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Partial observability in structured bilinear systems using a graphical approach

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***Abstract*—This work is dedicated to the study of partial observability for structured bilinear systems using a graph-theoretic approach. The proposed method assumes only the knowledge of the system structure. The graphical condition for partial observability is given and if it is not satisfied then solutions of sensor placement are given in order to recover the system partial observability. This sensor placement is easy to determinate because it consists of finding paths on a digraph and computing indices.**

***Index Terms*—Structured bilinear systems, partial observability, graph theory.**

I. Introduction

Bilinear systems (BLS) are amongst the simplest nonlinear systems and therefore are particularly adapted to analysis compared to more complicated nonlinear systems. They can be used to represent a wide range of physical, chemical, biological and social systems, as well as manufacturing processes which cannot be satisfactorily controlled under the linearity assumption. The properties and behavior of bilinear systems have been investigated and a number of useful results have been derived.

The observability of bilinear systems is an essential point. In fact, for an observable system, all the state variables can be estimated, that is important for control design, supervision or Fault Detection and Isolation (FDI). The observability of BLS was tackled in many works among which, can be cited [1], [2]. These results are essentially based on geometric or algebraic tools which need the exact knowledge of the state space matrices. To study the properties of uncertain systems, one idea consists to consider models where the fixed zeros are conserved while the non-zero entries are replaced by free parameters. Such considered systems are called structured systems.

Many results on structured systems are related to the graph-theoretic approach. This approach is mainly dedicated to linear systems for which many structural properties can be studied. Survey paper [3] reviews the most significant results in this area. From these studies, it results that the graph-theoretic approach provides simple and elegant solutions and so is very well-suited to analyse large scale or/and uncertain systems. Unfortunately, not so many works based on graph-theoretic methods deal with nonlinear systems. Among them, [4], [5] provide sufficient conditions for the uniform observability and [6], [7] deal with the observability of bilinear systems.

In this context, this paper is dedicated to the study of partial observability and to sensor placement for structured bilinear systems (SBLS). More precisely, we first recall the partial observability geometric condition and our aim is to obtain the equivalent graphical condition. Then, our purpose is to provide a sensor placement method enabling to recover the observability of a state variable set when its observability condition is not satisfied. This is very interesting in particular during the conception phase of a system because it enables to know where the sensors have to be located in order to ensure the observability of a state variable system.

The paper is organized as follows : after section 2, which is devoted to the problem formulation, graphical representation of SBLS and definitions are given in section 3. The main results are given in section 4 and are illustrated with an example in section 5. Finally, some concluding remarks are made.

II. Problem statement

In this paper, SBLS are considered in the form:

$$\Sigma_\Lambda : \left\{ \begin{array}{lcl} \dot{x}(t) & = & A_0 x(t) + \displaystyle\sum_{\ell=1}^{m} u_\ell(t) A_\ell x(t) \\ y(t) & = & C x(t) \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state vector, the input vector and the output vector. For $\ell = 0, \ldots, m$, $A_\ell \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{p\times n}$ are matrices which elements are either fixed to zero or assumed free non-zero parameters. These nonzero entries can be parameterized by scalar real (nonzero) parameters λ_i $(i = 1, \ldots, h)$ forming a parameter vector $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)^T \in \mathbb{R}^h$. If all the parameters λ_i are fixed, an admissible realisation of Σ_Λ is obtained. Theoretic properties of each realisation can be studied according to the values of λ_i. A property is true generically [8] if it is true for almost all the realizations of Σ_Λ. Here, "for almost all the realizations" is to be understood [3] as "for all parameter values ($\Lambda \in \mathbb{R}^h$) except for those in some proper algebraic variety in the parameter space". The proper algebraic variety for which the property is not true is the zero set of some nontrivial polynomial with real coefficients in the h system parameters, which can be written down explicitly.

To system (1) is associated X_Ω which denotes the state components having to be observed. The observability property is usually studied using the observability matrix defined by $\mathcal{O} = col\Big(C, CA_0, \ldots, CA_m, CA_0^2, \ldots, CA_0A_m, CA_1A_0, \ldots\Big)$ but it is not appropriate to study structured bilinear systems and more particularly large scale structured bilinear systems because of the complex computation of this observability matrix and of its rank. In the sequel, if e_{V_x} denotes the diagonal matrix of dimension $n \times n$ such that $e_{V_x}(i,i) = 1$ if $x_i \in V_x$ else $e_{V_x}(i,i) = 0$ and g_rank(M) denotes the generic rank of the matrix M. According to the fact that if the j-th column of the observability matrix is linearly independent from all the other columns then the j-th component x_j of X is observable iff :

$$\text{g_rank}(\mathcal{O}.e_X) > \text{g_rank}(\mathcal{O}.e_{X\setminus\{x_i\}}) \tag{2}$$

From the previous condition, the condition for X_Ω to be observable is simply obtained :

Corollary 1: The set X_Ω is observable iff

$$\forall\ x_i \in X_\Omega, \text{g_rank}(\mathcal{O}.e_X) > \text{g_rank}(\mathcal{O}.e_{X\setminus\{x_i\}}) \tag{3}$$

In this paper, considering structured bilinear system Σ_Λ and components set X_Ω of Σ_Λ, a necessary and sufficient condition of partial observability of X_Ω is given at first. In a second time, when this condition is not satisfied a sensor location method is given in order to recover it. More precisely, the first part is dedicated to the determination of the graphical conditions equivalent to those of inequality (2) and corollary 1. The second part provides graphic tools enabling the localization of additional sensors.

III. Graphical representation of structured bilinear systems

This section is devoted, in a first time, to the definition of a digraph which represents SBLS Σ_Λ. In a second time, some definitions associated to this graphical representation are given.

