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Solutions of some nonlinear parabolic equations with initial blow-up

Waad Al Sayed Laurent Véron

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract We study the existence and uniqueness of solutions of
\( \partial_t u - \Delta u + u^q = 0 \) in \( \Omega \times (0, \infty) \) where \( \Omega \subset \mathbb{R}^N \) is a domain with a compact boundary, subject to the conditions \( u = f \geq 0 \) on \( \partial \Omega \times (0, \infty) \) and the initial condition \( \lim_{t \to 0} u(x, t) = \infty \). By means of Brezis' theory of maximal monotone operators in Hilbert spaces, we construct a minimal solution when \( f = 0 \), whatever is the regularity of the boundary of the domain. When \( \partial \Omega \) satisfies the parabolic Wiener criterion and \( f \) is continuous, we construct a maximal solution and prove that it is the unique solution which blows-up at \( t = 0 \).

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Key words. Parabolic equations, singular solutions, semi-groups of contractions, maximal monotone operators, Wiener criterion.

1 Introduction

Let \( \Omega \) be a domain of \( \mathbb{R}^N (N \geq 1) \) with a compact boundary, \( Q^\Omega_\infty = \Omega \times (0, \infty) \) and \( q > 1 \). This article deals with the question of the solvability of the following Cauchy-Dirichlet problem \( P^\Omega,f \)

\[
\begin{align*}
\partial_t u - \Delta u + |u|^{q-1}u &= 0 \quad \text{in} \quad Q^\Omega_\infty \\
u &= f \quad \text{on} \quad \partial \Omega \times (0, \infty) \\
\lim_{t \to 0} u(x, t) &= \infty \quad \forall x \in \Omega.
\end{align*}
\]

If no assumption of regularity is made on \( \partial \Omega \), the boundary data \( u = f \) cannot be prescribed in sense of continuous functions. However, the case \( f = 0 \) can be treated if the vanishing condition on \( \partial \Omega \times (0, \infty) \) is understood in the \( H^1_0 \) local sense. We construct a positive solution \( u_{\Omega} \) of (1.1) with \( f = 0 \) belonging to \( C(0, \infty; H^1_0(\Omega) \cap L^{q+1}(\Omega)) \) thanks to Brezis results of contractions semigroups generated by subdifferential of proper convex functions in Hilbert spaces. We can also consider an internal increasing approximation of \( \Omega \) by smooth bounded domains \( \Omega^n \) such that \( \Omega = \bigcup \Omega^n \). For each of these domains, there exists a maximal solution \( \overline{\Omega}_n \) of \( \int_0^t \) with \( f = 0 \) belonging to \( C(0, \infty; H^1_0(\Omega) \cap L^{q+1}(\Omega)) \) thanks to Brezis results of contractions semigroups generated by subdifferential of proper convex functions in Hilbert spaces. We can also consider an internal increasing approximation of \( \Omega \) by smooth bounded domains \( \Omega^n \) such that \( \Omega = \bigcup \Omega^n \). For each of these domains, there exists a maximal solution \( \overline{\Omega}_n \) of problem \( P^{\Omega,n,0} \). Furthermore the sequence \( \{ \overline{\Omega}_n \} \) is increasing. The limit function \( u_\Omega := \lim_{n \to \infty} u_{\Omega,n} \) is the natural candidate to be the minimal positive solution of a solution of \( P^{\Omega,0} \). We prove that \( u_{\Omega} \) is the unique positive solution which blows-up at \( t = 0 \).
criterion \[\text{[8]}\], there truly exist solutions of \(P^{\Omega,0}\). We construct a maximal solution \(\bar{u}_\Omega\) of this problem. Our main result is the following:

**Theorem 1.** If \(\partial \Omega\) is compact and satisfies the parabolic Wiener criterion, there holds

\[
\bar{u}_\Omega = u_\Omega.
\]

In the last section, we consider the full problem \(P^{\Omega,f}\). Under the same regularity and boundedness assumption on \(\partial \Omega\) we construct a maximal solution \(\bar{u}_{\Omega,f}\) and we prove

**Theorem 2.** If \(\partial \Omega\) is compact and satisfies the parabolic Wiener criterion, and if \(f \in C((0,\infty;\partial \Omega))\) is nonnegative, \(\bar{u}_{\Omega,f}\) is the only positive solution to problem \(P^{\Omega,f}\).

These type of results are to be compared with the ones obtained by the same authors \([1]\) in which paper the following problem is considered

\[
\begin{align*}
\frac{\partial_t u}{\Delta} + |u|^{q-1}u & = 0 \quad \text{in } \Omega \\
\lim_{\text{dist}(x,\partial \Omega) \to 0} u(x,t) & = \infty \quad \text{locally uniformly on } (0,\infty) \\
u(x,0) & = f \quad \forall x \in \Omega.
\end{align*}
\]

In that case, the domain of \(A = \partial J_\Omega\) is \(D(A) := \{u \in H^1_0(\Omega) \cap L^{q+1}(\Omega) : \Delta u \in L^2(\Omega)\}\), and we endow \(D_\Omega(\Delta)\) with the graph norm of the Laplacian in \(H^1_0(\Omega)\)

\[
\|v\|_{D_\Omega(\Delta)} = \left(\int_\Omega ((\Delta v)^2 + |\nabla v|^2 + v^2) \, dx \right)^{1/2}.
\]

2 Minimal and maximal solutions

Let \(q > 1\) and \(\Omega\) be a proper domain of \(\mathbb{R}^N\), \(N > 1\) with a non-empty compact boundary. We set \(Q^\Omega_\infty = \Omega \times (0,\infty)\) and consider the following problem

\[
\begin{align*}
\frac{\partial_t u}{\Delta} + u^q & = 0 \quad \text{in } \Omega \times (0,\infty) \\
u(x,t) & = 0 \quad \text{on } \partial \Omega \times (0,\infty).
\end{align*}
\]

If there is no regularity assumption on \(\partial \Omega\), a natural way to consider the boundary condition is to impose \(u(.,t) \in H^1_0(\Omega)\). The Hilbertian framework for this equation has been studied by Brezis in a key article \([2]\) (see also the monography \([3]\) for a full treatment of related questions) in considering the maximal monotone operator \(v \mapsto A(v) := -\Delta v + |v|^{q-1}v\) seen as the subdifferential of the proper lower semi-continuous function

\[
J_\Omega(v) = \begin{cases} \int_\Omega \left(\frac{1}{2} |\nabla v|^2 + \frac{1}{q+1} |v|^{q+1}\right) \, dx & \text{if } v \in H^1_0(\Omega) \cap L^{q+1}(\Omega) \\
\infty & \text{if } v \notin H^1_0(\Omega) \cap L^{q+1}(\Omega).
\end{cases}
\]
Brezis’ result is the following.

**Theorem 2.1** Given $u_0 \in L^2(\Omega)$ there exists a unique function $v \in L^2_{\text{loc}}(0, \infty; D_\Omega(-\Delta)) \cap C(0, \infty; H^1_0(\Omega) \cap L^{q+1}(\Omega))$ such that $\partial_t v \in L^2_{\text{loc}}(0, \infty; L^2(\Omega))$ satisfying

$$
\begin{cases}
\partial_t v - \Delta v + |v|^{q-1}v = 0 \quad \text{a.e. in } Q^\Omega \\
v(., 0) = u_0 \quad \text{a.e. in } \Omega.
\end{cases}
$$

(2.3)

Furthermore the mapping $(t, u_0) \mapsto v(t, \cdot)$ defines an order preserving contraction semigroup in $L^2(\Omega)$, denoted by $S^{Q^\Omega}(t)[u_0]$, and the following estimate holds

$$
||\partial_t v(t, \cdot)||_{L^2(\Omega)} \leq \frac{1}{t^{\sqrt{2}}} ||u_0||_{L^2(\Omega)}. 
$$

(2.4)

From this result, we have only to consider solutions of (2.1) with the above regularity.

