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RANKIN-COHEN BRACKETS AND ASSOCIATIVITY

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Abstract. Don Zagier introduced and discussed in [21] a particular algebraic structure of the graded ring of modular forms. In this note we interpret it in terms of an associative deformation of this graded ring.

1. Rankin-Cohen brackets and Zagier RC-algebras

What differential operators does preserve modularity? That question was answered in 1956 by R.A. Rankin [13]. Nineteen years later H. Cohen [5] investigated this problem in the framework of bi-differential operators. Namely, he introduced a whole family of such operators transforming a given pair of modular forms into another modular form of a higher weight. These operators are usually referred to as *Rankin-Cohen brackets* and they have interesting applications in quantization theory as well as in the representation theory of the Lie group $SL(2,\mathbb{R})$. Notice that their generalizations for different series of simple Lie groups were studied in [1, 7, 8].

More precisely, let G be the simple Lie group $SL(2,\mathbb{R})$ and $K = SO(2)$ be its maximal compact subgroup. Throughout this note we fix one particular arithmetic subgroup of G, to wit $\Gamma = SL(2,\mathbb{Z}) \subset G$, so that all arithmetic notions should be understood as being defined with respect to Γ.

We say that a function f holomorphic in the upper half-plane $\Pi = \{z = x + iy \in \mathbb{C} \,|\, y > 0\}$ is a *modular form* of weight $k \in 2\mathbb{N}$ if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \forall z \in \Pi, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The set of such functions is denoted $M_k(\Gamma)$ and we define

$$M_*(\Gamma) = \oplus_{k\in 2\mathbb{N}} M_k(\Gamma)$$

to be the graded ring of modular forms.

For a pair of modular forms $f_1 \in M_{k_1}(\Gamma)$ and $f_2 \in M_{k_2}(\Gamma)$ their j-th Rankin-Cohen bracket is defined by

$$(1.1) \quad [f_1, f_2]_j(z) := \sum_{\ell=0}^{j} (-1)^\ell \begin{pmatrix} k_1 + j - 1 \\ \ell \end{pmatrix} \begin{pmatrix} k_2 + j - 1 \\ j - \ell \end{pmatrix} f_1^{(j-\ell)}(z) f_2^{(\ell)}(z),$$

where $f^{(\ell)}(z) = \left(\frac{\partial}{\partial z}\right)^{(\ell)} f(z)$.

Notice that the specific normalization is chosen in such a way that $[f_1, f_2]_j \in \mathbb{Z}[[q]]$, when $f_1, f_2 \in \mathbb{Z}[[q]]$, where $q = e^{2i\pi z}$.

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Let π_k denote a representation of G on the weighted Bergman space $\mathcal{H}^2_k(\Pi) := \mathcal{O}(\Pi) \cap L^2(\Pi, y^{k-2}dxdy)$ defined for any integer $k > 1$ by :

$$(\pi_k(g))f(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.2}$$

This is an irreducible unitary representation of G referred to as the *holomorphic discrete series* representation.

It turns out that for any $g \in G$ and any $f_i \in \mathcal{H}^2_{k_i}(\Pi)$, $i = 1, 2$ one has

$$[\pi_{k_1}(g)f_1, \pi_{k_2}(g)f_2]_j = \pi_{k_1+k_2+2j}(g)[f_1, f_2]_j. \tag{1.3}$$

Particularly, this property remains valid for any $g \in \Gamma$ and $f_i \in M_{k_i}(\Gamma)$, $i = 1, 2$ and it implies therefore that $[f_1, f_2]_j \in M_{k_1+k_2+2j}(\Gamma)$ for such functions.

The fact 1.3 can be proved in different manners: rather analytically by use of generating series or some properties of θ-functions [21] or purely algebraically by use of underlying highest weight Harish-Chandra modules [11, 12]. In either cases the main idea is to show that up to a constant the j-th Rankin-Cohen bracket is the only bi-differential intertwining operator sending modular forms of weights k_1 and k_2 into a modular form of weight $k_1 + k_2 + 2j$.