A. Digraph representation of structured bilinear systems

The digraph associated to Σ_Λ is noted $\mathcal{G}(\Sigma_\Lambda)$ and is constituted by a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$ *i.e.* $\mathcal{G}(\Sigma_\Lambda) = (\mathcal{V}, \mathcal{E})$. The vertices are associated to the state and the output components of Σ_Λ and the directed edges represent links between these variables.
More precisely, $\mathcal{V} = \mathbf{X} \cup \mathbf{Y}$, where $\mathbf{X} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ is the set of state vertices, $\mathbf{Y} = \Big\{\mathbf{y_1}, \ldots, \mathbf{y_p}\Big\}$ is the set of output vertices. For sake of clarity, vertices are written in bold fonts. The edge set is $\mathcal{E} = \bigcup_{\ell=0}^{m} A_\ell\text{-edges} \cup C\text{-edges}$, where for $\ell = 0, \ldots, m$, A_ℓ-edges $= \Big\{(\mathbf{x_j}, \mathbf{x_i}) | A_\ell(i,j) \neq 0\Big\}$ and C-edges $= \Big\{(\mathbf{x_j}, \mathbf{y_i}) | C(i,j) \neq 0\Big\}$. Here $M(i,j)$ is the $(i,j)th$ element of matrix M and $(\mathbf{v_1}, \mathbf{v_2})$ denotes a directed edge from vertex $\mathbf{v_1} \in \mathcal{V}$ to vertex $\mathbf{v_2} \in \mathcal{V}$.

The following notation is defined : $\bar{A}_0$-edges $=$ A_0-edges$\cup C$-edges and for $\ell = 1, \ldots, m$, $\bar{A}_\ell$-edges $= A_\ell$-edges. To each edge $e \in \bar{A}_\ell$-edges is associated an unique indice u_ℓ and u_ℓ is indicated over each edge e in the digraph representation. If an edge $e_1 \in \bar{A}_{\ell_1}$-edges and an edge $e_2 \in \bar{A}_{\ell_2}$-edges have the same begin and end vertices, only one edge is represented and the information over the edge will be u_{ℓ_1}, u_{ℓ_2} with $\ell_1 < \ell_2$.

Let us give an example of this representation.

Example 1. Consider SBLS defined by:

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_4 & 0 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_6 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \lambda_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & \lambda_8 \end{pmatrix}.$$

This model is associated to digraph 1 (fig. 1).

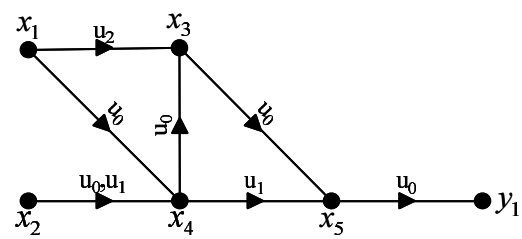

Figure 1: *Digraph 1 associated to Example 1*

Let us give some useful definitions associated to this graphical representation.

B. Definitions

- $P = \mathbf{v_0} \xrightarrow{u_{i_1}} \mathbf{v_1} \xrightarrow{u_{i_2}} \ldots \xrightarrow{u_{i_s}} \mathbf{v_s}$ denotes a path P which contains vertices $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_s}$ where $(\mathbf{v_{j-1}}, \mathbf{v_j}) \in \bar{A}_{i_j}$-edges for $j = 1, \ldots, s$. $\mathbf{v_0}$ (respectively $\mathbf{v_s}$) is the begin (respectively the end) vertex of P. P is an $\mathbf{Y}$-*topped* path if its end vertex is an element of $\mathbf{Y}$. To P is associated an unique indice noted $\sigma(P)$ and defined by the ordered monomial $\mathsf{u}_{i_1} \cdot \mathsf{u}_{i_2} \cdot \ldots \cdot \mathsf{u}_{i_s}$.

- Two paths $P_1 = \mathbf{v_0^1} \xrightarrow{u_{i_1}} \mathbf{v_1^1} \xrightarrow{u_{i_2}} \ldots \xrightarrow{u_{i_{k_1}}} \mathbf{v_{k_1}^1}$ and $P_2 = \mathbf{v_0^2} \xrightarrow{u_{j_1}} \mathbf{v_1^2} \xrightarrow{u_{j_2}} \ldots \xrightarrow{u_{j_{k_2}}} \mathbf{v_{k_2}^2}$ are A-disjoint iff :
  C1: $\mathbf{v_0^1} \neq \mathbf{v_0^2}$ and
  C2: $\forall\ k = 1, 2, \ldots, \min(k_1, k_2)$, if $\mathbf{v_k^1} = \mathbf{v_k^2}$ then $\mathbf{v_0^1} \xrightarrow{u_{i_1}} \ldots \xrightarrow{u_{i_k}} \mathbf{v_k^1}$ and $\mathbf{v_0^2} \xrightarrow{u_{j_1}} \ldots \xrightarrow{u_{j_k}} \mathbf{v_k^2}$ must have distinct indices (There exist ℓ such that $u_{i_\ell} \neq u_{j_\ell}$).

