Vertical and horizontal increments for the Brownian sheet, level curve's lenght and local time

Corinne Berzin-Joseph, José R. León

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VERTICAL AND HORIZONTAL INCREMENTS FOR THE BROWNIAN SHEET, LEVEL CURVE'S LENGHT AND LOCAL TIME

CORINNE BERZIN-JOSEPH
Université Versailles-Saint-Quentin en Yvelines.
email: Corinne.Berzin@math.u-psud.fr

JOSÉ R. LEÓN
Universidad Central de Venezuela,
Caracas. Venezuela.
A.P. 47197 Los Chaguaramos.
Caracas 1041-A. Venezuela.
email: jleon@euler.ciens.ucv.ve

Abstract

In this work we study the almost sure convergence for the vertical and horizontal increments of the Brownian sheet. Afterwards, a modification of these results for a regularization of this process, provides almost sure convergence in law for the length of the level curve, defined for the regularized process, towards a modification to the Brownian sheet's local time.

1 Introduction

In the study of the asymptotic behavior for the increments of one parameter processes $\{Z_t\}_{t\in\mathbb{R}}$, usually only a type of increments is defined, namely $Z_{t+\varepsilon} - Z_t$. In two parameter situations, as in the Brownian sheet case i.e. $\{W(t,s)_{(t,s)\in[0,1]^2}\}$, it is possible to introduce different types of increments. On the one hand the vertical and horizontal increments can be defined as: $W(t+\varepsilon,s)-W(t,s)$ and $W(t,s+\varepsilon)-W(t,s)$, and on the other hand one can consider the surface increments: $W(t+\varepsilon,s+\varepsilon)+W(t,s)-W(t+\varepsilon,s)-W(t,s+\varepsilon)$, and so on.
In this work we shall study the asymptotic behavior for the former increments. We will show in Theorem 1, denoting by λ the bi-dimensional Lebesgue measure, that for almost every ω (fixed) it holds:

$$\lambda\{(t,s)\in[0,1]^2; V_\varepsilon(t,s)\le x_1\ ,\ H_\varepsilon(t,s)\le x_2\}\to\int_{[0,1]^2}F_s(x_1)F_t(x_2)ds\,dt \text{ a.s. as } \varepsilon\to 0$$

This is the final form of the paper.

where $H_\varepsilon(t,s) = \frac{W(t,s+\varepsilon)-W(t,s)}{\sqrt{\varepsilon}}$, $V_\varepsilon(t,s) = \frac{W(t+\varepsilon,s)-W(t,s)}{\sqrt{\varepsilon}}$ and F_r is the Gaussian distribution with variance r. One could express this result by saying that there exits a null subset N of Ω, the probability space of the Brownian sheet such that if $\omega \notin N$ the random vector $(V_\varepsilon(\cdot,\cdot,\omega), H_\varepsilon(\cdot,\cdot,\omega))$ defined in the bidimensional Lebesgue space $([0,1]^2, \mathcal{M}, \lambda)$ converges in law to the random vector $(\sqrt{U_1}\xi_1, \sqrt{U_2}\xi_2)$ where $(U_i)_{i=1,2}$ are uniform and independent random variables and $(\xi_i)_{i=1,2}$ are independent standard Gaussian random variables independent also of the U_i.
A result in this direction, in the one parameter case, has been shown by M. Wschebor in [6]: Let $\{X_t\}_{t\in[0,1]}$ be a standard Brownian Motion, if $\Phi(x)$ denotes the standard Gaussian distribution and λ_1 is the one dimensional Lebesgue measure, we have for a.e. fixed ω

$$\lambda_1\{s \le 1 : \frac{X_{s+\varepsilon} - X_s}{\sqrt{\varepsilon}} \le x\} \to \Phi(x) \text{ as } \varepsilon \to 0$$

In the same article he also proves the following statement. Define the regularized process $X_t^\varepsilon =$
$\frac{1}{\varepsilon}\int_{-\infty}^{\infty} \varphi(\frac{t-s}{\varepsilon}) X_s ds$, where φ is a smooth density function, having bounded support. Let be f a continuous and bounded function, Wschebor's second result says that for a.e. fixed ω

$$\sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} f(x) \frac{\varepsilon^{1/2}}{||\varphi||} N^\varepsilon(x) dx \to \int_{-\infty}^{\infty} f(x) L(x) dx \text{ as } \varepsilon \to 0$$

where $N^\varepsilon(x) = \#\{s \le 1 : X_s^\varepsilon = x\}$ is the number of crossings of the process X^ε at level x, $||\ ||$ is the L^2 norm and $L(x)$ is the Brownian Motion's local time at level x. It is interesting to note that these two results establish a relationship between the asymptotic for increments and the asymptotics for crossings. This relation starts with the following Banach-Kac formula [1] [3], with f as before:

