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On Schrödinger equation outside strictly convex obstacles

Oana Ivanovici

Abstract

We prove Strichartz estimates without loss of derivatives for the Schrödinger equation on a Riemannian manifold with smooth, strictly geodesically concave boundary. We deduce global existence for the H^1 -critical (quintic) Schrödinger equation in $3D$.

1 Introduction

Let (Ω, g) be a Riemannian manifold of dimension $n \geq 2$. Strichartz estimates are a family of dispersive estimates on solutions $u(x, t) : \Omega \times [-T, T] \rightarrow \mathbb{C}$ to the Schrödinger equation

$$i\partial_t u + \Delta_g u = 0, \quad u(x, 0) = u_0(x), \quad (1.1)$$

where Δ_g denotes the Laplace-Beltrami operator on (Ω, g) . In their most general form, local Strichartz estimates state that

$$\|u\|_{L^q([-T, T], L^r(\Omega))} \leq C \|u_0\|_{H^s(\Omega)}, \quad (1.2)$$

where $H^s(\Omega)$ denotes the Sobolev space over Ω and $2 \leq q, r \leq \infty$ satisfy $(q, r, n) \neq (2, \infty, 2)$ (for the case $q = 2$ see [23]) and are given by the scaling admissibility condition

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}. \quad (1.3)$$

In \mathbb{R}^n and for $g_{ij} = \delta_{ij}$, Strichartz estimates in the context of the wave and Schrödinger equations have a long history, beginning with Strichartz pioneering work [33], where he proved the particular case $q = r$ for the wave and (classical) Schrödinger equations. This was later generalized to mixed $L_t^q L_x^r$ norms by Ginibre and Velo [15] for Schrödinger equations, where (q, r) is sharp admissible and $q > 2$; the wave estimates were obtained independently by Ginibre-Velo [17] and Lindblad-Sogge [25], following earlier work by Kapitanski [21]. The

remaining endpoints for both equations were finally settled by Keel and Tao [23]. In that case $s = 0$ and $T = \infty$; (see also Kato [22], Cazenave-Weissler [10]). Estimates for the flat 2-torus were shown by Bourgain [4] to hold for any $s > 0$.

In the variable coefficients case, even without boundaries, the situation is much more complicated: we simply recall the pioneering work of Staffilani and Tataru [32], dealing with compact, non trapping perturbations of the flat metric, and recent work of Bouclet and Tzvetkov [3] which considerably weakens the decay of the perturbation (retaining the non-trapping character at spatial infinity). On compact manifolds without boundaries, Burq, Gerard and Tzvetkov [7] established Strichartz estimates with $s = 1/p$, hence with a loss of derivatives when compared to the case of flat geometries. Recently, Blair, Smith and Sogge [2] improved on the current results for compact (Ω, g) where either $\partial\Omega = \emptyset$, or $\partial\Omega \neq \emptyset$ and g Lipschitz, by showing that Strichartz estimates hold with a loss of $s = 4/3p$ derivatives. This appears to be the natural analog of the estimates of [7] for the general boundaryless case.

In this paper we shall show that Strichartz estimates for the Euclidian Schrödinger equation also hold on Riemannian manifolds with smooth, strictly geodesically concave boundaries. By the last condition we understand that the second fundamental form on the boundary of the manifold is strictly positive definite and we assume moreover the manifold to be flat at infinity (though presumably one may use Bouclet and Tzvetkov [3] result to combine both situations).

Let (Ω, g) be a Riemannian manifold with C^∞ boundary, $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, where \mathcal{O} is compact with smooth boundary. We shall assume that $n \geq 2$ and that $\partial\Omega$ is strictly geodesically concave throughout. Let Δ_g be the Laplace-Beltrami operator associated to g on Ω , acting on $L^2(\Omega)$, with domain $H^2(\Omega) \cap H_0^1(\Omega)$. We assume that we can globally write

$$\Delta_g = \sum_{j,k=2}^n g^{j,k}(x) \partial_j \partial_k + \sum_{j=1}^n a_j(x) \partial_j, \quad (1.4)$$

where the coefficients belong to a bounded set of C^∞ and the principal part is uniformly elliptic. Our main result reads as follows

Theorem 1.1. *Given (q, r) satisfying the scaling condition (1.3), $q > 2$, there exists a constant C such that the solution $u(x, t)$ of the Schrödinger equation on $\Omega \times \mathbb{R}$ with Dirichlet boundary condition*

$$\begin{cases} i\partial_t u + \Delta_g u = 0, & \text{on } \Omega \times \mathbb{R} \\ u(x, 0) = u_0(x), \\ u|_{\partial\Omega} = 0 \end{cases} \quad (1.5)$$

satisfies

$$\|u\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq C \|u_0\|_{L^2(\Omega)}. \quad (1.6)$$

The proof of Theorem 1.1 combines several arguments: firstly, we perform a time rescaling, first used by Lebeau [24] in the context of control theory, which transforms the equation into a semiclassical problem: due to the finite speed of propagation we can use the arguments of Smith and Sogge [29] for the wave equation which consists of exploiting the L^2 continuity of certain operators to reduce consideration to operators which are similar to those on a manifold without boundary, which will allow to use the parametrix construction of Melrose and Taylor for the wave group. In order to prove (1.5) we only need to show that the "glancing" part of the parametrix satisfies the same bounds. Secondly, we adapt a result of Burq [5] which provides Strichartz estimates without loss for a non-trapping problem, with a metric that equals the identity outside a compact set. The proof relies on a local smoothing effect for the free evolution $\exp(it\Delta_g)$, first observed in the case of the flat space in the works of Constantin and Saut [13], Doi [14], [6] in the non-trapping case. Following a strategy suggested by Staffilani and Tataru [32], we prove that away from the obstacle the free evolution enjoys the Strichartz estimates exactly as for the free space.

We give two applications of Theorem 1.1 is a local existence result for the quintic Schrödinger equation in 3D:

Theorem 1.2. (*Local existence for the quintic Schrödinger equation*) *Let $T > 0$ and $u_0 \in H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is the complementary of a smooth, strictly convex obstacle \mathcal{O} . Then there exists a unique solution $u \in C([0, T], H_0^1(\Omega)) \cap L^5((0, T], W^{1,30/11}(\Omega))$, of the quintic nonlinear equation*

$$i\partial_t u + \Delta_g u = \pm |u|^4 u \text{ on } \Omega \times \mathbb{R}, \quad u|_{t=0} = u_0 \text{ on } \Omega, \quad u|_{\partial\Omega} = 0. \quad (1.7)$$

Moreover, for any $T > 0$, the flow $u_0 \rightarrow u$ is Lipschitz continuous from any bounded set of $H_0^1(\Omega)$ to $C([-T, T], H_0^1(\Omega))$. If the initial data u_0 has sufficiently small H^1 norm, then the solution is global in time.

Theorem 1.3. (*Scattering for subcritical Schrödinger equation*) *Let $3 \leq p < 5$ and $u_0 \in H_0^1(\Omega)$, with Ω the exterior of a strictly convex domain in \mathbb{R}^3 . Then the global in time solution of the defocusing Schrödinger equation*

$$i\partial_t u + \Delta_g u = |u|^{p-1} u, \quad u|_{t=0} = u_0 \text{ on } \Omega, \quad u|_{\partial\Omega} = 0 \quad (1.8)$$

scatters in $H_0^1(\Omega)$. If $p = 5$ and the initial data u_0 has sufficiently small H^1 norm, then the global solution of the critical Schrödinger equation scatters in $H_0^1(\Omega)$.

Results for the Cauchy problem associated to the critical wave equation outside a strictly convex obstacle were obtained by Smith and Sogge [29]. Their result was a consequence of the fact that the Strichartz estimates for the Euclidian wave equation also hold on Riemannian manifolds with smooth, compact and strictly concave boundaries.

In [8], Burq, Lebeau and Planchon proved that the defocusing quintic wave equation with Dirichlet boundary conditions is globally wellposed on $H^1(\Omega) \times L^2(\Omega)$ for any smooth, compact domain $\Omega \subset \mathbb{R}^3$. Their proof relies on L^p estimates for the spectral projector obtained by Smith and Sogge [30]. A similar result for the defocusing critical wave equation with Neumann boundary conditions was obtained in [9].

In the case of the Schrödinger equation outside a nontrapping obstacle in \mathbb{R}^3 , Burq, Gerard and Tzvetkov [6] proved global existence for subcubic defocusing nonlinearities and Anton [1] for the cubic case. Recently Planchon and Vega [27] improved the local well-posedness theory to H^1 -subcritical (subquintic) nonlinearities for $n = 3$. In $\mathbb{R}^3 \times \mathbb{R}_t$, Colliander, Keel, Staffilani, Takaoka and Tao [12] established global well-posedness and scattering for energy-class solutions to the quintic defocusing Schrödinger equation (1.7), which is energy-critical.

Theorem 1.3 is proved in [27] in the case of the exterior of a star-shaped domain, using

$$\|e^{it\Delta_g}u_0\|_{L^4_{t,x}} \lesssim \|u_0\|_{\dot{H}^{1/4}},$$

but since this estimate fails to provide control of $L^4_t L^\infty$ one has to use local smoothing estimates close to the boundary, and Strichartz estimates for the usual Laplacian on \mathbb{R}^3 away from it, the sub-criticality with respect to H^1 compensating the weakness of the local smoothing estimate. Here we give a much more simpler proof using the classical Strichartz estimates (1.6).

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2 Estimates for semiclassical Schrödinger equation

2.1 Preliminaries

We may assume that the metric g is extended smoothly across the boundary, so that Ω is a geodesically concave subset of a complete Riemannian manifold $\tilde{\Omega}$, which we assume to be \mathbb{R}^n . By the free Schrödinger equation we mean the Schrödinger equation on $\tilde{\Omega} = \mathbb{R}^n$, where the data u_0 has been extended to \mathbb{R}^n by an extension operator preserving the Sobolev spaces. This "mirror reflection" method consists in taking a copy of the domain Ω and "glue" it to the initial one by identifying the points of the boundary. For details see [1].