From example 1, $P_1 = \mathbf{x_1} \xrightarrow{u_2} \mathbf{x_3} \xrightarrow{u_0} \mathbf{x_5} \xrightarrow{u_0} \mathbf{y_1}$ and $P_2 = \mathbf{x_2} \xrightarrow{u_0} \mathbf{x_4} \xrightarrow{u_1} \mathbf{x_5} \xrightarrow{u_0} \mathbf{y_1}$ are A-disjoint. In fact, $\mathbf{x_1} \neq \mathbf{x_2}$ and considering the common vertex

$\mathbf{x_5}$, indices are different ($\mathsf{u}_2 \cdot \mathsf{u}_0 \neq \mathsf{u}_0 \cdot \mathsf{u}_1$). P_2 and $P_3 = \mathbf{x_1} \xrightarrow{u_0} \mathbf{x_4} \xrightarrow{u_1} \mathbf{x_5} \xrightarrow{u_0} \mathbf{y_1}$ are not A-disjoint because considering any common vertex, indices are the same.

• Some paths are A-disjoint if they are mutually A-disjoint.

• A set of k_l A-disjoint paths from $\mathbf{V_0}$ to $\mathbf{V_1}$ is called a A-linking of dimension k_l from $\mathbf{V_0}$ to $\mathbf{V_1}$. A A-linking composed of paths $P_{i_1}, P_{i_2}, \ldots, P_{i_{k_l}}$ is denoted $\mathfrak{m} = \bigcup_{j=1}^{k_l} P_{i_j}$. A A-linking from $\mathbf{V_0}$ to $\mathbf{V_1}$ is maximum when k_l is maximum.

• $\rho_A(\mathbf{V_0}, \mathbf{V_1})$ denotes the maximum number of A-disjoint $\mathbf{V_1}$-*topped* paths from $\mathbf{V_0}$. It gives the size of a maximum A-linking from $\mathbf{V_0}$ to $\mathbf{V_1}$.
From example 1, the A-linking $\mathfrak{m} = \{P_1, P_2, P_3, P_4, P_5\}$ is a maximum A-linking with $P_1 = \mathbf{x_1} \xrightarrow{u_2} \mathbf{x_3} \xrightarrow{u_0} \mathbf{x_5} \xrightarrow{u_0} \mathbf{y_1}$, $P_2 = \mathbf{x_2} \xrightarrow{u_0} \mathbf{x_4} \xrightarrow{u_1} \mathbf{x_5} \xrightarrow{u_0} \mathbf{y_1}$, $P_3 = \mathbf{x_3} \xrightarrow{u_0} \mathbf{x_5} \xrightarrow{u_0} \mathbf{y_1}$, $P_4 = \mathbf{x_4} \xrightarrow{u_1} \mathbf{x_5} \xrightarrow{u_0} \mathbf{y_1}$ and $P_5 = \mathbf{x_5} \xrightarrow{u_0} \mathbf{y_1}$. The maximum number of A-disjoint paths from $\mathbf{X}$ to $\mathbf{Y}$ is $\rho_A(\mathbf{X}, \mathbf{Y}) = 5$.

IV. Main results

The aim of this section is to give at first a graphical condition for the partial observability. In a second part, the purpose is to ensure the partial observability of X_Ω by additional sensor placement when the previous condition is not satisfied for X_Ω.

A. Partial observability condition

The first part consists in finding graphical condition equivalent to the observability matrix rank condition (inequality (2)). The notion of paths is clearly related to the elements of the observability matrix. In fact, if there exists a path P from x_i to y_j with $\sigma(P) = u_{i_1} \cdot u_{i_2} \cdot \ldots \cdot u_{i_k} \cdot u_0$, it means that $M(j,i) \neq 0$ with $M = CA_{i_k} \ldots A_{i_2} A_{i_1}$. In order to find this graphical condition, the two next lemma and the proposition 1 establish a relation between $\rho_A(\mathbf{V_X}, \mathbf{Y})$ and g_rank($O.e_{V_X}$) where V_X is a subset of X.

Lemma 1: Let Σ_Λ be the structured bilinear system defined by (1) with its associated digraph $\mathcal{G}(\Sigma_\Lambda)$, we have the following condition :

$$\text{if } \rho_A(\mathbf{V_x}, \mathbf{Y}) = \text{card}(\mathbf{V_x}) \text{ then g_rank}(O.e_{\mathbf{V_x}}) = \text{card}(\mathbf{V_x}) \quad (4)$$

Proof 1:
1. In a first time, the case of two A-disjoint paths from $\{\mathbf{v_0^1}, \mathbf{v_0^2}\}$ to $\mathbf{Y}$ is studied. $P_1 = \mathbf{v_0^1} \xrightarrow{u_{i_1}} \mathbf{v_1^1} \xrightarrow{u_{i_2}} \ldots \xrightarrow{u_{i_{k_1}}} \mathbf{v_{k_1}^1} \xrightarrow{u_0} \mathbf{y^1}$ and $P_2 = \mathbf{v_0^2} \xrightarrow{u_{j_1}} \mathbf{v_1^2} \xrightarrow{u_{j_2}} \ldots \xrightarrow{u_{j_{k_2}}} \mathbf{v_{k_2}^2} \xrightarrow{u_0} \mathbf{y^2}$.

Two different cases have to be distinguished :

