y)} \mathrm{d}y} \right| \leq \delta \right) = 1.$$

The statement of this Corollary is pretty close to the equation (2.16) in the proof of Lemma 2.1 in [1], in fact equation (2.16) with a few more arguments (already present in [1]) says that

$$\lim_{n \to +\infty} \mathbb{P}\left(\sup_{-K \leq x \leq K} \left| \frac{L_S(n, b_n + x)}{n} - \frac{e^{-V_{b_n}(x)} + e^{-V_{b_n}(x-1)}}{2 \sum_{x \in R_n} e^{-V_{b_n}(x)}} \right| \leq \delta \right) = 1$$

where $V_{b_n}(x)$ is the shifted Sinai's difference of potential and R_n Sinai's basic valley centered in 0. It looks like again that $H_n := \sum_{x \in R_n} \exp(-Z_y) + \exp(-Z_{y-1})$ is close to $\mathcal{R}_\alpha := \int_{-\theta_\alpha}^{\theta_\alpha} e^{-W_{m_\alpha}(y)} \mathrm{d}y$, but there is a huge difference between these two terms apart from their nature (discrete and continuous). Indeed $H_n \geq 2$ ($Z_0 = 0$) but nothing equivalent can be say $\mathcal{R}_\alpha$, in fact $\mathcal{R}_\alpha$ can be as close to 0 as we want with a positive probability (see Lemma 5.4 of [18]). This remark drives us back to the work of Z. Shi [18], showing that the almost sure rate of convergence of Sinai's walk and Brox's diffusion are totally different.
To finish with this discussion, it is interesting to notice that for the discrete time model once Lemma 2.1 (in [1]) is proved, a little more work (Lemma 3.1 and 3.2) leads quite easily to an almost sure asymptotic result of L_S^*. Here things are completely different, and the results we get in this paper do not conduct to the almost sure behavior of L_X^*.

1.3 Basic facts for diffusion with potential

In this section we recall basic definitions and tools traditionally used to study diffusion in random environment. For all $W \in \mathcal{W}$, define

$$\forall x \in \mathbb{R}, S_W(x) := \int_0^x e^{W(y)} \mathrm{d}y \tag{7}$$

and

$$\forall t \geq 0, T_W(t) := \int_0^t e^{-2W(S_W^{-1}(B(s)))} \mathrm{d}s. \tag{8}$$

As Brox points out in [3], the standard diffusion theory implies that the process

$$\begin{array}{rcl} X : \mathcal{W} \times \Omega & \longrightarrow & \Omega \\ (W, B) & \longmapsto & S_W^{-1} \circ B \circ T_W^{-1} \end{array} \tag{9}$$

is under $\mathbb{P}$ a diffusion in Brownian environment. To simplify notations, we write when there is no possible mistake S and T for respectively S_W and T_W.

Using Formula (9), we easily obtain that for any $x \in \mathbb{R}$ and $t \geq 0$,

$$L_X(t,x) = e^{-W(x)} L_B(T^{-1}(t), S(x)) \tag{10}$$

where L_B is the local time process of the Brownian motion B.

T. Brox ([3]) noticed also that it is more convenient to study the asymptotic behavior of the process X_α defined below instead of the one of X,

$$X_\alpha(W,\cdot) := X(\alpha W,\cdot).$$

For all $x \in \mathbb{R}$, let us denote

$$W^\alpha(x) := \frac{1}{\alpha} W(\alpha^2 x).$$

As $\mathcal{P}$ is invariant under the transformation $W \to W^\alpha$ for each $\alpha > 0$, we have a link between X_α and X given by:

Lemma 1.5 *For each $W \in \mathcal{W}$ and $\alpha > 0$. Then for a fixed $W \in \mathcal{W}$,*

$$(X_\alpha(W^\alpha, t))_{t\geq 0} := (X(\alpha W^\alpha, t))_{t\geq 0} \overset{\mathcal{L}_W}{=} \left(\frac{1}{\alpha^2} X(W, \alpha^4 t)\right)_{t\geq 0},$$

$$\left(L_{X_\alpha(W^\alpha,\cdot)}(t,x)\right)_{t\geq 0, x\in\mathbb{R}} \overset{\mathcal{L}_W}{=} \left(\frac{1}{\alpha^2} L_{X(W,\cdot)}(\alpha^4 t, \alpha^2 x)\right)_{t\geq 0, x\in\mathbb{R}}.$$

We do not give any detail of the proof of this Lemma, the first relation can be found in Brox (see [3], Lemma 1.3) and the second is a straightforward consequence of the first one.

Formulas (7), (8) and (9) for X_α are given by

$$\forall t \geq 0, X_\alpha(t) = S_\alpha^{-1}(B(T_\alpha^{-1}(t))) \tag{11}$$

where

$$\forall x \in \mathbb{R}, S_\alpha(x) := S_{\alpha W}(x) = \int_0^x e^{\alpha W(y)} \mathrm{d}y \tag{12}$$

and

$$\forall t \geq 0, T_\alpha(t) := T_{\alpha W}(t) = \int_0^t e^{-2\alpha W(S_\alpha^{-1}(B(s)))} \mathrm{d}s, \tag{13}$$

also for the local time we have,

$$\forall t \geq 0, \forall x \in \mathbb{R}, L_{X_\alpha}(t,x) = e^{-\alpha W(x)} L_B(T_\alpha^{-1}(t), S_\alpha(x)). \tag{14}$$

The rest of the paper is organized as follows: in the first part of Section 2 we get the asymptotic of the local time within a random amount of time, which is the inverse of the local time at $m_{\log t}$, in Section 2.2 the asymptotic of the inverse of the local time itself is studied. Note that Sections 2.1 and 2.2 can be read independently.
Propositions 2.1 and 2.5 of Sections 2 are the key results to get Theorem 1.3 proved at the beginning of Section 3. In the second and third subsection of Section 3, Theorems 1.1, 1.2 and Corollary 1.4 are proved. Note that the Theorems come from Theorem 1.3 together with the study of a functional of the random environment involving 3-dimensional Bessel-square processes. The Corollary 1.4 comes from Theorem 1.3 together with estimates on the random environment.

2 Asymptotics for the local time L_{X_α} and its inverse σ_{X_α}

We begin with some definitions that will be used all along the paper. For any process M we define the following stopping times with the usual convention $\inf \emptyset = +\infty$,

$$\forall x \in \mathbb{R}, \tau_M(x) := \inf\{t \geq 0 / M(t) = x\}, \tag{15}$$

$$\forall x \in \mathbb{R}, \forall r \geq 0, \sigma_M(r, x) := \inf\{t \geq 0 / L_M(t, x) \geq r\}. \tag{16}$$

We define for any $W \in \mathcal{W}$ and for all $x, y \in \mathbb{R}$,

$$\overline{W}(x, y) := \begin{cases} \sup_{[x,y]} W & \text{if } y \geq x, \\ \sup_{[y,x]} W & \text{if } y < x \end{cases}$$

and

$$\underline{W}(x, y) := \begin{cases} \inf_{[x,y]} W & \text{if } y \geq x, \\ \inf_{[y,x]} W & \text{if } y < x, \end{cases}$$

they represent respectively the maximum and the minimum of W between x and y. Finally we introduce the process starting in $x \in \mathbb{R}$,

$$(X_\alpha^x(W, t))_{t \geq 0} := (x + X_\alpha(W_x, t))_{t \geq 0} = (x + X(\alpha W_x, t))_{t \geq 0}$$

where W_x is the *shifted difference of potential* (see (6)). Notice that we have the equivalent of (11):

$$\forall t \geq 0, X_\alpha^x(t) = x + (S_\alpha^x)^{-1}(B((T_\alpha^x)^{-1}(t))) \tag{17}$$

where $S_\alpha^x := S_{\alpha W_x}$ and $T_\alpha^x := T_{\alpha W_x}$ and it is easy to establish that for a fixed $W \in \mathcal{W}$, X_α^x is a strong Markov process.

2.1 Asymptotic behaviour of L_{X_α} at time $\sigma_{X_\alpha}(m, e^{\alpha h(\alpha)})$

In this first sub-section we study the asymptotic behaviour of the local time at the inverse of the local time in $m := m_1$, recall that m_1 is the coordinate of the bottom of the basic valley defined page 5.

Proposition 2.1
Let $K > 0$, $W \in \mathcal{W}$ and let h be a function such that $\lim_{\alpha \to +\infty} h(\alpha) = 1$. Then, for all $\delta > 0$,

$$\lim_{\alpha \to +\infty} P\left(\sup_{-K \le x \le K} \left| \frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m), m + \alpha^{-2} x)}{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2} x)}} - 1 \right| \le \delta \right) = 1.$$

Proof :
For simplicity, we assume without loss of generality that $m = m_1(W) \ge 0$ and to lighten notations, we denote for all $x \in [-K, K]$, $x_\alpha := m + \alpha^{-2} x$.

