Hypergeometric Solutions of Systems of Linear Difference Equations

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Introduction

- This talk is based on a joint work with Mark van Hoeij (MvH) from FSU.
- We present a **direct** algorithm for computing hypergeometric solutions of systems of linear difference equations.
- "*direct algorithm*" means "*without first reducing the system to a single scalar equation by the use of a cyclic vector*"
- Motivation: finding d'th order ($d \geq 2$) right-hand factors of a given linear difference operator with polynomial coefficients.

The differential case

- Beke (1894) gave two algorithms for factoring linear differential operators.
- Given a differential operator L of order n, the first algorithm computes the exponential solutions of L, or equivalently, the first order right-hand factors of L.
- The second algorithm reduces the problem of finding d'th order right-hand factors of L to computing 1'st order right-hand factors of the so-called associated equations, which have order $\left(\begin{array}{c} n \\ d \end{array} \right)$.
- This was improved by M. Bronstein (1994).

- Bronstein's approach has several efficiency advantages:

  (i) Instead of computing exponential solutions of a number of associated equations $L_1(y_1) = 0$, $L_2(y_2) = 0$, ... one now computes the exponential solutions of just one associated system $Y' = AY$.

  (ii) Consequently: a combinatorial problem disappears, namely the problem of deciding which exponential solution of each L_i should be combined with each other.

  (iii) The most important advantage is: The coefficients of the associated system are much smaller than the coefficients of the associated equations.

- A consequence of advantage (iii) is that solving this one system $Y' = AY$ potentially takes less time, or even much less time, than solving even one of the associated equations L_i, let alone solving all of them.

- We would not have this advantage if we used the cyclic vector approach to solve $Y' = AY$, because the resulting operator would be an associated equation.

- Thus, to benefit fully from the advantages of Bronstein's approach, we need to be able to directly compute the exponential solutions of $Y' = AY$ without first reducing this system to an operator by the use of a cyclic vector.

- Algorithms for directly computing the exponential solutions are implemented in ISOLDE (a Maple package developed by E. Pfluegel and Bark.) .

The difference case

- First order right-hand factors of a difference operator L correspond to so-called hypergeometric solutions.
- Algorithms for computing hypergeometric solutions of difference operators are given in [Petkovšek (1992), Cluzeau-van Hoeij (2006)].
- For computing higher order factors, Bronstein's approach works for the difference case as well in much the same way.
- Again, one reduces the computation of a d'th order factor of L to solving (hypergeometric solutions) of an associated difference system

$$Y(x+1) = M(x)Y(x).$$

Main Contributions

- We show that Petkovšek's algorithm for hypergeometric solutions of operators can be extended to computing hypergeometric solutions of systems.
- As in the scalar case we need to compute first the local data at infinity (the integer slopes of the Newton polygon and the corresponding polynomials)
- In the operator case the local data can be easily obtained from the coefficients of the operator.
- However, in the case of systems, we need a reduction algorithm [Bark. 1989] to determine the local data.
- Algorithms for computing polynomial and rational solutions of difference systems are given in [Abramov&Bark. (1998), Barkatou (1999)].

Petkovšek's algorithm

Let $\mathbf{C}$ be a field of characteristic zero.

An operator

$$L = a_n\tau^n + \ldots + a_0\tau^0 \in \mathbf{C}(x)[\tau]$$

acts as follows:
If $u = u(x)$ is a function then

$$L(u) = a_n u(x+n) + \cdots + a_1 u(x+1) + a_0 u(x)$$

If

$$L(u) = 0 \quad \text{and} \quad \frac{u(x+1)}{u(x)} \in \mathbf{C}(x)$$

then u is called a **hypergeometric solution** and corresponds to a **right-hand factor** $\tau - r$ of L where

$$r = \frac{u(x+1)}{u(x)}.$$

A hypergeometric solution u of $L = a_n\tau^n + \cdots + a_0\tau^0$ can always be written as

$$u(x) = c^x P(x) \prod_i \Gamma(x - \alpha_i)^{e_i}$$

for some

$$c \in \mathbf{C}^*, \quad P(x) \in \mathbf{C}[x], \quad \alpha_i \in \mathbf{C}, \quad e_i \in \mathbb{Z}.$$

Write

$$A = \prod_{e_i > 0} (x - \alpha_i)^{e_i} \text{ and } B = \prod_{e_i < 0} (x - \alpha_i)^{-e_i}$$

so that

$$u(x) = c^x P(x) \mathrm{Sol}(\tau - \frac{A}{B}).$$

$$L = a_n(x)\tau^n + \cdots + a_0(x)\tau^0.$$

- Petkovšek (1992) gave a criterion for $A, B \in \mathbf{C}[x]$, namely:

$$A \text{ divides } a_0(x)$$

$$B \text{ divides } a_n(x - n + 1)$$

- This leaves only a finite (but exponential) number of potential A, B in the hypergeometric solutions:

$$\begin{aligned} u(x) &= c^x P(x) \mathrm{Sol}(\tau - \frac{A}{B}) \\ &= c^x P(x) \prod_i \Gamma(x - \alpha_i)^{e_i} \end{aligned}$$

