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# HOW TO INTEGRATE A POLYNOMIAL OVER A SIMPLEX

V. BALDONI, N. BERLINE, J. A. DE LOERA, M. KÖPPE, AND M. VERGNE

ABSTRACT. This paper settles the computational complexity of the problem of integrating a polynomial function  $f$  over a rational simplex. We prove that the problem is NP-hard for arbitrary polynomials via a generalization of a theorem of Motzkin and Straus. On the other hand, if the polynomial depends only on a fixed number of variables, while its degree and the dimension of the simplex are allowed to vary, we prove that integration can be done in polynomial time. As a consequence, for polynomials of fixed total degree, there is a polynomial time algorithm as well. We conclude the article with extensions to other polytopes and discussion of other available methods.

## 1. INTRODUCTION

Let  $\Delta$  be a  $d$ -dimensional rational simplex inside  $\mathbb{R}^n$  and let  $f \in \mathbb{Q}[x_1, \dots, x_n]$  be a polynomial with rational coefficients. We consider the problem of how to efficiently compute the *exact* value of the integral of the polynomial  $f$  over  $\Delta$ , which we denote by  $\int_{\Delta} f \, dm$ . We use here the *integral Lebesgue measure*  $dm$  on the affine hull  $\langle \Delta \rangle$  of the simplex  $\Delta$ , defined below in section 2.1. This normalization of the measure occurs naturally in Euler–Maclaurin formulas for a polytope  $P$ , which relate sums over the lattice points of  $P$  with certain integrals over the various faces of  $P$ . For this measure, the volume of the simplex and every integral of a polynomial function with rational coefficients are *rational numbers*. Thus the result has a representation in the usual (Turing) model of computation. This is in contrast to other normalizations, such as the induced Euclidean measure, where irrational numbers appear.

The main goals of this article are to discuss the computational complexity of the problem and to provide methods to do the computation that are both theoretically efficient and have reasonable performance in concrete examples.

Computation of integrals of polynomials over polytopes is fundamental for many applications. We already mentioned summation over lattice points of a polytope. They also make an appearance in recent results in optimization problems connected to moment matrices [20]. These integrals are also commonly computed in finite element methods, where the domain is decomposed into cells (typically simplices) via a

mesh and complicated functions are approximated by polynomials (see for instance [28]). When studying a random univariate polynomial  $p(x)$  whose coefficients are independent random variables in certain intervals, the probability distribution for the number of real zeros of  $p(x)$  is given as an integral over a polytope [7]. Integrals over polytopes also play a very important role in statistics, see, for instance, [22]. Remark that among all polytopes, simplices are the fundamental case to consider for integration since any convex polytope can be triangulated into finitely many simplices.

Regarding the computational complexity of our problem, one can ask what happens with integration over arbitrary polytopes. It is very educational to look first at the case when  $f$  is the constant polynomial 1, and the answer is simply a volume. It has been proved that already computing the volume of polytopes of varying dimension is #P-hard [13, 9, 17, 21], and that even approximating the volume is hard [14]. More recently in [25] it was proved that computing the centroid of a polytope is #P-hard. In contrast, for a simplex, the volume is given by a determinant, which can be computed in polynomial time. One of the key contributions of this paper is to settle the computational complexity of integrating a non-constant polynomial over a simplex.

Before we can state our results let us understand better the input and output of our computations. Our output will always be the rational number  $\int_{\Delta} f \, dm$  in the usual binary encoding. The  $d$ -dimensional input simplex will be represented by its vertices  $\mathbf{s}_1, \dots, \mathbf{s}_{d+1}$  (a  $V$ -representation) but note that, in the case of a simplex, one can go from its representation as a system of linear inequalities (an  $H$ -representation) to a  $V$ -representation in polynomial time, simply by computing the inverse of a matrix.

Thus the encoding size of  $\Delta$  is given by the number of vertices, the dimension, and the largest binary encoding size of the coordinates among vertices. Computations with polynomials also require that one specifies concrete data structures for reading the input polynomial and to carry on the calculations. There are several possible choices. One common representation of a polynomial is as a sum of monomial terms with rational coefficients. Some authors assume the representation is *dense* (polynomials are given by a list of the coefficients of all monomials up to a given total degree  $r$ ), while other authors assume it is *sparse* (polynomials are specified by a list of exponent vectors of monomials with non-zero coefficients, together with their coefficients). Another popular representation is by *straight-line programs*. A straight-line program which encodes a polynomial is, roughly speaking, a program without branches which enables us to evaluate it at any given point (see [11, 23] and references therein). As we explain in Section 2, general straight-line programs are *too compact* for our purposes, so instead we restrict to a subclass we call *single-intermediate-use (division-free) straight-line*

programs or *SIU straight-line programs* for short. The precise definition and explanation will appear in Section 2 but for now the reader should think that polynomials are represented as fully parenthesized arithmetic expressions involving binary operators  $+$  and  $\times$ .

Now we are ready to state our first result.

**Theorem 1** (Integrating general polynomials over a simplex is hard). *The following problem is NP-hard. Input:*

- (I<sub>1</sub>) numbers  $d, n \in \mathbb{N}$  in unary encoding,
- (I<sub>2</sub>) affinely independent rational vectors  $\mathbf{s}_1, \dots, \mathbf{s}_{d+1} \in \mathbb{Q}^n$  in binary encoding,
- (I<sub>3</sub>) an SIU straight-line program  $\Phi$  encoding a polynomial  $f \in \mathbb{Q}[x_1, \dots, x_n]$  with rational coefficients.

*Output, in binary encoding:*

- (O<sub>1</sub>) the rational number  $\int_{\Delta} f \, dm$ , where  $\Delta \subseteq \mathbb{R}^n$  is the simplex with vertices  $\mathbf{s}_1, \dots, \mathbf{s}_{d+1}$  and  $dm$  is the integral Lebesgue measure of the rational affine subspace  $\langle \Delta \rangle$ .

But we can also prove the following positive results.

**Theorem 2** (Efficient integration of polynomials of fixed effective number of variables). *For every fixed number  $D \in \mathbb{N}$ , there exists a polynomial-time algorithm for the following problem.*

*Input:*

- (I<sub>1</sub>) numbers  $d, n, M \in \mathbb{N}$  in unary encoding,
- (I<sub>2</sub>) affinely independent rational vectors  $\mathbf{s}_1, \dots, \mathbf{s}_{d+1} \in \mathbb{Q}^n$  in binary encoding,
- (I<sub>3</sub>) a polynomial  $f \in \mathbb{Q}[X_1, \dots, X_D]$  represented by either an SIU straight-line program  $\Phi$  of formal degree at most  $M$ , or a sparse or dense monomial representation of total degree at most  $M$ ,
- (I<sub>4</sub>) a rational matrix  $L$  with  $D$  rows and  $n$  columns in binary encoding, the rows of which define  $D$  linear forms  $\mathbf{x} \mapsto \langle \ell_j, \mathbf{x} \rangle$  on  $\mathbb{R}^n$ .

*Output, in binary encoding:*

- (O<sub>1</sub>) the rational number  $\int_{\Delta} f(\langle \ell_1, \mathbf{x} \rangle, \dots, \langle \ell_D, \mathbf{x} \rangle) \, dm$ , where  $\Delta \subseteq \mathbb{R}^n$  is the simplex with vertices  $\mathbf{s}_1, \dots, \mathbf{s}_{d+1}$  and  $dm$  is the integral Lebesgue measure of the rational affine subspace  $\langle \Delta \rangle$ .

In particular, the computation of the integral of a power of *one* linear form can be done by a polynomial time algorithm. This becomes false already if one considers powers of a quadratic form instead of powers of a linear form. Actually, we prove Theorem 1 by looking at powers  $Q^M$  of the Motzkin–Straus quadratic form of a graph.

As we will see later, when its degree is fixed, a polynomial has a polynomial size representation in either the SIU straight-line program encoding or the sparse or dense monomial representation and one can

switch between the two representations efficiently. The notion of formal degree of an SIU straight-line program will be defined in Section 2.

**Corollary 3** (Efficient integration of polynomials of fixed degree). *For every fixed number  $M \in \mathbb{N}$ , there exists a polynomial-time algorithm for the following problem. Input:*

- (I<sub>1</sub>) numbers  $d, n \in \mathbb{N}$  in unary encoding,
- (I<sub>2</sub>) affinely independent rational vectors  $\mathbf{s}_1, \dots, \mathbf{s}_{d+1} \in \mathbb{Q}^n$  in binary encoding,
- (I<sub>3</sub>) a polynomial  $f \in \mathbb{Q}[x_1, \dots, x_n]$  represented by either an SIU straight-line program  $\Phi$  of formal degree at most  $M$ , or a sparse or dense monomial representation of total degree at most  $M$ .