The proof is based on the decomposition of the local time into two terms, the first one is the contribution of the local time in x_α before $\tau_{X_\alpha}(m)$ (the first time X_α hits m) and the second one is the contribution of the local time between $\tau_{X_\alpha}(m)$ and $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)$ (the inverse of the local time in m):

$$\begin{aligned} L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m), x_\alpha) &= L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha) \\ &+ \left(L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m), x_\alpha) - L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha) \right). \end{aligned}$$

We treat this two terms in the Lemmata 2.2 and 2.3 below. Lemma 2.2 states that, asymptotically, the local time in a point x_α until the process reaches m is negligible compared to $e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2} x)}$. Thanks to the strong Markov property for X_α, it remains to study the asymptotic behaviour of

$$\left(L_{X_\alpha^m}(\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m), m + \alpha^{-2} x) \right)_{-K \le x \le K}$$

when α goes to infinity, this is what is done in Lemma 2.3 which says that the local time in x_α of X_α^m within the interval of time $[0, \sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)]$ is of the order of $e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2} x)}$. $\square$

Let us state and prove

Lemma 2.2 *For any $\delta > 0$,*

$$\lim_{\alpha\to+\infty} P\left(\sup_{-K\le x\le K} \frac{L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha)}{e^{\alpha h(\alpha)-\alpha W_m(\alpha^{-2}x)}} \le \delta\right) = 1.$$

Proof
First, as we have assumed that $m \ge 0$, for all $x > 0$ $L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha) = 0$ so we only have to consider non positive x. Notice also that for all $x \in [-K, 0]$, $L_{X_\alpha}(\tau_{X_\alpha}(m - \alpha^{-2}K), x_\alpha) = 0$, therefore

$$L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha) = L_{X_\alpha}(\tau_{X_\alpha}(m), x_\alpha) - L_{X_\alpha}(\tau_{X_\alpha}(m - \alpha^{-2}K), x_\alpha). \quad (18)$$

Let $\kappa_\alpha = m - \alpha^{-2}K$ thanks to (18) and the strong Markov property for X_α, we only need to prove that

$$\lim_{\alpha\to+\infty} P\left(\sup_{-K\le x\le 0} \frac{L_{X_\alpha^{\kappa_\alpha}}(\tau_{X_\alpha^{\kappa_\alpha}}(m), x_\alpha)}{e^{\alpha h(\alpha)-\alpha W_m(\alpha^{-2}x)}} \le \delta\right) = 1. \quad (19)$$

It follows from (17) with $x = \kappa_\alpha$ that

$$\tau_{X_\alpha^{\kappa_\alpha}}(m) = \tau_{X_\alpha(W_{\kappa_\alpha},\cdot)}(\alpha^{-2}K) = T_\alpha^{\kappa_\alpha}(\tau_B(S_\alpha^{\kappa_\alpha}(\alpha^{-2}K))),$$

so according to (14), we have for all $x \in \mathbb{R}$,

$$\begin{aligned} &L_{X_\alpha^{\kappa_\alpha}}(\tau_{X_\alpha^{\kappa_\alpha}}(m), x_\alpha) \quad = \\ &e^{-\alpha W_{\kappa_\alpha}(\alpha^{-2}(x+K))} L_B(\tau_B(S_\alpha^{\kappa_\alpha}(\alpha^{-2}K)), S_\alpha^{\kappa_\alpha}(\alpha^{-2}(x+K))). \end{aligned}$$

The classic scaling property of the local time of the Brownian motion given by:

$$\forall\lambda > 0, \forall y_1 > 0, \quad (\lambda L_B(\tau_B(y_1), y))_{y\in\mathbb{R}} \overset{\mathcal{L}_W}{=} (L_B(\tau_B(\lambda y_1), \lambda y))_{y\in\mathbb{R}}, \quad (20)$$

yields that the processes $\left(L_{X_\alpha^{\kappa_\alpha}}\left(\tau_{X_\alpha^{\kappa_\alpha}}(m), x_\alpha\right)\right)_{x\in\mathbb{R}}$ and

$$\left(S_\alpha^{\kappa_\alpha}(\alpha^{-2}K) e^{-\alpha W_{\kappa_\alpha}(\frac{x+K}{\alpha^2})} L_B\left(\tau_B(1), s_\alpha(\frac{x+K}{\alpha^2})\right)\right)_{x\in\mathbb{R}}$$

are equal in law, where $s_\alpha(z) := S_\alpha^{\kappa_\alpha}(z)/S_\alpha^{\kappa_\alpha}(\alpha^{-2}K)$.
We claim that for all $x \in [-K, 0]$

$$\begin{aligned} & S_\alpha^{\kappa_\alpha}(\alpha^{-2}K) e^{-\alpha W(x_\alpha)} L_B(\tau_B(1), s_\alpha(\alpha^{-2}(x+K))) \\ \le\ & \alpha^{-2}K e^{\alpha(\overline{W}_m(-\alpha^{-2}K,0) - W_m(\alpha^{-2}x))} \sup_{y\le 1} L_B(\tau_B(1), y). \end{aligned}$$

Indeed

$$S_\alpha^{\kappa_\alpha}(\alpha^{-2}K) = \int_0^{\frac{K}{\alpha^2}} e^{\alpha W_{\kappa_\alpha}(y)}\mathrm{d}y \leq \frac{K}{\alpha^2} e^{\alpha \overline{W}_{\kappa_\alpha}(0,\alpha^{-2}K)},$$

for all $x \in [-K, 0]$

$$\overline{W}_{\kappa_\alpha}(0, \alpha^{-2}K) - W_{\kappa_\alpha}(\alpha^{-2}(x+K)) = \overline{W}_m(-\alpha^{-2}K, 0) - W_m(\alpha^{-2}x)$$

and $s_\alpha(\alpha^{-2}(x+K)) \leq 1$.

Assembling the above estimates, we get for any $\delta > 0$,

$$\begin{aligned} & P\left(\sup_{-K\leq x\leq 0} \frac{L_{X_\alpha^{\kappa_\alpha}}(\tau_{X_\alpha^{\kappa_\alpha}}(m), x_\alpha)}{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}} \leq \delta\right) \\ \geq \quad & P\left(\alpha^{-2}Ke^{\alpha \overline{W}_m(-\alpha^{-2}K,0) - \alpha h(\alpha)} \sup_{y\leq 1} L_B(\tau_B(1), y) \leq \delta\right). \end{aligned}$$

As $\lim_{\alpha\to+\infty} \overline{W}_m(-\alpha^{-2}K, 0) = 0$, $\lim_{\alpha\to+\infty} h(\alpha) = 1$ and $\sup_{y\leq 1} L_B(\tau_B(1), y) < \infty$ P-a.s., the right hand side of the last inequality tends to 1 as α goes to infinity. (19) is proved together with the Lemma.□

We move to the proof of the second Lemma

Lemma 2.3 *For any $\delta > 0$,*

$$\lim_{\alpha\to+\infty} P\left(\sup_{-K\leq x\leq K} \left|\frac{L_{X_\alpha^m}(\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m), m + \alpha^{-2}x)}{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}} - 1\right| \leq \delta\right) = 1.$$

Proof

For simplicity we denote for any process M, $\sigma_M(r) := \sigma_M(r, 0)$, we will also assume without loss of generality that $m = 0$, Lemma 2.3 can therefore be rewritten in the following way:

$$\lim_{\alpha\to+\infty} P\left(\sup_{-K\leq x\leq K} \left|\frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}), \alpha^{-2}x)}{e^{\alpha h(\alpha) - \alpha W(\alpha^{-2}x)}} - 1\right| \leq \delta\right) = 1. \tag{21}$$