**Definition 2.2** We denote by $T(Q^\Omega_2)$ the set of positive functions $u \in L^2_{\text{loc}}(0, \infty; D_\Omega(-\Delta)) \cap C(0, \infty; H^1_0(\Omega) \cap L^{q+1}(\Omega))$ such that $\partial_t u \in L^2_{\text{loc}}(0, \infty; L^2(\Omega))$ satisfying

$$
\partial_t u - \Delta u + |u|^{q-1}u = 0
$$

(2.5)

in the semigroup sense, i.e.

$$
\frac{du}{dt} + \partial J_\Omega(u) = 0 \quad \text{a.e. in } (0, \infty).
$$

(2.6)

If $\Omega$ is not bounded it is useful to introduce another class which takes into account the Dirichlet condition on $\partial \Omega$: we assume that $\Omega^c \subset B_{R_0}$, denote by $\Omega_R = \Omega \cap B_R$ ($R \geq R_0$) and by $H^1_0(\Omega_R)$ the closure in $H^1_0(\Omega)$ of the restrictions to $\Omega_R$ of functions in $C^\infty_0(\Omega)$, thus we endow $D_{\Omega_R}(-\Delta)$ with the graph norm of the Laplacian in $H^1_0(\Omega_R)$.

$$
||v||_{D_{\Omega_R}(-\Delta)} = \left( \int_{\Omega_R} \left( (\Delta u)^2 + |\nabla u|^2 + u^2 \right) dx \right)^{1/2}.
$$

**Definition 2.3** If $\Omega$ is not bounded but $\Omega^c \subset B_{R_0}$, we denote by $T(Q^\Omega_{\infty, \text{loc}})$ the set of positive functions $u \in L^2_{\text{loc}}(Q^\Omega_{\infty})$ such that, for any $R > R_0$, $u \in L^2_{\text{loc}}(0, \infty; D_{\Omega_R}(-\Delta)) \cap C(0, \infty; H^1_0(\Omega_R) \cap L^{q+1}(\Omega_R))$, $\partial_t u \in L^2_{\text{loc}}(0, \infty; L^2(\Omega_R))$ and $u$ satisfies (2.5) in a.e. in $Q^\Omega_{\infty}$.

**Lemma 2.4** If $u \in T(Q^\Omega_2)$ or $T(Q^\Omega_{\infty, \text{loc}})$, its extension $\tilde{u}$ by zero outside $\Omega$ is a subsolution of (2.1) on $(0, \infty) \times \mathbb{R}^N$ such that $\tilde{u} \in C(0, \infty; H^1_0(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N))$ and $\partial_t \tilde{u} \in L^2(0, \infty; L^2(\mathbb{R}^N))$.

**Proof.** The proof being similar in the two cases, we assume $\Omega$ bounded. We first notice that $
 \tilde{u} \in C(0, \infty; H^1_0(\mathbb{R}^N))$ since $||\tilde{u}||_{H^1_0(\mathbb{R}^N)} = ||u||_{H^1_0(\Omega)}$. For $\delta > 0$ we set

$$
P_{\delta}(r) = \begin{cases}
r - 3\delta/2 & \text{if } r \geq 2\delta \\
 r^2/2\delta - r + \delta/2 & \text{if } \delta < r < 2\delta \\
0 & \text{if } r \leq \delta
\end{cases}
$$

3
When \( k \) for all in the sense that \( \int \int P u \) that \((2.7)\) holds with Lemma 2.5 For any \( \Omega \) is bounded, \( u \) thus \( s \) \( \lim \) holds that \( (2.7) \) holds with \( u \) replaced by \( \tilde{u} \). □

**Lemma 2.5** For any \( u \in \mathcal{I}(Q^\Omega_{\infty}) \), there holds

\[
u(x,t) \leq \left( \frac{1}{(q-1)t} \right)^{1/(q-1)} := \phi_q(t) \quad \forall (x,t) \in Q^\Omega_{\infty}.
\]

**Proof.** Let \( \tau > 0 \). Since the function \( \phi_{q,\tau} \) defined by \( \phi_{q,\tau}(t) = \phi_q(t-\tau) \) is a solution of

\[
\phi'_{q,\tau} + \phi^q_{q,\tau} = 0
\]

and \( (u - \phi_{q,\tau})_+ \in C(0, \infty; H^1_0(\Omega)) \), there holds

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u - \phi_{q,\tau})^2 \text{d}x + \int_{Q^\Omega_{\infty}} (\nabla u \cdot \nabla (u - \phi_{q,\tau})_+ + (u^q - \phi^q_{q,\tau})(u - \phi_{q,\tau})_+) \text{d}x \text{d}t = 0.
\]

Thus \( s \mapsto \|(u - \phi_{q,\tau})_+(s)\|_{L^2} \) is nonincreasing. By Lebesgue’s theorem,

\[
\lim_{s \downarrow \tau} \|(u - \phi_{q,\tau})_+(s)\|_{L^2} = 0,
\]

thus \( u(x,t) \leq \phi_{q,\tau}(t) \) a.e. in \( \Omega \). Letting \( \tau \downarrow 0 \) and using the continuity yields to \((2.8)\). □

**Theorem 2.6** For any \( q > 1 \), the set \( \mathcal{I}(Q^\Omega_{\infty}) \) admits a least upper bound \( \underline{u}_Q \) for the order relation. If \( \Omega \) is bounded, \( \underline{u}_Q \in \mathcal{I}(Q^\Omega_{\infty}) \); if it is not the case, then \( \underline{u}_Q \in \mathcal{I}(Q^\Omega_{\infty}) \).

**Proof.** Step 1- Construction of \( u_Q \) when \( \Omega \) is bounded. For \( k \in \mathbb{N}^* \) we consider the solution \( v = v_k \) (in the sense of Theorem 2.1) with the corresponding maximal operator in \( L^2(\Omega) \) of

\[
\begin{align*}
\partial_t v - \Delta v + v^q &= 0 \quad \text{in} \ \Omega \times (0, \infty) \\
v(x,t) &= 0 \quad \text{in} \ \partial \Omega \times (0, \infty) \\
v(x,0) &= k \quad \text{in} \ \Omega.
\end{align*}
\]

When \( k \to \infty \), \( v_k \) increases and converges to some \( \underline{u}_Q \). Because of \((2.8)\) and the fact that \( \Omega \) is bounded, \( \underline{u}_Q(t,.) \in L^2(\Omega) \) for \( t > 0 \). It follows from the closedness of maximal
monotone operators that \( u_\Omega \in L^2_{loc}(0, \infty; D_\Omega(-\Delta)) \cap C(0, \infty; H^1_0(\Omega) \cap L^{7+1}(\Omega)), \) \( \partial_t \bar{u}_\Omega \in L^2_{loc}(0, \infty; L^2(\Omega)) \) and \( \frac{\partial \bar{u}_\Omega}{\partial t} + \partial J_\Omega(\bar{u}_\Omega) = 0 \) a.e. in \( (0, \infty) \).

Thus \( u_\Omega \in \mathcal{I}(Q^\gamma_\Omega) \). For \( \gamma, \epsilon > 0 \), the function \( t \mapsto u_\Omega(x, t - \tau) + \epsilon \) is a supersolution of \((2.1)\). Let \( u \in \mathcal{I}(Q^\gamma_\Omega) \); for \( k > \phi_\Omega(\tau) \), the function \( (x, t) \mapsto (u(x, t) - u_\Omega(x; t - \tau) - \epsilon)_+ \) is a subsolution of \((2.1)\) and belongs to \( C(\tau, \infty; H^1_0(\Omega)) \). Since it vanishes at \( t = \tau \), it follows from Brezis’ result that it is identically zero, thus \( u(x, t) \leq u_\Omega(x, t - \tau) + \epsilon \). Letting \( \epsilon \downarrow 0 \) implies the claim.