*Output, in binary encoding:*

- (O<sub>1</sub>) the rational number  $\int_{\Delta} f(\mathbf{x}) \, dm$ , where  $\Delta \subseteq \mathbb{R}^n$  is the simplex with vertices  $\mathbf{s}_1, \dots, \mathbf{s}_{d+1}$  and  $dm$  is the integral Lebesgue measure of the rational affine subspace  $\langle \Delta \rangle$ .

Actually, we give two interesting algorithms that prove Corollary 3. First, we simply observe that a monomial with total degree  $M$  involves at most  $M$  variables. Our second algorithm is related to the polynomial Waring problem: we decompose a homogeneous polynomial of total degree  $M$  into a sum of  $M$ -th powers of linear forms.

In [19] Lasserre and Avrachenkov compute the integral  $\int_{\Delta} f(\mathbf{x}) \, dm$  when  $f$  is a homogeneous polynomial, in terms of the corresponding polarized symmetric multilinear form (Proposition 16). We show that their formula also leads to a proof of Corollary 3. Furthermore, several other methods can be used for integration of polynomials of fixed degree. We discuss them in Section 4.

This paper is organized as follows: After some preparation in Section 2, the main theorems are proved in Section 3. In Section 4, we discuss extensions to other convex polytopes and give a survey of the complexity of other algorithms. Finally, in Section 5, we report on a few computational experiments.

## 2. PRELIMINARIES

In this section we prepare for the proofs of the main results.

**2.1. Integral Lebesgue measure on a rational affine subspace of  $\mathbb{R}^n$ .** On  $\mathbb{R}^n$  itself we consider the standard Lebesgue measure, which gives volume 1 to the fundamental domain of the lattice  $\mathbb{Z}^n$ . Let  $L$  be a rational linear subspace of dimension  $d \leq n$ . We normalize the Lebesgue measure on  $L$ , so that the volume of the fundamental domain of the intersected lattice  $L \cap \mathbb{Z}^n$  is 1. Then for any affine subspace  $L + \mathbf{a}$  parallel to  $L$ , we define the *integral Lebesgue measure*  $dm$  by translation. For example, the diagonal of the unit square has length 1 instead of  $\sqrt{2}$ .

TABLE 1. The representation of  $(x_1^2 + \dots + x_n^2)^k$  as a straight-line program

Intermediate	Comment
$q_1 = 0$	
$q_2 = x_1$	
$q_3 = q_2 \cdot q_2$	
$q_4 = q_1 + q_3$	Thus $q_4 = x_1^2$ .
$q_5 = x_2$	
$q_6 = q_5 \cdot q_5$	
$q_7 = q_4 + q_6$	Now $q_7 = x_1^2 + x_2^2$ .
$\vdots$	
$q_{3n-1} = x_n$	
$q_{3n} = q_{3n-1} \cdot q_{3n-1}$	
$q_{3n+1} = q_{3n-2} + q_{3n}$	Now $q_{3n+1} = x_1^2 + \dots + x_n^2$ .
$q_{3n+2} = 1$	
$q_{3n+3} = q_{3n+2} \cdot q_{3n+1}$	
$q_{3n+4} = q_{3n+3} \cdot q_{3n+1}$	
$\vdots$	
$q_{3n+k+2} = q_{3n+k+1} \cdot q_{3n+1}$	Final result.

**2.2. Encoding polynomials for integration.** We now explain our encoding of polynomials as SIU straight-line programs and justify our use of this encoding. We say that a polynomial  $f$  is represented as a (division-free) *straight-line program*  $\Phi$  if there is a finite sequence of polynomial functions of  $\mathbb{Q}[x_1, \dots, x_n]$ , namely  $q_1, \dots, q_k$ , the so-called *intermediate results*, such that each  $q_i$  is either a variable  $x_1, \dots, x_n$ , an element of  $\mathbb{Q}$ , or either the sum or the product of two preceding polynomials in the sequence and such that  $q_k = f$ . A straight-line program allows us to describe in polynomial space polynomials which otherwise would need to be described with exponentially many monomial terms. For example, think of the representation of  $(x_1^2 + \dots + x_n^2)^k$  as monomials versus its description with only  $3n + k + 2$  intermediate results; see [Table 2](#). The number of intermediate results of a straight line program is called its *length*. To keep track of constants we define the *size* of an intermediate result as one, unless the intermediate result is a constant in which case its size is the binary encoding size of the rational number. The *size* of a straight-line program is the sum of the sizes of the intermediate results. The *formal degree* of an intermediate result  $q_i$  is defined recursively in the obvious way, namely as 0 if  $q_i$  is a constant of  $\mathbb{Q}$ , as 1 if  $q_i$  is a variable  $x_j$ , as the maximum of the formal degrees of the summands if  $q_i$  is a sum, and the sum of the formal degrees of the factors if  $q_i$  is a product. The *formal degree of*

TABLE 2. The representation of a straight-line program for  $x^{2^k}$ , using iterated squaring

Intermediate	Comment
$q_1 = x$	
$q_2 = q_1 \cdot q_1$	
$q_3 = q_2 \cdot q_2$	
$\vdots$	
$q_{k+1} = q_k \cdot q_k$	Final result.

the *straight-line program*  $\Phi$  is the formal degree of the final result  $q_k$ . Clearly the formal degree gives an upper bound on the degree of the polynomial represented by it.

A favorite example to illustrate the benefits of a straight-line program encoding is that of the symbolic determinant of an  $n \times n$  matrix. Its dense representation as monomials has size  $\Theta(n!)$  but it can be computed in  $O(n^3)$  operations by Gaussian elimination. See the book [11] as a reference for this concept.

From a monomial representation of a polynomial of degree  $M$  and  $n$  variables it is easy to encode it as a straight-line program: first, by going in increasing degree we can write a straight-line program that generates all monomials of degree at most  $M$  in  $n$  variables. Then for each of them compute the product of the monomial with its coefficient so the length doubles. Finally successively add each term. This gives a final length bounded above by four times the number of monomials of degree at most  $M$  in  $n$  variables.

Straight-line programs are quite natural in the context of integration. One would certainly not expand  $(x_1^2 + \dots + x_n^2)^k$  to carry on numeric integration when we can easily evaluate it as a function. More importantly, straight-line programs are suitable as an input and output encoding and data structure in certain symbolic algorithms for computations with polynomials, like factoring; see [11]. Since straight-line programs can be very compact, the algorithms can handle polynomials whose input and output encodings have an exponential size in a sparse monomial representation.

However, a problem with straight-line programs is that this input encoding can be so compact that the output of many computational questions cannot be written down efficiently in the usual *binary encoding*. For example, while one can encode the polynomial  $x^{2^k}$  with a straight-line program with only  $k+1$  intermediate results (see Table 2),

when we compute the value of  $x^{2^k}$  for  $x = 2$ , or the integral  $\int_0^2 x^{2^k} dx = 2^{2^k+1}/(2^k + 1)$ , the binary encoding of the output has a size of  $\Theta(2^k)$ . Thus the output, given in binary, turns out to be exponentially bigger

TABLE 3. The representation of a single-intermediate-use straight-line program for  $x^{2^k}$ ; note that the iterated squaring method cannot be used

Intermediate	Comment
$q_1 = x$	
$q_2 = x$	
$q_3 = q_1 \cdot q_2$	Now $q_3 = x^2$ , and $q_1$ and $q_2$ cannot be used anymore.
$q_4 = x$	
$q_5 = q_3 \cdot q_4$	Thus $q_5 = x^3$ .
$\vdots$	
$q_{2^{k+1}-2} = x$	
$q_{2^{k+1}-1} = q_{2^{k+1}-3} \cdot q_{2^{k+1}-2}$	Final result.

than the input encoding. We remark that the same difficulty arises if we choose a sparse input encoding of the polynomial where not only the coefficients but also the exponent vectors are encoded in binary (rather than the usual unary encoding for the exponent vectors).

This motivates the following variation of the notion of straight-line program: We say a (division-free) straight-line program is *single-intermediate-use*, or SIU for short, if every intermediate result is used only once in the definition of other intermediate results. (However, the variables  $x_1, \dots, x_n$  can be used arbitrarily often in the definition of intermediate results.) With this definition, all ways to encode the polynomial  $x^{2^k}$  require at least  $2^k$  multiplications. An example SIU straight-line program is shown in Table 3. Clearly single-intermediate-use straight-line programs are equivalent, in terms of expressiveness and encoding complexity, to fully parenthesized arithmetic expressions using binary operators  $+$  and  $\times$ .

### 2.3. Efficient computation of truncated product of an arbitrary number of polynomials in a fixed number of variables.

The following result will be used in several situations.