Like for τ in Lemma 2.2, we easily get that $\sigma_{X_\alpha}(t) = T_\alpha(\sigma_B(t))$, thus formula (14), together with the scale invariance for the local time of the Brownian motion

$$\forall r > 0, \forall \lambda > 0, (\lambda L_B(\sigma_B(r), y))_{y\in\mathbb{R}} \overset{\mathcal{L}_W}{=} (L_B(\sigma_B(\lambda r), \lambda y))_{y\in\mathbb{R}} \tag{22}$$

yields

$$\left(L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}),\alpha^{-2}x)\right)_{x\in\mathbb{R}} \overset{\mathcal{L}_W}{=} \left(e^{\alpha h(\alpha)-\alpha W(\alpha^{-2}x)}L_B(\sigma_B(1),\widetilde{s}_\alpha(\alpha^{-2}x))\right)_{x\in\mathbb{R}},$$

where $\widetilde{s}_\alpha(\alpha^{-2}x) := S_\alpha(\alpha^{-2}x)e^{-\alpha h(\alpha)}$. Let $K_\alpha := \alpha^{-2}Ke^{\alpha\overline{W}(-\alpha^{-2}K,\alpha^{-2}K)-\alpha h(\alpha)}$, we have for all $x\in[-K,K]$, $-K_\alpha \le \widetilde{s}_\alpha(\alpha^{-2}x) \le K_\alpha$. Collecting what we did above we get for any $\delta>0$,

$$\begin{aligned}
& P\left(\sup_{-K\le x\le K}\left|\frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}),\alpha^{-2}x)}{e^{\alpha h(\alpha)-\alpha W(\alpha^{-2}x)}}-1\right|\le\delta\right)\\
= \; & P\left(\sup_{-K\le x\le K}\left|L_B(\sigma_B(1),\widetilde{s}_\alpha(\alpha^{-2}x))-1\right|\le\delta\right),\\
\ge \; & P\left(\sup_{-K_\alpha\le y\le K_\alpha}|L_B(\sigma_B(1),y)-1|\le\delta\right).
\end{aligned}$$

Moreover $\lim_{\alpha\to+\infty} K_\alpha = 0$ and $y\to L_B(\sigma_B(1),y)$ is continuous at 0, so (21) and the Lemma are proved.□

2.2 Asymptotic behaviour of $\sigma_{X_\alpha}(m,e^{\alpha h(\alpha)})$

This section is devoted to the study of the asymptotic behavior of $\sigma_{X_\alpha}(e^{\alpha h(\alpha)})$, the main result is Proposition 2.5 below.
Before stating that Proposition we need a preliminary result on the random environment which gives precisions on the standard h-valley $\Delta_h(W) = (p_h,m_h,q_h)$ defined Section 1.2. We denote

$$W^{\#}(x,y) := \max_{[x,y]}\left(W(z)-\underline{W}(x,z)\right),$$

notice that the function $W^{\#}$ represents the largest barrier of potential we have to cross in the path from x to y. We call *depth of the valley* $\Delta_h(W)$ the quantity

$$D(\Delta_h(W)) := (W(p_h)-W(m_h))\wedge(W(q_h)-W(m_h))$$

and *inner directed ascent* the quantity

$$A(\Delta_h(W)) := W^{\#}(p_h,m_h)\vee W^{\#}(q_h,m_h).$$

Note that the above notions have already been introduced by Sinai [15], Brox [3], and Tanaka [7]. According to Brox, we have the following

Lemma 2.4 *There exists a subset $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ of $\mathcal{P}$-measure 1 such that for any $W \in \widetilde{\mathcal{W}}$, the standard 1-valley $\Delta_1(W) := (p_1, m_1, q_1)$ satisfies $A(\Delta_1(W)) < 1 < D(\Delta_1(W))$.*

Throughout this Section we write p, m, q, D and A for respectively $m_1(W)$, $p_1(W)$, $q_1(W)$, $D(\Delta_1(W))$ and $A(\Delta_1(W))$.

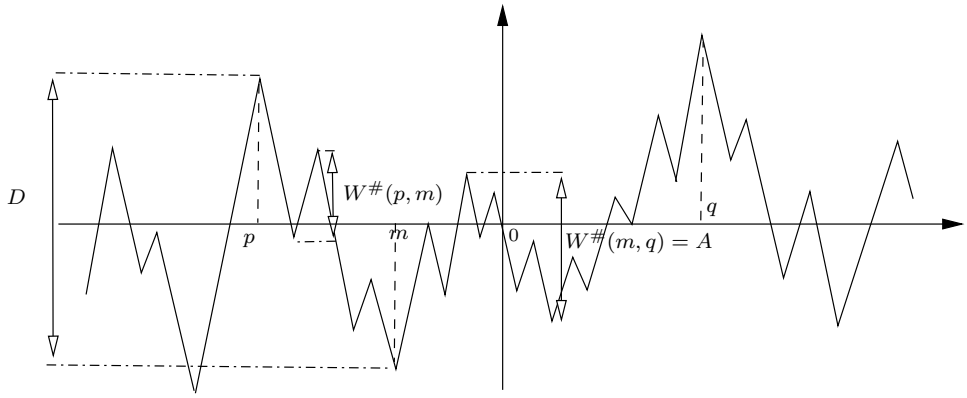

Figure 2: Example of 1-standard valley with its depth and its inner directed ascent.

We can now state the main result of this section:

Proposition 2.5
Let $W \in \widetilde{\mathcal{W}}$, $r \in (0,1)$ and h be a function such that $\lim_{\alpha \to +\infty} h(\alpha) = 1$, then for all $\delta > 0$,

$$\lim_{\alpha \to +\infty} P\left(\left| \frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)}{e^{\alpha h(\alpha)} \int_{a_r}^{b_r} e^{-\alpha W_m(x)} \mathrm{d}x} - 1 \right| \leq \delta \right) = 1$$

where a_r and b_r are defined in Theorem 1.3 page 6.

To lighten notations, in the rest of the paper we denote,

$$g(\alpha) := \int_{a_r}^{b_r} e^{-\alpha W_m(y)} \mathrm{d}y, \ \alpha > 0.$$

Proof :
We assume that $m > 0$, we get the other case by reflection, note that we

work at fixed W which belongs to $\widetilde{\mathcal{W}}$. We follow the same steps of the proof of Proposition 2.1: we decompose $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)$ into two terms,

$$\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m) = \tau_{X_\alpha}(m) + \left(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m) - \tau_{X_\alpha}(m)\right).$$

The first one $\tau_{X_\alpha}(m)$ is treated in Lemma 2.6, we show that its contribution is negligible comparing to $g(\alpha)e^{\alpha(\alpha)}$. Then thanks to the strong Markov property, it is enough to prove that $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)/g(\alpha)e^{\alpha(\alpha)}$ converge to 1 in P probability, this is what Lemma 2.7 tells. □

Let us state and prove the first Lemma

Lemma 2.6 *For any $\delta > 0$,*

$$\lim_{\alpha \to +\infty} P\left(\frac{\tau_{X_\alpha}(m)}{g(\alpha)e^{\alpha h(\alpha)}} \leq \delta\right) = 1.$$

Proof
This proof has the same outline of the proof of point (i) of Lemma 3.1 in [3], however because of some slight differences and for completeness we give some details.

By definition of the local time together with (10) and (20), the hitting time of m can be written

$$\begin{aligned}
\tau_{X_\alpha}(m) &= \int_{-\infty}^{m} L_{X_\alpha}(\tau_{X_\alpha}(m), z)\mathrm{d}z, \\
&= \int_{-\infty}^{m} e^{-\alpha W(z)} L_B(\tau_B(S_\alpha(m)), S_\alpha(z))\mathrm{d}z, \\
&\overset{\mathcal{L}_W}{=} S_\alpha(m) \int_{-\infty}^{m} e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z))\mathrm{d}z \qquad (23)
\end{aligned}$$

where $\hat{s}_\alpha(z) := S_\alpha(z)/S_\alpha(m)$. Let $n := \underset{[0,m]}{\operatorname{argmax}}\, W$ we denote

$$\begin{aligned}
I_{\alpha,1} &:= S_\alpha(m) \int_{-\infty}^{n} e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z))\mathrm{d}z, \\
I_{\alpha,2} &:= S_\alpha(m) \int_{n}^{m} e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z))\mathrm{d}z,
\end{aligned}$$

and formula (23) can be rewritten

$$\tau_{X_\alpha}(m) \overset{\mathcal{L}_W}{=} I_{\alpha,1} + I_{\alpha,2}. \qquad (24)$$

The rest of the proof consists essentially in finding an upper bound for $I_{\alpha,1}$ and $I_{\alpha,2}$.
We begin with $I_{\alpha,1}$, first we prove that, with a probability which tends to 1 when α goes to infinity, the process X_α does not visit coordinates smaller than p, where p is the left vertice of the standard 1-valley defined page 5. Thanks to this, the lower bound in the integral of $I_{\alpha,1}$ will be p and not $-\infty$, the upper bound for $I_{\alpha,1}$ follows almost immediatly.

