Singular metrics on the two-sphere in space mechanics

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Abstract

Riemannian metrics on the sphere of revolution with an equatorial singularity are introduced so as to study dynamics arising in space mechanics and two-body control. A homotopy from the round metric on the sphere is defined which gives evidence of classification by the order of singularity for such metrics. Symmetry, integrability and quasi-homogeneity properties near the singularity unveil the main features of geodesics and allow to compute asymptotics, in connection with optimality issues. This question is fully addressed in the end of the paper where cut and conjugate loci are analyzed, providing examples of bifurcation and collapsing.

Keywords. Two-body control, almost-Riemannian metrics, homotopy, cut and conjugate loci

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1 Two-body control

The aim of this paper is to put in the same framework two problems arising from space mechanics and studied in previous papers: Bi-entry orbit transfer (see [5]), and a single-input version of the problem as well (see [6]). We achieve this goal using a singular or *almost-Riemannian* setting, thus completing the study of the regular (*i.e.* classical Riemannian) situation initiated in [7]. This approach is motivated by the use of homotopy methods [3] in optimal control. We first recall the two-body optimal control problem which models orbit transfer in the bi-entry case.

Consider a $1/r^2$ controlled field in the plane,

$$\ddot{q} = -\frac{q}{|q|^3} + u, \quad |u| \leq \eta,$$

where the norm $r = |q| = \sqrt{|q_1|^2 + |q_2|^2}$ of the position vector, $q \in \mathbf{R}^2$, measures the distance between the two bodies—in practice a planet, and a spacecraft—,

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whereas $u \in \mathbf{R}^2$ (bi-entry system) is the two-dimensional control—the acceleration provided by the spacecraft thruster. For obvious technological reasons, the norm of the control is bounded over, $|u| = \sqrt{|u_1|^2 + |u_2|^2} \leq \eta$. One criterion of interest for real space missions is minimization of the L^1-norm of control, which is connected with fuel consumption minimization,

$$\int_0^{t_f} |u| \mathrm{d}t \to \min,$$

the final or *transfer* time t_f being fixed. Since boundary conditions typically consist in prescribing initial and final orbits for the spacecraft, trajectories are indeed referred to as transfers. The state of the spacecraft can be described by position and velocity vectors, $(q, \dot{q}) \in \mathbf{R}^4$, or by using coordinates in dimension three which account for the geometry of the *osculating conic* (*i.e.* the conic on which the spacecraft is bounded to remain if the control vanishes), together with a scalar parameter defining position on the conic itself. Restricting to negative mechanical energy

$$E = \frac{\dot{q}^2}{2} - \frac{1}{|q|} < 0,$$

we consider only the case of *elliptic* orbits and choose any set of three parameters describing the osculating ellipse (*e.g.*, semi-major axis, eccentricity and argument of perigee), then adding *longitude*, $l \in \mathbf{S}^1$, to determine the angular position on the ellipse. In the sequel, we shall not distinguish between longitude and *cumulated longitude* which measures the total angular length of the trajectory and can be used as a new time. This leads to a so-called *periodic sub-Riemannian* form of the dynamics.

Replacing time by longitude, the system is rewritten according to

$$\frac{\mathrm{d}x}{\mathrm{d}l} = u_1 F_1(l, x) + u_2 F_2(l, x), \quad |u| \leq \eta,$$

where F_1 and F_2 are vector fields over the three dimensional manifold, X_e, of elliptic orbits, parameterized by longitude in $\mathbf{S}^1$. As the original four dimensional state manifold which was trivially fibered over $\mathbf{S}^1$ (one coordinate being longitude), the submanifold of elliptic orbits has a natural product structure,

$$X_e = \mathbf{R}_+^* \times \mathbf{D}.$$

Here, $\mathbf{D}$ is the Poincaré open unit disk, and we can choose coordinates such as *parameter*, $P \in \mathbf{R}_+^*$, and cartesian or polar coordinates on the disk, that is either *eccentricity vector*, (e_x, e_y), or *eccentricity* and *argument of perigee*, (e, θ). In such coordinates,

$$\begin{aligned} F_1 &= \frac{P^2}{W^2}\left(\sin l \frac{\partial}{\partial e_x} - \cos l \frac{\partial}{\partial e_y}\right), \\ F_2 &= \frac{P^2}{W^2}\left[\frac{2P}{W}\frac{\partial}{\partial P}\right. \\ &\quad \left. + \left(\cos l + \frac{e_x + \cos l}{W}\right)\frac{\partial}{\partial e_x} + \left(\sin l + \frac{e_y + \sin l}{W}\right)\frac{\partial}{\partial e_y}\right], \end{aligned}$$

with $W = 1 + e_x \cos l + e_y \sin l$. The criterion becomes

$$\int_0^{l_f} |u| \frac{\mathrm{d}l}{\omega(l, x)} \to \min$$

where l_f is the final (cumulated) longitude, and ω the pulsation of the system,

$$\mathrm{d}l = \omega(l,x)\mathrm{d}t, \quad \omega = \frac{W^2}{P^{3/2}}. \tag{1}$$

Up to this point, the system has two parameters, $\varepsilon = 1/l_f$ and $\alpha = \varepsilon/\eta$. The first one is to be regarded as a small parameter since there is a practical interest for long orbit transfers (*i.e.* with many revolutions of one body around the other), while the second is a *saturation* parameter (the smaller it is, the more unlikely the activation of the constraint of the control, $|u| \leq \eta$). Before giving the Hamiltonian of the problem in normal form, we add a last and third parameter, namely of homotopy parameter connecting the L^1-cost to a more tractable L^2-cost,

$$\int_0^{l_f} [(1-\lambda)|u|^2 + \lambda|u|]\frac{\mathrm{d}l}{\omega(l,x)} \to \min, \quad \lambda \in [0,1].$$

Pontryagin maximum principle then tells us that so-called *normal* minimizing trajectories are projections on the state manifold of curves in the cotangent bundle T^*X_e, which are integral curves of the following Hamiltonian with parameters.

Proposition 1. *The normal maximized Hamiltonian of the problem is*

$$H_{\lambda,\alpha,\varepsilon}(s,z) = \frac{1}{1-\lambda}\Theta_{\mu(l,x)}\left[\frac{\omega(\varepsilon^{-1}s,x)}{2}\left(|\psi|(\varepsilon^{-1}s,z) - \frac{\lambda}{\varepsilon\omega(\varepsilon^{-1}s,x)}\right)_+^2\right].$$

In the previous expression, $(.)_+$ means that only the positive part is retained, and a second time change in the form of a rescaling has been performed setting $s = l/l_f = \varepsilon l$, so longitude acts as a *fast time* while s is the new, *slow* time. The variable $z = (x,p)$ denotes a state-costate point of the cotangent bundle, while function ψ is defined from Hamiltonian lifts $H_i(l,z) = \langle p, F_i(l,x)\rangle$ of vector fields,

$$\psi(l,z) = (H_1, H_2)(l,z).$$

The function Θ_μ finally is a deformation of identity as $\Theta_\mu(y) = (1/\mu)\Theta(\mu y)$ with $\mu = \mu(l,x) = \alpha^2\omega(l,x)/(1-\lambda)^2$, and Θ defined by

$$\Theta(y) = y \text{ when } y < 1/2, \quad \Theta(y) = \sqrt{2y} - 1 \text{ when } y \geq 1/2.$$

In particular, $\Theta_0 = \mathrm{id}$ and $\sqrt{\mu}\Theta_\mu \to \sqrt{2y}$ when $\mu \to \infty$. The following Hamiltonians are readily recovered as special instances of the general case. For $\lambda = 1$,

$$H_{1,\alpha,\varepsilon}(s,z) = \frac{1}{\alpha}\left(|\psi| - \frac{1}{\varepsilon\omega}\right)_+(\varepsilon^{-1}s,z)$$

is the original Hamiltonian of L^1-minimization while, for $\lambda = 0$,

$$H_{0,\alpha,\varepsilon}(s,z) = \Theta_{\alpha^2\omega}\left(\frac{\omega}{2}|\psi|^2\right)(\varepsilon^{-1}s,z)$$

is associated with (constrained) L^2-minimization. A fundamental issue with respect to homotopy methods in optimal control is to devise a set a parameters

providing an integrable case. This task is fulfilled by setting $\lambda = \alpha = \varepsilon = 0$, that is considering the averaged Hamiltonian of (unconstrained) L^2-minimization,

$$\begin{aligned} H_0(z) &= \overline{\left(\frac{\omega}{2}|\psi|^2\right)}(z), \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\omega}{2}|\psi|^2(l,z)\mathrm{d}l, \end{aligned}$$

since classical averaging is indeed obtained as a weak limit when $\varepsilon \to 0$. Using *mean motion*, n, instead of parameter, P, and polar coordinates on $\mathbf{D}$, one actually gets the following.

Proposition 2. *The averaged Hamiltonian of unconstrained* L^2*-minimization in the bi-entry case is*

$$H_0(z) = \frac{9n^{1/3}}{2}p_n^2 + \frac{1}{2n^{5/3}}\left[\frac{5(1-e^2)}{2}p_e^2 + \frac{5-4e^2}{2e^2}p_\theta^2\right].$$

This computation is already present in [11], and a detailed study of trajectories is initiated in the subsequent paper [12]. The analysis is completed in [5] thanks to the appropriate Riemannian setting that we now recall.