**Step 2. Construction of \( u_\Omega \) when \( \Omega \) is unbounded.** We assume that \( \partial \Omega \subset B_{R_0} \) and for \( n > R_0 \), we recall that \( \Omega_n = \Omega \cap \Omega_n \). For \( k > 0 \), we denote by \( 
abla \bar{u}_\Omega \) the solution obtained in Step 1. Then \( u_n = \lim_{k \to \infty} u_{n,k} \) where \( v_{n,k} \) is the solution, in the sense of maximal operators in \( \Omega_n \) of

\[
\begin{cases}
\frac{dv_{n,k}}{dt} + \partial J_{\Omega_n}(v_{n,k}) = 0 \quad \text{a.e. in } (0, \infty) \\
v_{n,k}(0) = k.
\end{cases}
\] (2.11)

It follows from Lemma 2.3 that the extension \( \tilde{v}_{n,k} \) by \( 0 \) of \( v_{n,k} \) in \( \Omega_{n+1} \) is a subsolution for the equation satisfied by \( v_{n+1,k} \), with a smaller initial data, therefore \( \tilde{v}_{n,k} \leq v_{n+1,k} \). This implies \( \bar{u}_\Omega \leq u_{n+1} \). Thus we define \( u_\Omega = \lim_{n \to \infty} u_{n,k} \). It follows from Lemma 2.3 and standard regularity results for parabolic equations that \( u = u_\Omega \) satisfies

\[
\partial_t u - \Delta u + u^q = 0
\] (2.12)

in \( Q^\Omega_\infty \). Multiplying

\[
\frac{du_{\Omega}}{dt} + \partial J_\Omega(u_{\Omega}) = 0
\] (2.13)

by \( \eta^2 \bar{u}_\Omega \) where \( \eta \in C^\infty_c(\mathbb{R}^N) \) and integrating over \( \Omega_n \), yields to

\[
2^{-1} \frac{d}{dt} \int_{\Omega_n} \eta^2 |\nabla u_{\Omega}|^2 + u_{\Omega}^q \, dx + \int_{\Omega_n} \eta^2 |\nabla u_{\Omega}|^2 + u_{\Omega}^{q+1} \, dx + 2 \int_{\Omega_n} \nabla u_{\Omega} \cdot \nabla \eta u_{\Omega} \, dx = 0.
\]

Thus, by Young’s inequality,

\[
2^{-1} \frac{d}{dt} \int_{\Omega_n} \eta^2 |\nabla u_{\Omega}|^2 + u_{\Omega}^q \, dx + \int_{\Omega_n} \left(2^{-1} |\nabla u_{\Omega}|^2 + u_{\Omega}^{q+1} \right) \eta^2 \, dx \leq 2 \int_{\Omega_n} |\nabla \eta|^2 |\nabla u_{\Omega}|^2 \, dx.
\]

If we assume that \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) on \( B_R \) \((R > R_0)\) and \( \eta = 0 \) on \( B^c_R \), we derive, for any \( 0 < \tau < t \),

\[
2^{-1} \int_{\Omega_n} \frac{d}{dt} \eta^2 |\nabla u_{\Omega}|^2 + \int_{\tau}^{t} \int_{\Omega_n} \left(2^{-1} |\nabla u_{\Omega}|^2 + u_{\Omega}^{q+1} \right) \eta^2 \, dx ds \\
\leq 2 \int_{\tau}^{t} \int_{\Omega_n} \eta^2 |\nabla \eta|^2 \, dx ds + 2^{-1} \int_{\Omega_n} \eta^2 |\nabla u_{\Omega}|^2 \, dx.
\] (2.14)

From this follows, if \( n > 2R \),

\[
2^{-1} \int_{\Omega \cap B_n} \frac{d}{dt} u_{\Omega}^2 + \int_{\tau}^{t} \int_{\Omega \cap B_n} \left(2^{-1} |\nabla u_{\Omega}|^2 + u_{\Omega}^{q+1} \right) \, dx ds \leq CR^N(t+1)^{-2/(q-1)}.
\] (2.15)
If we let \( n \to \infty \) we derive by Fatou’s lemma

\[
2^{-1} \int_{\Omega \setminus B_R} \omega_{\Omega}^2(\cdot, t) dx + \int_{\tau}^{t} \int_{\Omega \setminus B_R} \left( 2^{-1} |\nabla \omega_{\Omega}|^2 + \omega_{\Omega}^{q+1} \right) dx ds \leq C R^N (t + 1) \tau^{-2/(q-1)}.
\]

(2.16)

For \( \tau > 0 \) fixed, we multiply (2.13) by \( (t - \tau) \eta^2 \partial_t \omega_{\Omega} / \partial t \), integrate on \( (\tau, t) \times \Omega_\tau \) and get

\[
(t - \tau) \int_{\Omega_\tau} \left| \frac{d \omega_{\Omega}}{dt} \right|^2 \eta^2 dx + \frac{d}{dt} (t - \tau) \int_{\Omega_\tau} \left( \frac{|\nabla \omega_{\Omega}|^2}{2} + \frac{\omega_{\Omega}^{q+1}}{q + 1} \right) \eta^2 dx
\]

\[
= \int_{\Omega_\tau} \left( \frac{|\nabla \omega_{\Omega_\tau}|^2}{2} + \frac{\omega_{\Omega_\tau}^{q+1}}{q + 1} \right) \eta^2 dx - 2(t - \tau) \int_{\Omega_\tau} \nabla \omega_{\Omega_\tau} \cdot \nabla \frac{d \omega_{\Omega_\tau}}{dt} \eta dx.
\]

Since

\[
2(t - \tau) \int_{\Omega_\tau} \nabla \omega_{\Omega_\tau} \cdot \nabla \frac{d \omega_{\Omega_\tau}}{dt} \eta dx \leq \frac{(t - \tau)}{2} \int_{\Omega_\tau} \left| \frac{d \omega_{\Omega_\tau}}{dt} \right|^2 \eta^2 dx + 4(t - \tau) \int_{\Omega_\tau} |\nabla \omega_{\Omega_\tau}|^2 |\nabla \eta|^2 dx,
\]

we get, in assuming again \( n > 2R \),

\[
2^{-1} \int_{\tau}^{t} \int_{\Omega} (s - \tau) \left| \frac{d \omega_{\Omega}}{dt} \right|^2 \eta^2 dx ds + (t - \tau) \int_{\Omega} \left( \frac{|\nabla \omega_{\Omega_\tau}|^2}{2} + \frac{\omega_{\Omega_\tau}^{q+1}}{q + 1} \right) \eta^2 dx
\]

\[
\leq 4 \int_{\tau}^{t} (s - \tau) \int_{\Omega} |\nabla \omega_{\Omega_\tau}|^2 |\nabla \eta|^2 dx ds,
\]

(2.17)

from which follows,

\[
2^{-1} \int_{\tau}^{t} \int_{\Omega \setminus B_R} (s - \tau) \left| \frac{d \omega_{\Omega}}{dt} \right|^2 dx ds + (t - \tau) \int_{\Omega \setminus B_R} \left( \frac{|\nabla \omega_{\Omega_\tau}|^2}{2} + \frac{\omega_{\Omega_\tau}^{q+1}}{q + 1} \right) dx
\]

\[
\leq 4 \int_{\tau}^{t} (s - \tau) \int_{\Omega \setminus B_R} |\nabla \omega_{\Omega_\tau}|^2 dx ds.
\]

(2.18)

The right-hand side of (2.18) remains uniformly bounded by \( 8 C (2 R)^N (t - \tau) \tau^{-2/(q - 1)} \) from (2.15). Then

\[
2^{-1} \int_{\tau}^{t} \int_{\Omega \setminus B_R} (s - \tau) \left| \frac{d \omega_{\Omega}}{dt} \right|^2 dx ds + (t - \tau) \int_{\Omega \setminus B_R} \left( \frac{|\nabla \omega_{\Omega_\tau}|^2}{2} + \frac{\omega_{\Omega_\tau}^{q+1}}{q + 1} \right) dx
\]

\[
\leq 8 C (2 R)^N (t - \tau) \tau^{-2/(q - 1)}
\]

(2.19)

By Fatou’s lemma the same estimate holds if \( \omega_{\Omega_\tau} \) is replaced by \( \omega_{\Omega_0} \). Notice also that this estimate implies that \( \omega_{\Omega_\tau} \) vanishes in the \( H^1_0 \)-sense on \( \partial \Omega \) since \( \eta \omega_{\Omega_\tau} \in H^1_0(\Omega) \), where the function \( \eta \in C_0^\infty(\mathbb{R}^N) \) has value 1 in \( B_R \) and \( \Omega^c \subset B_R \). Moreover estimates (2.16) and (2.19) imply that \( \omega_{\Omega} \) satisfies (2.12) a.e., and thus it belongs to \( I(Q_{\infty}^0) \).