$$\int_{-\infty}^{\infty} f(x) N^\varepsilon(x) dx = \int_0^1 f(X_s^\varepsilon) |\dot{X}_s^\varepsilon| ds$$

where $\dot{X}_s^\varepsilon$ is the derivative of X_s^ε, that exists due to the smoothness of φ.
In the two parameter context there exists another formula, named the co-area formula ([2] pp. 249), that we will describe now (note that an elementary proof of this formula can be given using a slight generalization of the Divergence Theorem). Let the regularized sheet be defined as $W_\varepsilon(t,s) =$
$\frac{1}{\varepsilon^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\frac{t-u}{\varepsilon})\varphi(\frac{s-v}{\varepsilon}) W(u,v) du\, dv$ and $\mathcal{L}^\varepsilon(x) = \lambda_1\{(t,s) \in [0,1]^2 : W_\varepsilon(t,s) = x\}$ be the level curve's lenght. The co-area formula is:

$$\int_{-\infty}^{\infty} f(x) \mathcal{L}^\varepsilon(x) dx = \int_{[0,1]^2} f(W_\varepsilon(t,s)) \sqrt{(\partial_t W_\varepsilon(t,s))^2 + (\partial_s W_\varepsilon(t,s))^2}\, ds\, dt$$

In Theorem 2 we will prove, using this last expression, that for a.e. ω fixed

$$\int_{-\infty}^{\infty} f(x) \frac{\varepsilon^{1/2}}{||\varphi||} \mathcal{L}^\varepsilon(x) dx \to \int_{-\infty}^{\infty} f(x) \tilde{\mathcal{L}}(x) dx \text{ as } \varepsilon \to 0,$$

where $\tilde{\mathcal{L}}(x)$ is a modification of the Brownian sheet's local time defined below. We shall not consider at all the surface increments, nevertherless we must note that these increments have not related to any level functional. However, several statistical applications could be deduced from its asymptotic behavior. This matter will be considered elsewhere.

2 Hypothesis and notation

Let W be the Brownian sheet on $\mathbb{R} \times \mathbb{R}$, setting $W(t,s) = 0$ if t or s are negative. For each (t,s) and $\varepsilon > 0$ we define the regularized process $W_\varepsilon(t,s) = \varphi_\varepsilon * W(t,s)$ where $\varphi_\varepsilon(t,s) = \frac{1}{\varepsilon^2}\varphi(\frac{t}{\varepsilon})\varphi(\frac{s}{\varepsilon})$ and $*$ denotes the convolution. We have the following hypothesis

(H1) For the kernel φ: $\int_{-1}^{1} \varphi(t), dt = 1$, $\varphi \in C^2$, $\varphi \geq 0$ and the support of φ is a subset of $[-1,1]$.

We denote λ the Lebesgue measure in $\mathbb{R}^2$. F_s, $s \in [0,1]$, stands for the Gaussian distribution with variance s.

Let ξ_1 and ξ_2 be two standard independent Gaussian r.v. The standard Gaussian density will be denoted by ϕ. Also $\mathcal{L}_\varepsilon(x)$ is the length of the x-level curve of the process W_ε on $[0,1]^2$. We shall define $\tilde{\mathcal{L}}(x)$ as a modification of the local time for the Brownian sheet on $[0,1]^2$ that satisfies for all bounded and continuous function

$$\int_{-\infty}^{\infty} f(x)\tilde{\mathcal{L}}(x)dx = \int_{[0,1]^2} f(W(t,s))E[\sqrt{s\xi_1^2 + t\xi_2^2}]ds\,dt := \int_{[0,1]^2} f(W(t,s))\Gamma(t,s)ds\,dt.$$

Throughout this article, **C** shall stand for a generic constant, whose value may change during a proof.

Define for $(t,s) \in [0,1]^2$,

$$V_\varepsilon(t,s) = \frac{1}{\sqrt{\varepsilon}}[W(t+\varepsilon,s) - W(t,s)]$$

and

$$H_\varepsilon(t,s) = \frac{1}{\sqrt{\varepsilon}}[W(t,s+\varepsilon) - W(t,s)]$$

the vertical and horizontal increments for the Brownian sheet.