It is well known [14, 18] that the restriction of the tensor product of two holomorphic discrete series representations $\pi_{k_1} \otimes \pi_{k_2}$ to the diagonal subgroup $diag\, G \subset G \times G$ decomposes multiplicity free into a direct discrete sum of irreducible representations of the holomorphic discrete series:

$$\pi_{k_1} \otimes \pi_{k_2} = \bigoplus_{j \geq 0} \pi_{k_1+k_2+2j}.$$

Being chosen a concrete model for representations π_k, to wit the weighted Bergman spaces, the j-th Rankin-Cohen bracket may be interpreted as the orthogonal G-equivariant projector of the tensor product's representation space onto its j-th irreducible component $\pi_{k_1+k_2+2j}$.

D. Zagier pointed out in [21] that Rankin-Cohen brackets satisfy an infinite series of algebraic relations that he formalized as a RC-algebra structure. Let us recall first such identities for $f_i \in M_{k_i}(\Gamma)$, $i = 1, 2, 3$:

$$\begin{aligned}
[f_1, f_2]_j &= (-1)^j [f_2, f_1]_j, \quad \forall j \in \mathbb{N}; \\
[[f_1, f_2]_0, f_3]_0 &= [f_1, [f_2, f_3]_0]_0\,; \\
[f_1, 1]_0 &= [1, f_1]_0 = f_1,\ [f_1, 1]_j = [1, f_1]_j = 0,\ \forall j > 0; \\
0 &= [[f_1, f_2]_1, f_3]_1 + [[f_2, f_3]_1, f_1]_1 [[f_3, f_1]_1, f_2]_1; \\
0 &= k_3[[f_1, f_2]_1, f_3]_0 + k_1[[f_2, f_3]_1, f_1]_0 + k_2[f_3, f_1]_1, f_2]_0; \\
[[f_1, f_2]_0, f_3]_1 &= [[f_2, f_3]_1, f_1]_0 - [[f_3, f_1]_1\,, f_2]_0\,; \\
[[f_1, f_2]_1, f_3]_0 &= \frac{k_1}{k_1+k_2+k_3}[[f_3, f_1]_0, f_2]_1 - \frac{k_3}{k_1+k_2+k_3}[[f_2, f_3]_0, f_1]_1.
\end{aligned}$$

Theses (redundant) relations say that the first two brackets $[\ \ ,\ ]_0$ and $[\ \ ,\ ]_1$ define one $M_*(\Gamma)$, and more generally on $\mathcal{H} = \oplus_{k>1}\mathcal{H}^2_k(\Pi)$, the structure of a Poisson algebra.

Let us mention some other relations satisfied by Rankin-Cohen brackets that can be checked directly :

$$
\begin{aligned}
[[f_1,f_2]_0,f_3]_2 &= -\frac{k_3(k_3+1)}{(k_1+1)(k_2+1)}[[f_1,f_2]_2,f_3]_0 \\
&+ \frac{k_1+k_2+1}{k_2+1}[[f_2,f_3]_2,f_1]_0 + \frac{k_1+k_2+1}{k_1+1}[[f_3,f_1]_2,f_2]_0;
\end{aligned}
$$

$$
\begin{aligned}
[[f_1,f_2]_2,f_3]_0 &= \frac{(k_1+1)(k_2+1)}{(k_1+k_2+k_3+1)(k_1+k_2+k_3+2)}[[f_1,f_2]_0,f_3]_2 \\
&- \frac{(k_1+1)(k_1+k_2+1)}{(k_1+k_2+k_3+1)(k_1+k_2+k_3+2)}[[f_2,f_3]_0,f_1]_2 \\
&- \frac{(k_2+1)(k_1+k_2+1)}{(k_1+k_2+k_3+1)(k_1+k_2+k_3+2)}[[f_3,f_1]_0,f_2]_2;
\end{aligned}
$$

$$
\begin{aligned}
[[f_1,f_2]_1,f_3]_1 &= [[f_2,f_3]_0,f_1]_2 + [[f_2,f_3]_2,f_1]_0 \\
&- [[f_3,f_1]_0,f_2]_2 - [[f_3,f_1]_2,f_2]_0.
\end{aligned}
$$

These higher order relations might look complicated at the first glance but their nature will be much clearer once one will have adopted an operadic point of view and will have thought of the Rankin-Cohen brackets as of the bi-differential operators acting on sections of an appropriate homogeneous line bundle. In order to set up this framework we start by introducing a quantization map for a particular homogeneous space of $SL(2,\mathbb{R})$ and deducing some properties of Rankin-Cohen brackets.