As is clear from Proposition 2, H_0 is Liouville-integrable and associated to the Riemannian metric in orthogonal form

$$\mathrm{d}s^2 = \frac{\mathrm{d}n^2}{9n^{1/3}} + n^{5/3}\left[\frac{2}{5(1-e^2)}\mathrm{d}e^2 + \frac{2}{5-4e^2}e^2\mathrm{d}\theta^2\right].$$

Setting $n = (5r/2)^{6/5}$ and $e = \sin\varphi$, one gets

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \frac{r^2}{c^2}\left(\frac{\sin^2\varphi}{1-(1-\mu^2)\sin^2\varphi}\mathrm{d}\theta^2 + \mathrm{d}\varphi^2\right) \tag{2}$$

with $c = \sqrt{2/5}$ and $\mu = (\sqrt{5})^{-1}$. This metric has to be compared with the flat one in $\mathbf{R}^3$, and there are convexity issues related to the fact that $c < 1$ (we will come back on this when addressing the single-input case in §2). Changing coordinates from (e,θ) to (θ,φ) allows to extend the metric from $X_e = \mathbf{R}_+^* \times \mathbf{D}$ to $X = \mathbf{R}_+^* \times \mathbf{S}^2$, lifting the open manifold with boundary $\mathbf{D}$ to the two-sphere where (θ,φ) are the standard angular coordinates. This compactification step is crucial and will be discussed anew in §4.

By homogeneity in the r variable, analysis can be restricted to $\mathbf{S}^2$. The resulting metric on the sphere is shown to be a deformation of standard ones in §2. The deformation reveals a classification with respect to the order of singularity of appropriate metrics on $\mathbf{S}^2$ whose study encompasses bi-entry and single-input control of two bodies. Section 3 provides a detailed account of symmetry, homogeneity and asymptotic properties of such singular metrics on the sphere. All these aspects are connected with the optimality status of geodesics which is studied in §4 where the emphasis if finally laid on bifurcation and collapsing phenomena.

2 Construction of homotopy

The restriction of metric (2) to $\mathbf{S}^2$ writes

$$\mathrm{d}s^2 = \frac{\sin^2\varphi}{1-(1-\mu^2)\sin^2\varphi}\mathrm{d}\theta^2 + \mathrm{d}\varphi^2 \tag{3}$$

and is a deformation of the canonical metric of the ellipsoid. Indeed, we have the following.

Proposition 3. *The metric (3) associated with bi-entry two-body control is conformal to the canonical metric on an oblate ellipsoid of revolution with unit semi-major axis and semi-minor axis* $\mu = (\sqrt{5})^{-1}$.

Proof. Note that for any positive μ,

$$\frac{\sin^2\varphi}{1-(1-\mu^2)\sin^2\varphi}\mathrm{d}\theta^2 + \mathrm{d}\varphi^2 = [1-(1-\mu^2)\sin^2\varphi]^{-1} g_{\mathscr{E},\mu}$$

where

$$g_{\mathscr{E},\mu} = \sin^2\varphi\,\mathrm{d}\theta^2 + [\mu^2 + (1-\mu^2)\cos^2\varphi]\mathrm{d}\varphi^2 \tag{4}$$

is the aforementioned metric on the oblate ellipsoid of revolution as is clear when parameterizing the surface by

$$x = \sin\varphi\cos\theta, \quad y = \sin\varphi\sin\theta, \quad z = \mu\cos\varphi.$$

□

As we shall see in §4, this analogy with the oblate ellipsoid is quite strong and sheds some light not only on the optimality status of geodesics, but also on the way the sphere collapses when $\mu \to 0$.

Setting now $X = \sin^2\varphi$ and $\nu = 1-\mu^2$, we can also write (3)

$$\mathrm{d}s^2 = XR(\nu X)\mathrm{d}\theta^2 + \mathrm{d}\varphi^2,$$

the rational fraction R being the simplest possible with a pole at $X = 1$, that is at $\varphi = \pi/2$.

Proposition 4. *Setting*

$$g_\nu = XR(\nu X)\mathrm{d}\theta^2 + \mathrm{d}\varphi^2 \quad \textit{and} \quad R(X) = \frac{1}{1-X}\,,$$

where $X = \sin^2\varphi$*, defines a homotopy between the round metric on* $\mathbf{S}^2$ *(for* $\nu = 0$*) and a metric with an equatorial singularity (for* $\nu = 1$*). The metric (3) associated with bi-entry two-body control is obtained for* $\nu = 4/5$.

Before discussing the singular metric $g_1 = \tan^2\varphi\,\mathrm{d}\theta^2 + \mathrm{d}\varphi^2$ (see also [9] where the same metric shows up in the context of quantum control), we go back to orbit transfer and introduce the *tangential* single-input model. Consider the control of a $1/r^2$ field by a command whose direction is prescribed to be tangential to the trajectory, that is to say directed by the speed,

$$\ddot{q} = -\frac{q}{|q|^3} + u\frac{\dot{q}}{|\dot{q}|}\,, \quad |u| \le \eta.$$

The control is now scalar, $u \in \mathbf{R}$, and the same machinery as before applies. Reparameterizing by longitude, the system is written in periodic sub-Riemannian form

$$\frac{\mathrm{d}x}{\mathrm{d}l} = u_1 F_1(l,x), \quad |u| = |u_1| \leq \eta,$$

with, in coordinates (n, e, θ) on X_e,

$$F_1 = -\frac{3(1-e^2)w}{n^{1/3}(1+e\cos v)^2}\frac{\partial}{\partial n} + \frac{2(1-e^2)^2}{n^{4/3}(1+e\cos v)^2 w}\left[(e+\cos v)\frac{\partial}{\partial e} + \frac{\sin v}{e}\frac{\partial}{\partial \theta}\right]$$

and

$$v = l - \theta, \quad w = |(e_x + \cos l, e_y + \sin l)| = \sqrt{1 + 2e\cos(l-\theta) + e^2}.$$

The pulsation remains the same—compare with (1)—,

$$\mathrm{d}l = \omega(l,x)\mathrm{d}t, \quad \omega(l,x) = \frac{n(1+e\cos(l-\theta))^2}{(1-e^2)^{3/2}}.$$

It is then remarkable that averaging the unconstrained L^2-minimization Hamiltonian provides again an integrable system, still associated with a Riemannian metric in orthogonal form [6].

Proposition 5. *The averaged Hamiltonian of unconstrained* L^2*-minimization in the tangential single-input case is*

$$H_{t,0}(z) = \frac{9n^{1/3}}{2}p_n^2 + \frac{1}{2n^{5/3}}\left[\frac{4(1-e^2)^{3/2}}{1+\sqrt{1-e^2}}p_e^2 + \frac{4(1-e^2)}{(1+\sqrt{1-e^2})e^2}p_\theta^2\right].$$

The corresponding Riemannian metric on $X_e = \mathbf{R}_+^* \times \mathbf{D}$ is

$$\mathrm{d}s^2 = \frac{\mathrm{d}n^2}{9n^{1/3}} + n^{5/3}\left[\frac{1+\sqrt{1-e^2}}{4(1-e^2)^{3/2}}\mathrm{d}e^2 + \frac{1+\sqrt{1-e^2}}{4(1-e^2)}e^2\mathrm{d}\theta^2\right],$$

which we rewrite

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \frac{r^2}{c_t^2}\left[\frac{\sin^2\varphi\,(2-\sin^2\varphi)^2}{4\cos^4\varphi}\mathrm{d}\theta^2 + \mathrm{d}\varphi^2\right], \tag{5}$$

setting again $n = (5r/2)^{6/5}$, but changing the way the Poincaré disk is lifted to the sphere when replacing e by φ. Actually, the appropriate change of coordinate turns here to be

$$e = \sin\varphi\sqrt{1+\cos^2\varphi},$$

while the new constant is $c_t = 2/5$.

Remark 1. In comparison with the bi-entry case, $c_t = c^2$, so $c_t < c < 1$ and both three-dimensional metrics on $X = \mathbf{R}_+^* \times \mathbf{S}^2$ are non-convex, as illustrated in *meridian half-planes*, $\{\theta = \mathrm{cst}\}$. Since θ is cyclic, p_θ is constant and choosing $p_\theta = 0$ generates trajectories in such planes. There, the two-dimensional restricted metric writes in both cases

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\psi^2$$

with either $\psi = \varphi/c$ or $\psi = \varphi/c_t$, and the metric is actually flat. Geodesics are straight lines in a non-convex angular sector of the plane which is either $(-\pi/(2c), \pi/(2c))$ or $(-\pi/(2c_t), \pi/(2c_t))$. The smaller the constant, the higher the non-convexity, accounting for the relation between the loss of existence for boundary conditions with too different eccentricities (see [5]for a detailed analysis) and the number of inputs available to control the system.

As before, we restrict to $\mathbf{S}^2$ by homogeneity and consider

$$\mathrm{d}s^2 = \frac{\sin^2\varphi\,(2-\sin^2\varphi)^2}{4\cos^4\varphi}\mathrm{d}\theta^2 + \mathrm{d}\varphi^2. \tag{6}$$

The following clearly holds.

Proposition 6. *Setting*

$$g_\nu = XR(\nu X)\mathrm{d}\theta^2 + \mathrm{d}\varphi^2 \quad \textit{and}$$

$$R(X) = \left(\frac{1-X/2}{1-X}\right)^2 = \frac{1}{4}\left[1 + \frac{2}{1-X} + \frac{1}{(1-X)^2}\right],$$

where $X = \sin^2\varphi$, defines a homotopy between the round metric on $\mathbf{S}^2$ (for $\nu = 0$) and the equatorially singular metric (6) associated with tangential single-input two-body control (for $\nu = 1$).