- Petkovšek's algorithm (1992):

  For all possible combinations of:
  - a monic factor $A \in \mathbf{C}[x]$ of a_0,
  - a monic factor B of $a_n(x-n+1)$,
  - and c in some finite list

  compute a recurrence $L_{A,B,c}$ that has $P(x)$ as a solution, and solve it to find hypergeometric solutions:

$$c^x P(x) \mathrm{Sol}(\tau - \frac{A}{B})$$

- Another algorithm was given by Cluzeau and v.Hoeij (2006).

Petkovšek's algorithm extended to systems

- The system to solve looks like

$$\begin{pmatrix} y_1(x+1) \\ \vdots \\ y_n(x+1) \end{pmatrix} = M \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}$$

where M is a n by n invertible matrix over $\mathbf{C}(x)$.

- The solution we search for is hypergeometric, meaning that it can be written as

$$\begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix} = c^x \begin{pmatrix} P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} \mathrm{Sol}(\tau - \frac{A}{B})$$

where $A, B, P_1, \ldots, P_n \in \mathbf{C}[x]$ and $c \in \mathbf{C}^*$.

- The goal: Find a solution

$$Y = c^x \begin{pmatrix} P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} \mathrm{Sol}(\tau - \frac{A}{B})$$

  of the system $\tau(Y) = MY$.

- Once you have A, B and c, then you can compute $P_1, \ldots, P_n$ using Abramov-Barkatou algorithm for polynomial solutions of difference systems (1998).

How to find A, B?

$$Y = c^x \begin{pmatrix} P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} \mathrm{Sol}(\tau - \frac{A}{B})$$

We may assume that the gcd of $P_1, \ldots, P_n$ is 1.
Now look at

$$\tau(Y) = \frac{A}{B} c^{x+1} \begin{pmatrix} P_1(x+1) \\ \vdots \\ P_n(x+1) \end{pmatrix} \mathrm{Sol}(\tau - \frac{A}{B})$$

- The denominator B in $\tau(Y)$ does not appear in Y, and since $\tau(Y) = MY$ it follows that the denominator B must divide the denominator of M.
- A similar argument shows that A divides the denominator of M^{-1}.

So one can generalize Petkovšek's algorithm to systems:

- B has to divide denominator(M),
- and A has to divide denominator(M^{-1}).

How to find c?

- If the system $\tau Y = MY$ has a hypergeometric solution

$$c^x \mathrm{Sol}(\tau - \frac{A}{B})\, P(x), \quad P(x) \in \mathbf{C}[x]^n$$

then it has a formal solution of the form

$$Y(x) = \Gamma(x)^s c^x x^d F(x)$$

where

$(s, c, d) \in \mathbb{Z} \times \mathbf{C}^* \times \mathbf{C}$

$F(x)$ is an n–dimensional vector with entries in $\mathbf{C}[[x^{-1}]]$:

$$F(x) = \left(F_0 + x^{-1} F_1 + \dots\right) \quad F_0 \neq 0.$$

In fact, if we write A, B as

$$A = x^{\ell} + \alpha_{\ell-1}x^{\ell-1} + \cdots + \alpha_0$$

$$B = x^{m} + \beta_{m-1}x^{m-1} + \cdots + \beta_0$$

then

$$s = \ell - m$$

and

$$d = \alpha_{\ell-1} - \beta_{m-1} + \deg P.$$

Local Hypergeometric Solutions

Consider a difference system

$$\tau Y = MY.$$

We call *local hypergeometric solution* (lhs, in short) any nonzero formal solution of the form

$$Y(x) = \Gamma(x)^s c^x x^d F(x)$$

where $s \in \mathbb{Z}$, $c \in \mathbf{C}^*$, $d \in \mathbf{C}$ and where $F(x)$ is an n–dimensional vector with entries in $\mathbf{C}[[x^{-1}]]$:

$$F(x) = \left(F_0 + x^{-1}F_1 + \dots\right) \qquad F_0 \neq 0.$$

If a formal solution of the form

$$Y(x) = \Gamma(x)^s c^x x^d F(x), \quad F(x) \in \mathbf{C}[[x]]^n.$$

exists then

- s is a slope of the $\tau-$Newton polygon of the system $\tau Y = MY$,
- c is a root of the Newton polynomial associated with s
- and d is then a root of the indicial equation associated with the slope zero in the $\Delta-$Newton polygon of the system