- If the two $\mathbf{Y}$-*topped* paths are disjoint (they have no common vertices) then $M_1 = CA_{i_{k_1}} \ldots A_{i_2} A_{i_1}$ and $M_2 = CA_{j_{k_2}} \ldots A_{j_2} A_{j_1}$ are defined. If $\mathbf{v_0^1} = \mathbf{x_{c_1}}$, $\mathbf{v_0^2} = \mathbf{x_{c_2}}$, $\mathbf{y^1} = \mathbf{y}_{\ell_1}$ and $\mathbf{y^2} = \mathbf{y}_{\ell_2}$ then $M_1(\ell_1, c_1) \neq 0$ and $M_2(\ell_2, c_2) \neq 0$. According to the fact that $\mathbf{v_0^2} \neq \mathbf{v_0^1}$, $\mathbf{y^1} \neq \mathbf{y^2}$ and P_1, P_2 have no common vertices, from works done for linear systems can be deduced that g_rank(col(M_1, M_2).$e_{\{v_0^1, v_0^2\}}$) = 2.
- If the two paths are not disjoint then following cases can be distinguished :
  - If $\mathbf{v_0^1}$ is a vertex of P_2 ($\mathbf{v_0^1} = \mathbf{v_{j_{k_2'}}}$) then $M_1 = CA_{i_{k_1}} \ldots A_{i_1}$, $P_2' = \mathbf{v_0^2} \xrightarrow{u_{j_1}} \ldots \xrightarrow{u_{j_{k_2'}}} \mathbf{v_0^1}$, $M_2^1 = A_{j_{k_2'}} \ldots A_{j_1}$ and $M_2^2 = CA_{j_{k_2}} \ldots A_{j_{k_2'+1}}$ are defined. g_rank(col($M_2^2 M_2^1, M_1$).$e_{\{v_0^1, v_0^2\}}$) = 2 can be deduced.
  - If $\mathbf{v_0^2}$ is a vertex of P_1 ($\mathbf{v_0^2} = \mathbf{v_{i_{k_1'}}}$) then $M_2 = CA_{j_{k_2}} \ldots A_{j_1}$, $P_1' = \mathbf{v_0^1} \xrightarrow{u_{i_1}} \ldots \xrightarrow{u_{i_{k_1'}}} \mathbf{v_0^2}$, $M_1^1 = A_{i_{k_1'}} \ldots A_{i_1}$ and $M_1^2 = CA_{i_{k_1}} \ldots A_{i_{k_1'+1}}$ are defined. g_rank(col($M_1^2 M_1^1, M_2$).$e_{\{v_0^1, v_0^2\}}$) = 2 can be deduced.
  - Else, from the definition of A-disjoint paths, there exists a vertex $\mathbf{v_a} = \mathbf{v_{k_1'}^1} = \mathbf{v_{k_2'}^2}$ such that $P_1^1 = \mathbf{v_0^1} \xrightarrow{u_{i_1}} \ldots \xrightarrow{u_{i_{k_1'}}} \mathbf{v_a}$ and $P_2^1 = \mathbf{v_0^2} \xrightarrow{u_{j_1}} \ldots \xrightarrow{u_{j_{k_2'}}} \mathbf{v_a}$ have only the vertex $\mathbf{v_a}$ in common and $\sigma(P_1^1) \neq \sigma(P_2^1)$. $M_1^1 = A_{i_{k_1'}} \ldots A_{i_2} A_{i_1}$ and $M_2^1 = A_{j_{k_2'}} \ldots A_{j_2} A_{j_1}$ are defined. According to the fact that $\sigma(P_1^1) \neq \sigma(P_2^1)$ and $\mathbf{v_0^1} \neq \mathbf{v_0^2}$ then g_rank(col(M_1^1, M_2^1).$e_{\{v_0^1, v_0^2\}}$) = 2 is concluded. Now, considering the second part of each path P_1 and P_2, $M_1^2 = CA_{i_{k_1}} \ldots A_{i_{k_1'+1}}$ and $M_2^2 = CA_{j_{k_2}} \ldots A_{j_{k_2'+1}}$ are defined. It is known that g_rank(col(M_1^2, M_2^2).$e_{\{v_a\}}$) = 1 and with the same reasoning as before g_rank(col($M_1^2 M_1^1, M_2^2 M_2^1$).$e_{\{v_0^1, v_0^2\}}$) = 2.

So, in the case of two A-disjoint paths from $\mathbf{V_X}$, it is proved that g_rank($O.e_{V_X}$) = 2 = card($\mathbf{V_X}$).

2. In a second time, we consider a set S_P of q $\mathbf{Y}$-*topped* paths from $\mathbf{V_X}$ which are A-disjoint and a $\mathbf{Y}$-*topped* path P' from $\mathbf{v}'$ which is A-disjoint to each path included in S_P. Then, the problem considering two subsets of S_P is solved.

- Subset S_P^1 is composed of q_1 paths of S_P which are disjoint with P'. The begin vertices of S_P^1 are included in the set V_{X_1}. Step 1 that have been seen just before can be studied considering a path P_i of S_P^1 (with $\mathbf{v_i}$ as begin vertex) and P', then it can be concluded that g_rank($O.e_{\{v_i, v'\}}$) = 2. Then, g_rank($O.e_{V_{X_1} \cup \{v'\}}$) = $q_1 + 1$ is obtained.
- Subset S_P^2 is composed of q_2 paths of S_P which are not disjoint with P'. The begin vertices of S_P^2 are included in the set V_{X_2}. With the same reasoning as just before, it can be concluded that g_rank($O.e_{V_{X_2} \cup \{v_i\}}$) = $q_2 + 1$.

Then the final result is that g_rank($O.e_{V_X}$) = $q_1 + q_2 + 1 = q + 1$. △

The first lemma gives a relation between $\rho_A(\mathbf{V_X}, \mathbf{Y})$ and g_rank($O.e_{V_X}$), but it is not sufficient to establish the final result because this relation is only satisfied if $\rho_A(\mathbf{V_X}, \mathbf{Y})$ = card($\mathbf{V_X}$). This is the reason why the following lemma is used.