Let $\Psi \in C_0^\infty(\mathbb{R}^*)$ be compactly supported away from 0 such that for all $\lambda \in \mathbb{R}$

$$\sum_{k \in \mathbb{Z}} \Psi(2^{-2k}\lambda) = 1. \quad (2.1)$$

In what follows let $k \in \mathbb{N}^*$ and $h = 2^{-k}$. We study the semiclassical Schrödinger equation with initial data $\Psi(h^2\Delta_g)u_0$ localized at frequency h^{-1} and then we recover the behavior of the linear flow thanks to using Lemma 3.2.

2.2 Proof of Theorem 1.1

The first step in the proof of Theorem 1.1 is to prove usual estimates for small intervals of time (depending on the frequency). In order to do that we introduce the semiclassical time t by $v(x, t) = u(x, ht)$. If u is a solution of the equation

$$\begin{cases} i\partial_t u + \Delta_g u = 0, \\ u(x, 0) = \Psi(h^2\Delta_g)u_0(x), \end{cases} \quad (2.2)$$

on a time interval $[0, T]$, $T > 0$ then v solves the following semiclassical equation on $[0, h^{-1}T]$

$$\begin{cases} ih\partial_t v + h^2\Delta_g v = 0, \\ v(x, 0) = \Psi(h^2\Delta_g)u_0(x). \end{cases} \quad (2.3)$$

Theorem 2.1. (*Strichartz estimates for the semiclassical problem*) *Under the assumptions of Theorem 1.1, there exists $C > 0$ such that for any $h \in (0, 1]$ and any interval of time I_h of length $|I_h| \leq Th$ the solution $u(x, t)$ of (2.2) satisfies*

$$\left(\int_{I_h} \|e^{it\Delta_g} \Psi(h^2\Delta_g)u_0\|_{L^r(\mathbb{R}^n)}^q \right)^{1/q} \leq C \|\Psi(h^2\Delta_g)u_0\|_{L^2(\mathbb{R}^n)}. \quad (2.4)$$

Proof. In order to prove Theorem 2.1 it is enough to prove Strichartz estimates for the solution of the semiclassical problem (2.3). The arguments of this part of the proof are very similar to those given in [28]. We need to show the following

$$\|v(x, t)\|_{L^q([0, T], L^r(\mathbb{R}^n))} \leq C \|\Psi(h^2 \Delta_g) u_0\|_{L^2(\mathbb{R}^n)}. \quad (2.5)$$

It will be enough to consider T an arbitrary, fixed small positive number. Let $\delta > 0$ be a constant to be fixed below and set $T = \delta/2$,

$$S(\delta, T) := \{(x, t) \in \Omega \times [0, T] \mid \text{dist}(x, \Omega) < \delta\}.$$

On the complement of $S(\delta, T)$ in $\Omega \times [0, T]$, the solution $v(x, t)$ equals the solution of the semiclassical Schrödinger equation on \mathbb{R}^n for which Strichartz estimates are known, thus it suffices to establish each of the estimates for the norm over $S(\delta, T)$. The (semiclassical) Schrödinger group $\exp(iht\Delta_g)$, which maps data at time 0 to data at time t , is an isomorphism on $H^\alpha(\Omega)$, consequently it suffices to establish the estimates over $S(\delta, T)$ for the function $\exp(ih(t_0 + \cdot)\Delta_g)\Psi(h^2\Delta_g)u_0$ for some t_0 .

It follows from [19, Lemma 24.3.4] the existence of $c_0 > 0$ such that of t_0 is less than some fixed constant, then whenever $\gamma(s)$ is a unit speed broken geodesic in Ω with $\text{dist}(\gamma(0), \partial\Omega) \leq c_0 t_0^2$, one has

$$\text{dist}(\gamma(s), \partial\Omega) \geq c_0 t_0^2, \quad s \in [t_0/2, 4t_0].$$

We now define a set $\omega \subset \Omega$: a point p lies in ω if there is a unit speed broken bicharacteristic with $\gamma(0) = p$ and $\text{dist}(\gamma(t_0 + s), \partial\Omega) \leq 2\delta$ for some $s \in [-\delta, \delta]$. We assume $2\delta < c_0 t_0^2$, so that ω is a compact subset of Ω with $\text{dist}(\omega, \partial\Omega) \geq c_0 t_0^2$. There is a smooth function χ supported in ω such that the Schwartz kernel of $\exp(ih(t_0 + t)\Delta_g)(1 - \chi)$ lies in $C^\infty(S(\delta, T) \times \Omega)$. We are thus reduced to estimating $\exp(ih(t_0 + \cdot)\Delta_g)\Psi(h^2\Delta_g)u_0$ for data u_0 supported in ω .

Let Δ denote the Laplacian on \mathbb{R}^n . We let \mathcal{M} denote the outgoing solution to the Dirichlet problem for the semiclassical Schrödinger operator on $\Omega \times \mathbb{R}$. Thus, if g is a function on $\partial\Omega \times \mathbb{R}$ which vanishes for $t \leq -t_0$, then $\mathcal{M}g$ is the solution on $\Omega \times \mathbb{R}$ to

$$\begin{cases} ih\partial_t \mathcal{M}g + h^2 \Delta_g \mathcal{M}g = 0, \\ \mathcal{M}g = 0, \quad t \leq -t_0, \\ \mathcal{M}g|_{\partial\Omega \times \mathbb{R}} = g. \end{cases} \quad (2.6)$$

Then, for $t \in [-t_0, T]$ and data f supported in ω and localized at frequency $1/h$, i.e. such that $f = \Psi(h^2\Delta_g)f$ (modulo $O(h^\infty)$), we have we have

$$\exp(ih(t_0 + \cdot)\Delta_g)f = \exp(ih(t_0 + \cdot)\Delta)f - \mathcal{M}(\exp(ih(t_0 + \cdot)\Delta)f|_{\partial\Omega \times \mathbb{R}}).$$

The cotangent bundle of $\partial\Omega \times \mathbb{R}$ is divided into three disjoint, time independent sets: the hyperbolic and elliptic regions where the Dirichlet problem is respectively hyperbolic and elliptic, and the glancing region which is the boundary between the two. We decompose the identity operator into

$$Id = \Pi_h + \Pi_e + \Pi_g,$$

where Π_h, Π_e are time-independent pseudo-differential cutoffs, essentially supported inside the hyperbolic and elliptic regions, and Π_g is essentially supported in a small set about the glancing region, on which the Melrose and Taylor construction is valid (see Section 5.1). On $S(\delta, T)$ we can write $\exp(ih(t_0 + \cdot)\Delta_g)f$ as the sum of four terms

$$\begin{aligned} \exp(ih(t_0 + \cdot)\Delta_g)f &= \exp(ih(t_0 + \cdot)\Delta)f - \mathcal{M}\Pi_e(\exp(ih(t_0 + \cdot)\Delta)f|_{\partial\Omega \times \mathbb{R}}) \\ &\quad - \mathcal{M}\Pi_h(\exp(ih(t_0 + \cdot)\Delta)f|_{\partial\Omega \times \mathbb{R}}) - \mathcal{M}\Pi_g(\exp(ih(t_0 + \cdot)\Delta)f|_{\partial\Omega \times \mathbb{R}}). \end{aligned}$$

For the first term in the right hand side, $\exp(ih(t_0 + \cdot)\Delta)f$, the desired estimates follow as in the boundaryless case. The projection on any characteristic direction for the semiclassical Schrödinger operator onto $T^*(\partial\Omega \times \mathbb{R})$ is contained in the hyperbolic and glancing regions, so that

$$\mathcal{M}\Pi_e(\exp(ih(t_0 + \cdot)\Delta)f|_{\partial\Omega \times \mathbb{R}}) \simeq 0,$$

where \simeq denotes equality modulo smoothing operators. As far as the hyperbolic part is concerned, because of the finite speed of propagation of the semiclassical Schrödinger equation, we can use the same arguments as in [29] in order to "eliminate" the hyperbolic part. Precisely, on the essential support of Π_h the forward Dirichlet problem can be solved locally, modulo smoothing kernels, on an open set in $\mathbb{R} \times \tilde{\Omega}$ around $\partial\Omega$. In fact, there is an operator $\tilde{\mathcal{M}}$ from $\mathbb{R} \times \partial\Omega$ to $\mathbb{R} \times \tilde{\Omega}$, and a constant t_1 such that

$$ih\partial_t\tilde{\mathcal{M}} + h^2\Delta\tilde{\mathcal{M}} = 0, \quad t \in [-2t_1, 2t_1],$$

$$(\mathcal{M} - \tilde{\mathcal{M}})\Pi_h \simeq 0.$$

We assume that $t_0 \leq t_1$ and we set $\tilde{f} := \tilde{\mathcal{M}}\Pi_h \exp(ih(t_0 + \cdot)\Delta)f|_{t=-t_0}$. The operator $f \rightarrow \tilde{f}$ is a classical Fourier integral operator associated to the relation "reflection about $\partial\Omega$ ". We can take the data \tilde{f} to be compactly supported in $\tilde{\Omega} \setminus \Omega$ and hence

$$\|\tilde{f}\|_{H^s(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}.$$

On $S(\delta, T)$ we have

$$\mathcal{M}\Pi_h \exp(ih(t_0 + \cdot)\Delta)f = \exp(ih(t_0 + \cdot)\Delta)\tilde{f}$$

and the estimates for the third term follow as for the first one, thus we are left with estimating the last term.