We are thus led to consider the following family of metrics with singularity on the sphere. For $p \geq 1$, let

$$R(X) = \sum_{n=0}^{p} \frac{a_n}{(1-X)^n}\,, \quad a_0, \ldots, a_{n-1} \geq 0, \quad a_n > 0, \tag{7}$$

be a rational fraction with a pole of finite order p at $X = 1$. Without loss of generality, we assume that $R(0) = 1$. As revealed by homotopy from the round metric, $p = 1$ in the bi-entry case while $p = 2$ in the tangential single-input case, and there is a classification according to the order of the singularity. Define

$$g = XR(X)\mathrm{d}\theta^2 + \mathrm{d}\varphi^2 \quad \text{where} \quad X = \sin^2\varphi. \tag{8}$$

It is a Riemannian metric on both open hemispheres of $\mathbf{S}^2$ with a singularity at the equator $\{\varphi = \pi/2\}$, and it defines a metric—a so-called *almost-Riemannian* one [2]—on the whole sphere as is clear from next proposition.

Proposition 7. *Let $\Gamma(\varphi) = (XR(X))^{-1}$, $X = \sin^2\varphi$. The two vector fields*

$$F_1 = \sqrt{\Gamma(\varphi)}\frac{\partial}{\partial\theta}\,, \quad F_2 = \frac{\partial}{\partial\varphi}$$

define a complete sub-Riemannian[1] *metric on $\mathbf{S}^2$.*

Proof. Readily, $(\mathrm{ad}^k F_2)F_1 = (\sqrt{\Gamma})^{(k)}\partial/\partial\theta$ which is not zero at $\varphi = \pi/2$ for some order of derivation since $\sqrt{\Gamma}$ is not flat as R has a finite order pole at $X = 1$. So the distribution generated by F_1 and F_2 verifies the Hörmander condition at any point of the compact manifold $\mathbf{S}^2$, including the singular set (the equator). □

[1] Carnot-Carathéodory [4].

The distance between two arbitrary points of the sphere is thus defined, up to a multiplicative constant, as the value of the optimal control problem with dynamics

$$\dot{x} = u_1 F_1(x) + u_2 F_2(x),$$

cost functional

$$\int_0^{t_f} |u|^2 \mathrm{d}t \to \min$$

with final time t_f fixed, and boundary conditions the two points. Equivalently, one can consider the minimum time trajectory between the points with respect to

$$\dot{x} = u_1 F_1(x) + u_2 F_2(x), \quad |u| \leq 1.$$

Classification by the order is also reflected by transcendency of geodesics, as will be clear from next section.

3 Symmetry, quasi-homogeneity, asymptotics

Given a metric defined by (7-8), we restrict to the level set $\{H = 1/2\}$ of the Hamiltonian associated to the metric,

$$H = \frac{1}{2}(\Gamma(\varphi)p_\theta^2 + p_\varphi^2), \quad \Gamma = (XR(X))^{-1}, \quad X = \sin^2\varphi,$$

and so parameterize extremals by arc length. Consider the extremal departing from $\varphi_0 \neq 0\,(\pi)$ (not a pole), θ_0 being normalized to 0 (cyclic variable) and defined by a positive p_θ (the degenerate case $p_\theta = 0$ corresponding to meridians—which are the only extremals passing through the poles) and non-negative $p_{\varphi 0} = \sqrt{1 - \Gamma(\varphi_0)p_\theta^2}$. Along the extremal, $\dot{\varphi}$ first vanishes when $\varphi = \varphi_1 = \pi - \Gamma^{-1}(p_\theta^{-2})$, Γ being a one-to-one (strictly decreasing) mapping between $(0, \pi/2)$ and $\mathbf{R}_+^*$ by the definition (7) of R. As Γ' does not vanish,

$$1 - \Gamma(\varphi)p_\theta^2 = O(\varphi_1 - \varphi)$$

in the neighbourhood of $\pi - \Gamma^{-1}(p_\theta^{-2})$, and the following integral is well-defined,

$$t_1(p_\theta, \varphi_0) = \int_{\varphi_0}^{\pi - \Gamma^{-1}(p_\theta^{-2})} \frac{\mathrm{d}\varphi}{\sqrt{1 - \Gamma(\varphi)p_\theta^2}}.$$

Lemma 1. *The axial symmetry σ_1 with respect to $\theta_1 = \theta(t_1)$ is an inner symmetry of the extremal.*

Proof. Set

$$\begin{array}{rclcrcl} \hat{\theta} & = & 2\theta_1 - \theta(2t_1 - t), & & \hat{p}_\theta & = & p_\theta, \\ \hat{\varphi} & = & \varphi(2t_1 - t), & & \hat{p}_\varphi & = & -p_\varphi(2t_1 - t), \end{array}$$

and check that new curve is still an extremal, passing through the same point of the cotangent bundle at t_1 since $p_\varphi(t_1) = 0$. □

Necessarily, $\pi - \Gamma^{-1}(p_\theta^{-2}) \geq \pi - \varphi_0$, so there also exists $t_1' \leq t_1$ such that $\varphi(t_1') = \pi - \varphi_0$. Using the previous axial symmetry, we deduce the existence of $t_2 = 2t_1 - t_1' \geq t_1$ such that, again, $\varphi(t_2) = \pi - \varphi_0$. Using now the equatorial symmetry of Γ,

$$\Gamma(\pi - \varphi) = \Gamma(\varphi),$$

the following is clear.

Lemma 2. *The central symmetry s_2 with respect to $(\theta_2/2, \pi/2)$ defines another extremal with same initial condition.*

Proof. Set

$$\begin{array}{rclcrcl} \hat{\theta} & = & \theta_2 - \theta(t_2 - t), & & \hat{p}_\theta & = & p_\theta, \\ \hat{\varphi} & = & \pi - \varphi(t_2 - t), & & \hat{p}_\varphi & = & p_\varphi(t_2 - t). \end{array}$$

The new curve is still an extremal since

$$\dot{\hat{\theta}} = \Gamma(\pi - \hat{\varphi})p_\theta = \Gamma(\hat{\varphi})\hat{p}_\theta, \quad \dot{\hat{p}}_\varphi = \frac{1}{2}\Gamma'(\pi - \hat{\varphi})p_\theta^2 = -\frac{1}{2}\Gamma'(\hat{\varphi})\hat{p}_\theta^2,$$

and $\hat{\theta}_0 = 0 = \theta_0$, $\hat{\varphi}_0 = \pi - (\pi - \varphi_0) = \varphi_0$. □

Finally denote t_3 the point such that $\varphi(t_3) = \pi/2 \leq \pi - \varphi_0$, and remark that the central symmetry σ_2 with respect to $(\theta(t_3), \varphi(t_3))$ leaves the extremal invariant. Since the axial symmetry s_1 with respect to $\theta = 0$ obviously defines another extremal with same initial condition, we conclude that the group generated by s_1 and s_2 acts on the set of extremals with same initial condition, while the group generated by σ_1 and σ_2 defines inner symmetries of extremals. The composition rules indicate in both cases that the underlying group is the four-order abelian Klein group,

$$\mathbf{V} = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \simeq \{\mathrm{id}, s_1, s_2, s_1 s_2\} \simeq \{\mathrm{id}, \sigma_1, \sigma_2, \sigma_1\sigma_2\}.$$

Proposition 8. *Given any initial condition, the Klein group acts on the set of extremals issuing from the point. It also defines inner symmetries of any extremal.*

An extremal is said to be a *pseudo-equator* whenever $\dot{\varphi}(0) = p_\varphi(0)$ is equal to zero, whereas the equator itself cannot be an extremal since $\Gamma(\pi/2) = 0$ because of the singularity.

Lemma 3. *Every extremal which is not a meridian is a pseudo-equator.*

Proof. For p_θ positive and $p_{\varphi 0}$ nonnegative (the other cases are deduced by symmetry), there exists $\widetilde{\varphi}_0 = \Gamma^{-1}(p_\theta^{-2})$ such that, up to a time shift, the extremal is the pseudo-equator with initial condition $\widetilde{\varphi}_0$. □

Conversely, any pseudo-equator meets $\varphi = \pi/2$ as one understands from the analysis of symmetries. Taking $\widetilde{\varphi}_0 = \pi/2$ as new initial condition and retaining the same value for p_θ provides the same geodesic, up to a time shift again. As a result, rather than parameterizing extremals using both φ_0—we set $\theta_0 = 0$ thanks to the symmetry of revolution—and p_θ, one may either parameterize by $\varphi_0 \in (0, \pi/2)$ alone using the fact all geodesics (with the exception of meridians,

$p_\theta = 0$) are pseudo-equators (then, implicitly, $p_\theta^2 = 1/\Gamma(\varphi_0)$, $\varphi_0 \neq \pi/2$ since the equator is not a geodesic), or parameterize by their *Clairaut constant* $p_\theta \in \mathbf{R}_+^*$, considering only the initial condition at singularity, $\varphi_0 = \pi/2$. The second point of view reduces the study of geodesics to those starting at singularity.