**Lemma 4.** *For every fixed number  $D \in \mathbb{N}$ , there exists a polynomial time algorithm for the following problem.*

*Input: a number  $M$  in unary encoding, a sequence of  $k$  polynomials  $P_j \in \mathbb{Q}[X_1, \dots, X_D]$  of total degree at most  $M$ , in dense monomial representation.*

*Output: the product  $P_1 \cdots P_k$  truncated at degree  $M$ .*

*Proof.* We start with the product of the first two polynomials. We compute the monomials of degree at most  $M$  in this product. This takes  $O(M^{2D})$  elementary rational operations, and the maximum encoding



length of any coefficient in the product is also polynomial in the input data length. Then we multiply this truncated product with the next polynomial, truncating at degree  $M$ , and so on. The total computation takes  $O(kM^{2D})$  elementary rational operations.  $\square$

### 3. PROOFS OF THE MAIN RESULTS

Our aim is to perform an efficient computation of  $\int_{\Delta} f \, dm$  where  $\Delta$  is a simplex and  $f$  a polynomial. We will prove first that this is not possible for  $f$  of varying degree under the assumption that  $P \neq NP$ . More precisely, we prove that, under this assumption, an efficient computation of  $\int_{\Delta} Q^M \, dm$  is not possible, where  $Q$  is a quadratic form and  $M$  is allowed to vary.

In the next subsection we present an algorithm to efficiently compute the integral  $\int_{\Delta} f \, dm$  in some particular situations, most notably the case of arbitrary powers of linear forms.

**3.1. Hardness for polynomials of non-fixed degree.** For the proof of Theorem 1 we need to extend the following well-known result of Motzkin and Straus [24]. In this section, we denote by  $\Delta$  the  $(n-1)$ -dimensional canonical simplex  $\{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^n x_i = 1\}$ , and we denote by  $dm$  the Lebesgue measure on the hyperplane  $\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1\}$ , normalized so that  $\Delta$  has volume 1. For a function  $f$  on  $\Delta$ , denote as usual  $\|f\|_{\infty} = \max_{\mathbf{x} \in \Delta} |f(\mathbf{x})|$  and  $\|f\|_p = (\int_{\Delta} |f|^p \, dm)^{1/p}$ , for  $p \geq 1$ . Recall that the *clique number* of a graph  $G$  is the largest number of vertices of a complete subgraph of  $G$ .

**Theorem 5** (Motzkin–Straus). *Let  $G$  be a graph with  $n$  vertices and clique number  $\omega(G)$ . Let  $Q_G(\mathbf{x})$  be the Motzkin–Straus quadratic form  $\frac{1}{2} \sum_{(i,j) \in E(G)} x_i x_j$ . Then  $\|Q_G(\mathbf{x})\|_{\infty} = \frac{1}{2}(1 - \frac{1}{\omega(G)})$ .*

Our first result might be of independent interest as it shows that integrals of polynomials over simplices can carry very interesting combinatorial information. This result builds on the theorem of Motzkin and Straus, using the proof of the well-known relation  $\|f\|_{\infty} = \lim_{p \rightarrow \infty} \|f\|_p$ .

**Lemma 6.** *Let  $G$  be a graph with  $n$  vertices and clique number  $\omega(G)$ . Let  $Q_G(\mathbf{x})$  be the Motzkin–Straus quadratic form. Then for  $p \geq 4(e-1)n^3 \ln(32n^2)$ , the clique number  $\omega(G)$  is equal to  $\lceil \frac{1}{1-2\|Q_G\|_p} \rceil$ .*

To prove Lemma 6 we will first prove the following intermediate result.

**Lemma 7.** *For  $\varepsilon > 0$  we have*

$$(\|Q_G\|_{\infty} - \varepsilon) \left(\frac{\varepsilon}{4}\right)^{(n-1)/p} \leq \|Q_G\|_p \leq \|Q_G\|_{\infty}.$$

*Proof.* The right-hand side inequality follows from the normalization of the measure, as  $|Q(\mathbf{x})| \leq \|Q_G\|_{\infty}$ , for all  $\mathbf{x} \in \Delta$ .

In order to obtain the other inequality, we use Hölder's inequality  $\int_{\Delta} |fg| \, dm \leq \|f\|_p \|g\|_q$ , where  $q$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ . For any (say) continuous function  $f$  on  $\Delta$ , let us denote by  $\Delta(f, \varepsilon)$  the set  $\{\mathbf{x} \in \Delta : |f(\mathbf{x})| \geq \|f\|_{\infty} - \varepsilon\}$ , and take for  $g$  the characteristic function of  $\Delta(f, \varepsilon)$ . We obtain

$$(\|f\|_{\infty} - \varepsilon)(\text{vol } \Delta(f, \varepsilon))^{1/p} \leq \|f\|_p. \quad (1)$$

Let  $\mathbf{a}$  be a point of  $\Delta$  where the maximum of  $Q_G$  is attained. Since  $\frac{\partial Q_G}{\partial x_i} = \sum_{(i,j) \in E(G)} x_j$  we know that  $0 \leq \frac{\partial Q_G}{\partial x_i} \leq 1$  for  $\mathbf{x} \in \Delta$ . Since  $\Delta$  is convex, we conclude that for any  $\mathbf{x} \in \Delta$ ,

$$0 \leq Q_G(\mathbf{a}) - Q_G(\mathbf{x}) \leq \sum_{i=1}^n |a_i - x_i|.$$

Thus  $\Delta(Q_G, \varepsilon)$  contains the set  $C_{\varepsilon} = \{\mathbf{x} \in \Delta : \sum_{i=1}^n |a_i - x_i| < \varepsilon\}$ . We claim that  $\text{vol}(C_{\varepsilon}) \geq (\frac{\varepsilon}{4})^{n-1}$ . This claim proves the left inequality of the lemma when we apply it to (1).

Consider the dilated simplex  $\frac{\varepsilon/2}{1+\varepsilon/2}\Delta$  and the translated set  $P_{\varepsilon} = \frac{\mathbf{a}}{1+\varepsilon/2} + \frac{\varepsilon/2}{1+\varepsilon/2}\Delta$ . Clearly  $P_{\varepsilon}$  is contained in  $\Delta$ . Moreover, for  $\mathbf{x} \in P_{\varepsilon}$ , we have  $\sum_{i=1}^n |a_i - x_i| \leq \frac{\varepsilon}{1+\varepsilon/2} \leq \varepsilon$ , hence  $P_{\varepsilon}$  is contained in  $C_{\varepsilon}$ . Since  $\text{vol}(\Delta) = 1$  for the normalized measure, the volume of  $P_{\varepsilon}$  is equal to  $(\frac{\varepsilon/2}{1+\varepsilon/2})^{n-1}$ . Hence  $\text{vol}(P_{\varepsilon}) \geq (\varepsilon/4)^{n-1}$ . This finishes the proof.  $\square$

*Proof of Lemma 6.* In the inequalities of Lemma 7, we substitute the relation  $\|Q_G\|_{\infty} = \frac{1}{2}(1 - \frac{1}{\omega(G)})$ , given by Motzkin–Straus's theorem (Theorem 5). We obtain

$$\left(\frac{1}{2}\left(1 - \frac{1}{\omega(G)}\right) - \varepsilon\right)(\varepsilon/4)^{\frac{n-1}{p}} \leq \|Q_G\|_p \leq \frac{1}{2}\left(1 - \frac{1}{\omega(G)}\right).$$

Let us rewrite these inequalities as

$$\frac{1}{1 - 2\|Q_G\|_p} \leq \omega(G) \leq \frac{1}{1 - \frac{2\|Q_G\|_p}{(\varepsilon/4)^{(n-1)/p}} - 2\varepsilon}. \quad (2)$$

We only need to prove that for  $\varepsilon = \frac{1}{8n^2}$  and  $p \geq 4(e-1)n^3 \ln(32n^2)$  we have

$$0 \leq L(p) := \frac{1}{1 - \frac{2\|Q_G\|_p}{(\varepsilon/4)^{\frac{n-1}{p}}} - 2\varepsilon} - \frac{1}{1 - 2\|Q_G\|_p} < 1. \quad (3)$$

Let us write

$$\delta_p = \|Q_G\|_p \left( \frac{1}{(\varepsilon/4)^{\frac{n-1}{p}}} - 1 \right) = \|Q_G\|_p \left( (32n^2)^{\frac{n-1}{p}} - 1 \right).$$

Thus  $L(p)$  in Equation 3 becomes now

$$L(p) = \frac{1}{1 - 2\|Q_G\|_p} \left( \frac{1}{1 - 2\frac{\delta_p + \varepsilon}{1 - 2\|Q_G\|_p}} - 1 \right). \quad (4)$$

Since  $\|Q_G\|_p \leq \frac{1}{2}(1 - \frac{1}{\omega(G)}) \leq \frac{1}{2}$ , we have a bound for  $\delta_p$

$$0 \leq \delta_p \leq \frac{1}{2}((32n^2)^{\frac{n-1}{p}} - 1).$$

Let  $A = (\frac{4}{\varepsilon})^{n-1} = (32n^2)^{n-1}$ . Since we assumed  $p \geq 4(e-1)n^3 \ln(32n^2)$ , we have  $0 \leq \frac{\ln A}{p} < 1$ , hence  $0 \leq A^{1/p} - 1 < (e-1)\frac{\ln A}{p}$ . We obtain

$$0 \leq \delta_p \leq \frac{e-1}{2} \frac{(n-1) \log(32n^2)}{p} \leq \frac{1}{8n^2}.$$