Near the glancing region we use the Melrose and Taylor construction in order to write each solution $v(x, t)$ of the semiclassical Schrödinger equation (2.3) as a finite sum of pseudodifferential cutoffs, each essentially supported in a suitably small neighborhood of a glancing ray. Each term in the resulting sum can be written, modulo a smoothing operator

$$\begin{aligned} & \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\theta(x, \xi) + t\xi_1^2)} 2\xi_1 (a(x, \xi, h)A(\zeta(x, \xi/h)) + b(x, \xi, h)A'(\zeta(x, \xi/h))) \times \\ & \times \frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))} \hat{K}(\Psi(h^2\Delta_g)u_0)\left(\frac{\xi}{h}\right) d\xi, \end{aligned} \quad (2.7)$$

where the symbols a , b and the phases θ , ζ have the properties stated in Proposition 5.1: a and b are symbols of type $(1, 0)$ and order $1/6$ and $-1/6$, respectively, both of which are supported in a small neighborhood of the ξ_1 axis and where K is a classical Fourier integral operator of order 0, compactly supported. The phases θ and ζ are real, smooth and homogeneous of degree 1 and $2/3$, respectively. If local coordinates are chosen so that Ω is given by $x_n > 0$, the phases satisfy the eikonal equations

$$\begin{cases} \xi_1^2 - \langle d\theta, d\theta \rangle_g + \zeta \langle d\zeta, d\zeta \rangle_g = 0, \\ \langle d\theta, d\zeta \rangle_g = 0, \\ \zeta(x', 0, \xi) = \zeta_0(\xi) = -\xi_1^{-1/3}\xi_n, \end{cases} \quad (2.8)$$

in the region $\zeta \leq 0$. Here $x' = (x_1, \dots, x_{n-1})$ and $\langle \cdot, \cdot \rangle_g$ denotes the inner product given by the metric g . The phases also satisfy the eikonal equations (2.8) to infinite order at $x_n = 0$ in the region $\zeta > 0$. For further discussions and proofs we refer the reader to the manuscript [26] of Melrose and Taylor.

By the L^2 continuity of K , to finish the proof of Theorem 2.1 we need only to show that the operator A_h defined by

$$\begin{aligned} A_h f(x, t) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} 2\xi_1 (a(x, \xi/h)A_+(\zeta(x, \xi/h)) + b(x, \xi/h)A'_+(\zeta(x, \xi/h))) \times \\ & \times e^{\frac{i}{h}(\theta(x, \xi) + t\xi_1^2)} \frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))} \mathcal{F}(\Psi(h^2\Delta_g)f)\left(\frac{\xi}{h}\right) d\xi, \end{aligned} \quad (2.9)$$

where by \mathcal{F} we denote the Fourier transform, satisfies

$$\|A_h f\|_{L^q((0, T], L^r(\mathbb{R}^n))} \leq Ch^{-n/2(1/2-1/r)} \|f\|_{L^2(\mathbb{R}^n)}. \quad (2.10)$$

End of the proof of Theorem 2.1:

Proof. Coming back to the solution $u(x, t) = e^{it\Delta_x} \Psi(h^2 \Delta_x) u_0(x)$, we find, for $\frac{1}{q} = \frac{n}{2}(\frac{1}{2} - \frac{1}{r})$

$$\|u(x, t)\|_{L^q(0, hT], L^r(\mathbb{R}^n)} = h^{1/q} \|v(x, t)\|_{L^q(0, T], L^r(\mathbb{R}^n)} \leq C \|\Psi(h^2 \Delta_x) u_0\|_{L^2(\mathbb{R}^n)}. \quad (2.11)$$

□

Thus, the proof of Theorem 2.1 will be accomplished once we establish (2.10). In order to prove the Strichartz estimates (2.10), we split the operator A_h into two parts: a main term and a diffractive term. To this end, let $\chi(s)$ be a smooth function satisfying

$$\text{supp} \chi \subset (-\infty, -1], \quad \text{supp}(1 - \chi) \subset [-2, \infty).$$

We write this operator as a sum $A_h = M_h + D_h$, by decomposing

$$A_+(\zeta) = (\chi A_+)(\zeta) + ((1 - \chi) A_+)(\zeta),$$

and letting the "main term" be defined by

$$\begin{aligned} M_h f(x, t) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} 2\xi_1(a(x, \xi/h)(\chi A_+)(\zeta(x, \xi/h)) + b(x, \xi/h)(\chi A'_+)(\zeta(x, \xi/h))) \times \\ &\quad \times e^{\frac{i}{h}(\theta(x, \xi) + t\xi_1^2)} \frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))} \mathcal{F}(\Psi(h^2 \Delta_g) f)\left(\frac{\xi}{h}\right) d\xi. \end{aligned}$$

The "diffractive term" is then defined by

$$\begin{aligned} D_h f(x, t) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} 2\xi_1(a(x, \xi/h)((1 - \chi) A_+)(\zeta(x, \xi/h)) + b(x, \xi/h)((1 - \chi) A'_+)(\zeta(x, \xi/h))) \times \\ &\quad \times e^{\frac{i}{h}(\theta(x, \xi) + t\xi_1^2)} \frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))} \mathcal{F}(\Psi(h^2 \Delta_g) f)\left(\frac{\xi}{h}\right) d\xi. \end{aligned}$$

We analyze these operators separately:

2.2.1 The main term M_h

To estimate the "main term" M_h we first use the fact that

$$\left| \frac{Ai(s)}{A_+(s)} \right| \leq 2, \quad s \in \mathbb{R}. \quad (2.12)$$

Consequently, since the term $\frac{Ai(\zeta_0)}{A_+(\zeta_0)}$ acts like a multiplier, the estimates for M_h will follow from showing that the operator

$$f \rightarrow \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} 2\xi_1(a(x, \xi/h)(\chi A_+)(\zeta(x, \xi/h)) + b(x, \xi/h)(\chi A'_+)(\zeta(x, \xi/h))) \times \quad (2.13)$$

$$\times e^{\frac{i}{h}(\theta(x,\xi)+t\xi_1^2)} \hat{f}\left(\frac{\xi}{h}\right) d\xi$$

satisfies the same bounds like in (2.10). Following [35, Lemma 4.1], we write χA_+ and $(\chi A_+)'$ in terms of their Fourier transform to express the phase function of this operator,

$$\phi(t, x, \xi) = t\xi_1^2 + \theta(x, \xi) - \frac{2}{3}(-\zeta)^{3/2}(x, \xi) \quad (2.14)$$

which satisfies the eikonal equation (2.8). We denote its symbol $c_m(x, \xi/h)$, $c_m(x, \xi) \in \mathcal{S}_{2/3, 1/3}^0(\mathbb{R}^n \times \mathbb{R}^n)$ and we also denote the operator defined in (2.13) by W_h^m , thus

$$W_h^m f(x, t) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(t, x, \xi)} c_m(x, \xi/h) \hat{f}\left(\frac{\xi}{h}\right) d\xi.$$

We can use the Littlewood-Paley theory to see that if $\psi \in C_0^\infty(\mathbb{R}^*)$ is compactly supported away from 0 such that $\sum_k \psi(2^{-k}\lambda) = 1$, then it is sufficient to prove the following:

Proposition 2.2. *Let (q, r) be an admissible pair with $q > 2$, let $T > 0$ be sufficiently small and define*

$$W_h f(x, t) := \frac{1}{(2\pi h)^n} \int e^{i\frac{\phi(t, x, \xi)}{h}} c_m(x, \xi/h) \psi(|\xi|) \hat{f}\left(\frac{\xi}{h}\right) d\xi,$$

then the following estimates hold

$$\|W_h f\|_{L^q((0, T], L^r(\mathbb{R}^n))} \leq Ch^{-n/2(1/2-1/r)} \|f\|_{L^2(\mathbb{R}^n)}. \quad (2.15)$$

In the rest of this section we prove Proposition 2.2. The main step in the proof is to reduce matters to proving more symmetric inequalities involving dual space. Explicitly,

$$\mathcal{F}(W_h^*(F))\left(\frac{\xi}{h}\right) = \int e^{-\frac{i}{h}\phi(s, y, \xi)} F(y, s) \overline{c_m(y, \xi/h)} dy ds$$

and if we set

$$(T_h F)(x, t) = (W_h W_h^* F)(x, t) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi(t, x, \xi) - \phi(s, y, \xi))} c_m(x, \xi/h) \overline{c_m(y, \xi/h)} \psi^2(|\xi|) F(y, s) d\xi ds dy,$$

then inequality (2.15) is equivalent to

$$\|T_h F\|_{L^q((0, T], L^r(\mathbb{R}^n))} \leq Ch^{-n(1/2-1/r)} \|F\|_{L^{q'}((0, T], L^{r'}(\mathbb{R}^n))}, \quad (2.16)$$

where (q', r') satisfies $1/q + 1/q' = 1$, $1/r + 1/r' = 1$. To see, for instance, that (2.16) implies (2.15), notice that the dual version of (2.15) is

$$\|W_h^* F\|_{L^2(\mathbb{R}^n)} \leq Ch^{-n/2(1/2-1/r)} \|F\|_{L^{q'}((0, T], L^{r'}(\mathbb{R}^n))},$$

and we have

$$\|W_h^* F\|_{L^2(\mathbb{R}^n)}^2 = \int W_h W_h^* F \bar{F} dt dx \leq \|T_h F\|_{L^q((0, T], L^r(\mathbb{R}^n))} \|F\|_{L^{q'}((0, T], L^{r'}(\mathbb{R}^n))}.$$

Proof. Since the symbols are of type $(2/3, 1/3)$ and not of type $(1, 0)$, before starting the proof of (2.15) for the operator W_h we need to make a further decomposition: let $\rho \in C_0^\infty(\mathbb{R})$ satisfying $\rho(s) = 1$ near 0, $\rho(s) = 0$ if $|s| \geq 1$. Let

$$T_h F = W_h W_h^* F, \quad T_h F = T_h^f F + T_h^s F,$$

where

$$T_h^s F(x, t) = \int K_h^s(t, x, s, y) F(y, s) ds dy, \quad (2.17)$$

$$K_h^s(t, x, s, y) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi(t, x, \xi) - \phi(s, y, \xi))} (1 - \rho(h^{-1/3}|t - s|)) c_m(x, \xi/h) \overline{c_m(y, \xi/h)} \psi^2(|\xi|) d\xi, \quad (2.18)$$

while

$$T_h^f F(x, t) = \int K_h^f(t, x, s, y) F(y, s) ds dy, \quad (2.19)$$

$$K_h^f(t, x, s, y) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi(t, x, \xi) - \phi(s, y, \xi))} \rho(h^{-1/3}|t - s|) c_m(x, \xi/h) \overline{c_m(y, \xi/h)} \psi^2(|\xi|) d\xi. \quad (2.20)$$

Remark 2.3. *The two pieces will be handled differently. The kernel of T_h^f is supported in a suitable small set and it will be estimate by "freezing" the coefficients. To estimate T_h^s we shall use the stationary phase method for type $(1, 0)$ symbols. For type $(2/3, 1/3)$ symbols, these stationary phase arguments break down if $|t - s|$ is smaller than $h^{1/3}$, which motivates the decomposition. We use here the same arguments as in [29].*