Lemma 2: Let Σ_Λ be the structured bilinear system defined by (1) with its associated digraph $\mathcal{G}(\Sigma_\Lambda)$, the following relation is obtained :

$$\begin{aligned}&\text{if there exist a state vertex } \mathbf{x_k} \text{ and a subset } \mathbf{V_X}\\ &\text{such that } \rho_A(\mathbf{V_X} \cup \{\mathbf{x_k}\}, \mathbf{Y}) = \rho_A(\mathbf{V_X}, \mathbf{Y}) = \text{card}(\mathbf{V_X})\\ &\text{then g_rank}(\mathcal{O}.e_{\mathbf{V_X}\cup\{\mathbf{x_k}\}}) = \text{g_rank}(\mathcal{O}.e_{\mathbf{V_X}}) = \text{card}(\mathbf{V_X})\end{aligned} \tag{5}$$

Proof 2:

1. In a first time, the case of two $\mathbf{Y}$-*topped* paths from $\{\mathbf{v_0^1}, \mathbf{v_0^2}\}$ is studied. $P_1 = \mathbf{v_0^1} \xrightarrow{u_{i_1}} \mathbf{v_1^1} \xrightarrow{u_{i_2}} \dots \xrightarrow{u_{i_{k_1}}} \mathbf{v_{k_1}^1} \xrightarrow{u_0} \mathbf{y^1}$ and $P_2 = \mathbf{v_0^2} \xrightarrow{u_{j_1}} \mathbf{v_1^2} \xrightarrow{u_{j_2}} \dots \xrightarrow{u_{j_{k_2}}} \mathbf{v_{k_2}^2} \xrightarrow{u_0} \mathbf{y^2}$.
From the definition of two A-disjoint paths, it can be deduces that two paths are not A-disjoint iff

C1': $\mathbf{v_0^1} = \mathbf{v_0^2}$ or

C2': $\exists\ k = 1, 2, \dots, \min(k_1, k_2)$ such that if $\mathbf{v_k^1} = \mathbf{v_k^2}$ then $\mathbf{v_0^1} \xrightarrow{u_{i_1}} \dots \xrightarrow{u_{i_k}} \mathbf{v_k^1}$ and $\mathbf{v_0^2} \xrightarrow{u_{j_1}} \dots \xrightarrow{u_{j_k}} \mathbf{v_k^2}$ have same indices.

It is clear that if $\mathbf{v_0^1} = \mathbf{v_0^2}$ then $\text{g_rank}(\mathcal{O}.e_{\{v_0^1, v_0^2\}}) = \text{g_rank}(\mathcal{O}.e_{\{v_0^1\}}) = 1$.
Now, considering the second condition, there exists k such that $\mathbf{v_0^1} \xrightarrow{u_{i_1}} \dots \xrightarrow{u_{i_k}} \mathbf{v_k^1}$ and $\mathbf{v_0^2} \xrightarrow{u_{j_1}} \dots \xrightarrow{u_{j_k}} \mathbf{v_k^2}$ have same indices. According to the fact that $M_1 = A_{i_k} \dots A_{i_2} A_{i_1}$ and $M_2 = A_{j_k} \dots A_{j_2} A_{j_1}$, it is known that $M_1 = M_2$ and it is deduced that $\text{g_rank}(e_{v_k^1}.\text{col}(M_1, M_2).e_{\{v_0^1, v_0^2\}}) = 1$. Then, it is clear that $\text{g_rank}(\mathcal{O}.e_{\{v_0^1, v_0^2\}}) = 1$.

2. Now, this study is extended to several paths. Considering a set S_P of q $\mathbf{Y}$-*topped* A-disjoint paths from $\mathbf{V_X}$ with $q = \text{card}(V_X)$. Now, considering P' a $\mathbf{Y}$-*topped* path from $\mathbf{x_j} \notin \mathbf{V_X}$ which is not A-disjoint to each path included in S_P. The previous study can be made considering the path P' and each path $P_i \in \mathbf{V_X}$ which has $\mathbf{x_i}$ as begin vertex. It can be concluded that for each consideration, $\text{g_rank}(\mathcal{O}.e_{\{x_i, x_j\}}) = 1$. And the final result is $\text{g_rank}(\mathcal{O}.e_{V_X \cup \{x_j\}}) = \text{g_rank}(\mathcal{O}.e_{V_X})$.

3. In a third part, we still consider a set S_P of q $\mathbf{Y}$-*topped* A-disjoint paths from $\mathbf{V_X}$ with $q = \text{card}(V_X)$ and a vertex $x_k \notin V_X$ such that $\rho_A(\mathbf{V_X} \cup \{\mathbf{x_k}\}, \mathbf{Y}) = \rho_A(\mathbf{V_X}, \mathbf{Y}) = \text{card}(\mathbf{V_X})$. The last condition means that all $\mathbf{Y}$-*topped* paths from $\mathbf{x_k}$ are not A-disjoint with all the paths of S_P. S_P^k denotes the set of all $\mathbf{Y}$-*topped* paths from $\mathbf{x_k}$. Then, considering the point 2 to each path of S_P^k, it can be deduced that $\text{g_rank}(\mathcal{O}.e_{V_X \cup \{x_k\}}) = \text{g_rank}(\mathcal{O}.e_{V_X})$. △

From lemma 1 and 2, the following proposition gives the final result about the relation between the observability matrix rank and the maximum number of A-disjoint paths.

Proposition 1: Let Σ_Λ be the structured bilinear system defined by (1) with its associated digraph $\mathcal{G}(\Sigma_\Lambda)$, the following relation is obtained :

$$\rho_A(\mathbf{V_X}, \mathbf{Y}) = \text{g_rank}(\mathcal{O}.e_{V_X}) \tag{6}$$

Proof 3:

1. If $\rho_A(\mathbf{V_X}, \mathbf{Y}) = \text{card}(\mathbf{V_X})$ then directly from lemma 1 is concluded that $\rho_A(\mathbf{V_X}, \mathbf{Y}) = \text{g_rank}(\mathcal{O}.e_{V_X})$.