Since  $\omega(G) \leq n$ , we have  $1 - 2\|Q_G\|_p \geq 1/n$ . Hence we have

$$\frac{2(\delta_p + \varepsilon)}{1 - 2\|Q_G\|_p} \leq \frac{1}{2n} \leq \frac{1}{2}.$$

Finally for any number  $0 < \alpha < 1/2$  we have  $\frac{1}{1-\alpha} < 1 + 2\alpha$ , hence applying this fact to Equation 4 with  $\alpha = 2\frac{\delta_p + \varepsilon}{1 - 2\|Q_G\|_p}$  we get

$$L(p) < \frac{1}{1 - 2\|Q_G\|_p} \left( \frac{4(\delta + \varepsilon)}{1 - 2\|Q_G\|_p} \right) \leq 4n^2(\delta_p + \varepsilon) \leq 1.$$

This proves Equation 3 and the lemma.  $\square$

*Proof of Theorem 1.* The problem of deciding whether the clique number  $\omega(G)$  of a graph  $G$  is greater than a given number  $K$  is a well-known NP-complete problem [15]. From Lemma 6 we see that checking this is the same as checking that for  $p = 4(e-1)n^3 \ln(32n^2)$  the integral part of  $\int_{\Delta} (Q_G)^p dm$  is less than  $K^p$ . Note that the polynomial  $Q_G(\mathbf{x})^p$  is a power of a quadratic form and can be encoded as a SIU straight-line program of length  $O(n^3 \log n \cdot |E(G)|)$ . If the computation of the integral  $\int_{\Delta} f dm$  of a polynomial  $f$  could be done in polynomial time in the input size of  $f$ , we could then verify the desired inequality in polynomial time as well.  $\square$

**3.2. An extension of a formula of Brion.** In this section, we obtain several expressions for the integrals  $\int_{\Delta} e^{\ell} dm$  and  $\int_{\Delta} \ell_1^{M_1} \cdots \ell_D^{M_D} dm$ , where  $\Delta \subset \mathbb{R}^n$  is a simplex and  $\ell, \ell_j$  are linear forms on  $\mathbb{R}^n$ . The first formula, (5) in Lemma 8, is obtained by elementary iterated integration on the standard simplex. It leads to a computation of the integral  $\int_{\Delta} \ell_1^{M_1} \cdots \ell_D^{M_D} dm$  in terms of the Taylor expansion of a certain analytic function associated to  $\Delta$  (Corollary 14), hence to a proof of the complexity result of Theorem 2.

In the case of one linear form  $\ell$  which is regular, we recover in this way the “short formula” of Brion as Corollary 11. This result was first obtained by Brion as a particular case of his theorem on polyhedra [10].

**Lemma 8.** *Let  $\Delta$  be the simplex that is the convex hull of  $(d+1)$  affinely independent vertices  $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{d+1}$  in  $\mathbb{R}^n$ , and let  $\ell$  be an arbitrary*

linear form on  $\mathbb{R}^n$ . Then

$$\int_{\Delta} e^{\ell} dm = d! \operatorname{vol}(\Delta, dm) \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \frac{\langle \ell, \mathbf{s}_1 \rangle^{k_1} \cdots \langle \ell, \mathbf{s}_{d+1} \rangle^{k_{d+1}}}{(|\mathbf{k}| + d)!}, \quad (5)$$

where  $|\mathbf{k}| = \sum_{j=1}^{d+1} k_j$ .

*Proof.* Using an affine change of variables, it is enough to prove (5) when  $\Delta$  is the  $d$ -dimensional standard simplex  $\Delta_{st} \subset \mathbb{R}^d$  defined by

$$\Delta_{st} = \left\{ \mathbf{x} \in \mathbb{R}^d : x_i \geq 0, \sum_{i=1}^d x_i \leq 1 \right\}.$$

The volume of  $\Delta_{st}$  is equal to  $\frac{1}{d!}$ . In the case of  $\Delta_{st}$ , the vertex  $\mathbf{s}_j$  is the basis vector  $\mathbf{e}_j$  for  $1 \leq j \leq d$  and  $s_{d+1} = 0$ . Let  $\langle \ell, x \rangle = \sum_{j=1}^d a_j x_j$ . Then (5) becomes

$$\int_{\Delta_{st}} e^{a_1 x_1 + \cdots + a_d x_d} d\mathbf{x} = \sum_{\mathbf{k} \in \mathbb{N}^d} \frac{a_1^{k_1} \cdots a_d^{k_d}}{(|\mathbf{k}| + d)!}.$$

We prove it by induction on  $d$ . For  $d = 1$ , we have

$$\int_0^1 e^{ax} dx = \frac{e^a - 1}{a} = \sum_{k \geq 0} \frac{a^k}{(k+1)!}.$$

Let  $d > 1$ . We write

$$\begin{aligned} \int_{\Delta_{st}} e^{a_1 x_1 + \cdots + a_d x_d} d\mathbf{x} &= \\ &= \int_0^1 e^{a_d x_d} \left( \int_{\substack{x_j \geq 0 \\ x_1 + \cdots + x_{d-1} \leq 1 - x_d}} e^{a_1 x_1 + \cdots + a_{d-1} x_{d-1}} dx_1 \cdots dx_{d-1} \right) dx_d. \end{aligned}$$

By the induction hypothesis and an obvious change of variables, the inner integral is equal to

$$(1 - x_d)^{d-1} \sum_{\mathbf{k} \in \mathbb{N}^{d-1}} (1 - x_d)^{|\mathbf{k}|} \frac{a_1^{k_1} \cdots a_{d-1}^{k_{d-1}}}{(|\mathbf{k}| + d - 1)!}.$$

The result now follows from the relation

$$\int_0^1 \frac{(1-x)^p}{p!} e^{ax} dx = \sum_{k \geq 0} \frac{a^k}{(k+p+1)!}.$$

□

**Remark 9.** Let us replace  $\ell$  by  $t\ell$  in (5) and expand in powers of  $t$ . We obtain the following formula.

$$\int_{\Delta} \ell^M dm = d! \operatorname{vol}(\Delta, dm) \frac{M!}{(M+d)!} \sum_{\mathbf{k} \in \mathbb{N}^{d+1}, |\mathbf{k}|=M} \langle \ell, \mathbf{s}_1 \rangle^{k_1} \cdots \langle \ell, \mathbf{s}_{d+1} \rangle^{k_{d+1}}. \quad (6)$$

This relation is a particular case of a result of Lasserre and Avrachenkov, Proposition 16, as we will explain in section 4.3 below.

**Theorem 10.** *Let  $\Delta$  be the simplex that is the convex hull of  $(d + 1)$  affinely independent vertices  $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{d+1}$  in  $\mathbb{R}^n$ .*

$$\sum_{M \in \mathbb{N}} t^M \frac{(M + d)!}{M!} \int_{\Delta} \ell^M \, dm = d! \operatorname{vol}(\Delta, dm) \frac{1}{\prod_{j=1}^{d+1} (1 - t \langle \ell, \mathbf{s}_j \rangle)}. \quad (7)$$

*Proof.* We apply Formula (6). Summing up from  $M = 0$  to  $\infty$ , we recognize the expansion of the right-hand side of (7) into a product of geometric series:

$$\begin{aligned} \sum_{M \in \mathbb{N}} t^M \frac{(M + d)!}{M!} \int_{\Delta} \ell^M \, dm = \\ d! \operatorname{vol}(\Delta, dm) \sum_{M \in \mathbb{N}} t^M \sum_{\mathbf{k} \in \mathbb{N}^{d+1}, |\mathbf{k}|=M} \langle \ell, \mathbf{s}_1 \rangle^{k_1} \cdots \langle \ell, \mathbf{s}_{d+1} \rangle^{k_{d+1}}. \end{aligned}$$

□

**Corollary 11** (Brion). *Let  $\Delta$  be as in the previous theorem. Let  $\ell$  be a linear form which is regular w.r.t.  $\Delta$ , i.e.,  $\langle \ell, \mathbf{s}_i \rangle \neq \langle \ell, \mathbf{s}_j \rangle$  for any pair  $i \neq j$ . Then we have the following relations.*

$$\int_{\Delta} \ell^M \, dm = d! \operatorname{vol}(\Delta, dm) \frac{M!}{(M + d)!} \left( \sum_{i=1}^{d+1} \frac{\langle \ell, \mathbf{s}_i \rangle^{M+d}}{\prod_{j \neq i} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle} \right). \quad (8)$$

$$\int_{\Delta} e^{\ell} \, dm = d! \operatorname{vol}(\Delta, dm) \sum_{i=1}^{d+1} \frac{e^{\langle \ell, \mathbf{s}_i \rangle}}{\prod_{j \neq i} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle}. \quad (9)$$

*Proof.* We consider the right-hand side of (7) as a rational function of  $t$ . The poles  $t = 1/\langle \ell, \mathbf{s}_i \rangle$  are simple precisely when  $\ell$  is regular. In this case, we obtain (8) by taking the expansion into partial fractions. The second relation follows immediately by expanding  $e^{\ell}$ . □

When  $\ell$  is regular, Brion's formula is very short, it is a sum of  $d + 1$  terms. When  $\ell$  is not regular, the expansion of (7) into partial fractions leads to an expression of the integral as a sum of residues. Let  $K \subseteq \{1, \dots, d + 1\}$  be an index set of the different poles  $t = 1/\langle \ell, \mathbf{s}_k \rangle$ , and for  $k \in K$  let  $m_k$  denote the order of the pole, i.e.,

$$m_k = \#\{i \in \{1, \dots, d + 1\} : \langle \ell, \mathbf{s}_i \rangle = \langle \ell, \mathbf{s}_k \rangle\}.$$

With this notation, we have the following formula.