Let us define

$$l := \inf\left\{x \leq 0 / L_B(\tau_B(1), x) > 0\right\}, \tag{25}$$

we claim that,

$$P.a.s,\ \exists \alpha_0 \text{ such that } \forall \alpha > \alpha_0, \hat{s}_\alpha^{-1}(l) > p. \tag{26}$$

Indeed

$$\hat{s}_\alpha^{-1}(l) \geq p \Longleftrightarrow l \geq \hat{s}_\alpha(p) = -\frac{\int_p^0 e^{\alpha W(x)}\mathrm{d}x}{\int_0^m e^{\alpha W(x)}\mathrm{d}x},$$

moreover Laplace's method gives

$$\begin{aligned}
\lim_{\alpha\to+\infty} \frac{1}{\alpha}\log\int_p^0 e^{\alpha W(x)}\mathrm{d}x &= \overline{W}(p,0) = W(p) \text{ and} \\
\lim_{\alpha\to+\infty} \frac{1}{\alpha}\log\int_0^m e^{\alpha W(x)}\mathrm{d}x &= \overline{W}(0,m) = W(n),
\end{aligned}$$

so

$$\hat{s}_\alpha(p) = -\exp\left(\alpha(W(p) - W(n)) + o(\alpha)\right),$$

finally according to the definition of the standard valley $W(p) > W(n)$, therefore

$$\lim_{\alpha\to+\infty} \hat{s}_\alpha(p) = -\infty$$

and (26) is true.

On the event $\left\{\hat{s}_\alpha^{-1}(l) > p\right\}$, we have

$$\begin{aligned}
I_{\alpha,1} &= \int_p^n e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z))\mathrm{d}z, \\
&\leq (n-p)e^{-\alpha \underline{W}(p,n)} \max_{x\leq 1} L_B(\tau_B(1), x),
\end{aligned}$$

moreover

$$S_\alpha(m) \leq m e^{\alpha \overline{W}(0,m)} \leq (q-p)e^{\alpha W(n)},$$

and we get the upper bound

$$I_{\alpha,1} \leq (q-p)^2 e^{\alpha A} \max_{x \leq 1} L_B(\tau_B(1), x) \tag{27}$$

where A is the inner direct ascent of the valley defined at the beginning of this section.

We continue with $I_{\alpha,2}$, the main ingredient to get an upper bound in this case is to use the first Ray-Knight which leads to the study of an integral involving a two-dimensional Bessel process: first we rewrite $I_{\alpha,2}$ in the following way

$$I_{\alpha,2} = S_\alpha(m) \int_n^m e^{-\alpha W(z)} L(\tau_B(1), 1 - \bar{s}_\alpha(z)) \mathrm{d}z \tag{28}$$

where

$$\bar{s}_\alpha := 1 - \hat{s}_\alpha(z) = \frac{1}{S_\alpha(m)} \int_z^m e^{\alpha W(x)} \mathrm{d}x.$$

Let R be a two-dimensional Bessel squared process starting from the origin, according to the First Ray-Knight theorem

$$\left(L_B(\tau_B(1), 1 - \bar{s}_\alpha(z))\right)_{z \in [0,m]} \overset{\mathcal{L}_W}{=} \left(R(\bar{s}_\alpha(z))\right)_{z \in [0,m]},$$

together with the scale invariance $(t^2 R_{\frac{1}{t}})_{t \in \mathbb{R}_+} \overset{\mathcal{L}_W}{=} (R_t)_{t \in \mathbb{R}_+}$ we get

$$\int_n^m e^{-\alpha W(z)} R(\bar{s}_\alpha(z)) \mathrm{d}z \overset{\mathcal{L}_W}{=} \int_n^m \left\{ e^{-\alpha W(z)} \bar{s}_\alpha(z) \right\} \bar{s}_\alpha(z) R(\frac{1}{\bar{s}_\alpha(z)}) \mathrm{d}z. \tag{29}$$

We are now able to get a preliminary upper bound for $I_{\alpha,2}$:

$$\begin{aligned}
& S_\alpha(m) \int_n^m \left\{ e^{-\alpha W(z)} \bar{s}_\alpha(z) \right\} \bar{s}_\alpha(z) R(\frac{1}{\bar{s}_\alpha(z)}) \mathrm{d}z, \\
\leq \quad & \max_{n \leq z \leq m} \left[e^{-\alpha W(z)} \int_z^m e^{\alpha W(x)} \mathrm{d}x \right] \int_n^m \bar{s}_\alpha(z) R(\frac{1}{\bar{s}_\alpha(z)}) \mathrm{d}z, \\
\leq \quad & (q-p) \exp\left\{ \alpha \max_{n \leq z \leq m} \left[-W(z) + \overline{W}(z,m) \right] \right\} \int_n^m \bar{s}_\alpha(z) R(\frac{1}{\bar{s}_\alpha(z)}) \mathrm{d}z, \\
\leq \quad & (q-p)^2 e^{\alpha A} \underbrace{\frac{1}{m-n} \int_n^m \bar{s}_\alpha(z) R(\frac{1}{\bar{s}_\alpha(z)}) \mathrm{d}z}_{J_\alpha}
\end{aligned} \tag{30}$$

and the last inequality comes from the relation $\max_{n \leq z \leq m} \left[-W(z) + \overline{W}(z,m) \right] = W^{\#}(n,m) \leq A$.

According to Jensen's inequality and Fubini's theorem the expectation of $(J_\alpha)^2$ satisfies

$$\begin{aligned} E_W\left[(J_\alpha)^2\right] &\leq \frac{1}{m-n}\int_n^m E_W[\bar{s}_\alpha^2(z)(R)^2(\frac{1}{\bar{s}_\alpha(z)})]\mathrm{d}z, \\ &= \frac{1}{m-n}\int_n^m \int_0^{+\infty} \frac{\bar{s}_\alpha(z)^3 y^2}{2} e^{-\frac{\bar{s}_\alpha(z)y}{2}}\mathrm{d}y\mathrm{d}z = 8. \end{aligned} \tag{31}$$

End of the proof of the Lemma: using (24), we obtain for all $\alpha > 0$

$$\begin{aligned} P\left(\frac{\tau_{X_\alpha}(m)}{g(\alpha)e^{\alpha h(\alpha)}} \leq \delta\right) &= P\left(\frac{S_\alpha(m)(I_{\alpha,1}+I_{\alpha,2})}{g(\alpha)e^{\alpha h(\alpha)}} \leq \delta\right), \\ \geq\ P\left(\frac{S_\alpha(m)I_{\alpha,1}}{g(\alpha)e^{\alpha h(\alpha)}} \leq \frac{\delta}{2}\ ;\ \hat{s}_\alpha^{-1}(l) \geq p\right) &+ P\left(\frac{S_\alpha(m)I_{\alpha,2}}{g(\alpha)e^{\alpha h(\alpha)}} \leq \frac{\delta}{2}\right) - 1. \end{aligned}$$

For the first term in the above expression, (27) yields

$$\begin{aligned} &P\left(\frac{S_\alpha(m)I_{\alpha,1}}{g(\alpha)e^{\alpha h(\alpha)}} \leq \frac{\delta}{2}\ ;\ \hat{s}_\alpha^{-1}(l) \geq p\right) \geq \\ &P\left(\max_{x\leq 1} L_B(\tau_B(1),x) \leq G(\alpha)\ ;\ \hat{s}_\alpha^{-1}(l) \geq p\right) \end{aligned} \tag{32}$$

where

$$G(\alpha) := \frac{\delta g(\alpha)}{2(q-p)^2} e^{\alpha(h(\alpha)-A)}.$$

By Laplace's method we know that $\lim_{\alpha\to+\infty} \frac{\log g(\alpha)}{\alpha} = 0$, so $g(\alpha) = e^{\circ(\alpha)}$, therefore as $\lim_{\alpha\to+\infty} h(\alpha) = 1 > A$, $G(\alpha)$ tends to infinity when α does. Using that $y \to L_B(\tau_B(1), y)$ is P-a.s. finite and (26) we get that (32) tends to 1 when α goes to infinity.