2. Else $\rho_A(\mathbf{V_X}, \mathbf{Y})$ is necessarily lower than $\text{card}(\mathbf{V_X})$. In this case, there exists $\mathbf{V'_X} \subseteq \mathbf{V_X}$ such that $\rho_A(\mathbf{V'_X}, \mathbf{Y}) = \rho_A(\mathbf{V_X}) = \text{card}(\mathbf{V'_X})$. From lemma 2 can be written for each vertex $\mathbf{x_k}$ such that ($\mathbf{x_k} \in \mathbf{V_X}$ and $\mathbf{x_k} \notin \mathbf{V'_X}$) that $\rho_A(\mathbf{V'_X} \cup \{\mathbf{x_k}\}, \mathbf{Y}) = \rho_A(\mathbf{V'_X}, \mathbf{Y}) = \text{card}(\mathbf{V'_X})$ and $\text{g_rank}(\mathcal{O}.e_{V'_X \cup \{x_k\}}) = \text{g_rank}(\mathcal{O}.e_{V'_X}) = \text{card}(\mathbf{V'_X})$.
If $\rho_A(\mathbf{V'_X} \cup \{\mathbf{x_{k_1}}\}, \mathbf{Y}) = \text{card}(\mathbf{V'_X})$ and $\rho_A(\mathbf{V'_X} \cup \{\mathbf{x_{k_2}}\}, \mathbf{Y}) = \text{card}(\mathbf{V'_X})$ with $\mathbf{x_{k_1}} \notin \mathbf{V'_X}$, $\mathbf{x_{k_2}} \notin \mathbf{V'_X}$ and $\mathbf{x_{k_1}} \neq \mathbf{x_{k_2}}$ then $\rho_A(\mathbf{V'_X} \cup \{\mathbf{x_{k_1}}, \mathbf{x_{k_2}}\}, \mathbf{Y}) = \text{card}(\mathbf{V'_X})$. In the same way, if $\text{g_rank}(\mathcal{O}.e_{V'_X \cup \{x_{k_1}\}}) = \text{g_rank}(\mathcal{O}.e_{V'_X})$ and $\text{g_rank}(\mathcal{O}.e_{V'_X \cup \{x_{k_2}\}}) = \text{g_rank}(\mathcal{O}.e_{V'_X})$ then $\text{g_rank}(\mathcal{O}.e_{V'_X \cup \{x_{k_1}, x_{k_2}\}}) = \text{g_rank}(\mathcal{O}.e_{V'_X}) = \text{card}(\mathbf{V'_X})$.
By recurrence, if $\rho_A(\mathbf{V'_X} \cup \mathbf{V''_X}, \mathbf{Y}) = \text{card}(\mathbf{V'_X})$ and $\rho_A(\mathbf{V'_X} \cup \{\mathbf{x_{k_2}}\}, \mathbf{Y}) = \text{card}(\mathbf{V'_X})$ with $\mathbf{V''_X} \cap \mathbf{V'_X} = \emptyset$ and $\mathbf{x_{k_2}} \notin \mathbf{V'_X} \cup \mathbf{V''_X}$ then $\rho_A(\mathbf{V'_X} \cup \mathbf{V''_X} \cup \{\mathbf{x_{k_2}}\}, \mathbf{Y}) = \text{card}(\mathbf{V'_X})$. In the same way, if $\text{g_rank}(\mathcal{O}.e_{V'_X \cup V''_X}) = \text{g_rank}(\mathcal{O}.e_{V'_X})$ and $\text{g_rank}(\mathcal{O}.e_{V'_X \cup \{x_{k_2}\}}) = \text{g_rank}(\mathcal{O}.e_{V'_X})$ then $\text{g_rank}(\mathcal{O}.e_{V'_X \cup V''_X \cup \{x_{k_2}\}}) = \text{g_rank}(\mathcal{O}.e_{V'_X}) = \text{card}(\mathbf{V'_X})$.
The final result is : if $\rho_A(\mathbf{V_X}, \mathbf{Y}) = \rho_A(\mathbf{V'_X}, \mathbf{Y}) = \text{card}(\mathbf{V'_X})$ then $\text{g_rank}(\mathcal{O}.e_{V'_X}) = \text{g_rank}(\mathcal{O}.e_{V_X}) = \text{card}(\mathbf{V'_X})$. △

Remark 1: If $\rho_A(\mathbf{X}, \mathbf{Y})$ is equal to n then all the state components are observable and X_Ω is necessarily observable.

From the previous proposition, the graphical condition for the partial observability can be easily determined. In fact, we have $\rho_A(\mathbf{X}, \mathbf{Y}) = \text{g_rank}(\mathcal{O}.e_{V_X})$ and $\rho_A(\mathbf{X}/\{\mathbf{x_i}\}, \mathbf{Y}) = \text{g_rank}(\mathcal{O}.e_{X/\{x_i\}})$ and then the condition of component observability is given by :

Corollary 2: Let Σ_Λ be the structured bilinear system defined by (1) with its associated digraph $\mathcal{G}(\Sigma_\Lambda)$, the state component x_i is observable iff

$$\rho_A(\mathbf{X}, \mathbf{Y}) = \rho_A(\mathbf{X} \backslash \{\mathbf{x_i}\}, \mathbf{Y}) + 1 \tag{7}$$

Proof 4: $\rho_A(\mathbf{X}, \mathbf{Y}) = \rho_A(\mathbf{X} \backslash \{\mathbf{x_i}\}, \mathbf{Y}) + 1$ means that the rank of the observability matrix decreases if it is computed without taking the i-th column in account. It means that the i-th column is independent from the others and then that x_i is observable.△

The following corollary can also be determined.