Proposition 9. *On every extremal, coordinate φ is periodic with period*

$$T(p_\theta) = 4 \int_{\Gamma^{-1}(p_\theta^{-2})}^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - \Gamma(\varphi) p_\theta^2}},$$

and $\theta(t + T) = \theta(t) \pm \Delta\theta$ (the sign depending on the sign of p_θ) with

$$\Delta\theta(p_\theta) = 4 \int_{\Gamma^{-1}(p_\theta^{-2})}^{\pi/2} \frac{\Gamma(\varphi) p_\theta \mathrm{d}\varphi}{\sqrt{1 - \Gamma(\varphi) p_\theta^2}}.$$

Proof. According to the previous analysis, it is enough to check the result on pseudo-equators. But then, $t_1 = t_2 = t_3 = 2t_4$, so setting $T = 2t_1$ and using the axial symmetry with respect to θ_1 gives the result since $\varphi(T) = \varphi(0)$, $p_\varphi(T) = -p_\varphi(0) = 0 = p_\varphi(0)$. Hence $\dot{\theta} = \Gamma(\varphi) p_\theta$ is also periodic, which concludes the proof. □

As functions of φ_0,

$$T(\varphi_0) = 4 \int_{\varphi_0}^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - \Gamma(\varphi)/\Gamma(\varphi_0)}}, \tag{9}$$

and

$$\Delta\theta(\varphi_0) = 4 \int_{\varphi_0}^{\pi/2} \frac{\Gamma(\varphi) \mathrm{d}\varphi}{\sqrt{\Gamma(\varphi_0) - \Gamma(\varphi)}}. \tag{10}$$

These relations actually cover the case of meridians $p_\theta = 0$ (*i.e.* $\varphi_0 = 0$) for which $T = 2\pi$ and $\Delta\theta = 2\pi$ (two instantaneous rotations of angle π when crossing poles at $t = \pi$ and $t = 2\pi$). The limits when approaching the equator are also known.

Proposition 10. *Both T and $\Delta\theta$ vanish when $|p_\theta| \to \infty$ (*i.e. *$\varphi_0 \to \pi/2$).*

Proof. Directly follows from estimates of integrals (9-10) using the fact that, because of (7), Γ does not vanish identically at $\varphi = \pi/2$. □

Refined estimates of T and $\Delta\theta$ will be given in the end of the section. We now focus of quasi-homogeneity properties of the local model at singularity. Setting $x = \pi/2 - \varphi$ and $y = \theta$, since $1 - \sin^2\varphi \sim (\pi/2 - \varphi)^2$ when φ tends to $\pi/2$, a local model for the metric (8) is

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \frac{\mathrm{d}y^2}{x^{2p}} \tag{11}$$

where p is the order of the pole. The equatorial symmetry of Γ is approximated by $(-x)^{2p} = x^{2p}$, so the discrete symmetry group is preserved. Such almost-Riemannian metrics are related to sub-Riemannian distributions. For $p = 1$, the local model is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2/x^2$, and the metric is actually obtained by projecting the *Heisenberg* sub-Riemannian distribution [8]. This distribution is

indeed defined, up to a renormalization, by the following two vector fields on $\mathbf{R}^3$,

$$F_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}\,, \quad F_2 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\,,$$

and the corresponding sub-Riemannian Hamiltonian is

$$H = \frac{1}{2}\left[(p_x^2 + p_y^2) + 2p_z(xp_y - yp_x) + (x^2+y^2)p_z^2\right],$$

which strongly suggests to use cylindrical coordinates. In these variables,

$$H = \frac{1}{2}\left[p_r^2 + (p_\theta/r + rp_z)^2\right].$$

As θ and z are cyclic, the system is integrable in dimension three, and projects onto a Hamiltonian in the (r,z)-space with the desired singularity,

$$H = \frac{1}{2}(p_r^2 + r^2p_z^2),$$

when restricting to $p_\theta = 0$. For $p = 2$, the local model is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2/x^4$, which is connected to the *flat Martinet* sub-Riemannian distribution. Consider indeed the two vector fields on $\mathbf{R}^3$ (see [1])

$$F_1 = \frac{\partial}{\partial x} + y^2\frac{\partial}{\partial z}\,, \quad F_2 = \frac{\partial}{\partial y}\,,$$

so the sub-Riemannian Hamiltonian is

$$H = \frac{1}{2}\left[p_y^2 + (p_x + y^2p_z)^2\right].$$

The two coordinates x and z are cyclic, and the Hamiltonian projects onto $H = (1/2)(p_y^2 + y^4p_z^2)$ in the (y,z)-space when restricting to $p_x = 0$, providing the higher order singularity.

Going back to the general case, we restrict the computation to geodesics issuing from the origin on the level set $\{H = 1/2\}$ of the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + x^{2p}p_y^2)$$

so that the initial adjoint state belongs to the union of the two lines, $\{p_x = \pm 1\}$. We set $\lambda = p_y$ and restrict to positive λ by symmetry (the trivial geodesics $(\pm t, 0)$ being obtained for $\lambda = 0$). The coordinate x is then

$$x = \frac{1}{\sqrt[p]{\lambda}}q(t\sqrt[p]{\lambda})\,,$$

where q is defined through the quadrature

$$q^{-1}(u) = \int_0^u \frac{\mathrm{d}v}{\sqrt[p]{1-v^{2p}}}\cdot$$

For $p = 1$, q is harmonic, elliptic for $p = 2$, hyperelliptic and reciprocal to an hypergeometric function in general. More precisely,

$$q^{-1}(u) = u\cdot\,{}_2\mathbf{F}_1(1/2, 1/(2p); 1 + 1/(2p); u^{2p}) \tag{12}$$

where ${}_2\mathbf{F}_1(a,b;c;z)$ is the hypergeometric series

$$ {}_2\mathbf{F}_1(a,b;c;z) = \sum_{n\geq 0} \alpha_n \frac{z^n}{n!}\,, $$

and

$$ \begin{array}{rcl} \alpha_0 & = & 1, \\ \alpha_{n+1}/\alpha_n & = & (n+a)(n+b)/(n+c). \end{array} $$

The reciprocal of q is hence equal to

$$ q^{-1}(u) = \sum_{n\geq 0} \alpha_n \frac{u^{2np+1}}{n!} $$

with $\alpha_n = (a)_n(b)_n/(c)_n$ for $a = 1/2$, $b = 1/(2p)$ and $c = 1 + 1/(2p)$, the notation $(a)_n$ standing for the Pochhammer symbol

$$ (a)_n = a(a+1)\cdots(a+n-1). $$

Here,

$$ \begin{array}{rcl} \alpha_0 & = & 1, \\ \alpha_1 & = & \dfrac{1}{2}\dfrac{1}{2p+1}\,, \\ \alpha_2 & = & \dfrac{3}{4}\dfrac{2p+1}{8p^2+6p+1}\,, \\ \alpha_3 & = & \dfrac{15}{8}\dfrac{8p^2+6p+1}{48p^3+44p^2+12p+1}\,, \\ \ldots & & \end{array} $$

which gives the usual Taylor series of the reciprocal of the sine function for $p = 1$, $\operatorname{asin} u = u + u^3/6 + 3x^5/40 + 5x^7/112...$

Eventually, $\dot{y} = \lambda x^{2p}$, so

$$ y = \frac{1}{(\sqrt[p]{\lambda})^{p+1}} r(t\sqrt[p]{\lambda}), $$

where r is defined by a second quadrature,

$$ r(t) = \int_0^t q^{2p}\mathrm{d}s. $$

In the Heisenberg case ($p = 1$), one has

$$ q = \sin t, \quad r = \frac{t}{2} - \frac{1}{4}\sin 2t. $$

In the flat Martinet case ($p = 2$), one has (see [1])

$$ q = -\mathrm{cn}(t\sqrt{2} + K), \quad r = (3\sqrt{2})^{-1}[t\sqrt{2} + 2\,\mathrm{sn}\,\mathrm{cn}\,\mathrm{dn}(t\sqrt{2} + K(k), k)], $$

where cn, sn and dn are Jacobi elliptic functions, here for the parameter $k = 1/\sqrt{2}$. The constant K is the complete elliptic integral of the first kind for the same value of the parameter,

$$ K(k) = \int_0^1 \frac{\mathrm{d}v}{\sqrt{1-v^2}\sqrt{1-k^2v^2}}. $$

The transcendency of geodesics of the global metric on $\mathbf{S}^2$ is similar. Namely, the quadrature on φ is

$$\pm \mathrm{d}t = \frac{\mathrm{d}\varphi}{\sqrt{1 - p_\theta^2 \Gamma(\varphi)}}, \tag{13}$$

which essentially amounts to

$$\pm\, \mathrm{d}t = \frac{\mathrm{d}X}{\sqrt{P(X, p_\theta)}}$$

where P is a polynomial in $X = \sin^2 \varphi$ and p_θ. The transcendency is then given by the genus, $\mathbf{g}$, of the Riemann surface associated with the algebraic curve $Y^2 = P(X, p_\theta)$ parameterized by the adjoint vector. In the bi-entry case, $p = 1$,

$$P(X, p_\theta) = 4(1 - X)(X - p_\theta^2(1 - X)),$$

so $\mathbf{g} = 0$, and we have harmonic functions. In the tangential single-input case where $p = 2$,

$$P(X, p_\theta) = (1 - X)(X(2 - X)^2 - 4p_\theta^2(1 - X)^2),$$

so $\mathbf{g} = 1$, and we have elliptic functions. When $p \geq 4$, $\mathbf{g} = [p/2]$, and hyper-elliptic functions are to be considered.

For any $p \geq 1$, symmetry reasons imply that non-trivial minimizing geodesics emanating from the origin first intersect on the y-axis. The *cut locus* at the origin, that is the set of points where geodesics cease to be minimizing, is the axis minus the origin itself. The *conjugate locus* is the set of first critical values of the exponential mapping on $\{H = 1/2\}$,

$$\exp_t(\lambda) = \exp_{(0,0),t}(\lambda) = (x(t, \lambda), y(t, \lambda)).$$

Let s_p denote the first positive root of

$$qr' - (p + 1)q'r = 0. \tag{14}$$

Because of quasi-homogeneity with respect to λ, first conjugate times are readily $t_{1c}(\lambda) = s_p / \sqrt[p]{\lambda}$. Since the conjugate locus can obviously not be contained into one of the two axes, the following holds.