2. Covariant quantization of co-adjoint orbits.

Denote $H = SO(1,1) \subset SL(2,\mathbb{R}) = G$ and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ the Lie algebra of G which is identified with its dual space $\mathfrak{g}^*$ by the Killing form $B(X,Y) = \mathrm{tr}(XY)$. The set $\mathcal{X} = G/H$ is an adjoint G-orbit passing through $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{g}$ endowed with the canonical Kostant- Kirillov symplectic form.

This symmetric space, which is nothing else but the one-sheeted hyperboloid in $\mathbb{R}^3 \simeq \mathfrak{g}^*$, posses a para-Hermitian structure in the sense that its tangent bundle splits into a G-equivariant direct sum of isomorphic sub-bundles : $T(\mathcal{X}) = T(\mathcal{X})^+ \oplus T(\mathcal{X})^-$. We shall then distinguish local coordinates (s,t) on $\mathcal{X}$ according to this splitting.

In [20] authors constructed and studied a G-covariant symbolic calculus on the phase space $\mathcal{X}$ and showed that the Rankin-Cohen brackets arise from the composition formula of symbols of particular type.

More precisely, for any $\lambda \in \mathbb{R}$ the map Op_λ defined by:

$$
Op_\lambda(f)u(s) = c_{-\lambda}\int_{\mathcal{X}} f(s,t)|s-t|^{-1-i\lambda}u(\tau)|\tau-t|^{-1+i\lambda}d\tau dt,
$$

where c_λ is a normalizing constant, is an isometry from the Hilbert space of square integrable functions $f \in L^2(\mathcal{X})$, called symbols, into the space of Hilbert-Schmidt operators on the configuration space $L^2(\mathbb{R})$. This construction respects symmetries. Indeed, the Lie group G acts on $L^2(\mathbb{R})$ through the so-called principal series representations:

$$\pi_{i\lambda}(g)u(s) = |cs+d|^{-1-i\lambda} u\left(\frac{as+b}{cs+d}\right),$$

with $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and one easily checks that

$$\pi_{i\lambda}(g) Op_\lambda(f) \pi_{i\lambda}(g^{-1}) = Op_\lambda(f \circ g^{-1}).$$

One might notice that this phenomenon is rather natural because the map Op_λ is build up from the intertwining operators of the unitary principal series representations.

Composition of linear operators gives rise to a naturally associative non-commutative product $\sharp_\lambda$ on $L^2(\mathcal{X})$ such that:

$$Op_\lambda(f_1 \sharp_\lambda f_2) = Op_\lambda(f_1) \circ Op_\lambda(f_2).$$

Moreover, there is an integral representation for this product:

$$f_1 \sharp_\lambda f_2(s,t) = |c_\lambda|^2 \int_{\mathcal{X}} f_1(s,x) f_2(y,t) \left|\frac{(s-x)(y-t)}{(s-t)(x-y)}\right|^{-1-i\lambda} d\mu(x,y),$$

where $d\mu(x,y) = (x-y)^2 dxdy$ is the G-invariant measure on X. Notice that this integral is not well defined in general and must be understood in the sense of distributions.

The regular action of G on the Hilbert space $L^2(\mathcal{X})$ is reducible. Its decomposition into irreducible representations is equivalent to the study of eigenspaces of the Laplace-Beltrami operator $\Box$ of the pseudo-Riemannian manifold $\mathcal{X}$ and it is well known [9, 19], that the spectrum of the G-invariant differential operator $\Box$ contains a continuous part with multiplicity two, corresponding to principal series representations of G, and two infinite discrete parts: $\{\mathcal{D}_n^\pm, n > 1\}$, corresponding to discrete holomorphic and anti-holomorphic series representations of G.