**Step 3: Comparison.** At end, assume \( u \in I(Q_{\infty}^0) \). For \( R > n_0 \) let \( W_R \) be the maximal solution of

\[
- \Delta W_R + W_R^{q+1} = 0 \quad \text{in} \quad B_R.
\]

(2.20)
Therefore Green’s formula is valid, which implies \( R \to \infty \implies v \zeta \) for any \( s \) is the extension of \( v \). Next, \( \supp v \) is the set \( \{ t \geq 0, v \geq \eta \} \) for \( \eta > 0 \). The restriction to \( \partial B \times [0, \tau] \) is integrable for the corresponding Hausdorff measure. Therefore Green’s formula is valid, which implies

\[
- \int_{t}^{t'} \int_{\Omega \cap B_{R}} \Delta v_{+} \, dx \, dt = \int_{t}^{t'} \int_{\Omega \cap B_{R}} |\nabla v_{+}|^{2} \, dx \, dt \quad \forall t > \tau.
\]

Therefore

\[
\int_{\Omega \cap B_{R}} v_{+}^{2}(x, t) \, dx + \int_{t}^{t'} \int_{\Omega \cap B_{R}} (|\nabla v_{+}|^{2} + (u - \eta u_{\Omega} \zeta(x, t - \tau) + W_{R}(x)) v_{+}) \, dx \, dt \leq \int_{\Omega \cap B_{R}} v_{+}^{2}(x, s) \, dx.
\]

We let \( s \downarrow \tau \) and get \( v_{+} = 0 \), equivalently \( u(x, t) \leq u_{\Omega} \zeta(x, t - \tau) + W_{R}(x) \). Then we let \( R \to \infty \) and \( \tau \to 0 \) and obtain \( u(x, t) \leq u_{\Omega}(x, t) \), which is the claim.

**Corollary 2.7** Assume \( \Omega^{1} \subset \Omega^{2} \subset \mathbb{R}^{N} \) are open domains, then \( u_{\Omega^{1}} \leq u_{\Omega^{2}} \). Furthermore, if \( \Omega = \cup \Omega^{n} \) where \( \Omega^{n} \subset \Omega^{n+1} \), then

\[
\lim_{n \to \infty} u_{\Omega^{n}} = u_{\Omega},
\]

locally uniformly in \( Q_{c}^{\infty} \).

**Proof.** The first assertion follows from the proof of Theorem 2.4. It implies

\[
\lim_{n \to \infty} u_{\Omega^{n}} = u_{\Omega}^{*} \leq u_{\Omega},
\]

and \( u_{\Omega}^{*} \) is a positive solution of (2.5) in \( Q_{c}^{\infty} \). There exists a sequence \( \{ u_{0,m} \} \subset L^{2}(\Omega) \) such that \( S^{\alpha, \lambda_{n}}(u_{0,m}) \uparrow u_{\Omega} \) as \( n \to \infty \), locally uniformly in \( Q_{c}^{\infty} \). Set \( u_{0,m,n} = u_{0,m} \chi_{\Omega^{n}} \), since \( u_{0,m,n} \to u_{0,m} \) in \( L^{2}(\Omega) \) then \( S^{\alpha, \lambda_{n}}(\cdot)[u_{0,m,n}] \uparrow S^{\alpha, \lambda_{n}}(\cdot)[u_{0,m}] \) in \( L^{\infty}(0, \infty; L^{2}(\Omega)) \). If \( \tilde{v}_{m,n} \) is the extension of \( v_{m,n} := S^{\alpha, \lambda_{n}}(\cdot)[u_{m,n}] \) by zero outside \( Q_{c}^{\infty} \) it is a subsolution smaller than \( S^{\alpha, \lambda_{n}}(\cdot)[u_{0,m,n}] \) and \( n \to \tilde{v}_{m,n} \) is increasing; we denote by \( \tilde{v}_{m} \) its limit as \( n \to \infty \). Since for any \( \zeta \in C_{c}^{0}(0, \infty \times \Omega) \) we have, for \( n \) large enough and \( s > 0 \),

\[
- \int_{0}^{s} \int_{\Omega} (\tilde{v}_{m,n}(\partial_{t} \zeta + \Delta \zeta)) \, dx \, dt = \int_{\Omega} u_{0,m,n} \zeta(x, 0) \, dx - \int_{\Omega} \tilde{v}_{m,n}(x, s) \zeta(x, t) \, dx,
\]

it follows

\[
- \int_{0}^{s} \int_{\Omega} (\tilde{v}_{m}(\partial_{t} \zeta + \Delta \zeta)) \, dx \, dt = \int_{\Omega} u_{0,m} \zeta(x, 0) \, dx - \int_{\Omega} \tilde{v}_{m}(x, s) \zeta(x, t) \, dx.
\]
Clearly $\tilde{v}_m$ is a solution of (2.5) in $Q^{2m}_\infty$. Furthermore

$$
\lim_{t \to 0} \tilde{v}_m(t,.) = u_{0,m} \quad \text{a.e. in } \Omega.
$$

Because

$$
\|\tilde{v}_m(t,.) - u_{0,m}\|_{L^2(\Omega)} \leq 2 \|u_0,m\|_{L^2(\Omega)},
$$

it follows from Lebesgue’s theorem that $t \mapsto \tilde{v}_m(t,.)$ is continuous in $L^2(\Omega)$ at $t = 0$. Furthermore, for any $t > 0$ and $h \in (-t, t)$, we have from 2.4,

$$
\|\tilde{v}_{m,n}(t + h, .) - \tilde{v}_{m,n}(t, .)\|_{L^2(\Omega^n)} \leq \frac{|h|}{t^{\sqrt{2}}} \|u_{0,m,n}\|_{L^2(\Omega^n)}.
$$

Thus $\tilde{v}_m \in C([0, \infty); L^2(\Omega))$. By the contraction principle, $\tilde{v}_m = S^{j_0}(t)[u_{0,m}]$ is the unique generalized solution to (2.3) such that for any $\epsilon > 0$, and $\tau > 0$,

$$
0 < u_{\Omega} - S^{j_0}(t)[u_{0,m}] \leq \epsilon/2
$$
on $[\tau, \infty) \times \Omega$. For any $m$, there exists $n_m$ such that

$$
0 < S^{j_0}(t)[u_{0,m}] - \tilde{v}_{m,n} \leq \epsilon/2
$$

Therefore

$$
0 < u_{\Omega} - u_{\Omega^n} \leq \epsilon,
$$
on $[\tau, \infty) \times \Omega$. This implies (2.23). □

We can also construct a minimal solution with conditional initial blow-up in the following way. Assuming that $\Omega = \bigcup \Omega^m$ where $\Omega^m$ are smooth bounded domains and $\Omega^m \subset \Omega^{m+1}$. We denote by $u_m$ the solution of

$$
\begin{cases}
\partial_t u_m - \Delta u_m + |u_m|^{q-1}u_m = 0 & \text{in } Q^{2m}_\infty \\
u_m = 0 & \text{in } \partial \Omega^m \times (0, \infty) \\
\lim_{t \to 0} u_m(x, t) = \infty & \text{locally uniformly on } \Omega^m.
\end{cases}
$$

Such a $u_m$ is the increasing limit as $k \to \infty$ of the solutions $u_{m,k}$ of the same equation, with same boundary data and initial value equal to $k$. Since $\Omega^m \subset \Omega^{m+1}$, $u_m < u_{m+1}$. We extend $u_m$ by zero outside $\Omega^m$ and the limit of the sequence $\{u_m\}$, when $m \to \infty$ is a positive solution of (2.5) in $Q_\infty^\Omega$. We denote it by $u_{\Omega}$. The next result is similar to Corollary 2.7, although the proof is much simpler.