Lemma 4. *The conjugate locus at the origin of* $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2/x^{2p}$ *is the set* $y = \pm C_p x^{p+1}$ *minus the origin where*

$$C_p = \frac{1}{p + 1} \sqrt{\frac{q^{2p}(s_p)}{1 - q^{2p}(s_p)}}$$

is nonzero, s_p *being the first positive root of (14),* q *defined by (12).*

Remark 2. The Hamiltonian of the local model (11) is

$$H = \frac{1}{2}(p_x^2 + x^k p_y^2), \quad k = 2p \geq 2,$$

so $\dot{x}^2 + \lambda x^k = 1$ on $\{H = 1/2\}$ (still with $\lambda = p_y^2$) and

$$\ddot{x} = -\frac{\lambda}{2} k x^{k-1}.$$

The associated variational equation is

$$\delta\ddot{x} = -\frac{\lambda}{2}k(k-1)x(t)^{k-2}\delta x,$$

which may alternatively be reduced to a linear second order differential equation using the so-called *Yosida transformation*. It consists in the time change $s = x^k(t)$, and denoting $' = \mathrm{d}/\mathrm{d}s$ one gets

$$s(1-\lambda s)\delta x'' + \eta[1-(1+\frac{1}{2\eta})\lambda s]\delta x' + \eta\frac{\lambda}{2}\delta x = 0, \quad \eta = \frac{k-1}{k}\cdot$$

See [10] where the same Jacobi equation is investigated using this transformation.

As a byproduct of the previous analysis, we get the following result.

Proposition 11. *For a singular metric on the sphere defined by (7-8), the conjugate locus of a point on the equator has two contacts of order $p+1$ with the meridian passing through the point.*

We end the section by providing asymptotics of T and $\Delta\theta$ in the neighbourhood of meridians, when $p_\theta \to 0$, and in the neighbourhood of the equator, when $p_\theta \to \infty$ (we only consider $p_\theta \geq 0$ because of symmetries). In the first case, the asymptotics measure how fast the metric (8) converges to the round one since both metrics restrict to $\mathrm{d}s^2 = \mathrm{d}\varphi^2$ along meridians—the only geodesics always shared by the two of them. Conversely, as T and $\Delta\theta$ vanish when $p_\theta \to \infty$ by Proposition 10, geodesics converge to the equator so the asymptotics tell us how fast cut points (see §4) accumulate towards the initial point, $\varphi_0 = \pi/2$ (as previously mentioned, parameterization by p_θ allows to restrict to geodesics starting at singularity). One may then consider that the equator is retrieved as a geodesic by augmenting the level set $\{H = 1/2\}$. When $\varphi_0 = \pi/2$, it is the union of two lines, $\{p_\theta \in \mathbf{R},\ p_{\varphi_0} = \pm 1\}$, and in contrast to the regular case discussed in [7], the singularity moves away to infinity the Clairaut constant of the equator so two points at infinity must be added to the level set, $\{p_\theta = \pm\infty,\ p_{\varphi_0} = 0\}$.

Next result implies that computing asymptotics of T suffices to obtain those of $\Delta\theta$.

Lemma 5. $\Delta\theta' = T'/p_\theta$.

Proof. Write as in [7]

$$T = p_\theta\Delta\theta + 4\int_{\Gamma^{-1}(p_\theta^{-2})}^{\pi/2}\sqrt{1-\Gamma(\varphi)p_\theta^2}\,\mathrm{d}\varphi,$$

so

$$\begin{aligned} T' &= \Delta\theta + p_\theta\Delta\theta' + 4\int_{\Gamma^{-1}(p_\theta^{-2})}^{\pi/2}\frac{\partial}{\partial p_\theta}\left(\sqrt{1-\Gamma(\varphi)p_\theta^2}\right)\mathrm{d}\varphi \\ &\quad -8(\Gamma^{-1})'(p_\theta^{-2})p_\theta\underbrace{\sqrt{1-\Gamma(\varphi)p_\theta^2}\,|_{\varphi=\Gamma^{-1}(p_\theta^{-2})}}_{0} \\ &= p_\theta\Delta\theta'. \end{aligned}$$

□

In the bi-entry case, $p = 1$ with $R = (1 - X)^{-1}$, and we get the quadratures hereafter,

$$T = \frac{2\pi}{\sqrt{1 + p_\theta^2}}, \quad \Delta\theta = 2\pi \left(1 - \frac{p_\theta}{\sqrt{1 + p_\theta^2}}\right). \tag{15}$$

In the tangential single-input case, $p = 2$ with $R = (1 - X/2)^2(1 - X)^{-2}$, and

$$T = \frac{4}{\sqrt{A_2 B_1}} \left[\Pi(\nu, k) + \frac{2 - p}{p - q} K(k)\right] \tag{16}$$

where K and Π are respectively complete elliptic integrals of first and third kind,

$$\Pi(\nu, k) = \int_0^1 \frac{\mathrm{d}v}{(1 - \nu v^2)\sqrt{1 - v^2}\sqrt{1 - k^2 v^2}},$$

with

$$\Delta = 4(\beta - \alpha)(\beta - \delta)(\gamma - \alpha)(\gamma - \delta), \quad \sigma = (\alpha + \delta)(\beta + \gamma) - 2(\alpha\delta + \beta\gamma),$$

$$l_1 = \frac{\sigma - \sqrt{\Delta}}{(\beta - \gamma)^2}, \quad l_2 = \frac{\sigma + \sqrt{\Delta}}{(\beta - \gamma)^2},$$

$$p = \frac{(\alpha + \delta) - l_1(\beta + \gamma)}{2(1 - l_1)}, \quad q = \frac{(\alpha + \delta) - l_2(\beta + \gamma)}{2(1 - l_2)},$$

$$A_1 = -l_2 \frac{1 - l_1}{l_2 - l_1}, \quad B_1 = -l_1 \frac{1 - l_2}{l_2 - l_1}, \quad A_2 = \frac{1 - l_1}{l_2 - l_1}, \quad B_2 = \frac{1 - l_2}{l_2 - l_1},$$

$$a = \sqrt{\frac{A_2}{B_2}}, \quad b = \sqrt{\frac{A_1}{B_1}}, \quad k = \frac{b}{a}, \quad \nu = b^2.$$

Remark 3. For the sake of completeness, we also provide the quadrature on φ in the tangential single-input case, which subsumes the computation of T (for the bi-entry case, see [5] from where computations in the singular case are easily deduced). According to (13), with the same notation as before,

$$\begin{aligned} \pm t \;=\; & \frac{1}{2\sqrt{A_2 B_1}} \left[\Pi(v, \nu, k) + \frac{2 - p}{p - q} \mathrm{sn}^{-1}(v, k)\right. \\ & \left. + \sqrt{A_2 B_1}\, \mathrm{atan}\left(\sqrt{A_1 A_2}\sqrt{(1 - v^2)(1 - k^2 v^2)} - \sqrt{B_1 B_2}(1 - \nu v^2)\right)\right]_v^1, \end{aligned}$$

where the elliptic integral of third kind is now incomplete, and where

$$v = b^{-1} \frac{X - q}{p - X} \in [-1, 1] \quad \text{with} \quad X = \sin^2 \varphi.$$

Similarly,

$$\Delta\theta = \frac{4 p_\theta}{\sqrt{A_2 B_1}} \left[\frac{2\Pi(\kappa, k)}{pq} - \frac{2\Pi(\mu, k)}{(2 - p)(2 - q)} + \frac{4(1 - p)^2}{p(2 - p)(p - q)} K(k)\right] \tag{17}$$

with, moreover,

$$c = \frac{q}{p}, \quad d = \frac{2 - q}{2 - p}, \quad \kappa = \frac{\nu}{c^2}, \quad \mu = \frac{\nu}{d^2}.$$

Here before, α, β, γ and $\delta = 1$ are roots of the degree four polynomial $P(X, p_\theta)$ involved in the computation,

$$P(X, p_\theta) = (1 - X)(X(2 - X)^2 - 4p_\theta^2(1 - X)^2).$$

We just need expansions of these roots in the neighbourhood of $p_\theta = 0$ and $p_\theta = \infty$. Such expansions are available in $\sqrt{\varepsilon}$-scale as both

$$Q(X, \varepsilon) = X(2 - X)^2 - 4\varepsilon(1 - X)^2 \quad \text{and} \quad \widetilde{Q}(X, \varepsilon) = 4(1 - X)^2 - \varepsilon X(2 - X)^2$$

possess either simple or order two roots for $\varepsilon = 0$.

Lemma 6. *When $p_\theta \to 0$,*

$$\alpha = p_\theta^2 + o(p_\theta^2),$$

$$\beta = 2 - p_\theta\sqrt{2} + \frac{3}{2}p_\theta^2 + o(p_\theta^2), \quad \gamma = 2 + p_\theta\sqrt{2} + \frac{3}{2}p_\theta^2 + o(p_\theta^2).$$

When $p_\theta \to \infty$,

$$\alpha = 1 - \frac{1}{2}p_\theta^{-1} - \frac{1}{8}p_\theta^{-2} + o(p_\theta^{-2}), \quad \beta = 1 + \frac{1}{2}p_\theta^{-1} - \frac{1}{8}p_\theta^{-2} + o(p_\theta^{-2}),$$

$$\gamma = 4p_\theta^2 + 2 + \frac{1}{4}p_\theta^{-2} + o(p_\theta^{-2}).$$

Plugging these expressions into quadrature (16) and using Lemma 5 leads to the desired asymptotics, those in the bi-entry case being obvious.