For the second term we collect (28), (29) and (30), we get

$$P\left(\frac{S_\alpha(m)I_{\alpha,2}}{g(\alpha)e^{\alpha h(\alpha)}} > \frac{\delta}{2}\right) \leq P\left(J_\alpha > G(\alpha)\right),$$

then by Tchebytchev's inequality and (31)

$$P\left(J_\alpha > G(\alpha)\right) \leq \frac{E_W\left[(J_\alpha)^2\right]}{(G(\alpha))^2} \leq \frac{8}{(G(\alpha))^2},$$

by using once again that $G(\alpha)$ tends to infinity we get the Lemma. □

Next step is to prove

Lemma 2.7 *For any $\delta > 0$,*

$$\lim_{\alpha\to+\infty} P\left(\left|\frac{\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m)}{g(\alpha)e^{\alpha h(\alpha)}} - 1\right| \leq \delta\right) = 1.$$

Proof:
Just like for the proof of Lemma 2.3 we assume without loss of generality that $m = 0$, as a consequence $\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m) = \sigma_{X_\alpha}(e^{\alpha h(\alpha)})$ and we simply have to establish that :

$$\lim_{\alpha\to+\infty} P\left(\left|\frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)})}{g(\alpha)e^{\alpha h(\alpha)}} - 1\right| \leq \delta\right) = 1. \tag{33}$$

In the same way we get (23), one can prove that

$$\sigma_{X_\alpha}(e^{\alpha h(\alpha)}) \overset{\mathcal{L}_W}{=} e^{\alpha h(\alpha)} \int_{-\infty}^{+\infty} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x))\mathrm{d}x, \tag{34}$$

recall that $\widetilde{s}_\alpha(y) = S_\alpha(y)e^{-\alpha h(\alpha)}$. The rest of the proof is devoted to estimate the integral and the main difficulty is to get the upper bound.
We begin with the lower bound, we easily get that

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x))\mathrm{d}x &\geq \int_{a_r}^{b_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x))\mathrm{d}x, \\ &\geq \inf_{y\in[\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y)g(\alpha),\end{aligned}$$

where a_r and b_r are defined at the end of Theorem 1.3, therefore

$$\begin{aligned}&P\left(\frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)})}{g(\alpha)e^{\alpha h(\alpha)}} \geq 1 - \delta\right) \geq \\ &P\left(\inf_{y\in[\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) \geq 1 - \delta\right).\end{aligned} \tag{35}$$

Also we recall that $r \in (0, 1)$ therefore we can prove easily by using the Laplace transform that $\lim_{\alpha\to+\infty} \widetilde{s}_\alpha(a_r) = \lim_{\alpha\to+\infty} \widetilde{s}_\alpha(b_r) = 0$. We conclude by noticing that $P - a.s.$, $\lim\limits_{\alpha\to+\infty} \inf_{y\in[\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) = 1$, thanks to the continuity of the function $y \to L_B(\sigma_B(1), y)$ at 0.
We continue with the upper bound. First we use the same idea of the proof of Lemma 2.6 when we had to deal with $I_{\alpha,1}$: we establish that with a

probability which tends to 1 as α goes to infinity, X_α does not exit from the standard valley (p, m, q). Define

$$L := \inf\{x \leq 0 / L_B(\sigma_B(1), x) > 0\}, U := \sup\{x \geq 0 / L_B(\sigma_B(1), x) > 0\},$$

we claim that,

$$P - a.s.\ \exists \alpha_0,\ \forall \alpha > \alpha_0, p < \widetilde{s}_\alpha^{-1}(L) < 0 < \widetilde{s}_\alpha^{-1}(U) < q. \tag{36}$$

Indeed, we have

$$\widetilde{s}_\alpha^{-1}(L) \geq p \Longleftrightarrow L \geq \widetilde{s}_\alpha(p) = -e^{-\alpha h(\alpha)} \int_p^0 e^{\alpha W(x)} \mathrm{d}x,$$

and by Laplace's method we get $\widetilde{s}_\alpha(p) = -e^{\alpha(W(p)-h(\alpha))+o(\alpha)}$. It follows from the fact that $W \in \widetilde{\mathcal{W}}$ and $\lim\limits_{\alpha \to +\infty} h(\alpha) = 1 < D \leq W(p)$ that $\lim_{\alpha \to +\infty} \widetilde{s}_\alpha(p) = -\infty,\ P$-a.s.. In a similar way we obtain $\lim_{\alpha \to +\infty} \widetilde{s}_\alpha(q) = +\infty,\ P$-a.s. and (36) is satisfied.

On the event $\left\{p < \widetilde{s}_\alpha^{-1}(L) < 0 < \widetilde{s}_\alpha^{-1}(U) < q\right\}$, we can write

$$\begin{aligned}
\int_{-\infty}^{+\infty} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x &= \int_p^{a_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x \\
&+ \int_{a_r}^{b_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x \\
&+ \int_{b_r}^{q} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x.
\end{aligned}$$

We only have to found an upper bound for these integrals, first we have

$$\begin{aligned}
&\int_p^{a_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x + \int_{b_r}^{q} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x \\
&\leq (q-p) \exp(-\alpha \min_{x \in I_r} W(x)) \sup_{y \in \mathbb{R}} L_B(\sigma_B(1), y)
\end{aligned}$$

where $I_r := [p, a_r] \cup [b_r, q]$, and moreover

$$\begin{aligned}
\int_{a_r}^{b_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x &\leq \sup_{y \in [\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) \int_{a_r}^{b_r} e^{-\alpha W(x)} \mathrm{d}x, \\
&= g(\alpha) \sup_{y \in [\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y).
\end{aligned}$$

Therefore, assembling the last two inequalities and the equality in law (34)

$$P\left(\frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)})}{g(\alpha)e^{\alpha h(\alpha)}}+1\leq\delta\right)\geq$$

$$P\left(\sup_{y\in[\widetilde{s}_\alpha(a_r),\widetilde{s}_\alpha(b_r)]}L_B(\sigma_B(1),y)-1+\frac{(q-p)}{g(\alpha)}e^{-\alpha\min_{I_r}W}\sup_{y\in\mathbb{R}}L_B(\sigma_B(1),y)\leq\delta\ ;\right.$$

$$\left.p<\widetilde{s}_\alpha^{-1}(L)<0<\widetilde{s}_\alpha^{-1}(U)<q\right). \tag{37}$$

By hypothesis $r<1$, so $\lim_{\alpha\to+\infty}\widetilde{s}_\alpha(a_r)=\lim_{\alpha\to+\infty}\widetilde{s}_\alpha(b_r)=0$, moreover $y\to L_B(\sigma_B(1),y)$ is P-a.s. continuous at 0, it follows

$$\lim_{\alpha\to+\infty}\sup_{y\in[\widetilde{s}_\alpha(a_r),\widetilde{s}_\alpha(b_r)]}L_B(\sigma_B(1),y)=1\quad P\text{-a.s..}$$

We also know that $\lim_{\alpha\to+\infty}\frac{\log g(\alpha)}{\alpha}=0$, moreover according to the definition of $\widetilde{\mathcal{W}}$, $\min_{x\in I_r}W(x)>0$, and finally $\sup_{y\in\mathbb{R}}L_B(\sigma_B(1),y)$ is P-a.s. finite, so

$$\lim_{\alpha\to+\infty}\frac{(q-p)}{g(\alpha)}e^{-\alpha\min_{I_r}W}\sup_{y\in\mathbb{R}}L_B(\sigma_B(1),y)=0\quad P\text{-a.s..}$$

Putting the last two assertions together with (37) we get the upper bound and finally the Lemma. □

3 Proof of the main results

One of the key result of this paper is Theorem 1.3, the other results can be deduced from that theorem together with estimates on the random environment, so we naturally start with the

3.1 Proof of Theorem 1.3

We begin with a Proposition which resume Propositions 2.1 and 2.5, we get the asymptotic behaviour of L_{X_α} within a deterministic interval of time :

Proposition 3.1
Let $K>0$, $r\in(0,1)$, $W\in\widetilde{\mathcal{W}}$ and h a real function such that $\lim_{\alpha\to\infty}h(\alpha)=1$. For all $\delta>0$, we have

$$\lim_{\alpha\to+\infty}P\left(\sup_{-K\leq x\leq K}\left|\frac{L_{X_\alpha}(e^{\alpha h(\alpha)},m+\alpha^{-2}x)}{e^{\alpha h(\alpha)}}\frac{\int_{a_r}^{b_r}e^{-\alpha W_m(y)}\mathrm{d}y}{e^{-\alpha W_m(\alpha^{-2}x)}}-1\right|\leq\delta\right)=1.$$

Proof :
Let $\delta > 0$, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{\alpha \to +\infty} f(\alpha) = 1$, define

$$A_{\alpha,f} := \left\{ \sup_{-K \le x \le K} \left| \frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha f(\alpha)}, m), m + \alpha^{-2}x)}{e^{\alpha f(\alpha) - \alpha W_m(\alpha^{-2}x)}} - 1 \right| \le \delta \right\}$$

and

$$B_{\alpha,f} := \left\{ \left| \frac{\sigma_{X_\alpha}(e^{\alpha f(\alpha)}, m)}{g(\alpha) e^{\alpha f(\alpha)}} - 1 \right| \le \delta \right\}$$

where, as in the previous section, $g(\alpha) = \int_{a_r}^{b_r} e^{-\alpha W_m(y)} \mathrm{d}y$. We also define two functions

$$\begin{aligned} h^+(\alpha) &:= h(\alpha) - \alpha^{-1} \log\left(g(\alpha)(1-\delta)\right), \\ h^-(\alpha) &:= h(\alpha) - \alpha^{-1} \log\left(g(\alpha)(1+\delta)\right). \end{aligned}$$