**Corollary 12.**

$$\int_{\Delta} \ell^M dm = d! \operatorname{vol}(\Delta, dm) \frac{M!}{(M+d)!} \sum_{k \in K} \operatorname{Res}_{\varepsilon=0} \frac{(\varepsilon + \langle \ell, \mathbf{s}_k \rangle)^{M+d}}{\varepsilon^{m_k} \prod_{\substack{i \in K \\ i \neq k}} (\varepsilon + \langle \ell, \mathbf{s}_k - \mathbf{s}_i \rangle)^{m_i}} \quad (10)$$

**Remark 13.** It is worth remarking that Corollaries 11 and 12 can be seen as a particular case of the localization theorem in equivariant cohomology (see for instance [6]), although we did not use this fact and instead gave a simple direct calculation. In our situation, the variety is the complex projective space  $\mathbb{C}\mathbb{P}^d$ , with action of a  $d$ -dimensional torus, such that the image of the moment map is the simplex  $\Delta$ . Brion's formula corresponds to the generic case of a one-parameter subgroup acting with isolated fixed points. In the degenerate case when the set of fixed points has components of positive dimension, the polar parts in (10) coincide with the contributions of the components to the localization formula.

Finally, we can extend the formula of Theorem 10 on integrating the power of a linear form to the case of a product of powers of *several* linear forms.

**Corollary 14.** *Let  $\ell_1, \dots, \ell_D$  be  $D$  linear forms on  $\mathbb{R}^n$ . We have the following Taylor expansion:*

$$\sum_{\mathbf{M} \in \mathbb{N}^D} t_1^{M_1} \dots t_D^{M_D} \frac{(|\mathbf{M}| + d)!}{d! \operatorname{vol}(\Delta, dm)} \int_{\Delta} \frac{\ell_1^{M_1} \dots \ell_D^{M_D}}{M_1! \dots M_D!} dm = \frac{1}{\prod_{i=1}^{d+1} (1 - t_1 \langle \ell_1, \mathbf{s}_i \rangle - \dots - t_D \langle \ell_D, \mathbf{s}_i \rangle)}. \quad (11)$$

*Proof.* Replace  $t\ell$  with  $t_1\ell_1 + \dots + t_D\ell_D$  in (7) and take the expansion in powers  $t_1^{M_1} \dots t_D^{M_D}$ .  $\square$

**3.3. Polynomial time algorithms for polynomial functions of a fixed number of linear forms.** We will present an algorithm which, given a polynomial of the particular form  $f(\langle \ell_1, \mathbf{x} \rangle, \dots, \langle \ell_D, \mathbf{x} \rangle)$  where  $f$  is a polynomial depending on a fixed number  $D$  of variables, and  $\langle \ell_j, \mathbf{x} \rangle = L_{j1}x_1 + \dots + L_{jn}x_n$ , for  $j = 1, \dots, D$ , are linear forms on  $\mathbb{R}^n$ , computes its integral on a simplex, in time polynomial on the input data. This algorithm relies on Corollary 14.

*Proof of Theorem 2.* The number of monomials of degree  $M$  in  $D$  variables is equal to  $\binom{M+D-1}{D-1}$ . Therefore, when  $D$  is fixed, the number of monomials of degree at most  $M$  in  $D$  variables is  $O(M^D)$ . When the number of variables  $D$  of a straight-line program  $\Phi$  is fixed, it is

possible to compute a sparse or dense representation of the polynomial represented by  $\Phi$  in polynomial time, by a straight-forward execution of the program. Indeed, all intermediate results can be stored as sparse or dense polynomials with  $O(M^D)$  monomials. Since the program  $\Phi$  is single-intermediate-use, the binary encoding size of all coefficients of the monomials can be bounded polynomially by the input encoding size. Thus it is enough to compute the integral of a monomial,

$$\int_{\Delta} \langle \ell_1, \mathbf{x} \rangle^{M_1} \cdots \langle \ell_D, \mathbf{x} \rangle^{M_D} dm. \quad (12)$$

From Theorem 14, it follows that

$$\frac{(|\mathbf{M}| + d)!}{d! \operatorname{vol}(\Delta, dm)} \int_{\Delta} \frac{\ell_1^{M_1} \cdots \ell_D^{M_D}}{M_1! \cdots M_D!} dm$$

is the coefficient of  $t_1^{M_1} \cdots t_D^{M_D}$  in the Taylor expansion of

$$\frac{1}{\prod_{i=1}^{d+1} (1 - t_1 \langle \ell_1, \mathbf{s}_i \rangle - \cdots - t_D \langle \ell_D, \mathbf{s}_i \rangle)}.$$

Since  $D$  is fixed, this coefficient can be computed in time polynomial with respect to  $\mathbf{M}$  and the input data, by multiplying series truncated at degree  $|\mathbf{M}|$ , as explained in Lemma 4.

Finally,  $\operatorname{vol}(\Delta, dm)$  needs to be computed. If  $\Delta = \operatorname{conv}\{\mathbf{s}_1, \dots, \mathbf{s}_{d+1}\}$  is full-dimensional ( $d = n$ ), we can do so by computing the determinant of the matrix formed by difference vectors of the vertices:

$$\operatorname{vol}(\Delta, dm) = \frac{1}{n!} |\det(\mathbf{s}_1 - \mathbf{s}_{n+1}, \dots, \mathbf{s}_n - \mathbf{s}_{n+1})|.$$

If  $\Delta$  is lower-dimensional, we first compute a basis  $B \in \mathbb{Z}^{n \times d}$  of the intersection lattice  $\Lambda = \operatorname{lin}(\Delta) \cap \mathbb{Z}^n$ . This can be done in polynomial time by applying an efficient algorithm for computing the Hermite normal form [16]. Then we express each difference vector  $\mathbf{v}_i = \mathbf{s}_i - \mathbf{s}_{d+1} \in \operatorname{lin}(\Delta)$  for  $i = 1, \dots, d$  using the basis  $B$  as  $\mathbf{v}_i = B\mathbf{v}'_i$ , where  $\mathbf{v}_i \in \mathbb{Q}^d$ . We obtain

$$\operatorname{vol}(\Delta, dm) = \frac{1}{d!} |\det(\mathbf{v}'_1, \dots, \mathbf{v}'_d)|,$$

thus the volume computation is reduced to the calculation of a determinant. This finishes the proof of Theorem 2.  $\square$

**3.4. Polynomial time algorithms for polynomials of fixed degree.** In the present section, we assume that the total degree of the input polynomial  $f$  we wish to integrate is a constant  $M$ .

*Proof of Corollary 3.* First of all, when the formal degree  $M$  of a straight-line program  $\Phi$  is fixed, it is possible to compute a sparse or dense representation of the polynomial represented by  $\Phi$  in polynomial time, by a straight-forward execution of the program. Indeed, all intermediate results can be stored as sparse or dense polynomials with  $O(n^M)$

monomials. Since the program  $\Phi$  is single-intermediate-use, the binary encoding size of all coefficients of the monomials can be bounded polynomially by the input encoding size.

Now, the key observation is that a monomial of degree at most  $M$  depends effectively on  $D \leq M$  variables  $x_{i_1}, \dots, x_{i_D}$ , thus it is of the form

$$\ell_1^{M_1} \dots \ell_D^{M_D},$$

where the linear forms  $\ell_j(\mathbf{x}) = x_{i_j}$  are the coordinates that effectively appear in the monomial. Thus, [Corollary 3](#) follows immediately from [Theorem 2](#).  $\square$

In the following, we give another proof of [Corollary 3](#), based on decompositions of polynomials as sums of powers of linear forms.