3 Results

Theorem 1

$$(i)\ \lambda\{(t,s) \in [0,1]^2; V_\varepsilon(t,s) \leq x_1\ ,\ H_\varepsilon(t,s) \leq x_2\} \to \int_{[0,1]^2} F_s(x_1)F_t(x_2)ds\,dt \ \ a.s.\ \ as\, \varepsilon \to 0$$

$$(ii)\ \int_{[0,1]^2} \sqrt{V_\varepsilon^2(t,s) + H_\varepsilon^2(t,s)}ds\,dt \to \int_{[0,1]^2} \Gamma(t,s)ds\,dt$$

Proof: The proof will follow the steps of [6]. We divide $\mathbb{R}^2$ in four regions as is shown in figure 1.

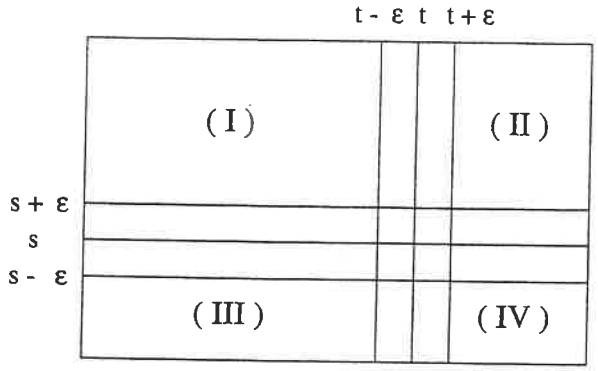

Figure 1

We need the correlations between $(V_\varepsilon(t,s), H_\varepsilon(t,s))$ and $(V_\varepsilon(t',s'), H_\varepsilon(t',s'))$ for $(t,s) \in [0,1]^2$ and (t',s') in regions (I), (II), (III) and (IV).

Lemma 1 *For $(t,s) \in [0,1]^2$ and $(t',s') \in [0,1]^2$,*
if $(t',s') \in (I)$, (II), (III) or (IV) $E[V_\varepsilon(t,s)V_\varepsilon(t',s')] = 0 = E[H_\varepsilon(t,s)H_\varepsilon(t',s')]$
if $(t',s') \in (II)$, (III) or (IV), $E[H_\varepsilon(t,s)V_\varepsilon(t',s')] = 0$ and if $(t',s') \in (I)$, $E[H_\varepsilon(t,s)V_\varepsilon(t',s')] = \varepsilon$
if $(t',s') \in (I)$, (II) or (III), $E[V_\varepsilon(t,s)H_\varepsilon(t',s')] = 0$ and if $(t',s') \in (IV)$, $E[V_\varepsilon(t,s)H_\varepsilon(t',s')] = \varepsilon$

Proof: The proof is the result of a straighforward computation.
Consider, for $\omega \in \Omega$ fixed, $(V_\varepsilon(\cdot,\cdot), H_\varepsilon(\cdot,\cdot))$ as a random vector on $([0,1]^2, \lambda)$. We shall prove in what follows that this vector converges in lawalmost surely to $(\sqrt{U_1}\xi_1, \sqrt{U_2}\xi_2)$ where $(U_i)_{i=1,2}$ are two independent uniform r.v.'s on $[0,1]$, independent also of $(\xi_i)_{i=1,2}$. For this we consider a linear combination of V_ε and H_ε, namely $\zeta_\varepsilon(t,s) = c_1 V_\varepsilon(t,s) + c_2 H_\varepsilon(t,s)$.

Let us compute the moments of this r.v. First, letting $\zeta_k^\varepsilon = \int_{[0,1]^2} \xi_\varepsilon^k(t,s)ds\,dt$, we have

$$E[\zeta_k^\varepsilon] = \int_{[0,1]^2} E[c_1\sqrt{s}\xi_1 + c_2\sqrt{t}\xi_2]^k ds\,dt.$$

Computing the variance we get

$$Var(\zeta_k^\varepsilon) = \int_{[0,1]^4} \{E[\xi_\varepsilon^k(t,s)\xi_\varepsilon^k(u,v)] - E[\xi_\varepsilon^k(t,s)]E[\xi_\varepsilon^k(u,v)]\} dv\,du\,ds\,dt.$$

We split the integrals into different regions $(I_i)_{i=1,4}$ defined as

$$I_1 = \{(u,v) \in [0,1]^2 \ , \ u \leq t \ , \ v \geq s\} \ , \ I_2 = \{(u,v) \in [0,1]^2 \ , \ u \geq t \ , \ v \geq s\}$$

$$I_3 = \{(u,v) \in [0,1]^2 \ , \ u \leq t \ , \ v \leq s\} \ , \ I_4 = \{(u,v) \in [0,1]^2 \ , \ u \geq t \ , \ v \leq s\}$$