**Corollary 2.8** Assume $\Omega^1 \subset \Omega^2 \subset \mathbb{R}^N$ are open domains, then $u_{\Omega^1} \leq u_{\Omega^2}$. Furthermore, if $\Omega = \bigcup \Omega^m$ where $\Omega^m \subset \Omega^{m+1}$, then

$$
\lim_{n \to \infty} u_{\Omega^n} = u_{\Omega},
$$

locally uniformly in $Q^{\Omega}_\infty$. 8
Proposition 2.9 There holds $u_\Omega = u_\Omega^\Omega$.

Proof. For any $m_k > 0$, the extension of $u_{m_k}$ by zero in $Q^\Omega_{\infty}$ is a subsolution, thus it is dominated by $u_\Omega$. Letting successively $k \to \infty$ and $m \to \infty$ implies $u_\Omega \leq u_\Omega$. In order to prove the reverse inequality, we consider an increasing sequence $\{u_\ell\} \subset \mathcal{I}(Q^\Omega_{\infty})$ converging to $u_\Omega$ locally uniformly in $Q^\Omega_{\infty}$. If $\Omega$ is bounded there exists a bounded sequence $\{u_{\ell,0,k}\}$ which converges to $u_\ell(\cdot,0) = u_{\ell,0}$ in $L^2(\Omega)$ and $S^{\partial J_\Omega}(\cdot)[u_{\ell,0,k}] \to S^{\partial J_\Omega}(\cdot)[u_{\ell,0}]$ in $L^\infty(0,\infty;L^2(\Omega))$. Therefore

$$S^{\partial J_\Omega}(\cdot)[u_{\ell,0,k}] \leq u_\Omega \implies S^{\partial J_\Omega}(\cdot)[u_{\ell,0}] \leq u_\Omega \implies u_\Omega \leq u_\Omega. \quad (2.27)$$

Next, if $\Omega$ is unbounded, $\Omega = \cup \Omega^n$, with $\Omega^n \subset \Omega^{n+1}$ are bounded, we have

$$\lim_{n \to \infty} u_{\Omega^n} = u_\Omega$$

and

$$\lim_{n \to \infty} u_{\Omega^n} = u_\Omega$$

by Corollary 2.9 and Corollary 2.8. Since $u_{\Omega^n} = u_{\Omega^n}$ from the first part of the proof, the result follows. \square

Remark. By construction $u_\Omega$ is dominated by any positive solution of (2.12) which satisfies the initial blow-up condition locally uniformly in $\Omega$. Therefore, $u_\Omega = u_\Omega$ is the minimal solution with initial blow-up.

If $\Omega$ has the minimal regularity which allows the Dirichlet problem to be solved by any continuous function $g$ given on $\partial \Omega \times [0,\infty)$, we can consider another construction of the maximal solution of (2.1) $\Omega_\infty$ in $Q^\Omega_{\infty}$. The needed assumption on $\partial \Omega$ is known as the parabolic Wiener criterion [9] (abr. PWC).

Definition 2.10 If $\partial \Omega$ is compact and satisfies PWC, we denote by $\mathcal{J}_{Q^\Omega_{\infty}}$ the set of $v \in C((0,\infty) \times \Omega) \cap C^2(\Omega_{\infty})$ satisfying (2.1).

Theorem 2.11 Assume $q > 1$ and $\Omega$ satisfies PWC. Then $\mathcal{J}_{Q^\Omega_{\infty}}$ admits a maximal element $\pi_{\Omega}$.

Proof. Step 1- Construction. We shall directly assume that $\Omega$ is unbounded, the bounded case being a simple adaptation of our construction. We suppose $\Omega^n \subset B_{R_0}$, and for $n > R_0$ set $\Omega^n = \Omega \cap B_n$. The construction of $u_n$ is standard: for $k \in \mathbb{N}$, we denote by $v^n_k = v^n_k$ the solution of (2.9). Lemma 2.3 is valid for $v^n_k$. Notice that uniqueness follows from the maximum principle. When $k \to \infty$ the sequence $\{v^n_k\}$ increases and converges to a solution $u_n$ of (2.12) in $Q^\Omega_{\infty}$. Because the exterior boundary of $\Omega_n$ is smooth, the standard equi-continuity of the sequence of solutions applies, thus $u_n(x,t) = 0$ for all $(x,t)$ s.t. $|x| = n$ and $t > 0$. In order to see that $u_n(x,t) = 0$ for all $(x,t)$ s.t. $x \in \partial \Omega$ and $t > 0$, we see that $u_n(x,t) \leq \phi_\tau(x,t)$ on $(\tau,\infty) \times \Omega_n$, where

$$\begin{cases}
\partial_t \phi_\tau - \Delta \phi_\tau + \phi_\tau \tau = 0 & \text{in } Q^\Omega_{\infty} \\
\phi_\tau(x,\tau) = \phi_\tau(\tau) & \text{in } \Omega \\
\phi_\tau(x,\tau) = 0 & \text{in } \partial \Omega \times [\tau,\infty) \quad (2.28)
\end{cases}$$
Such a solution exists because of PWC assumption. Since \( v_{n,k}^* \) is an increasing function of \( n \) (provided the solution is extended by 0 outside \( \Omega_n \)) and \( k \), there holds \( \bar{u}_n \leq u_{n+1} \) in \( \Omega^{n+1} \). If we set

\[
\overline{u}_\Omega = \lim_{n \to \infty} \bar{u}_n,
\]

then \( \overline{u}_\Omega \leq \phi_r \) for any \( r > 0 \). Clearly \( \overline{u}_\Omega \) is a solution of \((2.12)\) in \( Q^\Omega_{\infty} \). This implies that \( \overline{u}_\Omega \) is continuous up to \( \partial \Omega \times (0, \infty) \), with zero boundary value. Thus it belongs to \( J_{Q^\Omega_{\infty}} \).

**Step 2. Comparison.** In order to compare \( \overline{u}_\Omega \) to any other \( u \in J_{Q^\Omega_{\infty}} \), for \( R > R_0 \) we set

\[
v_{R,\tau}(x,t) = \overline{u}_\Omega(x,t-\tau) + W_R(x),\]

where \( W_R \) is the maximal solution of \((2.15)\) in \( B_R \). The function \( (u - v_{R,\tau}) \) is a subsolution of \((2.12)\) in \( \Omega \cap B_R \times (\tau, \infty) \). It vanishes in a neighborhood of \( \partial(\Omega \cap B_R) \times (\tau, \infty) \) and \( \Omega \cap B_R \times \{\tau\} \). Thus it is identically zero. If we let \( R \to \infty \) in the inequality \( u \leq v_{R,\tau} \) and \( \tau \to 0 \), we derive \( u \leq \overline{u}_\Omega \), which is the claim.

\[\square\]

**Proposition 2.12** Under the assumptions of Theorem 2.11, \( \overline{u}_\Omega \in I(Q^\Omega_{\infty}) \) if \( \Omega \) is bounded and \( \overline{u}_\Omega \in I(Q^\Omega_{\infty,\infty}) \) if \( \Omega \) is not bounded.