On B_{α,h^+} the following inequality holds:

$$\begin{aligned} \sigma_{X_\alpha}(e^{\alpha h^+(\alpha)}, m) &\ge (1-\delta) g(\alpha) e^{\alpha h^+(\alpha)}, \\ &\ge e^{\alpha h(\alpha)}, \end{aligned}$$

moreover in its first coordinate the local time is an increasing function, therefore on $A_{\alpha,h^+} \cap B_{\alpha,h^+}$, $\forall x \in [-K, K]$,

$$\begin{aligned} L_{X_\alpha}(e^{\alpha h(\alpha)}, m + \alpha^{-2}x) &\le L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h^+(\alpha)}, m), m + \alpha^{-2}x), \\ &\le e^{\alpha h^+(\alpha) - \alpha W_m(\alpha^{-2}x)}(1+\delta), \\ &\le \frac{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}}{g(\alpha)} \frac{1+\delta}{1-\delta}. \end{aligned}$$

In the same way, on $A_{\alpha,h^-} \cap B_{\alpha,h^-}$ we obtain

$$L_{X_\alpha}(e^{\alpha h(\alpha)}, m + \alpha^{-2}x) \ge \frac{e^{\alpha h(\alpha) - \alpha W_m(\alpha^{-2}x)}}{g(\alpha)} \frac{1-\delta}{1+\delta}.$$

By Laplace's method, $\lim_{\alpha \to +\infty} \frac{\log g(\alpha)}{\alpha} = 0$, so h^+ and h^- tend to 1 when α goes to infinity and we can apply Propositions 2.1 and 2.5, finally

$$\lim_{\alpha \to +\infty} P\left(A_{\alpha,h^+} \cap B_{\alpha,h^+} \cap A_{\alpha,h^-} \cap B_{\alpha,h^-}\right) = 1$$

and the Proposition is proved.□

We turn back to the proof of the Theorem, notice that the difference between Theorem 1.3 and Proposition 3.1 above is the process itself: one deals with X whereas the other deals with X_α. To finish the proof we need to show that thanks to Lemma 1.5 we can get the theorem from the proposition.

Let $\alpha > 0$, recall that $W^\alpha(.) := \alpha^{-1}W(\alpha^2\cdot)$. First, remark that for all $W \in \mathcal{W}$,

$$\begin{aligned} m_1(W^\alpha) &= \alpha^{-2}m_\alpha(W), \\ a_r(W^\alpha_{m_1(W^\alpha)}) &= \alpha^{-2}a_{\alpha r}(W_{m_\alpha(W)}), \\ b_r(W^\alpha_{m_1(W^\alpha)}) &= \alpha^{-2}b_{\alpha r}(W_{m_\alpha(W)}), \end{aligned}$$

and for any $x \in \mathbb{R}$,

$$W^\alpha_{m_1(W^\alpha)}(x) = \frac{1}{\alpha}W_{m_\alpha(W)}(\alpha^2 x).$$

Now replacing t by $\alpha^{-4}e^\alpha$ in the second part of Lemma 1.5 page 8, we obtain for all $W \in \mathcal{W}$,

$$\left(L_{X(\alpha W^\alpha,\cdot)}(\alpha^{-4}e^\alpha, m_1(W^\alpha) + \alpha^{-2}x)\right)_{x\in\mathbb{R}} \overset{\mathcal{L}_W}{=} \left(\frac{1}{\alpha^2}L_{X(W,\cdot)}(e^\alpha, m_\alpha(W) + x)\right)_{x\in\mathbb{R}}.$$

Therefore, for any $\alpha > 0$, $\delta > 0, K > 0$ and $W \in \mathcal{W}$,

$$\begin{aligned} &P\left(\sup_{-K\le x\le K}\left|\frac{L_{X(W,\cdot)}(e^\alpha, m_\alpha(W) + x)}{e^\alpha}\frac{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)}\mathrm{d}y}{e^{-W_{m_\alpha}(x)}} - 1\right| < \delta\right) = \\ &P\left(\sup_{-K\le x\le K}\left|\frac{L_{X(\alpha W^\alpha,\cdot)}(\alpha^{-4}e^\alpha, m_1(W^\alpha) + \alpha^{-2}x)}{\alpha^{-2}e^\alpha}\frac{\int_{\alpha^2 a_r}^{\alpha^2 b_r} e^{-\alpha W^\alpha_{m_1}(\alpha^{-2}y)}\mathrm{d}y}{e^{-\alpha W^\alpha_{m_1}(\alpha^{-2}x)}} - 1\right| < \delta\right). \end{aligned}$$

Moreover, for all $\alpha > 0$, $\mathcal{P}$ is invariant under the transformation $W \mapsto W^\alpha$, we get that

$$\begin{aligned} &\mathbb{P}\left(\sup_{-K\le x\le K}\left|\frac{L_{X(W,\cdot)}(e^\alpha, m_\alpha(W) + x)}{e^\alpha}\frac{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)}\mathrm{d}y}{e^{-W_{m_\alpha}(x)}} - 1\right| < \delta\right) = \\ &\mathbb{P}\left(\sup_{-K\le x\le K}\left|\frac{L_{X(\alpha W,\cdot)}(\alpha^{-4}e^\alpha, m_1(W) + \alpha^{-2}x)}{\alpha^{-4}e^\alpha}\frac{\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)}\mathrm{d}y}{e^{-\alpha W_{m_1}(\alpha^{-2}x)}} - 1\right| < \delta\right) \end{aligned}$$

and we recall that $\mathbb{P} = \mathcal{P} \otimes P$. To finish the proof we notice that $\mathcal{P}(\widetilde{\mathcal{W}}) = 1$, $\alpha^{-4}e^\alpha = e^{\alpha(1-\frac{4}{\alpha}\log\alpha)}$ and $\lim_{\alpha\to+\infty}(1 - \frac{4}{\alpha}\log\alpha) = 1$, so applying Proposition

3.1 we get Theorem 1.3. □

As we said at the begining of the section, once Theorem 1.3 is proved, we get the other results by studying in details some properties of the random environment. We continue with the

3.2 Proof of Theorems 1.1 and 1.2

We recall that Theorem 1.1 is a straight forward consequence of Theorem 1.2, so we are left to prove Theorem 1.2. The main ingredients to get this theorem are Theorem 1.3, Lemmata 3.2 and 3.3 below. We begin with the proof of the first Lemma:

We recall that c is a real number greater than 6 and θ_α stands for $(\log \alpha)^c$.

Lemma 3.2 *For all $\delta > 0$,*

$$\lim_{\alpha \to +\infty} \mathcal{P}\left(\left| \frac{\int_{a_{\alpha/2}}^{b_{\alpha/2}} e^{-W_{m_\alpha}(y)} \mathrm{d}y}{\int_{-\theta_\alpha}^{\theta_\alpha} e^{-W_{m_\alpha}(y)} \mathrm{d}y} - 1 \right| \leq \delta \right) = 1. \tag{38}$$

Moreover, for all positive and increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\alpha \to} \rho(\alpha) = +\infty$,

$$\lim_{\alpha \to +\infty} \mathcal{P}\left(\int_{a_{\alpha/2}}^{b_{\alpha/2}} e^{-W_{m_\alpha}(y)} \mathrm{d}y \geq \frac{1}{\rho(\alpha)} \right) = 1. \tag{39}$$

Proof :
First notice that by the iterated logarithm law, $\mathcal{P}$ almost surely for α large enough,

$$\theta_\alpha \leq |a_{\alpha/2}| \leq \alpha^3, \text{ and } \theta_\alpha \leq b_{\alpha/2} \leq \alpha^3.$$

Let us define $\bar{I}_\alpha = [a_{\alpha/2}, -\theta_\alpha] \cup [\theta_\alpha, b_{\alpha/2}]$, to get the Lemma, we decompose the integral in the following way $\int_{a_{\alpha/2}}^{b_{\alpha/2}} = \int_{\bar{I}_\alpha} + \int_{-\theta_\alpha}^{\theta_\alpha}$. We begin with the