- The "stationary phase admissible" term T_h^s

Proposition 2.4. *There is a constant $1 < C_0 < \infty$ such that the kernel K_h^s of T_h^s satisfies*

$$|K_h^s(t, x, s, y)| \leq C_N h^N \quad \forall N, \text{ if } \frac{|t - s|}{|x - y|} \notin [C_0^{-1}, C_0]. \quad (2.21)$$

On the other hand, there is a function $\xi_c(t, x, s, y)$ which is smooth in the variables (t, s) , uniformly over (x, y) , so that if $C_0^{-1} \leq \frac{|t - s|}{|x - y|} \leq C_0$, then

$$|K_h^s(t, x, s, y)| \simeq h^{-n} \left(1 + \frac{|t - s|}{h}\right)^{-n/2}, \text{ for } |t - s| \geq h^{1/3}. \quad (2.22)$$

Proof. We shall use stationary phase lemma to evaluate the kernel K_h^s of T_h^s . The critical points occur when $|t - s| \simeq |x - y|$. For some constant C_0 and for $|\xi| \in \text{supp}\psi$, ξ_1 in a small neighborhood of 1, we have

$$|\nabla_\xi(\phi(t, x, \xi) - \phi(s, y, \xi))| \simeq |t - s| + |x - y| \geq h^{1/3}, \text{ if } \frac{|t - s|}{|x - y|} \notin [C_0^{-1}, C_0].$$

Since $c \in S_{2/3, 1/3}^0$, an integration by parts leads to (2.21). If $|t - s| \simeq |x - y|$ we introduce a cutoff function $\kappa(\frac{|x-y|}{|t-s|})$ for some bump function κ . The phase function can be written as

$$\phi(t, x, \xi) - \phi(s, y, \xi) = (t - s)\Theta(t, x, s, y, \xi) \text{ for } |t - s| \simeq |x - y| \geq h^{1/3}.$$

We want to apply the stationary phase method with parameter $|t - s|/h \geq h^{-2/3} \gg 1$ to estimate K_h^s . For x, y, t, s fixed we must show that the critical points of Θ are non-degenerate.

Lemma 2.5. *The phase function $\Theta(t, x, s, y, \xi)$ admits a unique, non-degenerate critical point ξ_c . Moreover, for T sufficiently small and $0 \leq t, s \leq T$, the function $\xi_c(t, x, s, y)$ solving $\nabla_\xi \Theta(t, x, s, y, \xi_c) = 0$ is smooth in t and s , with uniform bounds on derivatives as x and y vary and we have*

$$|\partial_{t,s}^\alpha \partial_{x,y}^\gamma \xi_c(t, x, s, y)| \leq C_{\alpha,\gamma} h^{-|\alpha|/3}, \text{ if } |x - y| \geq h^{1/3}. \quad (2.23)$$

Proof. Setting $z = (x - y)/|t - s|$, the equation $\nabla_\xi \Theta(t, x, s, y, \xi) = 0$ writes

$$0 = \nabla_\xi(\xi_1^2 + \frac{1}{(t - s)}(\phi(0, x, \xi) - \phi(0, y, \xi))) =$$

$$\nabla_\xi(\langle \nabla_x \phi, \nabla_x \phi \rangle_g(0, x, \xi) - \frac{(x - y)}{(t - s)} \nabla_x \phi(0, x, \xi) + O(|x - y|)).$$

Let us write $\langle \nabla_x \phi, \nabla_x \phi \rangle_g = \sum_{j,k} g^{j,k}(x) \partial_{x_j} \phi \partial_{x_k} \phi$ and compute explicitly $\nabla_\xi \Theta$. For each $l \in \{1, \dots, n\}$ we have

$$0 = \partial_{\xi_l} \Theta = 2pos(l)(\nabla_x \nabla_\xi \phi) \begin{pmatrix} \sum_k g^{1,k}(x) \partial_{x_k} \phi - \frac{x_1 - y_1}{2(t-s)} \\ \vdots \\ \sum_k g^{n,k}(x) \partial_{x_k} \phi - \frac{x_n - y_n}{2(t-s)} \end{pmatrix} + O(|x - y|), \quad (2.24)$$

if $|x - y| \simeq |t - s|$ is small enough (notice that it is sufficient to take T small enough).

Here we have denoted $v_l = pos(l) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ the l -th position in the vector (v_1, \dots, v_n) .

We shall prove that there is a unique solution ξ_c to (2.24) and that

$$\begin{pmatrix} \sum_k g^{1,k}(x) \partial_{x_k} \phi - \frac{x_1 - y_1}{2(t-s)} \\ \vdots \\ \sum_k g^{n,k}(x) \partial_{x_k} \phi - \frac{x_n - y_n}{2(t-s)} \end{pmatrix} \Big|_{\xi_c} = O(|x - y|). \quad (2.25)$$

Notice that we will be finished once we show (2.25), since then for $q \in \{1, \dots, n\}$

$$\begin{aligned} \partial_{\xi_q} \partial_{\xi_l} \Theta &= 2 \text{pos}(l) (\nabla_x \nabla_\xi \partial_{\xi_q} \phi) \begin{pmatrix} \sum_k g^{1,k}(x) \partial_{x_k} \phi - \frac{x_1 - y_1}{2(t-s)} \\ \vdots \\ \sum_k g^{n,k}(x) \partial_{x_k} \phi - \frac{x_n - y_n}{2(t-s)} \end{pmatrix} + O(|x - y|) + \\ &+ \text{pos}(l) \left((\nabla_x \nabla_\xi \phi) (g^{ij}) \text{col}(q) (\nabla_x \nabla_\xi \phi) \right), \end{aligned} \quad (2.26)$$

where $\text{col}(q)M$ denotes the q -th column of a matrix $M \in \mathcal{M}_n$. Using (2.26) we can compute $\nabla_\xi^2 \Theta|_{\xi_c}$,

$$\partial_{\xi_q} \partial_{\xi_l} \Theta|_{\xi_c} = \left((\nabla_x \nabla_\xi \phi) (g^{ij})_{i,j} (\nabla_x \nabla_\xi \phi) \right)_{q,l} + O(|x - y|),$$

consequently for T small enough the critical point ξ_c is non-degenerate.

Let come back to equation (2.24). In order to prove (2.25) it is enough to show that $\nabla_x \nabla_\xi \phi(0, x, \xi)$ is invertible. To deduce this we need to use the precise form of the initial phase function in (2.13). Writing again χA_+ and $(\chi A_+)'$ in terms of their Fourier transform, we express the phase function of (2.13) in the form

$$t\xi_1^2 + \theta(x, \xi) + \sigma\xi_1^{-2/3} \zeta(x, \xi) + \sigma^3 / 3\xi_1^2 - \langle z, \xi \rangle, \quad (2.27)$$

where $\langle z, \xi \rangle$ denotes the scalar product. Notice that at $t = 0$ this phase is homogeneous of degree 1 in ξ and the following holds

Lemma 2.6. (see [35, Lemma 3.1], [29, Lemma A.2]) *At $t = 0$, the phase function defined in (2.27) is the graph of a canonical transformation.*

Proof. The proof reduces to checking that the Jacobian J of the mapping

$$(\xi, \sigma) \rightarrow (\nabla_x(\theta(x, \xi) + \sigma\zeta(x, \xi)), \zeta(x, \xi) + \sigma^2)$$

does not vanish at the glancing point ρ defined by $x = \bar{x}$, $\xi_1 = 1$, $z = 0$ (for the notations see Section 5.1). At this (critical) point $\sigma = \zeta(x, \xi) = 0$ and $\nabla_x \zeta(x, \xi) = 0$. Since

$\partial_{x_n}\zeta(x, \xi) \neq 0$ and $\partial_{\xi_n}\zeta(x, \xi) \neq 0$ there, the result follows by the nonvanishing of $|\nabla_{x'}\nabla_{\xi'}\theta(x, \xi)|$. In fact

$$J|_{\rho} = \det \left| \begin{pmatrix} \nabla_{x'}\nabla_{\xi'}\theta & \nabla_{\xi'}\partial_{x_n}\theta & \nabla_{\xi'}\zeta = 0 \\ \partial_{\xi_n}\nabla_{x'}\theta & \partial_{\xi_n}\partial_{x_n}\theta = 0 & \partial_{\xi_n}\zeta \neq 0 \\ \nabla_{x'}\zeta = 0 & \partial_{x_n}\zeta \neq 0 & 2\sigma = 0 \end{pmatrix} \right|_{\rho} \neq 0.$$

□

This geometric property can be restated in an equivalent form by saying that at $t = 0$, $\phi(0, x, \xi)$ satisfies

$$\det \left(\nabla_{\xi} \nabla_x \phi \right) (0, x, \xi) \neq 0$$

and thus equation (2.24) admits a critical point which satisfies (2.23) for $|x - y| \geq h^{1/3}$ and, on the other hand the same argument allows to obtain (2.25). □

On the support of κ it follows that the kernel K_h^s writes

$$K_h^s(t, x, s, y) = \frac{1}{(2\pi h)^n} \int e^{i|t-s|\Theta(t, x, s, y, \xi)} \psi^2(|\xi|) (1 - \rho(h^{-1/3}|t-s|)) c_m(x, \xi/h) \overline{c_m(y, \xi/h)} d\xi, \quad (2.28)$$

where, if $\omega = |t - s|/h$ and $\xi_1 \simeq 1$, the symbol satisfies

$$|\partial_{t,s}^{\alpha} \partial_{\omega}^k \sigma_h(t, x, s, y, \omega\xi/|t-s|)| \leq C_{\alpha,k} h^{-|\alpha|/3} (|t-s|^{3/2}/h)^{-2k/3},$$

where we set

$$\sigma_h(t, x, s, y, \omega\xi/|t-s|) = (1 - \rho(h^{-1/3}|t-s|)) c_m(x, \omega\xi/|t-s|) \overline{c_m(y, \omega\xi/|t-s|)}.$$

Indeed, since $c_m \in S_{2/3, 1/3}^0$, for $\alpha = 0$ one has

$$|\partial_{\omega}^k \sigma_h| \leq |\xi| |t-s|^{-k} |(\partial_{\xi}^k c)(t, x, \omega\xi/|t-s|)| \leq C_{0,k} |t-s|^{-k} (\omega/|t-s|)^{-2k/3} = C_{0,k} |t-s|^{-k} h^{2k/3}.$$

We conclude using the next lemma with $\omega = \frac{|t-s|}{h}$ and $\delta = |t-s|^{3/2} \geq h^{1/2} \gg h$.