Let J_i be the integral corresponding to the domain I_i, $i = 1,4$, we get

$$J_3 \simeq \int_{\Delta_\varepsilon^1} \{E[\xi_\varepsilon^k(t,s)\xi_\varepsilon^k(u,v)] - E[\xi_\varepsilon^k(t,s)]E[\xi_\varepsilon^k(u,v)]\} dv\, du\, ds\, dt$$

where $\simeq$ means asymptotically equivalent and $\Delta_\varepsilon^1 = [\varepsilon,1] \times [\varepsilon,1] \times [0,t-\varepsilon] \times [0,s-\varepsilon]$, all other terms are $O(\varepsilon)$.
Using Lemma 1 one can see, since (u,v) belongs to (III), that all the Gaussian r.v.'s in the integrand are independent so $J_3 = O(\varepsilon)$ also and the same procedure gives $J_2 = O(\varepsilon)$.
We shall study the asymptotic behavior of J_1. The same arguments apply for J_4. The computation here is more involved because the corresponding Gaussian r.v. are not independent. We obtain as before

$$J_1 \simeq \int_{\Delta_\varepsilon^2} \{E[(c_1\sqrt{s}\xi_1 + c_2\sqrt{t}\xi_2)^k (c_1\sqrt{v}\xi_1' + c_2\sqrt{u}\xi_2')^k]$$

$$- E[c_1\sqrt{s}\xi_1 + c_2\sqrt{t}\xi_2]^k E[c_1\sqrt{v}\xi_1 + c_2\sqrt{u}\xi_2]^k\} dv du ds dt$$

where $(\xi_i')_{i=1,2}$ are independent standard Gaussian r.v. with correlation $E(\xi_i\xi_j') = 0$ except for $i = 2$ and $j = 1$ and in this case $E(\xi_2\xi_1') = \frac{\varepsilon}{\sqrt{vt}}$ (see Lemma 1) and $\Delta_\varepsilon^2 = [\varepsilon,1] \times [0,1-\varepsilon] \times [0,t-\varepsilon] \times [s+\varepsilon,1]$. The other terms of J_1 are $O(\varepsilon)$. We now fix (t,s,u,v) and make a change of variables to transform $(\xi_1,\xi_2,\xi_1',\xi_2')$ into a standard Gaussian random vector, obtaining

$$J_1 \simeq \int_{\Delta_\varepsilon^2} \int_{\mathbb{R}^4} (c_1\sqrt{v}x_2 + c_2\sqrt{u}y_2)^k$$

$$\times \{[c_1\sqrt{s}x_1 + c_2\sqrt{t}(\sqrt{1 - \frac{\varepsilon^2}{tv}}y_1 + \frac{\varepsilon}{\sqrt{vt}}x_2)]^k - (c_1\sqrt{s}x_1 + c_2\sqrt{t}y_1)^k\}$$

$$\times \phi(x_1)\phi(y_1)\phi(x_2)\phi(y_2) dx_1 dy_1 dx_2 dy_2 dv\, du\, ds\, dt$$

To prove that this integral is $O(\varepsilon)$ we use the identity

$$A^k - B^k = (A-B) \sum_{j=0}^{k-1} A^j B^{k-j-1} \begin{pmatrix} k-1 \\ j \end{pmatrix}$$

and the fact that $\frac{1}{\sqrt{v}} \in L^1((0,1), dv)$.
So we have proved that in $L^2(\Omega)$, $\zeta_k^\varepsilon \to \int_{[0,1]^2} E[c_1\sqrt{s}\xi_1 + c_2\sqrt{t}\xi_2]^k ds\, dt$ as $\varepsilon \to 0$ and

furthermore $Var(\zeta_k^\varepsilon) = O(\varepsilon)$.
We want to prove now that this convergence takes place almost surely. Using the Borel-Cantelli lemma and taking $\varepsilon_\nu = \nu^{-a}$, $a > 1$, it holds that a.s. in ω

$$\zeta_k^{\varepsilon_\nu} \to \int_{[0,1]^2} E[c_1\sqrt{s}\xi_1 + c_2\sqrt{t}\xi_2]^k ds\, dt.$$

To finish the proof we have to consider the oscillation

$$\sup_{\varepsilon_{\nu+1}\le\varepsilon\le\varepsilon_\nu} |\int_{[0,1]^2} \xi_\varepsilon^k(t,s)ds\, dt - \int_{[0,1]^2} \xi_{\varepsilon_\nu}^k(t,s)ds\, dt|.$$