*Alternative proof of [Corollary 3](#).* From [Corollaries 11](#) and [12](#), we derive another efficient algorithm, as follows. The key idea now is that one can decompose the polynomial  $f$  as a sum  $f := \sum_{\ell} c_{\ell} \ell_j^M$  with at most  $2^M$  terms in the sum. To handle the case where  $f$  is a monomial, we use the well-known identity

$$\begin{aligned} & x_1^{M_1} x_2^{M_2} \dots x_n^{M_n} \\ &= \frac{1}{|\mathbf{M}|!} \sum_{0 \leq p_i \leq M_i} (-1)^{|\mathbf{M}| - (p_1 + \dots + p_n)} \binom{M_1}{p_1} \dots \binom{M_n}{p_n} (p_1 x_1 + \dots + p_n x_n)^{|\mathbf{M}|}, \end{aligned} \tag{13}$$

where  $|\mathbf{M}| = M_1 + \dots + M_n \leq M$ .

The number of terms in the sum is the product  $(M_1 + 1) \dots (M_n + 1)$ , which is bounded by  $2^{M_1 + \dots + M_n} = 2^{|\mathbf{M}|} \leq 2^M$ . Since the number of monomials of degree  $M$  in  $n$  variables is  $O(n^M)$ , we have a polynomial time algorithm.  $\square$

Actually, in the implementation of this method, we group together proportional linear forms, thus we often obtain a smaller number of summands.

The problem of finding a decomposition with the smallest possible number of summands is known as the *polynomial Waring problem*: What is the smallest integer  $r(M, n)$  such that a generic homogeneous polynomial  $f(x_1, \dots, x_n)$  of degree  $M$  in  $n$  variables is expressible as the sum of  $r(M, n)$   $M$ -th powers of linear forms? This problem was solved by Alexander and Hirschowitz (see [\[1\]](#), and [\[8\]](#) for an extensive survey), but there is no computational or constructive version of this result.

**Theorem 15.** *A generic homogeneous polynomial of degree  $M$  in  $n$  variables is expressible as the sum of*

$$r(M, n) = \left\lceil \frac{\binom{n+M-1}{M}}{n} \right\rceil$$



$M$ -th powers of linear forms, with the exception of the cases  $r(3, 5) = 8$ ,  $r(4, 3) = 6$ ,  $r(4, 4) = 10$ ,  $r(4, 5) = 15$ , and  $M = 2$ , where  $r(2, n) = n$ .

In the extreme case, when the polynomial  $f$  happens to be the power of one linear form  $\ell$ , one should certainly avoid applying the above decomposition formula to each of the monomials of  $f$ . We remark that, when the degree is fixed, we can decide in polynomial time whether a polynomial  $f$ , given in sparse or dense monomial representation, is a power of a linear form  $\ell$ , and, if so, construct such a linear form.

#### 4. OTHER ALGORITHMS FOR INTEGRATION AND EXTENSIONS TO OTHER POLYTOPES

We conclude with a discussion of how to extend integration to other polytopes and a review of the complexity of other methods to integrate polynomials over polytopes.

**4.1. A formul of Lasserre–Avrachenkov.** Another nice formula is the Lasserre–Avrachenkov formula for the integration of a homogeneous polynomial [19] on a simplex. As we explain below, this yields a polynomial-time algorithm for the problem of integrating a polynomial of fixed degree over a polytope in varying dimension, thus providing an alternative proof of [Corollary 3](#).

**Proposition 16** ([19]). *Let  $H$  be a symmetric multilinear form defined on  $(\mathbb{R}^d)^M$ . Let  $\mathbf{s}_1, \mathbf{s}_1, \dots, \mathbf{s}_{d+1}$  be the vertices of a  $d$ -dimensional simplex  $\Delta$ . Then one has*

$$\int_{\Delta} H(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x}) d\mathbf{x} = \frac{\text{vol}(\Delta)}{\binom{M+d}{M}} \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_M \leq d+1} H(\mathbf{s}_{i_1}, \mathbf{s}_{i_2}, \dots, \mathbf{s}_{i_M}). \quad (14)$$

**Remark 17.** By reindexing the summation in (14), as  $H$  is symmetric, we obtain

$$\int_{\Delta} H(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x}) d\mathbf{x} = \frac{\text{vol}(\Delta)}{\binom{M+d}{M}} \sum_{k_1 + \dots + k_{d+1} = M} H(\mathbf{s}_1, \dots, \mathbf{s}_1, \dots, \mathbf{s}_{d+1}, \dots, \mathbf{s}_{d+1}), \quad (15)$$

where  $\mathbf{s}_1$  is repeated  $k_1$  times,  $\mathbf{s}_2$  is repeated  $k_2$  times, etc. When  $H$  is of the form  $H = \prod_{i=1}^M \langle \ell, \mathbf{x}_i \rangle$ , for a single linear form  $\ell$ , then (15) coincides with Formula (6) in [Remark 9](#).

Now any polynomial  $f$  which is homogeneous of degree  $M$  can be written as  $f(\mathbf{x}) = H_f(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})$  for a unique multilinear form  $H_f$ . If  $f = \ell^M$  then  $H_f = \prod_{i=1}^M \langle \ell, \mathbf{x}_i \rangle$ . Thus for fixed  $M$  the computation of  $H_f$  can be done by decomposing  $f$  into a linear combination of powers

of linear forms, as we did in the proof of Corollary 3. Alternatively one can use the well-known polarization formula,

$$H_f(\mathbf{x}_1, \dots, \mathbf{x}_M) = \frac{1}{2^M M!} \sum_{\varepsilon \in \{\pm 1\}^M} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_M f\left(\sum_{i=1}^M \varepsilon_i \mathbf{x}_i\right), \quad (16)$$

Thus from (14) we get the following corollary.

**Corollary 18.** *Let  $f$  be a homogeneous polynomial of degree  $M$  in  $d$  variables, and let  $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{d+1}$  be the vertices of a  $d$ -dimensional simplex  $\Delta$ . Then*

$$\int_{\Delta} f(\mathbf{y}) \, d\mathbf{y} = \frac{\text{vol}(\Delta)}{2^M M! \binom{M+d}{M}} \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_M \leq d+1} \sum_{\varepsilon \in \{\pm 1\}^M} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_M f\left(\sum_{k=1}^M \varepsilon_k \mathbf{s}_{i_k}\right). \quad (17)$$

We remark that when we fix the degree  $M$  of the homogeneous polynomial  $f$ , the length of the polarization formula (thus the length of the second sum in (17)) is a constant. The length of the first sum in (17) is  $O(n^M)$ . Thus, for fixed degree in varying dimension, we obtain another polynomial-time algorithm for integrating over a simplex.

**4.2. Traditional conversion of the integral as iterated univariate integrals.** Let  $P \subseteq \mathbb{R}^d$  be a full-dimensional polytope and  $f$  a polynomial. The traditional method we teach our calculus students to compute multivariate integrals over a bounded region requires them to write the integral  $\int_P f \, dm$  as a sum of sequences of one-dimensional integrals

$$\sum_{j=1}^K \int_{a_{1j}}^{b_{1j}} \int_{a_{2j}}^{b_{2j}} \cdots \int_{a_{dj}}^{b_{dj}} f \, dx_{i_1} \, dx_{i_2} \cdots dx_{i_d} \quad (18)$$

for which we know the limits of integration  $a_{ij}, b_{ij}$  explicitly. The problem of finding the limits of integration and the sum has interesting complexity related to the well-known Fourier-Motzkin elimination method (see Chapter One in [27] for a short introduction).

Given a system of linear inequalities  $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ , describing a polytope  $P \subset \mathbb{R}^d$ , Fourier-Motzkin elimination is an algorithm that computes a new larger system of inequalities  $\hat{\mathbf{A}}\mathbf{x} \leq \hat{\mathbf{b}}$  with the property that those inequalities that do not contain the variable  $x_d$  describe the projection of  $P$  into the hyperplane  $x_d = 0$ . We will not explain the details, but Fourier-Motzkin elimination is quite similar to Gaussian elimination in the sense that the main operations necessary to eliminate the last variable  $x_d$  require to rearrange, scale, and add rows of the matrix  $(\mathbf{A}, \mathbf{b})$ , but unlike Gaussian elimination, new inequalities are added to the system.