$$\Delta Z(x) := Z(x+1) - Z(x) = (x^{-s} c^{-1} M(x) - I)Z(x).$$

$$Y(x) = \Gamma(x)^s c^x x^d F(x), \quad F(x) \in \mathbf{C}[[x]]^n.$$

- The pairs (s, c) are invariants under *gauge transformations*:

$$Y(x) \mapsto T(x)Y(x) \quad T \in \mathrm{GL}(n, \mathbf{C}((x^{-1}))).$$

- However, the d's may change into $d + \nu$, for some $\nu \in \mathbb{Z}$.

- We are interested in the computation of all the triplets $(s, c, d) \in \mathbb{Z} \times \mathbf{C}^* \times \mathbf{C}$ that appear in the lhs of our system.
- In general, the triplets (s, c, d) cannot be obtained directly from the matrix system $M(x)$.
- However, we show that all the possible pairs (s, c) can be obtained from a *supper-irreducible form* of the matrix $\widehat{M} := M - I_n$.
- Once a pair (s, c) is obtained, the corresponding d's can be computed from a supper-irreducible form of the matrix

$$(x^{-s} c^{-1} M(x) - I_n).$$

Super–irreducible Forms

The notion of super–reduced forms of linear difference systems had been developed in [Bark 1989].

Notation

- $\Delta = \tau - 1$ denotes the difference operator defined by

$$\Delta f(x) = f(x+1) - f(x)$$

for any function f of x.

- Any difference system

$$\tau Y = MY,$$

can be rewritten as

$$\Delta Y = \widehat{M} Y$$

where

$$\widehat{M} := M - I_n.$$

Gauge transformations

Let $T(x)$ be an invertible matrix.

The gauge transformation

$$Y = TZ$$

yields the following *equivalent* system

$$\tau Z = NZ \qquad N = T_\tau[M] := \tau T^{-1} M T$$

and in the $\Delta-$form

$$\Delta Z = \widehat{N} Z \qquad \widehat{N} = T_\Delta[\widehat{M}] := \tau T^{-1} \widehat{M} T - \tau T^{-1} \Delta T.$$

Super–irreducible Forms

- Consider a difference system of the form :

$$\Delta Y(x) = A(x)Y(x),$$

with

$$A(x) = x^q \sum_{\nu=0}^{\infty} x^{-\nu} A_\nu, \quad q \in \mathbb{Z}, \;\; A_0 \neq 0.$$

- For $k = 1, \ldots, q+1$, put

$$m_k(A) = \begin{cases} 0 & \text{if} \quad q \leq -1 \\ q + 1 + \frac{n_0}{n} + \frac{n_1}{n^2} + \cdots + \frac{n_{k-1}}{n^k} & \text{if} \quad q > -1 \end{cases}$$

where n_i is the number of rows of A with valuation $-q + i$.

- Define
$$\mu_k(A) = \min_T m_k(T_\Delta[A]).$$

- The matrix A is said to be k-irreducible if
$$m_k(A) = \mu_k(A).$$
Otherwise A is called k-reducible.

- A is $k-$reducible $\iff$ there exits T s.t. $m_k(T_\Delta[A]) < m_k(A)$.

- The matrix A is said to be super–irreducible, if it is k-irreducible for every k, or equivalently if
$$m_{q+1}(A) = \mu_{q+1}(A).$$

A Criterion for $k-$reducibility

- For $1 \leq k \leq q+1$ define

$$\theta_k(\lambda) = x^{-r_k(A)} \det\left(x^{k-q}A(x) - \lambda I_n\right)_{|x=\infty}$$

where

$$r_k(A) = kn_0 + (k-1)n_1 + \cdots + n_{k-1}.$$

- One can check that $\theta_k(\lambda)$ belongs to $\mathbf{C}[\lambda]$.

Theorem[Bark 1989]
The matrix A is k–irreducible, if and only if the polynomials

$$\theta_1(\lambda), \theta_2(\lambda), \ldots, \theta_k(\lambda)$$

do not vanish identically in λ.

Theorem[Bark 1989]
Given a difference system

$$\Delta Y = AY$$

one can construct a nonsingular matrix T, which is **polynomial** in x, such that the change of variable

$$Z = TY$$

leads to an equivalent system

$$\Delta Z = BZ$$

which is super-irreducible.

- This algorithm is implemented in Maple.
  Given a difference system it computes an equivalent super-reduced form as well as the corresponding polynomials $\theta_j(\lambda)$.