Let us start with

$$\begin{aligned}
&|\int_{[0,1]^2} \xi_\varepsilon^k(t,s)ds\, dt - \int_{[0,1]^2} \xi_{\varepsilon_\nu}^k(t,s)ds\, dt| \\
&\qquad \le |\varepsilon^{-k/2} - \varepsilon_\nu^{-k/2}| \int_{[0,1]^2} \{[c_1\sqrt{\varepsilon_\nu}V_{\varepsilon_\nu}(t,s) + c_2\sqrt{\varepsilon_\nu}H_{\varepsilon_\nu}(t,s)]^k\} ds\, dt| \\
&\qquad +\varepsilon^{-k/2}| \int_{[0,1]^2} \{[c_1\sqrt{\varepsilon}V_\varepsilon(t,s) + c_2\sqrt{\varepsilon}H_\varepsilon(t,s)]^k \\
&\qquad -[c_1\sqrt{\varepsilon_\nu}V_{\varepsilon_\nu}(t,s) + c_2\sqrt{\varepsilon_\nu}H_{\varepsilon_\nu}(t,s)]^k\} ds\, dt| \equiv l_1 + l_2.
\end{aligned}$$

If $\varepsilon_{\nu+1} \le \varepsilon \le \varepsilon_\nu$ then $|\varepsilon^{-k/2} - \varepsilon_\nu^{-k/2}| \le \varepsilon_\nu^{-k/2}[\left(\frac{\varepsilon_\nu}{\varepsilon_{\nu+1}}\right)^{k/2} - 1]$, we have $l_1 \le [\left(\frac{\varepsilon_\nu}{\varepsilon_{\nu+1}}\right)^{k/2} - 1]|\zeta_k^{\varepsilon_\nu}|$. Then by the choice of the sequence ε_ν it holds that $\left(\frac{\varepsilon_\nu}{\varepsilon_{\nu+1}}\right)^{k/2} \to 1$ showing that l_1 tends to zero a.s.
For l_2 we shall use as before the binomial identity for $A^k - B^k$, taking $A_\varepsilon(t,s) = \sqrt{\varepsilon}\xi_\varepsilon(t,s)$ and $B_{\varepsilon_\nu}(t,s) = \sqrt{\varepsilon_\nu}\xi_{\varepsilon_\nu}(t,s)$, we get

$$l_2 \le \varepsilon_{\nu+1}^{-k/2} \int_{[0,1]^2} |A_\varepsilon(t,s) - B_{\varepsilon_\nu}(t,s)| \sum_{j=0}^{k-1} \binom{k-1}{j} |A_\varepsilon(t,s)|^j |B_{\varepsilon_\nu}(t,s)|^{k-j-1} ds\, dt.$$

Using the modulus of continuity for the Brownian sheet ([4], pag 154) i.e. $|W(t+\varepsilon, s) - W(t,s)| \le c(\omega)\varepsilon^{(1/2)-\delta}$ for any $\delta > 0$ as small as we wish, we obtain

$$l_2 \le c^k(\omega)(|c_1| + |c_2|)^k \varepsilon_{\nu+1}^{-k/2} (\varepsilon_\nu - \varepsilon)^{((1/2)-\delta)} \sum_{j=0}^{k-1} \binom{k-1}{j} \varepsilon^{((1/2)-\delta)j} \varepsilon_\nu^{((1/2)-\delta)(k-1-j)}.$$

This implies that

$$l_2 \le \mathbf{C}\ \varepsilon_{\nu+1}^{-k/2} (\varepsilon_\nu - \varepsilon_{\nu+1})^{((1/2)-\delta)} \varepsilon_\nu^{((1/2)-\delta)(k-1)}$$

where the constant depends on k, ω and δ. Finally one gets

$$l_2 \leq \mathbf{C} \left(\frac{\varepsilon_\nu}{\varepsilon_{\nu+1}}\right)^{k/2} (1 - \frac{\varepsilon_{\nu+1}}{\varepsilon_\nu})^{((1/2)-\delta)} \varepsilon_\nu^{-k\delta} \to 0$$

as $\nu \to +\infty$ if δ is small enough i.e. $\delta < \frac{1}{2(k\alpha+1)}$.
Thus we have shown that

$$\int_{[0,1]^2} (c_1 V_\varepsilon(t,s) + c_2 H_\varepsilon(t,s))^k ds\, dt \to \int_{[0,1]^2} E[c_1\sqrt{s}\xi_1 + c_2\sqrt{t}\xi_2]^k ds\, dt \text{ a.s. as } \varepsilon \to 0.$$