Lemma 2.7. *Suppose that $\Theta(z, \xi) \in C^{\infty}(\mathbb{R}^{2(n+1)} \times \mathbb{R}^n)$ is real, $\nabla_{\xi}\Theta(z, \xi_c(z)) = 0$, $\nabla_{\xi}\Theta(z, \xi) \neq 0$ if $\xi \neq \xi_c(z)$, and*

$$|\det \nabla_{\xi\xi}^2 \Theta| \geq c_0 > 0, \text{ if } |\xi| \leq 1.$$

Suppose also that

$$|\partial_z^{\alpha} \partial_{\xi}^{\beta} \Theta(z, \xi)| \leq C_{\alpha,\beta} h^{-|\alpha|/3}, \quad \forall \alpha, \beta.$$

In addition, suppose that the symbol $\sigma_h(z, \xi, \omega)$ vanishes when $|\xi| \geq 1$ and satisfies

$$|\partial_z^\alpha \partial_\xi^\gamma \partial_\omega^k \sigma_h(z, \xi, \omega)| \leq C_{k, \alpha, \gamma} h^{-(|\alpha| + |\gamma|)/3} (\delta/h)^{-2k/3}, \quad \forall k, \alpha, \gamma,$$

where on the support of σ_h we have $\omega \geq h^{-2/3}$ and $\delta > 0$. Then we can write

$$\int_{\mathbb{R}^n} e^{i\omega\Theta(z, \xi)} \sigma_h(z, \xi, \omega) d\xi = \omega^{-n/2} e^{i\omega\Theta(z, \xi_c(z))} b_h(z, \omega),$$

where b_h satisfies

$$|\partial_\omega^k \partial_z^\alpha b_h(z, \omega)| \leq C_{k, \alpha} h^{-|\alpha|/3} (\delta/h)^{-2k/3}$$

and where each of the constants depend only on c_0 and the size of finitely many of the constants $C_{\alpha, \beta}$ and $C_{k, \alpha, \gamma}$ above. In particular, the constants are uniform in δ if $1 \geq \delta \geq h$.

This Lemma is used in [29, Lemma 2.6] and also in the thesis of Grieser [18] and it follows easily from the proof of standard stationary phase lemma (see [31, pag. 45]). Proposition 2.4 is thus proved. \square

For each t, s , let $T_h^s(t, s)$ be the "frozen" operator defined by

$$T_h^s(t, s)g(x) = \int K_h^s(t, x, s, y)g(y)dy.$$

From Proposition 2.4 we deduce

$$\|T_h^s(t, s)g\|_{L^\infty(\mathbb{R}^n)} \leq C \max(h^{-n}, (h|t - s|)^{-n/2}) \|g\|_{L^1(\mathbb{R}^n)}. \quad (2.29)$$

We need the following

Lemma 2.8. *For t, s fixed the frozen operator $T_h^s(t, s)$ is bounded on $L^2(\mathbb{R}^n)$,*

$$\|T_h^s(t, s)g\|_{L^2(\mathbb{R}^n)} \leq C \|g\|_{L^2(\mathbb{R}^n)}. \quad (2.30)$$

Proof. Explicitly we have

$$T_h^s(t, s)g(x) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi(t, x, \xi) - \phi(s, y, \xi))} (1 - \rho(h^{-1/3}|t - s|)) c_m(x, \xi/h) \overline{c_m(y, \xi/h)} \psi^2(|\xi|) g(y) d\xi dy$$

and (since $0 \leq \rho(h^{-1/3}|t - s|) \leq 1$) it is enough to prove that the frozen operator $W_h(\cdot, t)$ is bounded on $L^2(\mathbb{R}^n)$. This follows from the energy conservation $\|A_h f(\cdot, t)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$ together with the fact that the $L^2(\mathbb{R}^n)$ norms of $A_h f$, $M_h f$ and thus $W_h f$ are equivalent. \square

Interpolation between (2.29) and (2.30) with weights $1 - 2/r$ and $2/r$ respectively yields

$$\|T_h^s(t, s)g\|_{L^r(\mathbb{R}^n)} \leq Ch^{-n(1-2/r)} \left(1 + \frac{|t-s|}{h}\right)^{-n(1/2-1/r)} \|g\|_{L^{r'}(\mathbb{R}^n)} \quad (2.31)$$

and hence

$$\|T_h^s F\|_{L^q(0,T), L^r(\mathbb{R}^n)} \leq Ch^{-n/2(1-2/r)} \left\| \int_{1 \ll \frac{|t-s|}{h}}^T |t-s|^{-n/2(1-2/r)} \|F(\cdot, s)\|_{L^{r'}(\mathbb{R}^n)} ds \right\|_{L^{q'}((0,T))}.$$

Since $n(\frac{1}{2} - \frac{1}{r}) = \frac{2}{q} < 1$ the application $|t|^{-2/q} : L^{q'} \rightarrow L^q$ is bounded and by Hardy-Littlewood-Sobolev inequality we deduce

$$\|T_h^s F\|_{L^q((0,T), L^r(\mathbb{R}^n))} \leq Ch^{-n(1/2-1/r)} \|F\|_{L^{q'}((0,T), L^{r'}(\mathbb{R}^n))}. \quad (2.32)$$

- The "frozen" term T_h^f

To estimate T_h^f it suffices to obtain bounds for its kernel K_h^f with both the variables (t, x) and (s, y) restricted to lie in a cube of \mathbb{R}^{n+1} of sidelength comparable to $h^{1/3}$. Let us decompose S_T into disjoint cubes $Q = Q_x \times Q_t$ of sidelength $h^{1/3}$. We then have

$$\|T_h^f F\|_{L^q([0,T], L^r(\mathbb{R}^n))}^q = \int_0^T \left(\sum_{Q=Q_t \times Q_x} \|\chi_Q T_h^f F\|_{L^r(Q_x)}^r \right)^{q/r} dt = \sum_Q \|\chi_Q T_h^f F\|_{L^q([0,T], L^r(\mathbb{R}^n))}^q.$$

In fact, by the definition, the integral kernel $K_h^f(t, x, s, y)$ of T_h^f vanishes if $|t-s| \geq h^{1/3}$. If $|t-s| \leq h^{1/3}$ and $|x-y| \geq C_0 h^{1/3}$, then the phase

$$\phi(t, x, \xi) - \phi(s, y, \xi)$$

has no critical points with respect to ξ_1 (on the support of ψ), so that

$$|K_h^f(t, x, s, y)| \leq C_N h^N \quad \forall N, \text{ if } |x-y| \geq C_0 h^{1/3}.$$

It therefore suffices to estimate $\sum_Q \|\chi_Q T_h^f \chi_{Q^*} F\|_{L^q([0,T], L^r(\mathbb{R}^n))}$, where Q^* is the dilate of Q by some fixed factor independent of h (and where by χ_Q we denoted the characteristic function of the cube Q). Since $q > 2 > q'$, $r \geq 2 \geq r'$, where q' , r' are such that $1/q + 1/q' = 1$, $1/r + 1/r' = 1$, then we shall obtain

$$\sum_Q \|\chi_Q T_h^f \chi_{Q^*} F\|_{L^q([0,T], L^r(\mathbb{R}^n))}^q \leq C_1 \sum_Q \|\chi_{Q^*} F\|_{L^{q'}([0,T], L^{r'}(\mathbb{R}^n))}^q \leq C_2 \|F\|_{L^{q'}([0,T], L^{r'}(\mathbb{R}^n))}^q. \quad (2.33)$$

In order to prove (2.33) we shall use the following:

Proposition 2.9. *Let $b(\xi) \in L^\infty(\mathbb{R}^n)$ be elliptic near $\xi_1 \simeq 1$, $b_h(\xi) := b(\xi/h)$, then for $h \ll |t - s| \leq h^{1/3}$, $h \ll |x - y| \leq h^{1/3}$ the operator defined by*

$$B_h f(x, t) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t, x, \xi)} \psi(|\xi|) b_h(\xi) \hat{f}\left(\frac{\xi}{h}\right) d\xi \quad (2.34)$$

satisfies

$$\|B_h f\|_{L^q((0, T], L^r(\mathbb{R}^n))} \leq C h^{-n/2(1/2-1/r)} \|f\|_{L^2(\mathbb{R}^n)}. \quad (2.35)$$

Proof. We use the TT^* argument. Since $b(\xi)$ acts as an L^2 multiplier we can apply the stationary phase theorem in the integral

$$\int e^{\frac{i}{h}(\phi(t, x, \xi) - \phi(s, y, \xi))} \psi(|\xi|) d\xi$$

in order to obtain

$$\|B_h B_h^* F\|_{L^q((0, T], L^r(\mathbb{R}^n))} \lesssim h^{-n(1/2-1/r)} \|F\|_{L^{q'}((0, T], L^{r'}(\mathbb{R}^n))}.$$

Noticed that we haven't used the special properties of the phase function at $t = 0$. \square

Let now Q be a fixed cube in \mathbb{R}^{n+1} of sidelength $h^{1/3}$. Let

$$b_h(t, x, s, y, \xi) = \rho(h^{-1/3}|t - s|) c_m(x, \xi/h) \overline{c_m(y, \xi/h)},$$

and write

$$\begin{aligned} b_h(t, x, s, y, \xi) &= b_h(0, 0, s, y, \xi) + \int_0^t \partial_t b_h(r, 0, s, y, \xi) dr + \dots \\ &+ \dots + \int_0^t \dots \int_0^{x_n} \partial_t \dots \partial_{x_n} b_h(r, z_1, \dots, z_n, s, y, \xi) dr dz. \end{aligned}$$

If the symbol c is independent of t, x then the estimates (2.15) follow from Proposition 2.9. We use this, for instance, to deduce

$$\begin{aligned} &\|T_h^f F\|_{L^q((0, T], L^r(\mathbb{R}^n))} \leq \\ &C h^{-n/2(1/2-1/r)} (\| \int \int e^{\frac{i}{h}(x\xi - \phi(s, y, \xi))} \psi(|\xi|) b_h(0, 0, s, y, \xi) F(s, y) d\xi ds dy \|_{L^2(\mathbb{R})} + \dots \\ &\dots + \| \int_0^{h^{1/3}} \dots \int_0^{h^{1/3}} \int \int e^{\frac{i}{h}(x\xi - \phi(s, y, \xi))} \partial_t \dots \partial_{x_n} \psi(|\xi|) b_h(r, z, s, y, \xi) F(s, y) d\xi ds dy \|_{L^2(\mathbb{R}^n)} dr dz. \end{aligned}$$

Each derivative of $b_h(t, x, s, y, \xi)$ loses a factor of $h^{-1/3}$, but this is compensated by the integral over (r, z) , so that it suffices to establish uniform estimates for fixed (r, z) . By duality, we have to establish the estimate

$$\left\| \int \int e^{\frac{i}{h}\phi(s, y, \xi)} \psi(|\xi|) b_h(0, 0, s, y, \xi) \hat{f}\left(\frac{\xi}{h}\right) d\xi \right\|_{L^q((0, T], L^r(\mathbb{R}^n))} \leq C \|f\|_{L^2(\mathbb{R}^n)},$$

which follows by using the same argument of freezing the variables (s, y) together with Proposition 2.9. □

2.2.2 The diffractive term D_h

In order to estimate the diffractive term we shall reason again like in [29, Sect.2].