Particular geometric structure of para-Hermitian symmetric spaces implies that they are densely embedded in the Cartesian product of two conjugate (in an appropriate sense) maximal flag varieties as, for instance, $\mathcal{X} \subset \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$. On the other hand specific spectral properties of tensor products of highest weight Harish-Chandra modules (underlying holomorphic discrete series representations), namely their discrete decomposability in the sense of T. Kobayashi, guarantee that the set $\oplus_{n>1}\mathcal{D}_n^+$ is an algebra with respect to the non-commutative and certainly associative product $\sharp_\lambda$ and more precisely, that

$$f_1 \sharp_\lambda f_2 = \sum_{n \geq 0} c_\lambda(n)[f_1, f_2], \quad \forall f_i \in \mathcal{D}_{k_i}^+, i = 1, 2, \tag{2.1}$$

where $c_\lambda(n)$ are combinatorial expressions involving the spectral parameter λ and depending on n but which should not be interpreted as some kind of powers of the Plank constant. Their explicit form was obtained in [20] and by different methods in [6, 22]. It is noteworthy that the right hand side in the above formula defines an absolutely convergent series of holomorphic functions. The same phenomenon remains valid for a wilder class of so-called conformal Lie groups. It was studied in [7].

Notice that in this approach the associativity of 2.1 is granted from the very beginning and actually it implies the identities referred to as the *RC-algebra structure*[1].

In order to make our statement clearer we give, in the next section, another realization of the product $\sharp_\lambda$ and show that its associativity governs the Zagier RC-algebra axioms.

3. Main observation

The Plancherel formula for real reductive Lie groups says that "essentially" all unitary irreducible representations are obtained by induction from finite dimensional representations of appropriate subgroups and according to the Mackey theory an induced representation must be thought of as an action on sections of vector bundles rather then on functions. Adopting this approach we will see that the $\sharp_\lambda$-product on $\mathcal{H}$ is related to the Moyal star-product on a flat symplectic vector space.

Let $G_{\mathbb{C}} = SL(2,\mathbb{C})$ be the complexification of the Lie group G and let B denote its Borel subgroup. Let τ_k with $k \in \mathbb{N}^+$ be a holomorphic character of B and let $\mathcal{L}_k$ be the homogeneous holomorphic line bundle over $G_{\mathbb{C}}/B$ corresponding to τ_k. Therefore a function $f \in \mathcal{H}^2_k(\Pi)$ can be identified with a square integrable, holomorphic section $F \in \Gamma(\mathcal{L}_k)$.

Proposition 3.1. *Let $f_1(z_1,w_1)$ and $f_2(z_2,w_2)$ be two holomorphic functions on $\mathbb{C}^2$ homogeneous of degree k_1 and k_2 respectively (with $k_1,k_2 \in \mathbb{N}^+$). Let $\widetilde{f}_i(z) := f_i(z,1)$, $i=1,2$ be their projectivisations.*

Let $\Omega = \dfrac{\partial^2}{\partial z_1 \partial w_2} - \dfrac{\partial^2}{\partial z_2 \partial w_1}$ be the second order differential operator corresponding to the canonical symplectic form on $\mathbb{C}^2 \times \mathbb{C}^2$. Then,

$$\Omega(f_1(z_1,w_1)\otimes f_2(z_2,w_2))\Big|_{\substack{z=z_1=z_2\\ w_1=w_2=1}} = [\widetilde{f}_1,\widetilde{f}_2]_1(z).$$

More generally,

$$\Omega^n\widetilde{(f_1\otimes f_2)} = n![\widetilde{f}_1,\widetilde{f}_2]_n.$$

The proof is straightforward. Indeed, for $i=1,2$ we have $f_i(z,w) = w^{-k_i}\widetilde{f}_i\left(\frac{z}{w}\right)$ and

$$\frac{\partial f_i}{\partial w}(z,w) = -k_i w^{-k_i-1}\widetilde{f}_i\left(\frac{z}{w}\right) - z w^{-k_i-2}\widetilde{f}_i'\left(\frac{z}{w}\right), \quad \frac{\partial f_i}{\partial z}(z,w) = w^{-k_i-1}\widetilde{f}_i'\left(\frac{z}{w}\right).$$