Proposition 12. *In the neighbourhood of meridians,*

$$T \sim 2\pi(1 - \frac{1}{2}p_\theta^2), \quad \Delta\theta \sim 2\pi(1 - p_\theta), \quad p_\theta \to 0,$$

in the by-entry case, and

$$T \sim 2\pi(1 - \frac{3\sqrt{2}}{8}p_\theta^2), \quad \Delta\theta \sim 2\pi(1 - \frac{3\sqrt{2}}{4}p_\theta), \quad p_\theta \to 0,$$

in the tangential single-input case. In the neighbourhood of the equator,

$$T \sim \frac{2\pi}{p_\theta}, \quad \Delta\theta \sim \frac{\pi}{p_\theta^2}, \quad p_\theta \to \infty,$$

in the by-entry case, and

$$T \sim 4(2-\sqrt{2})K(3-2\sqrt{2})p_\theta^{-1/2}, \quad \Delta\theta \sim \frac{4}{3}(2-\sqrt{2})K(3-2\sqrt{2})p_\theta^{-3/2}, \quad p_\theta \to \infty,$$

in the tangential single-input case.

The influence of the order on the rate of accumulation of cut points is clear, and the matter of cut and conjugate loci is addressed in details in last section. The condition for a geodesic to be closed is $\Delta\theta \in \pi\mathbf{Q}$ (rationality of $\Delta\theta/\pi$), so that asymptotics when $p_\theta \to \infty$ also measure the density of closed curves in the neighbourhood of the equator. This is related to optimality conditions, too, through injectivity radius (see [5] in the Riemannian case).

4 Bifurcation and collapsing

Singular metrics (8) are complete but not Riemannian on the whole sphere while Riemannian but incomplete on hemispheres, so standard arguments of global Riemannian geometry have to be adapted. The structure result hereafter holds unchanged, where *separating lines* are sets of intersections of distinct minimizing geodesics with same initial condition.

Proposition 13. *Cut points of a singular metric on the sphere defined by (7-8) are either conjugate points, or points in separating lines.*

Proof. First assume $\varphi_0 \neq \pi/2$. If the cut point is not a conjugate point, the exponential mapping is a diffeomorphism in the neighbourhood of the time, t_l, and adjoint vector which generate the cut point. Since the metric is complete by Proposition 7, there are minimizing extremals γ_n joining the initial point to $\gamma(t_l+1/n)$, $n \geq 1$, where γ is the curve in the state space defining the cut point, $\gamma(t_l)$. As $\varphi_0 \neq \pi/2$, the set $\{p = (p_\theta, p_\varphi) \,|\, H(0, \varphi_0, p) = 1/2\}$ is compact, and one can extract a converging subsequence of the $(p_n)_n$ generating the extremals γ_n, and thus get the standard contradiction [13].

When $\varphi_0 = \pi/2$, though $\{p \,|\, H = 1/2\} = \mathbf{R} \times \{p_\varphi = \pm 1\}$ is not bounded anymore, $(p_{\theta_n})_n$ still has to be bounded otherwise there would exist a subsequence such that $|p_{\theta_n}| \to \infty$, and $\gamma_n(t_l + 1/n)$ would tend to $(0, \pi/2)$ according to Lemma 10, that is to the initial point, not to the cut point $\gamma(t_l)$. The sequence being thus bounded, we can conclude as before. □

Conversely, geodesics of such metrics are extremals of a minimum time control problem (see §2), and results of optimal control ensure that optimality is lost after conjugate points [8]. Besides, extremals of such problems have to be smooth so that broken curves which are concatenations of minimizing geodesics cannot be minimizing, entailing that optimality classically cannot be preserved after points in separating lines.

Theorem 1. *Consider a singular metric on the sphere defined by (7-8). Under the assumption that $\Delta\theta$ is strictly decreasing for $p_\theta > 0$, cut loci are antipodal subarcs. The cut locus of a pole is reduced to the opposite pole, is equal to the equator minus the point for an equatorial point, and to a proper closed subarc of the antipodal parallel otherwise.*

Proof. The case of poles is obvious since the only extremals through them are meridians.

Consider now the situation $\varphi_0 = \pi/2$, and show that the exponential mapping is injective on the quadrant

$$D = \cup_{p_\theta > 0} [0, T(p_\theta/2)] \times \{p_\theta, 1\},$$

that is show that subarcs of extremals defined by $t \in [0, T(p_\theta)/2]$, positive p_θ and $p_\varphi = +1$ do not intersect. If $p'_\theta > p_\theta$, the arc associated with p'_θ is strictly below the one associated with p_θ. Indeed, note that on the first half of such an arc $t \in [0, T/4)$ and $\dot{\varphi}$ does not vanish so that the curve can be parameterized by φ instead of time. There,

$$f(\varphi, p_\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}\varphi} = \frac{\Gamma(\varphi) p_\theta}{\sqrt{1 - \Gamma(\varphi) p_\theta^2}}$$

is an increasing function of p_θ since

$$\frac{\partial f}{\partial p_\theta} = \frac{\Gamma(\varphi)}{(1-\Gamma(\varphi)p_\theta^2)^{3/2}} > 0.$$

As geodesics starting from $\varphi_0 = \pi/2$ cross again the equator at $\Delta\theta/2$, the assumption ensures that the aforementioned subarcs do not intersect. We conclude by remarking that the full set of extremals is obtained by considering the action of the Klein group on geodesics with same initial condition (see §3). First, the central symmetry s_2 which generates intersections at $t = T/2$, then the axial symmetry s_1 with respect to $\theta = 0$ which generates intersections at $\theta = \pi$, thus not prior to the previous ones since $\theta(T/2) = \Delta\theta/2$, and since $\Delta\theta < 2\pi$ for $p_\theta > 0$ (by assumption, $\Delta\theta$ is decreasing, and equal to 2π on meridians, *i.e.* when $p_\theta = 0$). So extremals are optimal up to $t = T/2$, and the corresponding point belongs to the separating line. Since the metric is complete, each point of the equator is reached by such an extremal and the separating line, hence the cut locus, is the equator minus the initial point itself.

Consider finally the case when the initial point is neither polar nor equatorial. Then p_θ^2 belongs to $(0, 1/\Gamma(\varphi_0))$, and extremals are again optimal up to $t = T/2$. Indeed, there would otherwise exist shorter extremals which would lead to the existence of shorter extremals for the initial condition $\varphi_0 = \pi/2$ too, contradicting the previous fact. The central symmetry s_2 still generates an intersection at $t = T/2$, and $\varphi(T/2) = \pi - \varphi_0$ so the corresponding point in the separating line belongs to the antipodal parallel of the starting point. Since $\Delta\theta$ is decreasing, the extremities of the cut are obtained letting p_θ tend to $\pm(\Gamma(\varphi_0))^{-1/2}$ (now finite, since $\varphi_0 \neq \pi/2$), and the subarc is closed. □

To get the result on conjugate loci, we finally also assume that $\Delta\theta$ is convex.

Theorem 2. *Consider a singular metric on the sphere defined by (7-8). Under the assumption that $\Delta\theta$ is strictly decreasing and convex for $p_\theta > 0$, conjugate loci are reduced to the opposite pole for poles, double-heart shaped with four meridional cusps for equatorial points, and astroidal with two meridional and two equatorial cusps otherwise.*

Proof. The analysis outside the equator being a direct extension of the result in [7], we focus on the proof for $\varphi_0 = \pi/2$. Consider an extremal defined by a positive p_θ and $p_{\varphi_0} = +1$. For t in $(T/4, 3T/4)$, $\dot\varphi \neq 0$ and the extremal can be parameterized by φ according to

$$\theta(\varphi, p_\theta) = \frac{\Delta\theta(p_\theta)}{2} + \int_\varphi^{\pi/2} f(\varphi', p_\theta)\mathrm{d}\varphi',$$

where, as before,

$$f(\varphi, p_\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}\varphi} = \frac{\Gamma(\varphi)p_\theta}{\sqrt{1-\Gamma(\varphi)p_\theta^2}}.$$

The conjugacy condition writes $\partial\theta/\partial p_\theta = 0$, so the coordinate $\varphi_{1c}(p_\theta)$ of the first conjugate point is solution of

$$\int_{\varphi_{1c}(p_\theta)}^{\pi/2} \frac{\partial f}{\partial p_\theta}(\varphi', p_\theta)\mathrm{d}\varphi' = -\frac{\Delta\theta'(p_\theta)}{2} > 0,$$

in order that $\varphi_{1c}(p_\theta) < \pi/2$ (since $\partial f/\partial p_\theta > 0$). By differentiating the previous equality, one gets

$$\varphi'_{1c}(p_\theta) = \left[\frac{\partial f}{\partial p_\theta}(\varphi_{1c}(p_\theta), p_\theta)\right]^{-1} \left[\frac{\Delta\theta''}{2}(p_\theta) + \int_{\varphi_{1c}(p_\theta)}^{\pi/2} \frac{\partial^2 f}{\partial p_\theta^2}(\varphi', p_\theta)\mathrm{d}\varphi'\right],$$

which is positive first because $\varphi_{1c}(p_\theta) < \pi/2$, then because

$$\frac{\partial^2 f}{\partial p_\theta^2} = \frac{3\Gamma^2(\varphi)p_\theta}{(1-\Gamma(\varphi)p_\theta^2)^{5/2}} > 0,$$

and by virtue of positiveness of $\Delta\theta''$. In particular, the parameterization $p_\theta \mapsto (\varphi_{1c}(p_\theta), \theta(\varphi_{1c}(p_\theta), p_\theta))$ of the conjugate locus is regular, and the set has no cusp for positive p_θ.