It was first observed by Schechter [26] that Fourier-Motzkin elimination provides a way to generate the traditional iterated integrals. More precisely, let us call  $P_d$  the projection of  $P$  into the hyperplane  $x_d = 0$ . Clearly when integrating over a polytopal region we expect that the limits of integration will be affine functions. From the output of Fourier-Motzkin  $\hat{A}\mathbf{x} \leq \hat{\mathbf{b}}$ , we have  $\mathbf{x} \in P$  if and only if  $(x_1, \dots, x_{d-1}) \in P_d$  and for the first  $k+r$  inequalities of the system

$$x_d \leq \hat{b}_i - \sum_{j=1}^{d-1} \hat{a}_{ij} x_j = A_i^u(x_1, \dots, x_{d-1})$$

for  $i = 1, \dots, k$  as well as

$$x_d \geq \hat{b}_{k+i} - \sum_{j=1}^{d-1} \hat{a}_{k+ij} x_j = A_i^l(x_1, \dots, x_{d-1})$$

for  $i = 1, \dots, r$ . Then, if we define

$$m(x_1, \dots, x_d) = \max\{A_j^l(x_1, \dots, x_{d-1}), j = 1, \dots, r\}$$

and

$$M(x_1, \dots, x_d) = \min\{A_j^u(x_1, \dots, x_{d-1}), j = 1, \dots, r\},$$

we can write

$$\int_P f(\mathbf{x}) dm = \int_{P_d} \int_m^M f(\mathbf{x}) dx_1 dx_2 \cdots dx_{d-1}$$

Finally the convex polytope  $P_d$  can be decomposed into polyhedral regions where the functions  $m, M$  become simply affine functions from among the list. Since the integral is additive we get an expression

$$\int_P f(\mathbf{x}) dm = \sum_{i,j} \int_{P_d^{ij}} \int_{A_j^l}^{A_j^u} f(\mathbf{x}) dx_1 dx_2 \cdots dx_{d-1}$$

Finally by repeating the elimination of variables we recover the full iterated list in (18). As it was observed in [26], this algorithm is unfortunately not efficient because the iterated Fourier-Motzkin elimination procedure can produce exponentially many inequalities for the description of the projection (when the dimension  $d$  varies). Thus the number of summands considered can in fact grow exponentially.

**4.3. Two formulas for integral computation.** We would like to review two formulas that are nice and could speed up computation in particular cases although they do not seem to yield efficient algorithms just on their own.

First, one may reduce the computation of  $\int_P f dm$  to integrals on the facets of  $P$ , by applying Stokes formula. We must be careful to use a rational Lebesgue measure on each facet. As shown in ([4]), we have the following result.

**Theorem 19.** *Let  $\{F_i\}_{i=1,\dots,m}$  be the set of facets of a full-dimensional polytope  $P \subseteq \mathbb{R}^n$ . For each  $i$ , let  $\mathbf{n}_i$  be a rational vector which is transverse to the facet  $F_i$  and pointing outwards  $P$  and let  $d\mu_i$  be the Lebesgue measure on the affine hull of  $F_i$  which is defined by contracting the standard volume form of  $\mathbb{R}^n$  with  $\mathbf{n}_i$ . Then*

$$I_P(\mathbf{a}) = \int_P e^{\langle \mathbf{a}, \mathbf{x} \rangle} d\mathbf{x} = \frac{1}{\langle \mathbf{a}, \mathbf{y} \rangle} \sum_{i=1}^m \langle \mathbf{n}_i, \mathbf{y} \rangle \int_{F_i} e^{\langle \mathbf{a}, \mathbf{x} \rangle} d\mu_i$$

for all  $\mathbf{a} \in \mathbb{C}^n$  and  $\mathbf{y} \in \mathbb{R}^n$  such that  $\langle \mathbf{y}, \mathbf{a} \rangle \neq 0$ .

It is clear that, by considering the expansion of the analytic function  $\int_P e^{\langle \mathbf{a}, \mathbf{x} \rangle} d\mathbf{x}$ , we can again obtain an analogous result for polynomials. An alternative proof was provided by [18]. The above theorem, however, does not necessarily reduce the computational burden because, depending on the representation of the polytope, the number of facets can be large and also the facets themselves can be complicated polytopes. Yet, together with our results we obtain the following corollary for two special cases.

**Corollary 20.** *There is a polynomial-time algorithm for the following problem. Input:*

- (I<sub>1</sub>) *the dimension  $n \in \mathbb{N}$  in unary encoding,*
- (I<sub>2</sub>) *a list of rational vectors in binary encoding, namely*
  - (i) *either vectors  $(\mathbf{h}_1, h_{1,0}), \dots, (\mathbf{h}_m, h_{m,0}) \in \mathbb{Q}^{n+1}$  that describe the facet-defining inequalities  $\langle \mathbf{h}_i, \mathbf{x} \rangle \leq h_{i,0}$  of a simplicial full-dimensional rational polytope  $P$ ,*
  - (ii) *or vectors  $\mathbf{s}_1, \dots, \mathbf{s}_N \in \mathbb{Q}^n$  that are the vertices of a simple full-dimensional rational polytope  $P$ ,*
- (I<sub>3</sub>) *a rational vector  $\mathbf{a} \in \mathbb{Q}^n$  in binary encoding,*
- (I<sub>4</sub>) *an exponent  $M \in \mathbb{N}$  in unary encoding.*

*Output, in binary encoding,*

- (O<sub>1</sub>) *the rational number*

$$\int_P f(\mathbf{x}) d\mathbf{m} \quad \text{where} \quad f(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle^M$$

*and where  $d\mathbf{m}$  is the standard Lebesgue measure on  $\mathbb{R}^n$ .*

*Proof.* In the case (i) of simplicial polytopes  $P$  given by facet-defining inequalities, we can use linear programming to compute in polynomial time a  $V$ -representation for each simplex  $F_i$  that is a facet of  $P$ . By applying **Theorem 19** with  $t\mathbf{a}$  in place of  $\mathbf{a}$  and extracting the coefficient of  $t^M$  in the Taylor expansion of the analytic function  $t \mapsto I_P(t\mathbf{a})$ , we obtain the formula

$$\int_P \langle \mathbf{a}, \mathbf{x} \rangle^M d\mathbf{x} = \frac{1}{(M+1)\langle \mathbf{y}, \mathbf{a} \rangle} \sum_{i=1}^m \langle \mathbf{y}, \mathbf{n}_i \rangle \int_{F_i} \langle \mathbf{a}, \mathbf{x} \rangle^{M+1} d\mu_i,$$

which holds for all  $\mathbf{y} \in \mathbb{R}^n$  with  $\langle \mathbf{y}, \mathbf{a} \rangle \neq 0$ . It is known that a suitable  $\mathbf{y} \in \mathbb{Q}^n$  can be constructed in polynomial time. The integrals on the right-hand side can now be evaluated in polynomial time using [Theorem 2](#).

In the case (ii) of simple polytopes  $P$  given by their vertices, we make use of the fact that a variant of Brion's formula [\(9\)](#) actually holds for arbitrary rational polytopes. For a simple polytope  $P$ , it takes the following form.

$$\int_P \ell^M dx = \frac{M!}{(M+n)!} \sum_{i=1}^N \Delta_i \frac{\langle \ell, \mathbf{s}_i \rangle^{M+n}}{\prod_{\mathbf{s}_j \in N(\mathbf{s}_i)} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle}, \quad (19)$$

where  $N(\mathbf{s}_i)$  denotes the set of vertices adjacent to  $\mathbf{s}_i$  in  $P$ , and  $\Delta_i = |\det(\mathbf{s}_i - \mathbf{s}_j)_{j \in N(\mathbf{s}_i)}|$ . The right-hand side is a sum of rational functions of  $\ell$ , where the denominators cancel out so that the sum is actually polynomial. If  $\ell$  is regular, that is to say  $\langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle \neq 0$  for any  $i$  and  $j \in N(\mathbf{s}_i)$ , then the integral can be computed by [\(19\)](#) which is a very short formula. However it becomes difficult to extend the method which we used in the case of a simplex. Instead, we can do a perturbation. In [\(19\)](#), we replace  $\ell$  by  $\ell + \varepsilon \ell'$ , where  $\ell'$  is such that  $\ell + \varepsilon \ell'$  is regular for  $\varepsilon \neq 0$ . The algorithm for choosing  $\ell'$  is bounded polynomially. Then we do expansions in powers of  $\varepsilon$  as explained in [Lemma 4](#).  $\square$

**4.4. Triangulation of arbitrary polytopes.** It is well-known that any convex polytope can be triangulated into finitely many simplices. Thus we can use our result to extend the integration of polynomials over any convex polytope. The complexity of doing it this way will directly depend on the number of simplices in a triangulation. This raises the issue of finding the smallest triangulation possible of a given polytope. Unfortunately this problem was proved to be NP-hard even for fixed dimension three (see [\[12\]](#)). Thus it is in general not a good idea to spend time finding the smallest triangulation possible. A cautionary remark is that one can naively assume that triangulations help for non-convex polyhedral regions, while in reality it does not because there exist nonconvex polyhedra that are not triangulable unless one adds new points. Deciding how many new points are necessary is an NP-hard problem [\[12\]](#).

## 5. SOME COMPUTATIONAL EXPERIMENTS

We have written Maple programs to perform some initial experiments with the methods described in this paper. The programs are available at [\[3\]](#).<sup>1</sup>

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<sup>1</sup>All algorithms are implemented in the file `integration.lib` and illustrated in files called `test-*.maple`. The tables in this section have been created using the files `tables-*.maple`.