This result can be expressed saying that for a.s. ω $(V_\varepsilon(\cdot,\cdot), H_\varepsilon(\cdot,\cdot))$ converges in law to $(\sqrt{U_1}\xi_1, \sqrt{U_2}\xi_2)$ where $(U_i)_{i=1,2}$ are two independent uniform r.v. on $[0,1]$ also independent of $(\xi_i)_{i=1,2}$, hence (i) and (ii) follow. □

Remark. Result (ii) is closely linked with the convergence of the level curve length $\mathcal{L}_\varepsilon$ to the Brownian sheet local time, a fact that we shall study next.
The surface $W_\varepsilon(t,s)$ is almost surely C^2 and we know, thanks to the level curve's theory for differentiable surfaces [5], that for all x the set $\{(t,s) : W_\varepsilon(t,s) = x\}$ has finite length a.s. Federer ([2], pp. 249) proved the following change of variable formula, called the co-area formula: If f is a bounded continuous function then

$$\int_{-\infty}^{\infty} f(x)\mathcal{L}_\varepsilon(x)dx = \int_{[0,1]^2} f(W_\varepsilon(t,s))\sqrt{(\partial_t W_\varepsilon(t,s))^2 + (\partial_s W_\varepsilon(t,s))^2}\, ds\, dt.$$

This formula can be viewed as the bi-dimensional analoguous to the well known Banach-Kac formula [1] [3] for the number of crossings of a process. We have the following

Theorem 2 *Under* **(H1)** *and if f is a bounded continuous function then a.s.*

$$\sqrt{\varepsilon}||\varphi||^{-1} \int_{-\infty}^{\infty} f(x)\mathcal{L}_\varepsilon(x)dx \to \int_{-\infty}^{\infty} f(x)\tilde{\mathcal{L}}(x)dx \text{ when } \varepsilon \to 0.$$

Proof: First we shall consider the case where $f \equiv 1$. We want to prove that a.s.

$$\sqrt{\varepsilon}||\varphi||^{-1} \int_{[0,1]^2} \sqrt{(\partial_t W_\varepsilon(t,s))^2 + (\partial_s W_\varepsilon(t,s))^2}\, ds\, dt \to \int_{[0,1]^2} \Gamma(t,s) ds\, dt \text{ as } \varepsilon \to 0$$

The proof will follow the same lines of the previous one for the increments. We have the following

Lemma 2 *If $s > \varepsilon$ and $t > \varepsilon$*

$$E[\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s)\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s)] = \frac{1}{4}\varepsilon$$

$$\varepsilon E[\partial_t W_\varepsilon(t,s)]^2 = ||\varphi||^2 \lambda_{1,\varepsilon}^2(s) := ||\varphi||^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(u)\varphi(v)[(s-\varepsilon u)\wedge(s-\varepsilon v)]du\,dv$$

(where $\wedge$ means the minimum between the two numbers)

$$\varepsilon E[\partial_s W_\varepsilon(t,s)]^2 = \lambda_{1,\varepsilon}^2(t)||\varphi||^2$$

Proof: The proof is an easy consequence of the definitions.
So we obtain that if $s>0$ and $t>0$,

$$||\varphi||^{-2}\varepsilon E[\partial_t W_\varepsilon(t,s)]^2 \to s \ , \ ||\varphi||^{-2}\varepsilon E[\partial_s W_\varepsilon(t,s)]^2 \to t$$

and $\varepsilon E[\partial_t W_\varepsilon(t,s)\partial_s W_\varepsilon(t,s)] \to 0$ as $\varepsilon \to 0$. Hence this implies that if $(c_i)_{i=1,2}$ are two real constants it follows

$$\int_{[0,1]^2} E[c_1||\varphi||^{-1}\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s) + c_2||\varphi||^{-1}\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s)]^k ds\,dt$$

$$\to \int_{[0,1]^2} E[c_1\sqrt{s}\xi_1 + c_2\sqrt{t}\xi_2]^k ds\,dt \text{ as } \varepsilon \to 0.$$

We shall compute the correlations, for different points (t,s), between the Gaussian r.v. that appear in the above integral. Thus we get as in lemma 1 the following