Contribution of $\int_{\bar{I}_\alpha}$
Let $c_0 \geq 10$, for all $\alpha > 0$ we denote $u_\alpha := c_0 \log \alpha$. According to Cheliotis, [27] Lemma 13, $\mathcal{P}$-almost surely, for α large enough,

$$\min_{\bar{I}_\alpha} W_{m_\alpha} > u_\alpha,$$

so, $\mathcal{P}$-a.s for α large enough,

$$\begin{aligned}\int_{[a_{\alpha/2},b_{\alpha/2}]} e^{-W_{m_\alpha}(y)}\mathrm{d}y - \int_{[-\theta_\alpha,\theta_\alpha]} e^{-W_{m_\alpha}(y)}\mathrm{d}y &= \int_{[a_{\alpha/2},-\theta_\alpha]\cup[\theta_\alpha,b_{\alpha/2}]} e^{-W_{m_\alpha}(y)}\mathrm{d}y,\\ &\leq (b_{\alpha/2}-a_{\alpha/2})e^{-u_\alpha},\\ &\leq 2\alpha^{3-c_0}.\end{aligned}$$

A lower bound for $\int_{a_{\alpha/2}}^{b_{\alpha/2}}$ We define for all $\alpha > 0$, $u, v > 0$,

$$H_\alpha^+ := \inf\left\{x \geq 0/W^{\#}(0,x) \geq \alpha\right\},\ m_\alpha^+ := \underset{[0,H_\alpha^+]}{\mathrm{argmin}} W,$$

$$d_{\alpha/2}^+ := \inf\left\{x \geq 0/W_{m_\alpha^+}(x) \geq \alpha/2\right\},\ \text{ and } E_{u,v}(\alpha) := \left\{\max_{[0,v]} W_{m_\alpha^+} > u\right\},$$

we recall that $W^{\#}(x,y) := \max_{[x,y]}(W(z) - \underline{W}(x,z))$. Then, if we denote by R_0 a three dimensional Bessel process starting from 0, we have

$$\begin{aligned}\mathcal{P}(E_{u,v}(\alpha)) &= \mathcal{P}(\max_{[0,v]} W_{m_\alpha^+} > u),\\ &= \mathcal{P}(\max_{[0,v]} R_0 > u),\\ &= \mathcal{P}(\tau_R(u) < v).\end{aligned}$$

So, according to Borodin [28], Property 2.0.2 Part II section 5,

$$\begin{aligned}\mathcal{P}(E_{u,v}(\alpha)) &= \int_0^v \frac{2u}{\sqrt{2\pi}y^{3/2}} \sum_{n=0}^{+\infty} e^{-(2k+1)^2u^2/(2y)}\left(\frac{(2k+1)^2u^2}{y} - 1\right)\mathrm{d}y\\ &= \frac{4u}{\sqrt{2\pi v}} \sum_{n=0}^{+\infty} e^{-(2n+1)^2u^2/(2v)} \leq \frac{4u}{\sqrt{2\pi v}} \frac{e^{-u^2/(2v)}}{1-e^{-u^2/v}}.\end{aligned}$$

Taking $v_\alpha := \frac{u^2}{2\rho(\alpha)}$, where ρ is a positive and increasing function tending to $+\infty$ with α, we get

$$\mathcal{P}(E_{u,v_\alpha}(\alpha)) \leq c_1\sqrt{\rho(\alpha)}\exp(-\rho(\alpha)), \tag{40}$$

where $c_1 > 0$. $\mathcal{P}$ almost surely for α large enough

$$\begin{aligned}\int_0^{d_{\alpha/2}^+} e^{-W_{m_\alpha^+}(y)}\mathrm{d}y &\geq \int_0^{v_\alpha} e^{-W_{m_\alpha^+}(y)}\mathrm{d}y,\\ &\geq v_\alpha \exp\left(-\max_{[0,v_\alpha]} W_{m_\alpha^+}\right).\end{aligned} \tag{41}$$

(40) together with (41) yields

$$\mathcal{P}\left(\int_0^{d^+_{\alpha/2}} e^{-W_{m^+_\alpha}(y)}\mathrm{d}y < 1/\rho(\alpha)\right) \quad \leq \quad c_1\sqrt{\rho(\alpha)}\exp(-\rho(\alpha)).$$

Defining in the same way

$$H^-_\alpha := \sup\left\{x \leq 0/W^{\#}(x,0) \geq \alpha\right\},\ m^-_\alpha := \underset{[H^-_\alpha,0]}{\mathrm{argmin}} W \text{ and}$$
$$d^-_\alpha := \sup\left\{x \leq 0/W_{m^-_\alpha}(x) \geq \alpha/2\right\},$$

we easily proved that

$$\mathcal{P}\left(\int_{d^-_{\alpha/2}}^0 e^{-W_{m^-_\alpha}(y)}\mathrm{d}y < 1/\rho(\alpha)\right) \quad \leq \quad c_1\sqrt{\rho(\alpha)}\exp(-\rho(\alpha)). \tag{42}$$

According to Kesten [12], $m_\alpha \in \{m^-_\alpha, m^+_\alpha\}$, therefore

$$\begin{aligned}
&\mathcal{P}\left(\int_{a_{\alpha/2}}^{b_{\alpha/2}} e^{-W_{m_\alpha}(y)}\mathrm{d}y < 1/\rho(\alpha)\right)\\
\leq\ &\mathcal{P}\left(\int_0^{d^+_{\alpha/2}} e^{-W_{m^+_\alpha}(y)}\mathrm{d}y < 1/\rho(\alpha)\right)\mathcal{P}\left(m_\alpha = m^+_\alpha\right) +\\
&\mathcal{P}\left(\int_{d^-_{\alpha/2}}^0 e^{-W_{m^-_\alpha}(y)}\mathrm{d}y < 1/\rho(\alpha)\right)\mathcal{P}\left(m_\alpha = m^-_\alpha\right).
\end{aligned}$$

Assembling the above estimates we get a lower bound for $\int_{a_{\alpha/2}}^{b_{\alpha/2}}$ and therefore (39), and (38) follows by choosing for $\rho(.)$ a function asymptoticaly larger than $(\log .)^{3-c_0}$. $\square$

Considering Theorem 1.3 together with Lemma 3.2, the proof of the theorem will be finished once we will have shown

$$\left(\frac{e^{-W_{m_\alpha}(x)}}{\int_{-\theta_\alpha}^{\theta_\alpha} e^{-W_{m_\alpha}(y)}\mathrm{d}y}\right)_{x\in\mathbb{R}} \xrightarrow{\mathcal{L}} \left(\frac{e^{-R(x)}}{\int_{-\infty}^{\infty} e^{-R(y)}\mathrm{d}y}\right)_{x\in\mathbb{R}}.$$

We recall that$\forall x \in \mathbb{R}, R(x) := R_1(x)\mathbf{1}_{\{x\geq 0\}} + R_2(-x)\mathbf{1}_{\{x<0\}}$, R_1 and R_2 being two independent 3-dimensional Bessel processes starting from 0.
The proof is done in the following

Lemma 3.3 *For any positive constant K and any bounded continuous functional $F \equiv F_K$ on $C(\mathbb{R},\mathbb{R})$ such that $F(f)$ depends only on the values of the function f on $[-K,K]$,*

$$\lim_{\alpha\to+\infty} \mathbb{E}\left[F\left(\frac{e^{-W_{m_\alpha}}}{\int_{-\theta_\alpha}^{\theta_\alpha} e^{-W_{m_\alpha}(y)}\mathrm{d}y}\right)\right] = \mathbb{E}\left[F\left(\frac{e^{-R}}{\int_{-\infty}^{\infty} e^{-R(y)}\mathrm{d}y}\right)\right]. \tag{43}$$

Proof :
We start the proof by recalling a:
Lemma of Tanaka ([7]) First let us introduce some new notations. On a suitable probability space we consider a process $(\omega^+(x))_{x\in\mathbb{R}}$ such that $(\omega^+(x))_{x\geq 0}$ and $(\omega^+(-x))_{x\geq 0}$ are independent reflecting Brownian motions on $[0,+\infty)$ starting at 0. Let $(l^+(x))_{x\in\mathbb{R}}$ be the local time at 0 of ω^+. Let $\beta^+ := \omega^+ + l^+$, by Pitman's theorem (see [29]) $(\beta^+(x))_{x\geq 0}$ and $(\beta^+(-x))_{x\geq 0}$ are independent Bessel processes of dimension 3 starting at 0. For $\alpha > 0$ we define ρ_α^+ as the smallest zero of ω^+ in $(z,0]$ where $z := \max\{x < 0/\omega^+(x) = \alpha\}$. We also introduce another process $(\omega^-(x), x \in \mathbb{R})$ which is equal in law to $(\omega^+(x), x \in \mathbb{R})$. We assume that ω^+ and ω^- are defined on a common probability space $(\overline{\Omega}, \overline{P})$ and that they are independent. We denote by l^- the local time at 0 of ω^- and similarly β^- (resp. ρ_α^-) the equivalent of β^+ (resp. ρ_α^+). We now define quantities related to W and then to β^+ and β^-, recall the definition of $H_\alpha^\pm$ and $m_\alpha^\pm$ in the proof of Lemma 3.2, then

$$\begin{aligned}
b_\alpha^+ &:= H_\alpha^+ - m_\alpha^+,\ b_\alpha^- := H_\alpha^- - m_\alpha^-,\\
\overline{b}_\alpha^+ &:= \inf\{x > 0/\beta^+(x) = \alpha\} - \rho_\alpha^+,\\
\overline{b}_\alpha^- &:= -(\inf\{x > 0/\beta^-(x) = \alpha\} - \rho_\alpha^-),\\
\overline{m}_\alpha^+ &:= -\rho_\alpha^+,\\
\overline{m}_\alpha^- &:= \rho_\alpha^-.
\end{aligned}$$