TABLE 4. Decomposition of polynomials into powers of linear forms, monomial-by-monomial construction

$n$	Degree $M$							
	1	2	5	10	20	30	40	50
2	2	5	28	145	963	3047	7019	13503
3	3	12	162	2094	40041	252367		
4	4	22	572	15476				
5	5	35	1546	78786				
8	8	92	13168					
10	10	145	37192					
15	15	330	252813					
20	20	590						
30	30	1335						
40	40	2380						
50	50	3725						

### 5.1. Decomposition of polynomials into powers of linear forms.

We have written a Maple program that decomposes a homogeneous polynomial  $f$  as a sum  $f = \sum_{\ell} c_{\ell} \ell^M$  with at most  $2^M$  powers of linear forms  $\ell$  for each monomial, using the formula (13) from the construction in the alternative proof of Corollary 3.<sup>2</sup>

Actually, in the implementation of this method, we group together proportional linear forms, thus we often obtain a smaller number of summands. Table 4 shows the number of terms in the resulting formula when  $f$  is a dense polynomial, for different values of  $M$  (degree) and  $n$  (dimension). In Table 5, we compare these numbers with those guaranteed by the Alexander–Hirschowitz theorem. The numbers illustrate that there is a big space for improvement, using a possible constructive solution of the polynomial Waring problem rather than the simple explicit construction of formula (13).

### 5.2. Integration of a power of a linear form over a simplex.

We have written a Maple program that implements the method of Section 3.3 for the efficient integration of a power of one linear form over a simplex,  $\int_{\Delta} \ell^M dm$ .<sup>3</sup> In a computational experiment, for a given dimension  $n$  and degree  $M$ , we picked random full-dimensional simplices  $\Delta$  and random linear forms  $\ell$  and used the Maple program to compute the integral. Table 6 shows the computation times.<sup>4</sup>

<sup>2</sup>The decomposition is done by the Maple procedure *from\_monome\_to\_linear* and is illustrated by the program *test-decomposition-powerlinform.maple*.

<sup>3</sup>The integration is done by the Maple procedure *without\_basic\_simplex\_integral*.

<sup>4</sup>All experiments were done with Maple 10 on a Sun Fire V440 with UltraSPARC-IIIi processors running at 1.6 GHz. The computation times are given in CPU seconds. All experiments were subject to a time limit of 600 seconds per example.

TABLE 5. Decomposition of polynomials into powers of linear forms, Alexander–Hirschowitz bounds

$n$	Degree $M$							
	1	2	5	10	20	30	40	50
2	1	2	3	6	11	16	21	26
3	1	3	7	22	77	166	287	442
4	1	4	14	72	443	1364	3086	5857
5	1	5	26	201	2126	9276	27151	63251
8	1	8	99	2431	111004	$1.3 \cdot 10^6$	$7.9 \cdot 10^6$	$3.3 \cdot 10^7$
10	1	10	201	9238	$1.0 \cdot 10^6$	$2.1 \cdot 10^7$	$2.1 \cdot 10^8$	$1.3 \cdot 10^9$
15	1	15	776	130751	$9.3 \cdot 10^7$	$7.7 \cdot 10^9$	$2.2 \cdot 10^{11}$	$3.2 \cdot 10^{12}$
20	1	20	2126	$1.0 \cdot 10^6$	$3.4 \cdot 10^9$	$9.4 \cdot 10^{11}$	$7.0 \cdot 10^{13}$	$2.3 \cdot 10^{15}$
30	1	30	9276	$2.1 \cdot 10^7$	$9.4 \cdot 10^{11}$	$2.0 \cdot 10^{15}$	$7.9 \cdot 10^{17}$	$1.1 \cdot 10^{20}$
40	1	40	27151	$2.1 \cdot 10^8$	$7.0 \cdot 10^{13}$	$7.9 \cdot 10^{17}$	$1.3 \cdot 10^{21}$	$6.7 \cdot 10^{23}$
50	1	50	63251	$1.3 \cdot 10^9$	$2.3 \cdot 10^{15}$	$1.1 \cdot 10^{20}$	$6.7 \cdot 10^{23}$	$1.0 \cdot 10^{27}$

TABLE 6. Integration of powers of linear forms over simplices

$n$	Degree $M$						
	2	10	20	50	100	300	1000
10	0.0	0.0	0.0	0.1	0.0	0.0	0.0
20	0.1	0.2	0.1	0.2	0.1	0.2	0.2
50	1.0	1.4	1.4	1.6	1.6	1.6	1.7
100	5.1	8.4	8.7	9.2	9.5	10.0	11.0
200	36	71	84	88	97	110	120
300	150	320	400	470	520	530	
400	500						
1000							

**5.3. Integration of a monomial over a simplex.** Finally, we tested the algorithms discussed in this paper, for the case of the integration of a monomial over a simplex. For our experiments, we considered random simplices  $\Delta$  of dimension  $n$ . We consider a random monomial  $\mathbf{x}^{\mathbf{m}}$  where  $\mathbf{m} = (m_1, \dots, m_d)$  with exponents  $m_i$  between  $M$  and  $M + 1$ . For each choice of  $n$  and  $M$ , we picked 50 combinations of simplices  $\Delta$  and exponent vectors  $\mathbf{m}$ .

First we decompose a given monomial into a sum of powers of linear forms and integrate each summand using the Maple program discussed above.<sup>5</sup> Table 7 shows the minimum, average, and maximum computation times.

<sup>5</sup>This method is implemented in the Maple procedure *without\_from\_monome\_to\_linear\_integral*.

TABLE 7. Integration of a monomial by decomposition into a sum of powers of linear forms

$n$	Degree $M$										
	1	2	5	10	20	30	40	50	100	200	300
2	0.0	0.0	0.0	0.3	1.2	3.2	6.3	9.7	44	270	
	<b>0.0</b>	<b>0.0</b>	<b>0.1</b>	<b>0.3</b>	<b>1.5</b>	<b>3.5</b>	<b>6.9</b>	<b>11.0</b>	<b>52</b>	<b>320</b>	
	0.1	0.1	0.2	0.4	1.7	3.8	7.6	11.0	54	350	
3	0.0	0.1	0.9	7.1	58	260					
	<b>0.1</b>	<b>0.2</b>	<b>1.3</b>	<b>8.5</b>	<b>67</b>	<b>300</b>					
	0.2	0.3	1.8	10.0	74	320					
4	0.1	0.5	10	140							
	<b>0.2</b>	<b>1.0</b>	<b>14</b>	<b>180</b>							
	0.6	1.7	19	220							
5	0.3	2.4	99								
	<b>0.8</b>	<b>5.3</b>	<b>150</b>								
	2.3	8.4	230								
8	8.4										
	<b>37.0</b>										
	110.0										
10											

Second, we have implemented the Lasserre–Avrachenko polarization formula (17) of Corollary 18, as described in Section 4.1<sup>6</sup>. The running times are shown in Table 8.

## 6. CONCLUSIONS

We discussed various algorithms for the exact integration of polynomials over simplicial regions. Beside their theoretical efficiency, the simple rough experiments we performed clearly demonstrated that these methods are robust enough to attack rather difficult problems. Our investigations opened several doors for further development, which we will present in a forthcoming paper.

First, we have some theoretical issues expanding on our results. As in the case of volumes and the computation of centroids, it is likely that our hardness result, Theorem 1, can be extended into an inapproximability result as those obtained in [25]. Another goal is to study other families of polytopes for which exact integration can be done efficiently. Furthermore, we will present a natural extension of the computation of integrals, the efficient computation of the highest degree coefficients of

<sup>6</sup>This method is implemented in the Maple procedure `integrate_with_polarization`.



TABLE 8. Integration of a monomial using polarization

$n$	Degree $M$										
	1	2	5	10	20	30	40	50	100	200	300
2	0.0	0.0	15								
	<b>0.0</b>	<b>0.1</b>	<b>51</b>								
	0.1	0.4	110								
3	0.0	1.0									
	<b>0.3</b>	<b>7.2</b>									
	1.1	28.0									
4	0.2										
	<b>5.1</b>										
	31.0										
5											

a weighted Ehrhart quasipolynomial of a simplex. Besides the methods of the present article, these last computations are based on the results of [2] and [5].

Second, our intention has been all along to develop algorithms with a good chance of becoming practical and that allow for clear implementation. Thus we have also some practical improvements to discuss. Instead of Corollary 14, we can use Brion’s formula (9) and a proper use of iterated Laurent expansion to compute  $\int_{\Delta} f dm$  in polynomial time when the degree of  $f$  is fixed. We expect that it will speed up the current techniques we have implemented in practice. Finally, in order to develop practical integration software, it appears that our methods should be coupled with fast techniques for decomposing domains into polyhedral regions (e.g. triangulations).

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