Lemma 2.10. *For $x_n \geq 0$ and for ξ in a small conic neighborhood of the positive ξ_1 axis, the symbol q of S_h can be written in the form*

$$\begin{aligned} q(x, \xi) &:= (a(x, \xi)((1 - \chi)A_+)(\zeta(x, \xi)) + ib(x, \xi)((1 - \chi)A_+)'(\zeta(x, \xi))) \frac{Ai(\zeta_0(\xi))}{A_+(\zeta_0(\xi))} \\ &= p(x, \xi, \zeta(x, \xi)), \end{aligned}$$

where, for some $c > 0$

$$|\partial_\xi^\alpha \partial_\zeta^j \partial_x^\beta \partial_{x_n}^k p(x, \xi, \zeta(x, \xi))| \leq C_{\alpha, j, \beta, k} \xi_1^{1/6 - |\alpha| + 2k/3} e^{-cx_n^{3/2} \xi_1 - |\zeta|^{3/2}/2}.$$

Proof. Since

$$|\partial_\zeta^k ((1 - \chi)A_+)(\zeta)| \leq C_{k, \epsilon} e^{(2/3 + \epsilon)|\zeta|^{3/2}}, \quad \forall \epsilon > 0,$$

and the symbols a and b belong to $S_{1,0}^{1/6}$, the above fact will follow by showing that in the region $\zeta(x, \xi) \geq -2$,

$$\frac{Ai}{A_+}(\zeta_0(\xi)) = \tilde{p}(x, \xi', \zeta(x, \xi)),$$

where if $\xi' = (\xi_1, \dots, \xi_{n-1})$

$$|\partial_{\xi'}^\alpha \partial_\zeta^j \partial_x^\beta \partial_{x_n}^k \tilde{p}(x, \xi', \zeta)| \leq C_{\alpha, j, \beta, k, \epsilon} \xi_1^{-|\alpha| + 2k/3} e^{-cx_n^{3/2} \xi_1 - (4/3 - \epsilon)|\zeta|^{3/2}}. \quad (2.36)$$

At $x_n = 0$, one has $\zeta = \zeta_0$, $\partial_{x_n} \zeta < 0$. It follows that for some $c > 0$

$$\zeta_0(x, \xi) \geq \zeta(x, \xi) + cx_n \xi_1^{2/3}.$$

By the asymptotic behavior of the Airy function we have, in the region $\zeta(x, \xi) \geq -2$

$$\left| \left(\frac{Ai}{A_+} \right)^{(k)}(\zeta_0) \right| \leq C_{k,\epsilon} e^{-cx_n^{3/2}\xi_1 - (4/3 - \epsilon)|\zeta(x, \xi)|^{3/2}}. \quad (2.37)$$

We introduce a new variable $\tau(x, \xi) = \xi_1^{1/3}\zeta(x, \xi)$. At $x_n = 0$ one has $\tau = -\xi_n$, so that we can write $\xi_n = \sigma(x, \xi', \tau)$, where σ is homogeneous of degree 1 in (ξ', τ) . We set

$$\tilde{p}(x, \xi', \zeta) = \frac{Ai}{A_+}(-\xi_1^{-1/3}\sigma(x, \xi', \xi_1^{1/3}\zeta)).$$

The estimates (2.36) will follow by showing that

$$|\partial_{\xi'}^\alpha \partial_\tau^j \partial_{x'}^\beta \partial_{x_n}^k \frac{Ai}{A_+}(-\xi_1^{-1/3}\sigma(x, \xi', \tau))| \leq C_{\alpha,j,\beta,k,\epsilon} \xi_1^{-|\alpha| - j + 2k/3} e^{-cx_n^{3/2}\xi_1 - (4/3 - \epsilon)|\tau|^{3/2}\xi_1^{-1/2}}. \quad (2.38)$$

For $k = 0$, the estimates (2.38) follow from (2.37), together with the fact that

$$|\partial_{\xi'}^\alpha \partial_\tau^j \partial_{x'}^\beta \frac{Ai}{A_+}(-\xi_1^{-1/3}\sigma(x, \xi', \tau))| \leq C_{\alpha,\beta,j}(x_n \xi_1^{2/3} + \xi_1^{-1/3}|\tau|)\xi_1^{-|\alpha| - j},$$

which, in turn, holds by homogeneity, together with the fact that $\sigma(x, \xi', \tau) = 0$ if $x_n = \tau = 0$. If $k = 0$, the estimate (2.38) follows by observing that the effect of differentiating in x_n is similar to multiplying by a symbol of order $2/3$. This concludes the proof of Lemma 2.10. \square

We now write the Schwartz kernel of the diffractive term D_h in the form

$$\begin{aligned} & \int e^{i(\theta(x, \xi) + ht\xi_1^2)} \psi(h|\xi|) q(x, \xi) d\xi \\ &= \int e^{i(\theta(x, \xi) + ht\xi_1^2 + s\xi_1^{-2/3}\zeta(x, \xi) + s^3/3\xi_1^{-2} \langle y, \xi \rangle)} \psi(h|\xi|) a(x, \xi, s\xi_1^{-2/3}) ds d\xi, \end{aligned}$$

where

$$|\partial_\xi^\alpha \partial_s^j \partial_{x'}^\beta \partial_{x_n}^k a(x, \xi, s\xi_1^{-2/3})| \leq C_{\alpha,j,\beta,k,N} \xi_1^{-1/2 - |\alpha| - 2j/3 + 2k/3} e^{-cx_n^{3/2}\xi_1} \langle \xi^{-2/3} s \rangle^{-N}, \quad \forall N.$$

From now on we proceed as for the main term and we reduce the problem to considering the operator

$$W_h^d f(x, t) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t, x, \xi)} c_d(t, x, \xi/h) \psi(|\xi|) \hat{f}\left(\frac{\xi}{h}\right) d\xi,$$

where now $x_n^j \partial_{x_n}^k c_d(t, x, \xi) \in S_{2/3, 1/3}^{2(j-k)/3}(\mathbb{R}_{x', t}^n \times \mathbb{R}_\xi^n)$ uniformly over x_n . Using the freezing arguments behind the proof of the estimates for T_h^f and Minkovski inequality we have

$$\|W_h^d f\|_{L^q((0, T], L^r(\mathbb{R}^n))} \leq \left\| \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t, x, \xi)} c_d(t, x', 0, \xi/h) \psi(|\xi|) \hat{f}\left(\frac{\xi}{h}\right) d\xi \right\|_{L^q((0, T], L^r(\mathbb{R}^n))} +$$

$$h^{-2/3} \int_0^{h^{2/3}} \left\| \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t,x,\xi)} (h^{2/3} \partial_{x_n} c_d(t, x', r, \xi/h)) \psi(|\xi|) \hat{f}\left(\frac{\xi}{h}\right) d\xi \right\|_{L^q((0,T], L^r(\mathbb{R}^n))} dr +$$

$$h^{2/3} \int_{r>h^{2/3}} \frac{dr}{r^2} \left\| \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t,x,\xi)} (h^{-2/3} r^2 \partial_{x_n} c_d(t, x', r, \xi/h)) \psi(|\xi|) \hat{f}\left(\frac{\xi}{h}\right) d\xi \right\|_{L^q((0,T], L^r(\mathbb{R}^{n-1} \times \{x_n > r\}))}.$$

Since $c_d(t, x', 0, \xi)$ and $h^{2/3}(1 + h^{-4/3}r^2)\partial_{x_n}c_d(t, x', r, \xi)$ are symbols of order 0 and type $(2/3, 1/3)$ with uniform estimates over r , the estimates for the diffractive term also follow from Proposition 2.2. Indeed, the term in the second line loses a factor $h^{-2/3}$, but this is compensated by the integral over $r \leq h^{2/3}$. The term in the third line can be bounded by above by

$$\left\| \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t,x,\xi)} (h^{-2/3} r^2 \partial_{x_n} c_d(t, x', r, \xi/h)) \psi(|\xi|) \hat{f}\left(\frac{\xi}{h}\right) d\xi \right\|_{L^q((0,T], L^r(\mathbb{R}^n))} \times h^{2/3} \int_{r>h^{2/3}} \frac{dr}{r^2}$$

$$\leq \left\| \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t,x,\xi)} (h^{-2/3} r^2 \partial_{x_n} c_d(t, x', r, \xi/h)) \psi(|\xi|) \hat{f}\left(\frac{\xi}{h}\right) d\xi \right\|_{L^q((0,T], L^r(\mathbb{R}^n))}.$$

We conclude using the same arguments as in the proof of Proposition 2.2, where now W_h is replaced by operators with symbols $c_d(t, x', 0, \xi)$, $h^{2/3}\partial_{x_n}c_d(t, x', r, \xi)$ and $h^{-2/3}r^2\partial_{x_n}c_d(t, x', r, \xi)$ respectively. □

3 Classical Strichartz estimates

In this section we achieve the proof of Theorem 1.1 using essentially Theorem 2.1. We work here with the Laplace-Beltrami operator Δ_g defined in (1.4), under the assumption that its principal part is uniformly elliptic.