Lemma 3 *For $(t,s)\in[0,1]^2$ and $(t',s')\in[0,1]^2$, verifying $t>\varepsilon$, $t'>\varepsilon$, $s>\varepsilon$ and $s'>\varepsilon$, we have*

$$\begin{aligned}
&(a)\ E[\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s)\sqrt{\varepsilon}\partial_t W_\varepsilon(t',s')] &&= 0 \text{ if } |t-t'|\geq 2\varepsilon\\
&(b)\ E[\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s)\sqrt{\varepsilon}\partial_s W_\varepsilon(t',s')] &&= 0 \text{ if } |s-s'|\geq 2\varepsilon\\
&(c)\ E[\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s)\sqrt{\varepsilon}\partial_s W_\varepsilon(t',s')] &&= \varepsilon \text{ if } t'-t\geq 2\varepsilon \text{ and } s-s'\geq 2\varepsilon\\
&(d)\ E[\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s)\sqrt{\varepsilon}\partial_s W_\varepsilon(t',s')] &&= 0 \text{ if } t-t'\geq 2\varepsilon \text{ or } s'-s\geq 2\varepsilon\\
&(e)\ E[\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s)\sqrt{\varepsilon}\partial_t W_\varepsilon(t',s')] &&= \varepsilon \text{ if } t-t'\geq 2\varepsilon \text{ and } s'-s\geq 2\varepsilon\\
&(f)\ E[\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s)\sqrt{\varepsilon}\partial_t W_\varepsilon(t',s')] &&= 0 \text{ if } t'-t\geq 2\varepsilon \text{ or } s-s'\geq 2\varepsilon
\end{aligned}$$

Let us return to the proof of the theorem, the situation here is similar to the previous one for the increments.
As before we must consider the following linear combination

$$\xi_\varepsilon(t,s) = ||\varphi||^{-1}[c_1\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s) + c_2\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s)]$$

There exists some differences between this situation and that of the increments. The Gaussian r.v.'s now do not have variances equal to s or t but tending towards these numbers. Furthermore $\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s)$ and $\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s)$ are not independent (but

asymptotically independent).
We shall study $\zeta_k^\varepsilon = \int_{[0,1]^2} \xi_\varepsilon^k(t,s)ds\,dt$. Let us compute its variance:

$$Var(\zeta_k^\varepsilon) = \int_{[0,1]^4} \{E[\xi_\varepsilon^k(t,s)\xi_\varepsilon^k(u,v)] - E[\xi_\varepsilon^k(t,s)]E[\xi_\varepsilon^k(u,v)]\}dv\,du\,ds\,dt.$$

Splitting the domain of integration $[0,1]^4$ into four regions $(I_i)_{i=1,4}$ denoting by J_i the integrals over these regions and using lemma 3 it is easy to show, as in the previous theorem, that $J_i = O(\varepsilon)$ for $i = 2, 3$. Hence it is enough consider the integrals J_1, J_4. We will study only J_1 because J_4 is similar.

$$J_1 \simeq \int_{\Delta_\varepsilon^3} \{E[(c_1X_1+c_2X_2)^k(c_1X_3+c_2X_4)^k]-E[(c_1X_1+c_2X_2)^k]E[(c_1X_3+c_2X_4)^k]\}dvdudsdt.$$

where $\Delta_\varepsilon^3 = [(M+2)\varepsilon, 1] \times [2\varepsilon, 1-2\varepsilon] \times [M\varepsilon, t-2\varepsilon] \times [s+2\varepsilon, 1]$. We choose also M large enough such that $\lambda_{1,\varepsilon}^2(t) \geq \frac{t}{2}$ and $\lambda_{1,\varepsilon}^2(u) \geq \frac{u}{2}$. Here (X_1, X_2, X_3, X_4) is a Gaussian vector with correlations given by lemmas 2 and 3. The other terms of J_1 are $O(\varepsilon)$. We now fix (t, s, u, v) and make a change of variable in the first term transforming (X_1, X_2, X_3, X_4) into a standard Gaussian vector and another change of variable in the two remainder terms that reduce (X_1, X_2) (resp. (X_3, X_4)) into standard Gaussian vectors in $\mathbb{R}^2$. We get

$$J_1 \simeq \int_{\Delta_\varepsilon^3} \int_{\mathbb{R}^4} ((c_1 + ac_2)\lambda_{1,\varepsilon}(s)x_1 + bc_2x_2)^k$$

$$\times[(c_2dx_2+(c_1+c_2c)\lambda_{1,\varepsilon}(v)x_3+c_2ex_4)^k-((c_1+cc_2)\lambda_{1,\varepsilon}(v)x_3+c_2fx_4)^k]\phi(\hat{x})d\hat{x}dvdudsdt,$$

where $\phi(\hat{x})d\hat{x} = \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)dx_1dx_2dx_3dx_4$ and:

$$a = \frac{1}{4}\varepsilon||\varphi||^{-2}\lambda_{1,\varepsilon}^{-2}(s)\ ,\ b = [\lambda_{1,\varepsilon}^2(t) - a^2\lambda_{1,\varepsilon}^2(s)]^{1/2}\ ,\ c = \frac{1}{4}\varepsilon||\varphi||^{-2}\lambda_{1,\varepsilon}^{-2}(v)$$

$$d = -\frac{c}{b}\varepsilon||\varphi||^{-2}\ ,\ e = [\lambda_{1,\varepsilon}^2(u) - c^2\lambda_{1,\varepsilon}^2(v) - d^2]^{1/2}\ ,\ f = [\lambda_{1,\varepsilon}^2(u) - c^2\lambda_{1,\varepsilon}^2(v)]^{1/2}.$$