**Proof. Case 1:** \( \Omega \) bounded. Let \( \Omega^n \) be a sequence of smooth domains such that

\[
\Omega^n \subset \overline{\Omega^n} \subset \Omega^{n+1} \subset \Omega
\]

and \( \cup_n \Omega^n = \Omega \). For \( r > 0 \), let \( u_{n,\tau} \) be the solution of

\[
\begin{cases}
\partial_t u_{n,\tau} - \Delta u_{n,\tau} + u_{n,\tau}^q = 0 & \text{in } \Omega^n \times (\tau, \infty) \\
u_{n,\tau}(\cdot, \tau) = \overline{u}_\Omega(\cdot, \tau) & \text{in } \Omega^n \\
u_{n,\tau}(x,t) = 0 & \text{in } \partial \Omega^n \times [\tau, \infty)
\end{cases}
\]

(2.29)

Because \( \overline{u}_\Omega(\cdot, \tau) \in C^2(\overline{\Omega^n}) \), \( u_{n,\tau} \in C^{2,1}(\overline{\Omega^n} \times [\tau, \infty)) \). By the maximum principle,

\[
0 \leq \overline{u}_\Omega(\cdot, t) - u_{n,\tau}(\cdot, t) \leq \max\{\overline{u}_\Omega(x, s) : (x, s) \in \partial \Omega^n \times [\tau, t]\}
\]

(2.30)

for any \( t > \tau \). Because \( \overline{u}_\Omega \) vanishes on \( \partial \Omega \times [\tau, t] \), we derive

\[
\lim_{n \to \infty} \overline{u}_{n,\tau} = \overline{u}_\Omega
\]

(2.31)

uniformly on \( \overline{\Omega^n} \times [\tau, t] \) for any \( t \geq \tau \), where \( \overline{u}_{n,\tau} \) is the extension of \( u_{n,\tau} \) by zero outside \( \Omega_n \). Applying (2.15) and (2.19) with \( \eta = 1 \) to \( \overline{u}_{n,\tau} \) in \( \Omega \) yields to

\[
2^{-1} \int_{\Omega^n} \overline{u}_{n,\tau}^2(\cdot, t)dx + \int_{\tau}^t \int_{\Omega^n} \left(|\nabla \overline{u}_{n,\tau}|^2 + \tilde{\overline{u}}_{n,\tau}^{q+1} \right) dxds \leq C(t + 1)\tau^{-2/(q-1)}.
\]

(2.32)

and

\[
2^{-1} \int_{\tau}^t \int_{\Omega} (s - \tau)(\partial_s \overline{u}_{n,\tau})^2dxds + (t - \tau) \int_{\Omega} \left(\frac{|\nabla \overline{u}_{n,\tau}|^2}{2} + \frac{\tilde{\overline{u}}_{n,\tau}^{q+1}}{q+1}\right)(t, \cdot)dx \leq C(t - \tau)t\tau^{-2/(q-1)}.
\]

(2.33)
Letting $n \to \infty$ and using (2.31) yields to
\[
2^{-1} \int_{\Omega} \varpi_{\Omega}(\cdot, t)dx + \int_{\tau}^{t} \int_{\Omega} \left( |\nabla \varpi_{\Omega}|^2 + \frac{\varpi_{\Omega}^{q+1}}{q+1} \right) dx ds \leq C(t + 1)\tau^{-2/(q-1)},
\]
(2.34)
and
\[
2^{-1} \int_{\tau}^{t} \int_{\Omega} (s - \tau)(\partial_{s} \varpi_{\Omega})^2 dx ds + (t - \tau) \int_{\tau}^{t} \left( \frac{|\nabla \varpi_{\Omega}|^2}{2} + \frac{\varpi_{\Omega}^{q+1}}{q+1} \right)(t, \cdot)dx \leq C(t - \tau)t\tau^{-2/(q-1)}.
\]
(2.35)
Since $L^2(\tau, t; H^1(\Omega))$ is a closed subspace of $L^2(\tau, t; H^1(\Omega))$, for any $0 < \tau < t$, $\varpi_{\Omega} \in L^2_{loc}(0, \infty; H^1(\Omega))$. Furthermore $\partial_{t} \varpi_{\Omega} \in L^2_{loc}(0, \infty; L^2(\Omega))$. Because $\varpi_{\Omega}$ satisfies (2.12), it implies $\varpi_{\Omega} \in \mathcal{I}(Q_{\infty}^\Omega)$.

**Case 2: $\Omega$ unbounded.** We assume that $\Omega^c \subset B_{R_0}$. We consider a sequence of smooth unbounded domains $\{\Omega^n\} \subset \Omega$ ($n > 1$) such that $\sup\{\text{dist}(x, \Omega^c) : x \in \partial \Omega^n\} < 1/n$ as $n \to \infty$, thus $\cup_n \Omega^n = \Omega$. For $m > R_0$ we set $\Omega^m_n = \Omega^n \cap B_m$. Therefore $\Omega^m_n \subset \Omega^m_m \subset \Omega^m_{m+1}$ and $\cup_{m,n} \Omega^m_n = \Omega$. For $\tau > 0$, let $u = u_{m,n,\tau}$ be the solution of
\[
\begin{cases}
\partial_t u - \Delta u + u^q = 0 & \text{in } \Omega^m_n \times (\tau, \infty) \\
u(\cdot, \tau) = \varpi_{\Omega}(\cdot, \tau) & \text{in } \Omega^m_n \\
u(x, \tau) = 0 & \text{in } \partial \Omega^m_n \times [\tau, \infty) \\
u(\cdot, \tau) = \varpi_{\Omega}(\cdot, \tau) & \text{in } \partial B_m \times (\tau, \infty).
\end{cases}
\]
(2.36)
By the maximum principle,
\[
0 \leq \varpi_{\Omega}(\cdot, t) - u_{m,n,\tau}(\cdot, t) \leq \max\{\varpi_{\Omega}(x, s) : (x, s) \in \partial \Omega^n \times [\tau, t]\} \to 0,
\]
as $n \to 0$. Next we extend $u_{m,n,\tau}$ by zero in $\Omega \setminus \Omega_n$ and apply (2.15)-(2.19) with $\eta$ as in Theorem 2.4 and $m > 2R$. We get, with $\Omega_R = \Omega \cap B_R$,
\[
2^{-1} \int_{\Omega_R} u_{m,n,\tau}(\cdot, t)dx + \int_{\tau}^{t} \int_{\Omega_R} \left( 2^{-1} |\nabla u_{m,n,\tau}|^2 + u_{m,n,\tau}^{q+1} \right) dx ds \leq CR^N(t + 1)\tau^{-2/(q-1)},
\]
(2.38)
and
\[
2^{-1} \int_{\tau}^{t} \int_{\Omega_R} \left| \frac{du_{m,n,\tau}}{dt} \right|^2 dx ds + (t - \tau) \int_{\tau}^{t} \left( \frac{|\nabla u_{m,n,\tau}|^2}{2} + \frac{u_{m,n,\tau}^{q+1}}{q+1} \right) dx \leq 8C(2R)^N(t - \tau)t\tau^{-2/(q-1)}
\]
(2.39)
We let successively $m \to \infty$ and $n \to \infty$ and derive by Fatou’s lemma and (2.37) that inequalities (2.38) and (2.39) still hold with $\varpi_{\Omega}$ instead of $u_{m,n,\tau}$. If we denote by $H^1(\Omega_R)$ the closure of the space of $C^\infty(\Omega_R)$ functions which vanish in a neighborhood on $\partial \Omega$, then (2.38) is an estimate in $L^2(\tau, t; H^1(\Omega_R))$ which is a closed subspace of $L^2(\tau, t; H^1(\Omega_R))$. Therefore $\varpi_{\Omega} \in L^2_{loc}(0, \infty; H^1(\Omega_R))$. Using (2.39) and equation (2.12) we conclude that $\varpi_{\Omega} \in \mathcal{I}(Q_{\infty}^\Omega)$.

We end this section with a comparison result between $\varpi_{\Omega}$ and $\varpi_{\Omega}$. 

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Theorem 2.13 Assume $q > 1$ and $\Omega$ satisfies PWC. Then $\overline{u}_n = \overline{v}_n$.