Corollary 3: Let Σ_Λ be the structured bilinear system defined by (1) with its associated digraph $\mathcal{G}(\Sigma_\Lambda)$ and $y_s = \lambda_{k_i} x_i$, the state component x_i is observable iff

$$\rho_A(\mathbf{X}, \mathbf{Y}) = \rho_A(\mathbf{X}, \mathbf{Y} \cup \{\mathbf{y_s}\}) \tag{8}$$

Proof 5: y_s denotes the sensor placed to x_i. It is known that in this case, component x_i is observable. If $\rho_A(\mathbf{X}, \mathbf{Y}) = \rho_A(\mathbf{X}, \mathbf{Y} \cup \{\mathbf{y_s}\})$, it means that the dimension of the observable subspace has not change even if a sensor is placed to x_i then it can be deduced that x_i is observable even if there is no sensor placed to x_i. △

X_{obs} denotes the set of the observable state components, which is computed as follows $X_{\text{obs}} = \{x_i | \rho_A(\mathbf{X}, \mathbf{Y}) = \rho_A(\mathbf{X}, \mathbf{Y} \cup \{\mathbf{y_s}\})\}$ with $y_s = \lambda_{k_i} x_i$. The following corollary is given.

Corollary 4: Let Σ_Λ be the structured bilinear system defined by (1) with its associated digraph $\mathcal{G}(\Sigma_\Lambda)$, the set of state components X_Ω is observable iff

$$X_\Omega \subseteq X_{\text{obs}} \tag{9}$$

If the set X_Ω is not observable, sensors have to be placed in order to recover the previous corollary. The next part is dedicated to this sensor placement in order to ensure the observability of all the state components included in X_Ω.

B. Partial observability recovering with additional sensors

Associated to system Σ_Λ, a new output vector $z \in \mathbb{R}^q$ is defined which collects the new added measurements :

$$z(t) = Hx(t), \tag{10}$$

where z_i is the measurement obtained from the i-th additional sensor. It is to be noticed that a single sensor can measure several state components via structured matrix H.
According to this definition, composite system Σ^c_Λ is defined such that :

$$\Sigma^c_\Lambda : \begin{cases} \dot{x}(t) &= A_0 x(t) + \sum_{\ell=1}^{m} u_\ell(t) A_\ell x(t) \\ y(t) &= Cx(t) \\ z(t) &= Hx(t) \end{cases} \tag{11}$$

To this composite system can be associated digraph $\mathcal{G}(\Sigma^c_\Lambda)$ in the same manner as for Σ_Λ. The additional sensor components are represented by vertex set $\mathbf{Z}$ and edge set H-edges linking $\mathbf{X}$ to $\mathbf{Z}$. Representation of digraph $\mathcal{G}(\Sigma^c_\Lambda)$ is not necessary in the sequel of the paper.

Finally, our aim is to find z such that X_Ω is observable. In this way, we have to test the partial observability condition on the system.

At first, for each unobservable element of X_Ω all the potential sensors are determined which ensure the condition of partial observability. The set of these potential sensors ensuring the observability of element x_i are defined in sets denoted Z_i.
For all x_i such that $x_i \in X_\Omega$ and $x_i \notin X_{\text{obs}}$, we define $Z_i = \{x_\ell | \rho_A(\mathbf{X}, \mathbf{Y} \cup \{\mathbf{y_{s_1}}\}) = \rho_A(\mathbf{X}, \mathbf{Y} \cup \{\mathbf{y_{s_1}}, \mathbf{y_{s_2}}\})\}$ with $y_{s_1} = \lambda_{k_\ell} x_\ell$ and $y_{s_2} = \lambda_{k_i} x_i$.
It can be remarked that x_i is necessarily included in the set Z_i since placing a sensor in x_i makes the latter observable. It can also be remarked that placing a sensor in an already observable component cannot make component x_i observable.

The sets Z_i can be ordered with a natural partial order. Z_i and Z_j are such that $Z_i \subseteq Z_j$.
The infimal elements with this order correspond to the sets Z_i enabling to make observable all the elements of X_Ω and d_o denotes the number of such infimal sets.

Proposition 2: Let Σ^c_Λ be the structured bilinear system defined by (11) with its associated graph $\mathcal{G}(\Sigma^c_\Lambda)$. X_Ω is observable if :
there exists only one edge from each infimal set $\mathbf{Z_i}$ to a sensor $\mathbf{z_{k_i}}$. There is no edge from $\mathbf{Z_j}$ to a sensor $\mathbf{z_{k_i}}$ with $j \neq i$. $\mathbf{z_{k_i}}$ is the end vertex of any edge from $\mathbf{X}_{\text{obs}}$.

Remark 2: Elements Z_i have been determined considering that a sensor was connected to only one element of $\bar{\mathbf{X}}_{\text{obs}}$. If a sensor is connected to several elements of $\bar{\mathbf{X}}_{\text{obs}}$, the partial observability condition can be unsatisfied. This is the reason why a sensor z_j is restrictively connected (via an edge) to only one element of Z_i.

From the above proposition, it can be deduced that the number of state components which have to be measured by the additional sensors is at least equal to d_o and that more than d_o sensors is theoretically useless.

V. Example

Considering the system represented by the digraph 2 (fig. 2) and the components having to be observed are $X_\Omega = \{x_1, x_3, x_6, x_8, x_9\}$.