The tangent vector along the conjugate locus is

$$\left(\frac{\partial\theta}{\partial\varphi}\varphi'_{1c} + \underbrace{\frac{\partial\theta}{\partial p_\theta}}_{0}\right)\frac{\partial}{\partial\theta} + \varphi'_{1c}\frac{\partial}{\partial\varphi},$$

which is proportional to $f(\varphi_{1c}(p_\theta), p_\theta)\partial/\partial\theta + \partial/\partial\varphi$. On the one hand, as in the regular case discussed in [7], $f(\varphi_{1c}(p_\theta), p_\theta)$ tends to 0 when $p_\theta \to 0+$, and the locus has a first meridional cusp because of the axial symmetry s_1. On the other hand, when p_θ goes to $+\infty$, the analysis on the local model shows that, contrary to the regular case, the conjugate locus has two tangential contacts with the meridian (Proposition 11) which, combined again with symmetry s_1, form a second meridional cusp at the initial point where conjugate points accumulate, and we have a heart shaped conjugate locus. The central symmetry s_2 gives the symmetric part, whence the result. □

These two results directly apply to the simplest case of order one, $R = (1-X)^{-1}$, associated with the bi-entry case. Indeed, differentiating (15) one obtains

$$\Delta\theta' = -\frac{1}{(1+p_\theta^2)^{3/2}} < 0 \quad \text{and} \quad \Delta\theta'' = \frac{3p_\theta}{(1+p_\theta^2)^{5/2}} \geq 0$$

when $p_\theta > 0$. The computation is more intricate in the tangential single-input case, but there is numerical evidence that the structure is similar, and it is still possible to verify the required assumptions on $\Delta\theta$ in the neighbourhood of meridians and equator thanks to asymptotics. According to Proposition 12,

$$\Delta\theta \sim \frac{4}{3}(2-\sqrt{2})K(3-2\sqrt{2})p_\theta^{-3/2}, \quad p_\theta \to \infty,$$

and $\Delta\theta' < 0 \leq \Delta\theta''$ when $p_\theta \to \infty$. The analysis has to be refined when $p_\theta \to 0$ since a higher order expansion of $\Delta\theta$ is required, involving terms up to fourth order in the asymptotic expansions of roots β and γ.

Lemma 7. *In the neighbourhood of meridians,*

$$\beta = 2 - p_\theta\sqrt{2} + \frac{3}{2}p_\theta^2 - \frac{13\sqrt{2}}{16}p_\theta^3 + \frac{1}{2}p_\theta^4 + o(p_\theta^4),$$

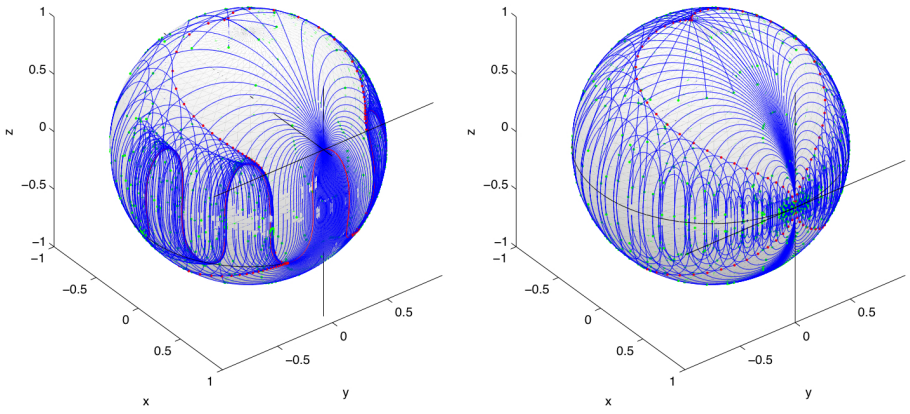

Figure 1: Bi-entry case. Bifurcation of the cut (in black) and conjugate (red dots) loci, respectively from a closed antipodal subarc to the pointed equator, and from an astroid to a double-heart, when the initial condition goes to the equator ($\varphi_0 < \pi/2$ on the left, $\varphi_0 = \pi/2$ on the right). Geodesics are portrayed in blue.

$$\gamma = 2 + p_\theta\sqrt{2} + \frac{3}{2}p_\theta^2 + \frac{13\sqrt{2}}{16}p_\theta^3 + \frac{1}{2}p_\theta^4 + o(p_\theta^4).$$

Moreover,

$$\Delta\theta \sim 2\pi(1 - \frac{3\sqrt{2}}{4}p_\theta + \frac{35\sqrt{2}}{128}p_\theta^3), \quad p_\theta \to 0.$$

The condition $\Delta\theta' < 0 \leq \Delta\theta''$ consequently also holds when $p_\theta \to 0$.

In such situations, we observe a first bifurcation as the initial condition tends to the equator. Indeed, the cut and conjugate loci are respectively a closed antipodal subarc and an astroid-shaped set outside polar and equatorial points, while they bifurcate to an open arc (the equator minus the initial point) and a double-heart set for an equatorial initial condition (see Fig. 1). A similar bifurcation may also occur in connection with the homotopy from round metric on the sphere discussed in §2, as will be clear from curvature estimates.

Given a metric $\mathrm{d}s^2 = G(\varphi)\mathrm{d}\theta^2 + \mathrm{d}\varphi^2$ on $\mathbf{S}^2$, the Gaussian curvature is

$$K = -(\sqrt{G})^{-1}\frac{\mathrm{d}^2\sqrt{G}}{\mathrm{d}\varphi^2}\cdot$$

Lemma 8. *Let R be a rational fraction verifying (7), and consider the homotopy from the round metric on the sphere defined by*

$$g_\nu = XR(\nu X)\mathrm{d}\theta^2 + \mathrm{d}\varphi^2, \quad X = \sin^2\varphi. \tag{18}$$

For any $\nu \in [0,1]$, the curvature K_ν of g_ν is a rational fraction in X homogeneously depending on R,

$$K_\nu = 1 + \nu(4X-3)\frac{R'}{R} + \nu^2X(1-X)\frac{R'^2}{R^2} - 2\nu^2X(1-X)\frac{R''}{R}\cdot$$

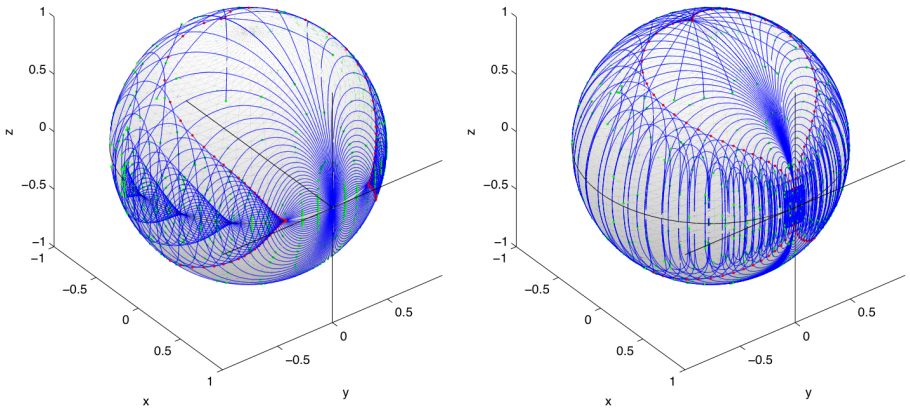

Figure 2: Tangential single-input case. Bifurcation of the cut (in black) and conjugate (red dots) loci, respectively from a closed antipodal subarc to the pointed equator, and from an astroid to a double-heart, when $\nu \to 1$ ($\nu < 1$ on the left, $\nu = 1$ on the right). Geodesics are portrayed in blue.

Proposition 14. *The following curvature estimates hold for metrics g_ν defined by (18),*

$$K_\nu(X) = 1 + \nu R'(0)(4X - 3) + O(\nu^2), \quad \nu \to 0,$$

$$K_1(X) \sim -\frac{p(p+1)}{1-X}, \quad X \to 1,$$

$$K_\nu(1) \sim \frac{p}{1-\nu}, \quad \nu \to 1,$$

where p is the order of the pole of R.

Corollary 1. *Cut loci of the metrics g_ν defined by (18) are closed antipodal subarcs for ν small enough.*

Proof. Since $K'_\nu(X) \sim 4\nu(R'/R)(0) + O(\nu^2)$ with $(R'/R)(0) > 0$ by virtue of (7), the curvature is monotone non-decreasing along half-meridians and the result follows from [14] main theorem. □

A second bifurcation is therefore obtained on cut loci, again from closed antipodal subarcs to the pointed equator, when $\nu \to 1$. The bifurcation may also occur on conjugate loci, as illustrated in the bi-entry case. For $R = (1-X)^{-1}$ indeed, it is proven in the Riemannian setting of [7] that conjugate loci are astroidal when $\nu < 1$. The same phenomenon is observed numerically for the tangential single-input case (see Fig. 2).