Note that we have used the same letters b and m for W and β because they typically play the same role. We also introduce processes allowing us to condition the values of m_α :

$$\begin{aligned}
J_\alpha^+ &:= \max\left(\overline{W}(0, m_\alpha^+), \alpha + W(m_\alpha^+)\right),\\
J_\alpha^- &:= \max\left(\overline{W}(m_\alpha^-, 0), \alpha + W(m_\alpha^-)\right),
\end{aligned}$$

and the analog one for $\beta^\pm$

$$\bar{J}_\alpha^\pm := (\max_{[\rho_\alpha^\pm, 0]} \beta^\pm - \beta^\pm(\rho_\alpha^\pm)) \vee (\alpha - \beta^\pm(\rho_\alpha^\pm)).$$

We recall that Kesten [12] shows that the process $m_\alpha \in \{m_\alpha^-, m_\alpha^+\}$ according to the values of J_α^+ and J_α^-, more precisely

$$m_\alpha = \begin{cases} m_\alpha^+ & \text{if} \quad J_\alpha^+ \leq J_\alpha^-, \\ m_\alpha^- & \text{if} \quad J_\alpha^+ > J_\alpha^-. \end{cases}$$

Tanaka proved in [16] that :

Lemma 3.4 *The processes*

$$\left(W_{m_\alpha^+}(x)\right)_{-m_\alpha^+ \leq x \leq b_\alpha^+},$$

and

$$\left(W_{m_\alpha^-}(-x)\right)_{-m_\alpha^- \leq x \leq b_\alpha^-}$$

are independent and have the same distribution as

$$\left(\beta^+(x)\right)_{-\overline{m}_\alpha^+ \leq x \leq \overline{b}_\alpha^+}.$$

End of the proof: Let us denote

$$F_\alpha := F\left(\frac{e^{-W_{m_\alpha}}}{\int_{-\theta_\alpha}^{\theta_\alpha} e^{-W_{m_\alpha}(y)}\mathrm{d}y}\right),$$

thanks to Kesten [12],

$$\begin{aligned} \mathbb{E}\left[F_\alpha\right] &= \frac{1}{2}\left(\mathbb{E}\left[F_\alpha | m_\alpha = m_\alpha^+\right] + \mathbb{E}\left[F_\alpha | m_\alpha = m_\alpha^-\right]\right), \\ &= \frac{1}{2}\left(\mathbb{E}\left[F_\alpha | J_\alpha^+ \leq J_\alpha^-\right] + \mathbb{E}\left[F_\alpha | J_\alpha^+ > J_\alpha^-\right]\right). \end{aligned}$$

Both terms in the above equation are treated in the same way, we only discuss the first one $\mathbb{E}\left[F_\alpha | J_\alpha^+ \leq J_\alpha^-\right]$, we denote

$$A_\alpha := \{K \vee \theta_\alpha < b_\alpha^+ \wedge m_\alpha^+\} \cap \{J_\alpha^+ \leq J_\alpha^-\},$$

as $\mathcal{P}$-a.s. for α large enough, $(K \vee \theta_\alpha) < (b_\alpha^+ \wedge m_\alpha^+)$, and F bounded the following equality holds

$$\mathbb{E}\left[F_\alpha | J_\alpha^+ \leq J_\alpha^-\right] = \mathbb{E}\left[F_\alpha | A_\alpha\right] + \circ(1)$$

where $\lim_{\alpha\to+\infty} \circ(1) = 0$. We can apply Lemma 3.4, and therefore move to the process β^+,

$$\mathbb{E}\left[F_\alpha | A_\alpha\right] = \mathbb{E}\left[F\left(\frac{e^{-\beta^+}}{\int_{-\theta_\alpha}^{\theta_\alpha} e^{-\beta^+(y)}\mathrm{d}y}\right)\middle| \overline{A}_\alpha\right].$$

In the same way we have defined A_α, let

$$\overline{A}_\alpha := \{K \vee \theta_\alpha < \overline{b}_\alpha^+ \wedge \overline{m}_\alpha^+\} \cap \{\bar{J}_\alpha^+ \leq \bar{J}_\alpha^-\}$$

and for the same reason as above,

$$\mathbb{E}\left[F\left(\frac{e^{-\beta^+(x)}}{\int_{-\theta_\alpha}^{\theta_\alpha} e^{-\beta^+(y)}\mathrm{d}y}\right)\middle| \overline{A}_\alpha\right] = \mathbb{E}\left[F\left(\frac{e^{-\beta^+(x)}}{\int_{-\theta_\alpha}^{\theta_\alpha} e^{-\beta^+(y)}\mathrm{d}y}\right)\middle| \bar{J}_\alpha^+ \leq \bar{J}_\alpha^-\right] + \circ(1).$$

Assembling the above estimates, we get

$$\mathbb{E}\left[F_\alpha | J_\alpha^+ \leq J_\alpha^-\right] = \mathbb{E}\left[F\left(\frac{e^{-\beta^+}}{\int_{-\theta_\alpha}^{\theta_\alpha} e^{-\beta^+(y)}\mathrm{d}y}\right)\middle| \bar{J}_\alpha^+ \leq \bar{J}_\alpha^-\right] + \circ(1),$$

in the same way,

$$\mathbb{E}\left[F_\alpha | J_\alpha^+ > J_\alpha^-\right] = \mathbb{E}\left[F\left(\frac{e^{-\beta^+}}{\int_{-\theta_\alpha}^{\theta_\alpha} e^{-\beta^+(y)}\mathrm{d}y}\right)\middle| \bar{J}_\alpha^+ > \bar{J}_\alpha^-\right] + \circ(1).$$

This two last expressions together with the continuity of F and the fact that it is bounded yields

$$\lim_{\alpha\to+\infty} \mathbb{E}\left[F_\alpha\right] = \mathbb{E}\left[F\left(\frac{e^{-R}}{\int_{-\infty}^{\infty} e^{-R(y)}\mathrm{d}y}\right)\right]$$

and the lemma is established. □

3.3 Proof of Corollary 1.4

First we notice that by modifying slightly our proof using Ray-Knight theorem, basic facts about Brownian motion and 0-dimensional Bessel process, we get the following modification of the statement of Theorem 1.3, taking $r = 1/2$

$$\lim_{\alpha\to+\infty} \mathbb{P}\left(\sup_{-K\leq x\leq K}\left|\frac{L_X(e^\alpha, m_\alpha + x)}{e^\alpha}\frac{g'(\alpha)}{e^{-W_{m_\alpha}(x)}} - 1\right| \leq \frac{1}{\rho_0(\alpha)}\right) = 1$$

where $\rho_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive, increasing function such that $\lim_{\alpha\to} \rho_0(\alpha) = +\infty$, and $g'(\alpha) = \int_{a_{\alpha/2}}^{b_{\alpha/2}} e^{-W_{m_\alpha}(y)}\mathrm{d}y$. It follows from the continuity of W that

$$\lim_{\alpha\to+\infty} \mathbb{P}\left(\sup_{-K\le x\le K} \left| \frac{L_X(e^\alpha, m_\alpha + x)}{e^\alpha} - \frac{e^{-W_{m_\alpha}(x)}}{g'(\alpha)} \right| \le \frac{\bar{c}}{\rho_0(\alpha)g'(\alpha)} \right) = 1$$

where $\bar{c}$ is a constant.
To get the corollary from the above expression we need two estimates on the random environment. The first one says that $g'(\alpha)$ is not too small, it is the second part of Lemma 3.2. The second one, given by (38), allows us to exchange the random upper and lower bounds in the integral $\int_\cdot^\cdot e^{-W_{m_\alpha}(y)}\mathrm{d}y$ with deterministic one. As conclusion, the corollary is a consequence of Theorem 1.3 with $r = 1/2$ and of Lemma 3.2.

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