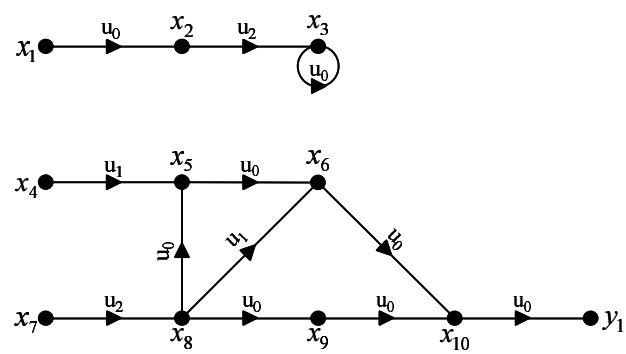

Figure 2: *Digraph* $\mathcal{G}(\Sigma_1)$

The first step consists in determining the observable components X_{obs}.

The following paths are used to find all the maximum A-linkings from $\mathbf{X}$ to $\mathbf{Y}$:

- $P_1 = \mathbf{x_4} \xrightarrow{u_1} \mathbf{x_5} \xrightarrow{u_0} \mathbf{x_6} \xrightarrow{u_0} \mathbf{x_{10}} \xrightarrow{u_0} \mathbf{y_1}$
- $P_2 = \mathbf{x_5} \xrightarrow{u_0} \mathbf{x_6} \xrightarrow{u_0} \mathbf{x_{10}} \xrightarrow{u_0} \mathbf{y_1}$
- $P_3 = \mathbf{x_7} \xrightarrow{u_2} \mathbf{x_8} \xrightarrow{u_1} \mathbf{x_6} \xrightarrow{u_0} \mathbf{x_{10}} \xrightarrow{u_0} \mathbf{y_1}$
- $P_4 = \mathbf{x_8} \xrightarrow{u_1} \mathbf{x_6} \xrightarrow{u_0} \mathbf{x_{10}} \xrightarrow{u_0} \mathbf{y_1}$
- $P_5 = \mathbf{x_7} \xrightarrow{u_2} \mathbf{x_8} \xrightarrow{u_0} \mathbf{x_5} \xrightarrow{u_0} \mathbf{x_6} \xrightarrow{u_0} \mathbf{x_{10}} \xrightarrow{u_0} \mathbf{y_1}$
- $P_6 = \mathbf{x_8} \xrightarrow{u_0} \mathbf{x_5} \xrightarrow{u_0} \mathbf{x_6} \xrightarrow{u_0} \mathbf{x_{10}} \xrightarrow{u_0} \mathbf{y_1}$
- $P_7 = \mathbf{x_6} \xrightarrow{u_0} \mathbf{x_{10}} \xrightarrow{u_0} \mathbf{y_1}$
- $P_8 = \mathbf{x_9} \xrightarrow{u_0} \mathbf{x_{10}} \xrightarrow{u_0} \mathbf{y_1}$
- $P_9 = \mathbf{x_{10}} \xrightarrow{u_0} \mathbf{y_1}$

Then, these maximum A-linkings are expressed by $\mathfrak{m}_i = \{P_1, P_2, P_9, P_3 \text{ or } P_5, P_4 \text{ or } P_6, P_7 \text{ or } P_8\}$. Then, the maximum number of A-disjoint paths is $\rho_A(\mathbf{X}, \mathbf{Y}) = 6$. It means that the rank of the observability matrix is equal to 6 but it doesn't mean that 6 state components are observable. In fact, from the corollary 2, we can conclude that the observable state components are $X_{\text{obs}} = \{x_4, x_5, x_7, x_8, x_{10}\}$. If

V_b^i denotes the set of the begin vertices of the paths belonging to $\mathfrak{m}_i$ then, all the components of X_{obs} belong to all sets V_b^i. Moreover, x_6 and x_9 are not observable although they belong to some sets V_b^i but not to all sets V_b^i. It can also be remarked that the dimension of X_{obs} is equal to 5 and not 6 as we have seen before.

Then, the unobservable components of X_Ω are $\bar{X}_\Omega = \{x_1, x_3, x_6, x_9\}$. Corollary 4 is not satisfied, then, the sets Z_k associated to each element of $\bar{X}_\Omega$ have to be determined. All the possible sensor placements are given by $\bar{X} = X \backslash X_{\text{obs}} = \{x_1, x_2, x_3, x_6, x_9\}$.

The sets of suitable sensors are $Z_1 = \{x_1, x_2, x_3\}$, $Z_3 = \{x_3\}$, $Z_6 = \{x_6, x_9\}$ and $Z_9 = \{x_6, x_9\}$.
Now, we have to order these sets. We have the two relations $Z_3 \subseteq Z_2$ and $Z_6 = Z_9$.
Finally, we conclude that the number of infimal elements is equal to 2. The infimal elements are Z_3 and Z_6. So, X_Ω is observable if :
z_1 is connected to x_3 and any combination of elements of X_Ω and z_2 is connected to x_6 or x_9 and any combination of elements of X_Ω

VI. Conclusion

In this paper, a new analysis tool is proposed to study the partial observability of structured bilinear systems. Using a new graphical representation of this class of nonlinear systems, a necessary and sufficient condition for generic partial observability is given and expressed in graphical terms. This condition needs few information about the system and is easy to check by means of combinatorial techniques or simply by hand for small systems. Furthermore, the use of graph-theoretic approach makes it easy to visualise the system structure. This paper also deals with sensor placement. The number and the location of additional sensors are studied in order to recover the partial observability of state variable set. This study is very useful during the design phase of a system. Indeed synthesis of the control law often assumes the observability of a given part of the state variables. Moreover, if a system cannot be made observable (all the state components) then it is interesting to make this study of partial observability applied to particular state components.
In further works, it could be interesting to extend the study of sensor placement to the case where the notion of cost is taken in account or to the case of a minimum number of sensors.

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