Proof. By Proposition 2.9 and Theorem 2.11 Step 2, $\overline{u}_n \leq \overline{v}_n$. If $\Omega$ is bounded, we can compare $\overline{u}_n(\cdot, t)$ and $\overline{v}_n(\cdot, + \tau, \cdot)$ on $\Omega \times (0, \infty)$. Since $\overline{u}_n$, the least upper bound of $I(Q^\infty_n)$ belongs to $I(Q^\infty_n)$, and $\overline{v}_n(\cdot, + \tau, \cdot) \in I(Q^\infty_n)$ we derive $\overline{v}_n(\cdot, + \tau, \cdot) \leq \overline{u}_n(\cdot, \cdot)$, from which follows $\overline{v}_n \leq \overline{u}_n$. Next, if $\Omega$ is not bounded, we can proceed as in the proof of Theorem 2.6 by comparing $\overline{u}_n(\cdot, \cdot) + W_R$ and $\overline{v}_n(\cdot, \cdot)$ on $\Omega_R \times (0, \infty)$, where $W_R$ is defined in (2.20). Because $(\overline{v}_n(\cdot, + \tau, \cdot) - \overline{u}_n(\cdot, \cdot) - W_R(t))_+$ is a subsolution of (2.12) in $Q^\infty$ which vanishes at $t = 0$ and near $\partial \Omega_R \times (0, \infty)$; it follows $\overline{v}_n(\cdot, + \tau, \cdot) \leq \overline{u}_n(\cdot, \cdot) + W_R(\cdot)$. Letting $R \rightarrow \infty$ and $\tau \rightarrow 0$ completes the proof. \hfill \Box

3 Uniqueness of large solutions

Definition 3.1 Let $q > 1$ and $\Omega \subset \mathbb{R}^N$ be any domain. A positive function $u \in C^{2,1}(Q^\infty_n)$ of (2.12) is a large initial solution if it satisfies

$$\lim_{t \rightarrow 0} u(x, t) = \infty \quad \forall x \in \Omega,$$

uniformly on any compact subset of $\Omega$.

We start with the following lemma

Lemma 3.2 Assume $u \in C^{2,1}(Q^\infty_n)$ is a large solution of (2.12), then for any open subset $G$ such that $\overline{G} \subset \Omega$, there holds

$$\lim_{t \rightarrow 0} t^{1/(q-1)} u(x, t) = c_q := \left( \frac{1}{q-1} \right)^{1/(q-1)} \text{ uniformly in } G.$$ 

Proof. By compactness, it is sufficient to prove the result when $G = B_\rho$ and $\overline{B_\rho} \subset B_{\rho'} \subset \Omega$. Let $\tau > 0$; by comparison, $u(x, t) \geq u_{B_{\rho'}}(x, t + \tau)$ for any $(x, t) \in Q^\infty_n$. Letting $\tau \rightarrow 0$ yields to $u \geq u_{B_{\rho'}}$. Next for $\tau > 0$,

$$\phi_q(t + \tau) \leq u_{B_{\rho'}}(x, t) + u_{B_{\rho'}}(x, t) + W_R(x) \quad \forall (x, t) \in Q^\infty_n.$$

Similarly

$$\max\{u_{B_{\rho'}}(x, t + \tau), u_{B_{\rho'}}(x, t + \tau)\} \leq \phi_q(t) + W_R(x) \quad \forall (x, t) \in Q^\infty_n.$$

Letting $R \rightarrow \infty$ and $\tau \rightarrow 0$,

$$\max\{u_{B_{\rho'}}, u_{B_{\rho'}}\} \leq \phi_q \leq u_{B_{\rho'}} + u_{B_{\rho'}} \quad \text{in } Q^\infty_n.$$

For symmetry reasons, $x \mapsto u_{B_{\rho'}}(x, t)$ is radially increasing for any $t > 0$, thus, for any $\rho < \rho'$ and $T > 0$, there exists $C_{\rho, T} > 0$ such that

$$u_{B_{\rho'}}(x, t) \leq C_{\rho, T} \quad \forall (x, t) \in B_\rho \times [0, T].$$

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Therefore
\[ \lim_{t \to 0} t^{1/(q-1)} u_{B,\rho}(x, t) = c_\rho \quad \text{uniformly on } B_\rho. \]
Because
\[ u_{B,\rho}(x, t) \leq u(x, t) \leq \phi_\rho(t) \quad \forall (x, t) \in Q_\infty^\Omega, \]
(3.41) follows. \( \square \)

As an immediate consequence of Lemma 3.2 and (2.23), we obtain

**Proposition 3.3** Assume \( q > 1 \) and \( \partial \Omega \) is compact. Then \( u_\Omega \) is a large solution.

We start with the following uniqueness result

**Proposition 3.4** Assume \( q > 1 \), \( \Omega \) satisfies PWC, \( \partial \Omega \) is bounded, and either \( \Omega \) or \( \Omega^c \) is strictly starshaped with respect to some point. Then \( \overline{u}_\Omega \) is the unique large solution belonging to \( J(Q_\infty^\Omega) \).

**Proof.** Without loss of generality, we can suppose that either \( \Omega \) or \( \Omega^c \) is strictly starshaped with respect to 0. By Theorem 2.11, \( \overline{u}_\Omega \) exists and, by (2.23) and Lemma 3.2, it is a large solution. Let \( u \in J(Q_\infty^\Omega) \) be another large solution. Clearly \( u \leq \overline{u}_\Omega \). If \( \Omega \) is starshaped, then for \( k > 1 \), the function \( u_k(x, t) := k^{2/(q-1)} u(kx, k^2 t) \) is a solution in \( Q_{\Omega_k} \), with \( \Omega_k := k^{-1} \Omega \). Clearly it is a large solution and it belongs to \( J(Q_{\Omega_k}) \). For \( \tau \in (0, 1) \), set \( u_{k,\tau}(x, t) = u_k(x, t-\tau) \). Because \( \partial \Omega \) is compact,
\[ \lim_{k \uparrow 1} d_H(\partial \Omega, \partial \Omega_k) = 0, \]
where \( d_H \) denotes the Hausdorff distance between compact sets. By assumption \( \overline{u}_\Omega \in C([\tau, \infty) \times \overline{\Omega}) \) vanishes on \( [\tau, \infty) \times \partial \Omega \), thus, for any \( \epsilon > 0 \), there exists \( k_0 > 1 \) such that for any
\[ k \in (1, k_0] \implies \sup \{ \overline{u}_\Omega(x, t) : (x, t) \in [\tau, 1] \times \partial \Omega_k \} \leq \epsilon. \]
Since \( u_{k,\tau} + \epsilon \) is a super solution in \( Q_{\Omega_k} \) which dominates \( \overline{u}_\Omega \) on \( [\tau, 1] \times \partial \Omega_k \) and at \( t = \tau \), it follows that \( u_{k,\tau} + \epsilon \geq \overline{u}_\Omega \) in \( (\tau, 1] \times \Omega_k \). Letting successively \( k \to 1 \), \( \tau \to 0 \) and using the fact that \( \epsilon \) is arbitrary, yields to \( u \geq \overline{u}_\Omega \) in \( (0, 1] \times \Omega \) and thus in \( Q_\infty^\Omega \). If \( \Omega^c \) is starshaped, then the same construction holds provided we take \( k < 1 \) and use the fact that, for \( R > 0 \) large enough, \( u_{k,\tau} + \epsilon + W_R \) is a super solution in \( Q_{\Omega_k} \cap B_R \) which dominates \( \overline{u}_\Omega \) on \( [\tau, 1] \times \partial \Omega_k \cap B_R \) and at \( t = \tau \). Letting successively \( R \to \infty \), \( k \to 1 \), \( \tau \to 0 \) and \( \epsilon \to 0 \) yields to \( u \geq \overline{u}_\Omega \). \( \square \)