Following [5], we split $u(x, t) = (e^{it\Delta_g}u_0)(x)$ as a sum of two terms $u = \chi u + (1 - \chi)u$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ equals 1 near the boundary $\partial\Omega$.

- Study of $v = \chi u$. We have

$$i\partial_t v - \Delta_g v = [\Delta_g, \chi]u, \quad v|_{t=0} = \chi u_0. \quad (3.1)$$

Let $\Psi \in C_0^\infty(\mathbb{R}_+)$ like before and let $\varphi \in C_0^\infty((-1, 2))$ equal to 1 on $[0, 1]$. Let $v_h = \Psi(h^2\Delta_g)v$ and for $l \in \mathbb{Z}$, $v_{h,l} = \varphi(t/h - l)v_h$, solution to

$$\begin{cases} i\partial_t v_{h,l} - \Delta_g v_{h,l} = \varphi(t/h - l)\Psi(h^2\Delta_g)[\Delta_g, \chi]u + i\frac{\varphi'(t/h-l)}{h}\Psi(h^2\Delta_g)\chi u, \\ v_{h,l}|_{t < hl-h} = 0, \quad v_{h,l}|_{t > hl+2h} = 0. \end{cases} \quad (3.2)$$

If we denote $V_{h,l}$ the right-hand side of (3.2), the Duhamel formula writes

$$v_{h,l}(x, t) = \int_{hl-l}^t e^{i(t-s)\Delta_g} V_{h,l}(x, s) ds.$$

Using Minkovski inequality we obtain

$$\|v_{h,l}\|_{L^q([hl-h, hl+2h], L^r(\Omega))} \leq C \int_{[hl-h, hl+2h]} \|e^{i(t-s)\Delta_g} V_{h,l}(s)\|_{L^q([hl-h, hl+2h], L^r(\Omega))} ds$$

and using Theorem 2.1

$$\begin{aligned} \|v_{h,l}\|_{L^q([hl-h, hl+2h], L^r(\Omega))} &\leq C \int_{[hl-h, hl+2h]} \|V_{h,l}(s)\|_{L^2(\Omega)} ds \\ &\leq Ch^{1/2} \|V_{h,l}\|_{L^2([hl-h, hl+2h], L^2(\Omega))} \end{aligned} \quad (3.3)$$

and since $q \geq 2$,

$$\begin{aligned} \|v_h\|_{L^q(\mathbb{R}, L^r(\Omega))}^q &\leq Ch^{q/2} \sum_{l=-\infty}^{\infty} \|V_{h,l}(s)\|_{L^2([hl-h, hl+2h], L^2(\Omega))}^q \\ &\leq Ch^{q/2} \left(\sum_{l=-\infty}^{\infty} \|V_{h,l}\|_{L^2([hl-h, hl+2h], L^2(\Omega))}^2 \right)^{q/2}. \end{aligned} \quad (3.4)$$

Thanks to the spectral cutoff Ψ , we obtain

$$\begin{aligned} &\|V_{h,l}\|_{L^2([hl-h, hl+2h], L^2(\Omega))} \\ &= \|\varphi(t/h - l)\Psi(h^2\Delta_g)[\Delta_g, \chi]u + i\frac{\varphi'(t/h - l)}{h}\Psi(h^2\Delta_g)\chi u\|_{L^2([hl-h, hl+2h], L^2(\Omega))} \\ &\leq Ch^{-1/2}\|\varphi(t/h - l)\Psi(h^2\Delta_g)[\Delta_g, \chi]u\|_{L^2([hl-h, hl+2h], H^{-1/2}(\Omega))} \\ &\quad + Ch^{1/2}\|\frac{\varphi'(t/h - l)}{h}\Psi(h^2\Delta_g)\chi u\|_{L^2([hl-h, hl+2h], H^{1/2}(\Omega))}. \end{aligned} \quad (3.5)$$

If $\tilde{\varphi} \in C_0^\infty(\mathbb{R})$, $\tilde{\psi} \in C_0^\infty(\mathbb{R}^*)$ and $\tilde{\chi} \in C_0^\infty(\Omega)$ are chosen such that they are equal to 1 on the supports of φ , Ψ , χ , we get, modulo $O(h^\infty)$

$$\|V_{h,l}\|_{L^2([hl-h, hl+2h], L^2(\Omega))} \leq Ch^{-1/2}\|\tilde{\varphi}(t/h - l)\tilde{\psi}(h^2\Delta_g)\tilde{\chi}u\|_{L^2([hl-h, hl+2h], H^{1/2}(\Omega))}, \quad (3.6)$$

from which we deduce, using also (3.4)

$$\|v_h\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq C\|\tilde{\psi}(h^2\Delta_g)\tilde{\chi}u\|_{L^2(\mathbb{R}, H^{1/2}(\Omega))}. \quad (3.7)$$

We also need the next result

Proposition 3.1. (*[6, Prop.2.7]*) Assume that $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, $\mathcal{O} \neq \emptyset$. Then for $T > 0$, for every $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, one has

$$\|\tilde{\chi}u\|_{L^2([-T,T],H^{s+1/2}(\Omega))} \leq C\|u_0\|_{H^s(\Omega)},$$

where $s \in [0, 1]$ and $u(x, t) = e^{it\Delta_g}u_0(x)$. Moreover, the constant C do not depend on T , i.e. the estimates are global in time.

In order to finish the proof in this case it suffices to use the next lemma:

Lemma 3.2. Let $\Psi_0 \in C_0^\infty(\mathbb{R})$, $\Psi \in C_0^\infty(\mathbb{R}^*)$ such that

$$\Psi_0(\lambda) + \sum_{j \geq 1} \Psi(2^{-2j}\lambda) = 1, \quad \forall \lambda \in \mathbb{R}.$$

Then for all $r \in [2, \infty)$ we have

$$\|f\|_{L^r(\Omega)} \leq C_r \left(\|\Psi_0(-\Delta_g)f\|_{L^r(\Omega)} + \left(\sum_{j=1}^{\infty} \|\Psi(-2^{-2j}\Delta_g)f\|_{L^r(\Omega)}^2 \right)^{1/2} \right). \quad (3.8)$$

A complete and elementary proof of this lemma is given in [20].

Remark 3.3. Notice that here we have defined the fractional Sobolev spaces through the spectral localization, following [27]; they do coincide with the usual ones $B_r^{0,2}(\Omega)$ in the range we are interested in and the proof follows from [34].

Now, applying Lemma 3.2 to $f = \chi u(\cdot, t)$ and taking the L^q norm in time yields

$$\|\chi u\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq \left\| \left(\|\Psi_0(-\Delta_g)u_0\|_{L^r(\Omega)} + \left(\sum_{j \geq 1} \|e^{it\Delta_g}\Psi(-2^{-2j}\Delta_g)u_0\|_{L^r(\Omega)}^2 \right)^{1/2} \right) \right\|_{L^q(\mathbb{R})} \quad (3.9)$$

which, by Minkowski inequality and Theorem 2.1 leads to

$$\|\chi u\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}. \quad (3.10)$$

- Study of $w = (1 - \chi)u$. We follow here the method suggested by Staffilani and Tataru [32]. We have

$$i\partial_t w - \Delta_g w = [\Delta_g, \chi]u, \quad w|_{t=0} = (1 - \chi)u_0. \quad (3.11)$$

Since on the support of w we have $g = (\delta_{i,j})$, i.e. the flat metric, we can replace in (3.11) Δ_g by the Euclidian Laplace operator Δ (on \mathbb{R}^n) and we find

$$w(x, t) = e^{it\Delta}(1 - \chi)u_0 - i \int_0^t e^{i(t-s)\Delta} [\Delta_g, \chi]u(s) ds. \quad (3.12)$$

The contribution of the first term can be easily evaluated. For the second term we shall use the next lemma, due to Christ and Kiselev [11]:

Lemma 3.4. (Christ and Kiselev) Consider a bounded operator

$$T : L^{q'}(\mathbb{R}, B_1) \rightarrow L^q(\mathbb{R}, B_2)$$

given by a locally integrable kernel $K(t, s)$ with values in bounded operators from B_1 to B_2 , where B_1 and B_2 are Banach spaces. Suppose that $q' < q$. Then the operator

$$\tilde{T}f(t) = \int_{s < t} K(t, s)f(s)ds$$

is bounded from $L^{q'}(\mathbb{R}, B_1)$ to $L^q(\mathbb{R}, B_2)$ and

$$\|\tilde{T}\|_{L^{q'}(\mathbb{R}, B_1) \rightarrow L^q(\mathbb{R}, B_2)} \leq C(1 - 2^{-(1/q - 1/q')})^{-1} \|T\|_{L^{q'}(\mathbb{R}, B_1) \rightarrow L^q(\mathbb{R}, B_2)}.$$

This lemma allows (since $q > 0$) to replace the study of the second term in the right hand side of (3.12) by that of

$$W(x, t) = \int_0^\infty e^{i(t-s)\Delta} [\Delta_g, \chi]u(s)ds = U_0 U_0^*(x, t),$$

where $U_0 = e^{it\Delta}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^q(\mathbb{R}, L^r(\mathbb{R}^n))$ and U_0^* is bounded from $L^2(\mathbb{R}, H_{comp}^{-1/2})$ to $L^2(\mathbb{R}^n)$, which gives the estimates for W and thus for w .

4 Applications

In this section we sketch the proofs of Theorem 1.2 and Theorem 1.3.

We start with Theorem 1.2. From Theorem 1.1 we have an estimate of the linear flow of the Schrödinger equation

$$\|e^{-it\Delta_g} u_0\|_{L^5(\mathbb{R}, L^{30/11}(\Omega))} \leq C \|u_0\|_{L^2(\Omega)}.$$

One may shift regularity by 1 and obtain

$$\|e^{-it\Delta_g} u_0\|_{L^5(\mathbb{R}, W^{1,30/11}(\Omega))} \leq C \|u_0\|_{H_0^1(\Omega)}.$$

For $T > 0$, let $X_T := L^5((0, T], W^{1,30/11}(\Omega))$, one may then set up the usual fixed point argument in X_T .