How to compute the pairs (s, c)

We look for lhs of the form

$$Y(x) = \Gamma(x)^s c^x x^d (F_0 + F_1 x^{-1} + \mathrm{O}(x^{-1})) \quad F_0 \neq 0$$

Write our system $\tau Y = MY$ in the Δ-form :

$$\Delta Y(x) = \widehat{M}(x) Y(x),$$

with

$$\widehat{M}(x) = x^q \sum_{\nu=0}^{\infty} x^{-\nu} \widehat{M}_\nu \quad \widehat{M}_0 \neq 0$$

We can always suppose that $q \geq 0$.
The possible values of s are then such that: $0 \leq s \leq q$.

We can suppose that our system is supper-irreducible and denote by

$$\theta_1(\lambda), \theta_2(\lambda), \ldots, \theta_{q+1}(\lambda)$$

the corresponding polynomials.

Let $0 \leq s \leq q$ be fixed.

Rewrite the system as

$$\mathcal{L}(Y) := D(x)\Delta Y(x) - N(x)Y(x) = 0.$$

with

$$D(x) = \mathrm{diag}(x^{\nu_1}, \ldots, x^{\nu_n}) \quad \text{and} \quad N(x) = D(x)\widehat{M}(x).$$

where

$$\nu_i = \min\left(\mathrm{v}(\widehat{M}_{i,.}), -s\right)$$

$\widehat{M}_{i,.}$ denoting the ith row of the matrix $\widehat{M}$.

The matrices D and N can be expanded as:

$$D(x) = x^{-s}\left(D_0 + \mathrm{O}(x^{-1})\right) \quad \text{and} \quad N(x) = N_0 + \mathrm{O}(x^{-1}).$$

Compute

$$\mathcal{L}(Y) = \Gamma(x)^s c^x x^d \left(((c - \varepsilon_s)D_0 - N_0)F_0 + \mathrm{O}(x^{-1})\right)$$

where $\varepsilon_s = 1$, when $s = 0$ and $\varepsilon_s = 0$, otherwise.

If the system has a local hypergeometric solution

$$Y(x) = \Gamma(x)^s c^x x^d (F_0 + F_1 x^{-1} + \mathrm{O}(x^{-2})) \quad F_0 \neq 0$$

then one must have

$$((c - \varepsilon_s)D_0 - N_0)F_0 = 0.$$

- If a nonzero local hypergeometric solution

$$Y(x) = \Gamma(x)^s c^x x^d F(x)$$

exists then c must annihilate the polynomial

$$E_s(\lambda) := \det\left(N_0 - (\lambda - \varepsilon_s) D_0\right).$$

Proposition

$$E_s(\lambda) = \theta_{q-s}(\lambda - \varepsilon_s)$$

where $\varepsilon_s = 1$, when $s = 0$ and $\varepsilon_s = 0$, otherwise.

Algorithm Δ-HS

Input: A field $\mathbf{C}$ of characteristic 0, a variable x, and an n by n matrix M with entries in $\mathbf{C}(x)$.
Output: A basis B of the hypergeometric solutions that are defined over $\mathbf{C}$.

- Let d_M resp. $d_{M^{-1}}$ be the denominator of M resp. M^{-1}.
- Compute the local data of $\tau(Y) = MY$.
  This produces a list L consisting of the triplets $(s, c, d) \in \mathbb{Z} \times \mathbf{C}^* \times \mathbf{C}$ that occur in the local hypergeometric solutions of $\tau(Y) = MY$.
- Form the *AB-list*.
  This produces a set $S = \{[A_1, B_1, s_1, c_1, d_1], \ldots, [A_l, B_l, s_l, c_l, d_l]\}$ of pairs (A, B) of monic factors $A | d_{M^{-1}}$ and $B | d_M$ with $A, B \in K[x]$ that match the local data in L.

Algorithm Δ-HS

- Let $B = \emptyset$.
- For all i from 1 to l do
  - $N_i := B_i/(c_i A_i) \cdot M$.
  - Compute a basis $P_1, \ldots, P_{m_i} \in K[x]^n$ of polynomial solutions for $\tau(Y) = N_i Y$.

    If $m_i > 0$ then for $j = 1, \ldots, m_i$ do:
    - Let $u = \mathrm{Sol}(\tau - c_i \frac{A_i}{B_i}) P_j$.
    - $B := B \bigcup \{u\}$.
- Return B and stop.

- The algorithm we implemented in Maple includes efficiency improvements obtained by adapting ideas from [Cluzeau& Hoeij 2006] to the case of systems.
- We use modular computations (namely, $p-$curvature test) to reduce the number of pairs (A, B) to check.
- We also use a method to partially desingularize systems.
- Examples show that this provides significant efficiency benefits.


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