As a consequence of Section 2, we have the more complete uniqueness theorem

**Theorem 3.5** Assume \( q > 1 \), \( \Omega \subset \mathbb{R}^N \) is a domain with a bounded boundary \( \partial \Omega \) satisfying PWC. Then for any \( f \in C(\partial \Omega \times [0, \infty)) \), \( f \geq 0 \), there exists a unique positive function \( u = \overline{u}_{\Omega,f} \in C(\overline{\Omega} \times (0, \infty)) \cap C^{2,1}(Q_\infty^\Omega) \) satisfying
\[
\begin{cases}
\partial_t u - \Delta u + |u|^{q-1} u = 0 & \text{in } Q_\infty^\Omega \\
u = f & \text{in } \partial \Omega \times (0, \infty) \\
lim_{t \to 0} u(x, t) = \infty & \text{locally uniformly on } \Omega.
\end{cases}
\]
(3.42)
Proof. Step 1: Existence. It is a simple adaptation of the proof of Theorem 2.11. For \( k, \tau > 0 \), we denote by \( u = u_{k, \tau, f} \) the solution of

\[
\begin{aligned}
\partial_t u - \Delta u + |u|^{q-1}u &= 0 & \text{in } \Omega \times (\tau, \infty) \\
\partial_t u + \partial \partial \Omega \times (\tau, \infty) \\
u(x, \tau) &= k & \text{on } \Omega.
\end{aligned}
\] (3.43)

Notice that \( u_{k, \tau, f} \) is bounded from above by \( \overline{m}_\Omega(., - \tau) + v_{f, \tau} \), where \( v_{f, \tau} = v \) solves

\[
\begin{aligned}
\partial_t v - \Delta v + |v|^{q-1}v &= 0 & \text{in } \Omega \times (\tau, \infty) \\
v &= f & \text{in } \partial \partial \Omega \times (\tau, \infty) \\
v(x, \tau) &= 0 & \text{on } \Omega.
\end{aligned}
\] (3.44)

If we let \( k \to \infty \) we obtain a solution \( u_{\infty, \tau, f} \) of the same problem except that the condition at \( t = \tau \) becomes \( \lim_{t \to \tau} u(x, t) = \infty \), locally uniformly for \( x \in \Omega \). Clearly \( u_{\infty, \tau, f} \) dominates in \( \Omega \times (\tau, \infty) \) the restriction to this set of any \( u \in C(\Omega \times \infty) \cap C^{2,1}(Q_\infty^W) \) solution of (3.42), in particular \( \overline{m}_\Omega \). Therefore \( u_{\infty, \tau, f} \geq u_{\infty, \tau', f} \) in \( \Omega \times (\tau, \infty) \) for any \( 0 < \tau' < \tau \). When \( \tau \to 0 \), \( u_{\infty, \tau, f} \) converges to a function \( \overline{m}_f \) which satisfies the lateral boundary condition \( \overline{m}_{\Omega, f} = f \). Therefore \( \overline{m}_{\Omega, f} \) satisfies (3.42).

Step 2: Uniqueness. Assume that there exists another positive function \( u := u_f \in C(\Omega \times (0, \infty)) \cap C^{2,1}(Q_\infty^W) \) solution of (3.42). Then \( u_f < \overline{m}_{\Omega, f} \). For \( \tau > 0 \), consider the solution \( v := v_\tau \) of

\[
\begin{aligned}
\partial_t v - \Delta v + |v|^{q-1}v &= 0 & \text{in } \Omega \times (\tau, \infty) \\
v &= 0 & \text{in } \partial \partial \Omega \times (\tau, \infty) \\
v(x, \tau) &= u_f(x, \tau) & \text{on } \Omega.
\end{aligned}
\] (3.45)

Then \( v_\tau \leq u_f \) in \( \Omega \times (\tau, \infty) \). In the same way, we construct a solution \( v \) of the same problem (3.45) except that the condition at \( t = \tau \) is now \( v(x, \tau) = \overline{m}_{\Omega, f}(x, \tau) \) for all \( x \in \Omega \). Furthermore \( v_\tau \leq \overline{v}_\tau \leq \overline{m}_{\Omega, f} \). Next we adapt a method introduced in [6], [7] in a different context. We denote

\[
Z_f = \overline{m}_{\Omega, f} - u_f \quad \text{and} \quad Z_{0, \tau} = \overline{v}_\tau - v_\tau.
\] (3.46)

and, for \((r, s) \in \mathbb{R}_+^2\),

\[
h(r, s) = \begin{cases} 
\frac{r^q - s^q}{r - s} & \text{if } r \neq s \\
0 & \text{if } r = s.
\end{cases}
\]

Since \( r \mapsto r^q \) is convex on \( \mathbb{R}_+ \), there holds

\[
\begin{cases} 
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\begin{cases} 
r_0 \geq s_0, r_1 \geq s_1 \\
r_1 \geq r_0, s_1 \geq s_0
\end{array}
\end{array}
\end{cases}
\end{array}
\end{cases}
\end{cases}
\end{array}
\end{cases}
\Rightarrow h(r_1, s_1) \geq h(r_0, s_0).
\]

This implies

\[
h(u_{\Omega, f}, u_f) \geq h(\overline{v}_\tau, v_\tau) \quad \text{in } \Omega \times [\tau, \infty).
\] (3.47)
Next we write

\[
0 = \partial_t(Z_f - Z_{0,\tau}) - \Delta(Z_f - Z_{0,\tau}) + \overline{\tau_{f,\Omega}} - u_f^\tau - (\overline{\mathcal{h}}_f^\tau - v_f^\tau)
\]

\[
= \partial_t(Z_f - Z_{0,\tau}) - \Delta(Z_f - Z_{0,\tau}) + h(\tau_{f,\Omega} - u_f)Z_f - h(\overline{\mathcal{h}}_f - v_f)Z_{0,\tau}.
\]

Combining (3.47), (3.48) with the positivity of \(v\), we derive

\[
\partial_t(Z_f - Z_{0,\tau}) - \Delta(Z_f - Z_{0,\tau}) + h(\tau_{f,\Omega} - u_f)(Z_f - Z_{0,\tau}) \leq 0,
\]

in \(\Omega \times (\tau, \infty)\). On \(\partial \Omega \times [\tau, \infty)\) there holds \(Z_f - Z_{0,\tau} = f - f = 0\). Furthermore, at
time \(t = \tau\), \(Z_f(\cdot, \tau) - Z_{0,\tau}(\cdot, \tau) = \tau_{f,\Omega}(\cdot, t) - u_f(\cdot, \tau) - \overline{\tau_{f,\Omega}(\cdot, t)} + u_f(\cdot, \tau) = 0\). By
the maximum principle, it follows \(Z_f \leq Z_{0,\tau}\) in \(\Omega \times [\tau, \infty)\). Since \(\tau > \tau' > 0\) implies
\(v_{\tau'}(\cdot, \tau) = u_f(\cdot, \tau) \geq v_{\tau'}(\cdot, \tau)\) and \(\overline{\mathcal{h}}_{\tau'}(\cdot, \tau) = \overline{\tau_{\tau',\Omega}(\cdot, \tau)} \geq \overline{\mathcal{h}}_{\tau'}(\cdot, \tau)\), the sequences \(\{v_{\tau'}\}\) and
\(\overline{\mathcal{h}}_{\tau'}\) converge to some functions \(\{\overline{v}_0\}\) and \(\overline{\mathcal{h}}_0\) which belong to \(C(\Omega \times (0, \infty)) \cap C^{2,1}(Q'_{\infty})\)
and satisfy (3.42) with \(f = 0\) on \(\partial \Omega \times (0, \infty)\). Furthermore

\[
\overline{\tau_{f,\Omega}} - u_f \leq \overline{v}_0 - v_0,
\]

Since \(\overline{\tau_{f,\Omega}} \geq \overline{\tau}, \overline{\mathcal{h}}_0 \geq \overline{\mathcal{h}}\), which implies that \(\overline{v}_0 = \overline{\tau}\) by the maximality of \(\overline{\tau}\). If \(\Omega'\) is
any smooth bounded open subset such that \(\Omega' \subset \Omega\) there holds by an easy approximation
argument \(v_0 \geq \overline{v}_0\) in \(\Omega' \times (0, \infty)\). Therefore \(v_0 \geq \overline{v}_0 = \overline{\tau} = \overline{\tau}\), by Proposition 2.13 and
Theorem 2.13. Applying again Theorem 2.13 we derive that the right-hand side of (3.50) is zero, which yields to \(\overline{\tau_{f,\Omega}} = u_f\)

\[\square\]

References