Using again the identity $A^k - B^k = (A-B)\sum_{j=0}^{k-1} A^jB^{k-j-1}\begin{pmatrix} k-1 \\ j \end{pmatrix}$ the conditions $\lambda_{1,\varepsilon}^2(t) \geq \frac{t}{2}$, $\lambda_{1,\varepsilon}^2(u) \geq \frac{u}{2}$ and the fact that both $\frac{1}{\sqrt{u}}$ and $\frac{1}{\sqrt{t}}$ are in L^1 for the one-dimensional Lebesgue measure, we obtain finally that for M large enough $J_1 = O(\varepsilon)$ and $Var(\xi_k^\varepsilon) = O(\varepsilon)$.
To finish the proof of Theorem 2 for $f \equiv 1$ we have to evaluate as in the proof of Theorem 1 the modulus of continuity for the derivatives of the regularization of the Brownian sheet. But

$$\varepsilon\partial_t W_\varepsilon(t,s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot{\varphi}(u)\varphi(v)W(t-\varepsilon u, s-\varepsilon v)du\,dv,$$

and a similar expression holds for $\varepsilon_\nu \partial_t W_{\varepsilon_\nu}(t,s)$. Thus we get

$$|\varepsilon \partial_t W_\varepsilon(t,s) - \varepsilon_\nu \partial_t W_{\varepsilon_\nu}(t,s)| \leq \mathbf{C}(\omega)|\varepsilon - \varepsilon_\nu|^{\frac{1}{2}-\delta}$$

Arguing in the same fashion as in the proof of Theorem 1 we obtain, a.s. in ω as ε tends to zero:

$$\int_{[0,1]^2} (c_1||\varphi||^{-1}\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s) + c_2||\varphi||^{-1}\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s))^k dt\, ds \to \int_{[0,1]^2} E[c_1\sqrt{s}\xi_1 + c_2\sqrt{s}\xi_2]^k ds dt.$$

Hence it holds that for a.s. ω the random vector $(||\varphi||^{-1}\sqrt{\varepsilon}\partial_t W_\varepsilon(\cdot,\cdot), ||\varphi||^{-1}\sqrt{\varepsilon}\partial_s W_\varepsilon(\cdot,\cdot))$ converges in law to $(\sqrt{U_1}\xi_1, \sqrt{U_2}\xi_2)$ so the Theorem is proved for $f \equiv 1$. Let us finish the proof of the Theorem, consider $g : [0,1]^2 \to \mathbb{R}$ a bounded continuous function, then for a.s. ω as ε tends to zero

$$\int_{[0,1]^2} g(t,s)\sqrt{[||\varphi||^{-1}\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s)]^2 + [||\varphi||^{-1}\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s)]^2}\, dt\, ds \to \int_{[0,1]^2} g(t,s)\Gamma(t,s) ds\, dt.$$

This can be proved by noting that if $I \times J$ is any rectangle included in $[0,1]^2$ the previous result, for $f \equiv 1$, is still true but with the integrals defined only in $I \times J$. Thus if $z(t,s)$ is a step function the above statement holds, we get then the result approximating uniformly g by step functions.
Now, to conclude the proof let us consider

$$\int_{[0,1]^2} f(W_\varepsilon(t,s))\sqrt{[||\varphi||^{-1}\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s)]^2 + [||\varphi||^{-1}\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s)]^2} ds\, dt.$$

We can show that the difference between this integral and the term

$$\int_{[0,1]^2} f(W(t,s))\sqrt{[||\varphi||^{-1}\sqrt{\varepsilon}\partial_t W_\varepsilon(t,s)]^2 + [||\varphi||^{-1}\sqrt{\varepsilon}\partial_s W_\varepsilon(t,s)]^2} ds\, dt$$

goes to zero when $\varepsilon \to 0$ for a.e. ω, and this last expression tends to the required limit, thus Theorem 2 holds. □

Remark. In [5] (pp. 106) Wschebor has shown that for each $x \in \mathbb{R}$ it holds $\sqrt{\varepsilon}||\varphi||^{-1}\mathcal{L}_\varepsilon(x) \to \tilde{\mathcal{L}}(x)$ in $L^p(\Omega)$.

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