The existence of conjugate points for such metrics as (3), in the bi-entry case, or (6), in the tangential single-input case, is typical of the almost-Riemannian setting where conjugate points may exist although curvature remains negative whenever defined [2]. Indeed, with $X = \sin^2\varphi$,

$$K = -\frac{2}{1-X} < 0 \quad \text{and} \quad K = -\frac{(1+X)(4-X)}{(2-X)(1-X)} < 0 \tag{19}$$

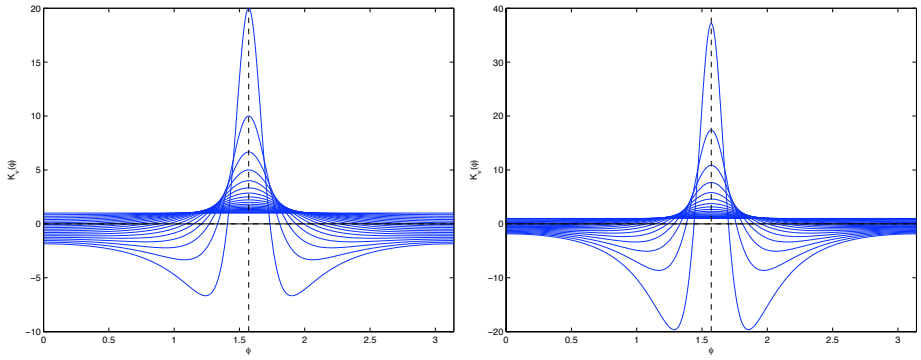

Figure 3: Curvature K_ν associated with homotopy from the round metric. On the left, $R = 1/(1-X)$ (bi-entry case), $R = (1/4)(1+2/(1-X)+1/(1-X)^2)$ (single-input tangential case) on the right.

for (3) and (6), respectively. An explanation of this fact is provided by homotopy from the round metric (18). Proposition 14 actually establishes that although curvature is negative in the neighbourhood of the singularity and even tends to $-\infty$ when $\varphi \to \pi/2$, there is a concentration of positive curvature on the singularity itself as $K_\nu(1) \to +\infty$ when $\nu \to 1-$, responsible for the existence of conjugate points after crossing the singularity (cut, and thus conjugate points being located after the equator by antipodality). See also Fig. 3 for portrays of the curvature.

We conclude by reinterpretating homotopy from the round metric in the light of Proposition 3. It asserts that the metric with an order one singularity associated with the bi-entry case is conformal to the canonical metric on an ellipsoid of revolution with unit semi-major axis and semi-minor axis $\mu = \sqrt{1-\nu}$. Now, as $\nu \to 1$, $\mu \to 0$ and the ellipsoid collapses on a two-sided Poincaré disk endowed with the flat metric since, by (4), $g_{\mathscr{E},\mu} \to g_{\mathscr{E},0} = \sin^2\varphi\,\mathrm{d}\theta^2 + \cos^2\varphi\,\mathrm{d}\varphi^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2$, which is the flat metric in polar coordinates (ρ,θ) on $\mathbf{D}$,

$$\rho = \sin\varphi, \quad \mathrm{d}\rho^2 = (1-X)\mathrm{d}\varphi^2, \quad X = \sin^2\varphi.$$

The standard astroid conjugate loci on the ellipsoid are then generated by reflections of straight lines on the boundary, as caustics formed by envelopes of reflecting rays in the usual analogy from optics (see Fig. 4). So we have three conformal metrics on the Poincaré disk, namely the flat one with zero curvature, $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2$, the metric associated with the bi-entry case which reads

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2}{1-\rho^2}$$

in polar coordinates and whose negative curvature $K = -2(1-\rho^2)^{-1}$ (see (19)) tends to $-\infty$ when $\rho \to 1$, and the canonical Poincaré metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2}{(1-\rho^2)^2} \tag{20}$$

with constant negative curvature $K = -1$. As illustrated by Fig. 4 in the bi-entry case, crossing the equatorial singularity is then interpretated as going

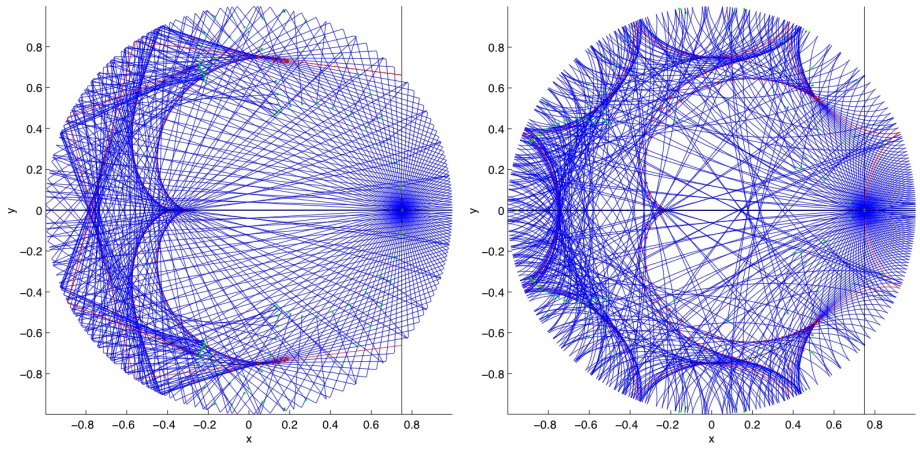

Figure 4: On the left, flat metric on the Poincaré disk. The conjugate locus formed by the envelope of straight lines (in blue) reflecting on the boundary is the standard astroid (red dots). On the right, the picture for the same initial condition of the singular metric defined by $R = (1-X)^{-1}$. As for the Poincaré metric, contacts with the boundary are orthogonal.

from one side of $\mathbf{D}$ to the other. This can also be seen, like in the flat case, as generating reflections of geodesics with the boundary. As for the canonical Poincaré metric (20), those reflections turn to be orthogonal in general.

Consider any metric defined by (7-8) and write the corresponding Hamiltonian in coordinates (ρ, θ) on $\mathbf{D}$,

$$H = \frac{1}{2}\left[(1-X)p_\rho^2 + \frac{p_\theta^2}{XR(X)}\right], \quad X = \sin^2\varphi = \rho^2.$$

In particular,

$$\dot\theta \sim \frac{p_\theta}{a_p}(1-X)^p \quad \text{and} \quad |\dot\rho| = (1-X)|p_\rho| \sim \sqrt{1-X}$$

when $X \to 1$ since

$$p_\rho^2 = (1-X)^{-1}\left[1 - \frac{p_\theta^2}{XR(X)}\right] \sim (1-X)^{-1}$$

on $\{H = 1/2\}$. Reparameterizing time according to $\mathrm{d}\tau = \mathrm{d}t\sqrt{1-X}$ we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} \sim \frac{p_\theta}{a_p}(1-X)^{p-1/2} \to 0 \quad \text{and} \quad \left|\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\right| \sim 1, \quad X \to 1,$$

so contacts with $\partial\mathbf{D}$ (*i.e.* for $\rho = X = 1$) are orthogonal reflections.

Conversely, using homotopy (18) to replace the singular metric by a Riemannian one when $\nu < 1$ changes the contact for $p_\theta \neq 0$ (meridians, obtained for $p_\theta = 0$, obviously remain perpendicular to the boundary). Now indeed,

$$H_\nu = \frac{1}{2}\left[(1-X)p_\rho^2 + \frac{p_\theta^2}{XR(\nu X)}\right],$$

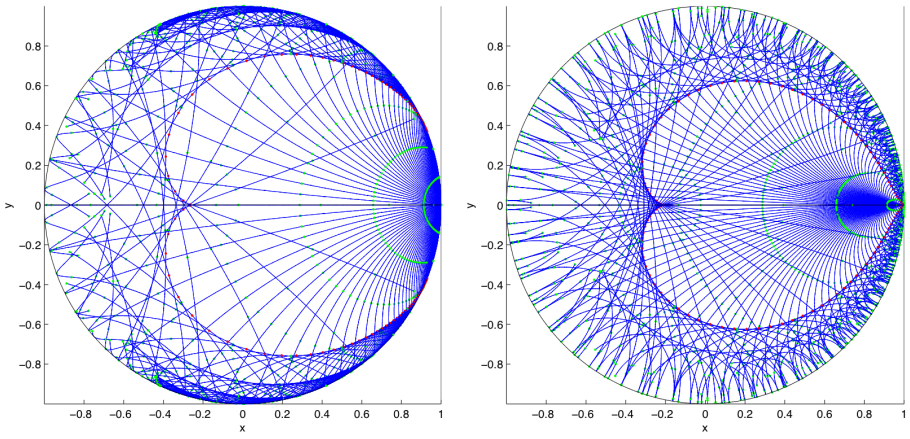

Figure 5: Tangential single-input case. On the left, metric obtained by homotopy from the round one for $\nu < 1$. The initial condition is on the boundary, tangential contacts with $\partial\mathbf{D}$ are observed. On the right, singular metric ($\nu = 1$) for the same initial condition. Contacts with $\partial\mathbf{D}$ are orthogonal and the conjugate locus is the caustic generated by reflections of geodesics on the boundary.

and, when $X \to 1$,

$$\dot{\theta} \sim \frac{p_\theta}{R(\nu)} \quad \text{and} \quad \dot{\rho} = p_\varphi \sqrt{1 - X}$$

since $p_\varphi = p_\rho \sqrt{1 - X}$, which remains finite. So $\dot{\theta} \neq 0$ if $p_\theta \neq 0$ while $\dot{\rho} = 0$ at $X = 1$, and contacts with the boundary are tangential outside meridians (see Fig. 5). The last proposition summarizes the analysis.

Proposition 15. *For a singular metric on the sphere defined by (7-8), crossing the equatorial singularity is interpretated on the Poincaré disk as reflecting on the boundary, and those reflections generate conjugate loci as caustics. Outside meridians, contacts with the boundary bifurcate from tangential to orthogonal through homotopy (18).*

Let us finally remark that the collapsing interpretation is also important since going back from $\mathbf{S}^2$ to $\mathbf{D}$ allows to analyze the two original problems of two-body control. With evidence of antipodal cut points and further conjugate points, that is points located on the other side of the Poincaré disk, we conclude that there are no such points in $\mathbf{D}$ for the original coordinates.

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