Thus

$$\begin{aligned}
&\Omega(f_1(z_1,w_1)\otimes f_2(z_2,w_2)) = \\
&w_1^{-k_1-1}\widetilde{f}_1'\left(\frac{z_1}{w_1}\right)\left(-k_2 w_2^{-k_2-1}\widetilde{f}_2\left(\frac{z_2}{w_2}\right) - z_2 w_2^{-k_2-1}\widetilde{f}_2'\left(\frac{z_2}{w_2}\right)\right)\\
-\ &w_2^{-k_2-1}\widetilde{f}_2'\left(\frac{z_2}{w_2}\right)\left(-k_1 w_1^{-k_1-1}\widetilde{f}_1\left(\frac{z_1}{w_1}\right) - z_1 w_1^{-k_1-1}\widetilde{f}_1'\left(\frac{z_1}{w_1}\right)\right)
\end{aligned}$$

[1]The remark on the fact that this infinite series of identities should be interpreted as the graded counter-part of the associativity of some non-commutative product was already made in the addendum of D. Zagier's original paper [21]. But the kind of star-product mentioned there was not associative and no proof was given.

Putting $z = z_1 = z_2$ and $w_1 = w_2 = 1$ in the above expression we get $k_1\widetilde{f}_1(z)\widetilde{f}_2{}'(z) - k_2\widetilde{f}_1{}'(z)\widetilde{f}_2(z)$. The last statement is proved in a similar way.

Notice that the successive powers of Ω, usually called *transvectants*, was already used by P. Gordan in 1887 in order to construct the so called binary invariants [10].

Let Λ be the canonical Poisson structure on $\mathbb{C}^2 \times \mathbb{C}^2$ given by

$$\Lambda = \sum \Lambda^{ij} \partial_i \wedge \partial_j,$$

with $\Lambda^{ij} = -\Lambda^{ji}$. Then the Moyal product associated with Λ is defined on the set of formal power series with smooth coefficients $C^\infty[[\hbar]](\mathbb{C}^4)$, by:

$$f \star_M g\,(z) = \exp(i\pi\hbar\,\Lambda^{rs}\partial_{x_r}\partial_{y_s})(f(x)g(y))_{|x=y=z}.$$

The previous proposition implies therefore the following statement.

Theorem 3.2. *For $f_1(z_1, w_1)$ and $f_2(z_2, w_2)$ being two holomorphic functions on $\mathbb{C}^2$ homogeneous of degree k_1 and k_2 respectively (with $k_1, k_2 \in \mathbb{N}^+$) we can specialize the formal parameter $\hbar = 1$ and then, for any $\lambda \in \mathbb{R}$ one has,*

$$\widetilde{f_1 \star_M f_2}(z) = \widetilde{f}_1 \sharp \widetilde{f}_2(z).$$

Notice that P. Olver did already mention in [15] the link between transvectants and star-products. The same idea, to wit the fact that the Rankin-Cohen brackets do define an associative deformation of the graded ring of modular forms were discussed in [4, 6, 3, 16, 17, 22].

The associativity of the Moyal product, granted by the previous theorem (it also can be proved in a different way, see for instance [2]) induces an infinite series of relations for bi-differential operators $\Lambda^{rs}\partial_{x_r}\partial_{y_s}$. Once restricted to sections of appropriate homogeneous line bundles these conditions, by use of the proposition 3.1 transform precisely into the identities defining the RC-algebra structure of the graded ring $M(\Gamma)$. More precisely we have:

Proposition 3.3. *The RC-algebra structure of the graded ring $M(\Gamma)$ is the prejectivization of the associative algebra structure on the set of corresponding sections of holomorphic homogeneous line bundles $\mathcal{L}_k$ over the flag manifold $G_{\mathbb{C}}/B$.*

In conclusion we should say that the similar approach to the quantization of pseudo-Riemannian symmetric spaces allows a generalization of the above construction for the whole class of conformal Lie groups for which the notion of generalized Rankin-Cohen brackets was defined in [7]. According to some arguments of the representation theory of underlying semi-simple Lie groups, these bi-differential operators are parameterized by integral lattices. The associativity of the corresponding $\sharp$-product implies therefore a very interesting, from the combinatorial point of view, family of identities that must be understood.

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