Let us proceed with Theorem 1.3. From [27], one has a global in time control on the solution u , at the level of $\dot{H}^{\frac{1}{4}}$ regularity:

$$u \in L^4((0, +\infty), L^4(\Omega)).$$

By interpolation with either mass or energy conservation, combined with the local existence theory, one may bootstrap this global in time control into

$$u \in L^{p-1}((0, +\infty), L^\infty(\Omega)),$$

from which scattering in $H_0^1(\Omega)$ follows immediately.

5 Appendix

5.1 Construction of Melrose and Taylor parametrices: general theory.

We provide in this appendix an outline of the construction of Melrose and Taylor parametrices for diffractive problems. The material is originally from the papers [28], [26], [35].

Let x denote the local geodesic normal coordinates on $\partial\Omega$ such that the normal coordinate $x_n > 0$ defines Ω and such that the principal symbol of $-\Delta_g$ on $T^*(\Omega)$ writes

$$\langle \eta, \eta \rangle_g = \eta_n^2 + r(x, \eta'). \quad (5.1)$$

The time variable and its dual are respective t and λ . Let

$$Q = \{(x, t, \eta, \lambda), x_n = 0\}, \quad P = \{(x, t, \eta, \lambda), \lambda^2 = \eta_n^2 + r(x, \eta')\}. \quad (5.2)$$

The cotangent bundle of $\partial\Omega \times \mathbb{R}$ is naturally the quotient of Q by the action of translation in η_n and we take as coordinates (x', t, η, λ) . Assume that $\partial\Omega$ is strictly convex, so that geodesics (light rays) which hit $\partial\Omega$ tangentially have exactly second order contact. The overlying bicharacteristic is called a glancing ray. A point $\rho \in Q \cap P$ is called *glancing point* if the bicharacteristic through ρ is a glancing ray. This is equivalent to the condition $\eta_n = 0$. Strict convexity implies that at such points $\partial_{x_n} r(x, \eta')|_\rho < 0$.

A point $(x', t, \eta', \lambda) \in T^*(\partial\Omega \times \mathbb{R})$ is classified as one of three distinct types. It is said to be *hyperbolic* if $\lambda^2 > r(x, \eta')$ so that there are two distinct nonzero real solutions η_n to $\lambda^2 - r(x, \eta') = \eta_n^2$. These two solutions yield two distinct bicharacteristics, one of which enters Ω as t increases (the *incoming ray*) and one which exits Ω as t increases (the *outgoing ray*). The point is *elliptic* if $\lambda^2 < r(x, \eta')$, so there are no real solutions η_n to $\lambda^2 - r(x, \eta') = \eta_n^2$. In the remaining case $\lambda^2 = r(x, \eta')$ there is a unique solution which yields a glancing ray, and the point is said to be a *glancing point*. Consider the problem

$$-(\lambda^2 + \Delta_g)w(x, \lambda) = 0, \quad w(x', 0, \lambda) = f(x', \lambda).$$

Let $(y, y_{n+1}, \xi, \xi_{n+1})$ denote the canonical coordinates on $T^*(\mathbb{R}^{n+1})$. The normal form for a pair of glancing hypersurfaces is the pair

$$Q_F = \{(y, y_{n+1}, \xi, \xi_{n+1}), y_{n+1} = 0\}, \quad P_F = \{(y, y_{n+1}, \xi, \xi_{n+1}), \xi_{n+1}^2 - y_{n+1}\xi_1^2 - \xi_1\xi_n = 0\}. \quad (5.3)$$

Let $\rho = (\bar{x}, 0, \bar{\eta}, 1)$ be a glancing point in $Q \cap P$. By the equivalence of glancing surfaces theorem of Melrose [26], there exists a homogeneous canonical transformation χ from a conical neighborhood of the point $\rho_F = (y = 0, y_{n+1} = 0, \xi_1 = 1, \xi'' = 0, \xi_{n+1} = 0)$ in $T^*(\mathbb{R}^{n+1})$ to a conical neighborhood of ρ in $T^*(\mathbb{R}^n \times \mathbb{R})$ such that χ maps ρ_F to ρ , Q_F to Q and P_F to P . By translation invariance of both forms, one can take $\lambda = \xi_1$.

In the model coordinates, the function

$$\Phi_F(y, y_{n+1}, \xi, \xi_{n+1}) = \langle y, \xi \rangle + \frac{2}{3} \frac{\xi_{n+1}^3}{\xi_1^2} \quad (5.4)$$

satisfies $d\Phi_F = \sum_{j=1}^{n+1} \xi_j dy_j$ when restricted to the Lagrangian manifold of the form

$$\xi = \text{const.}, \quad \xi_{n+1}^2 = y_{n+1}\xi_1^2 + \xi_1\xi_n.$$

Let $\Phi(x, \xi, \xi_{n+1}) + t\xi_1$ be this function in (x, t, ξ, ξ_{n+1}) coordinates. One seeks to use (x, t, ξ) as coordinates on P , since in this coordinates Φ solves the eikonal equation, that is

$$(x, t, \xi) \rightarrow (x, t, d_x \Phi(x, \xi, \xi_{n+1})) \in P$$

parameterizes the characteristic variety P . The hypersurface P is defined by

$$\xi_{n+1}^2 = x_n f(x, \xi, \xi_{n+1}) \xi_1^2 + \xi_1 \xi_n,$$

where $y_{n+1} = x_n f(x, \xi, \xi_{n+1})$, with $f(x, \xi, \xi_{n+1}) > 0$, homogeneous of degree 0. Consider the mapping

$$P \rightarrow \mathbb{R}^n, \quad P \ni p \rightarrow (\xi_j(\chi^{-1}(p)))_{j \in \{1, \dots, n\}} \in \mathbb{R}^n,$$

defined by sending p to the values of the $(\xi_j)_{j \in \{1, \dots, n\}}$ at the image point under the canonical transformation χ . Combining this map with the projection Y from P to the base gives (for small x_n) a map with Whitney folds at points in P for which $2\xi_{n+1} = x_n \partial_{\xi_{n+1}} f(x, \xi, \xi_{n+1}) \xi_1^2$ (see [26, Chp.4]). Near the positive ξ_1 axis, the image of these folding points under Y is defined by

$$\xi_n + x_n \phi(x, \xi') = 0$$

for some positive, elliptic ϕ homogeneous of degree one in ξ' , with image $Y(P) = \{\xi_n + x_n \phi(x, \xi') \geq 0\}$. In particular, on $\partial\Omega \times \mathbb{R}$, the hyperbolic, elliptic and glancing points are

respectively given by $\xi_n > 0$, $\xi_n < 0$ and $\xi_n = 0$. If we consider in P the submanifolds defined by the constancy of the $(\xi_j)_{j=1,\dots,n}$, $\Lambda_\xi = \{p \in P, Y(p) = (\cdot, \xi)\}$, (i.e. the Lagrangian Λ_ξ foliates P near ρ), then the function Φ satisfies $d(\Phi|_{\Lambda_\xi}) = \alpha|_{\Lambda_\xi}$, and, since Y is a fold, $\Phi|_P$ can be written uniquely in the form

$$\Phi = Y^* \left(\Theta \pm \frac{2}{3}(-\zeta)^{3/2} \right), \text{ where } \Theta, \zeta : Y(P) \rightarrow \mathbb{R} \text{ are } C^\infty. \quad (5.5)$$

The even part of Φ projects to a function $\theta(x, \xi) + t\xi_1$, smoothly extendable. The odd part vanishes to second order (and hence to third order) at the fold because the differential of Φ is the pullback of a smooth form. Therefore, the odd part of Φ can be written in the form

$$\pm \mu(x, \xi)(\xi_n + x_n \phi(x, \xi'))^{3/2},$$

where μ is a smoothly extendable function defined on $Y(P)$, homogeneous of degree $-1/2$. At $x_n = 0$, P is defined by $\xi_{n+1}^2 = \xi_1 \xi_n$ and Φ is given by

$$\langle y(x', t, \xi), \xi \rangle + \frac{2}{3} \frac{\xi_{n+1}^3}{\xi_1^2} = \theta_0(x', \xi) + t\xi_1 \pm \frac{2}{3} \frac{\xi_n^{3/2}}{\xi_1^{1/2}},$$

with $\theta_0(x', \xi) = \theta(x', 0, \xi)$, $\mu(x, \xi) \neq 0$. We take $\mu(x, \xi) > 0$ and define

$$\zeta(x, \xi) = -\mu^{2/3}(x, \xi)(\xi_n + x_n \phi(x, \xi')),$$

so that $Y(P) = \{\zeta(x, \xi) \leq 0\}$. At $x_n = 0$ we have $\zeta(x', 0, \xi) = \zeta_0(\xi) = -\xi_1^{-1/3} \xi_n$. The above imply that the phases Θ, ζ satisfy the eikonal equations (2.8).

Proposition 5.1. (see [35], [29], [28]) Let $w(x, \lambda)$ solve

$$\begin{cases} -(\lambda^2 + \Delta_g)w(x, \lambda) = 0, \\ w(x, \lambda)|_{\partial\Omega} = f(x', \lambda). \end{cases} \quad (5.6)$$

Near a glancing direction, the stationary wave $w(x, \lambda)$ can be written in the form

$$w(x, \lambda) = \int \left(a(x, \lambda, \xi') A_+(\zeta) + b(x, \lambda, \xi') A'_+(\zeta) \right) \frac{Ai(\zeta_0)}{A_+(\zeta_0)} e^{i\theta(x, \lambda, \xi')} \hat{K}(f)(\lambda, \xi') d\xi',$$

where here $\xi' = (\xi_2, \dots, \xi_n)$, the phase functions $\theta(x, \lambda, \xi') = \lambda\theta(x, 1, \xi'/\lambda)$, $\zeta(x, \lambda, \xi') = \lambda^{2/3}\zeta(x, 1, \xi'/\lambda)$ are smooth and satisfy the eikonal equations (2.8) in the region $\zeta \leq 0$, K is a Fourier integral operator of order 0 and $a \in S_{phm}^{1/6}$, $b \in S_{phm}^{-1/6}$ are polyhomogeneous symbols defined on a conic neighborhood of the glancing point.

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