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Convergence of non-linear functionals of smoothed empirical processes and kernel density estimates

Corinne Berzin ^{*} José R. León [†] Joaquín Ortega [‡]

Abstract. We consider regularizations by convolution of the empirical process and study the asymptotic behaviour of non-linear functionals of this process. Using a result for the same type of non-linear functionals of the Brownian Bridge, shown in a previous paper [4], and a strong approximation theorem, we prove several results for the p -deviation in estimation of the derivatives of the density. We also study the asymptotic behaviour of the number of crossings of the Smoothed Empirical Process defined by Yukich [17] and of a modified version of the Kullback deviation.

KEY WORDS: Non-linear functionals, Empirical process, kernel density estimation, crossings, Kullback-Leibler deviation, regularization by convolution.

1 Introduction

The problem of estimating the density $s(x)$ of simple random sample X_1, X_2, \dots, X_n is a classical problem in non-parametric statistics. The most popular method is kernel estimation, introduced by Parzen and Rosenblatt in the 50's. The study of consistency properties and asymptotic normality of these estimators was one of the important subjects of study during the following two decades.

Among these works was Bickel & Rosenblatt [5] where the authors use strong convergence results to establish the asymptotic normality of the L^2 distance between the estimator and the real density. This work, source of inspiration for many others, allows the substitution of the Empirical Process by the Brownian Bridge in the functional under study, that is, the L^2 distance between the estimator and the real density. This replacement can be controlled using the strong approximation theorem of Komlós, Major and Tusnády [8]. One of the important consequences of their result is that it gives L^2 confidence bands and provides the tools for hypothesis tests for contiguity alternatives for functions close to the density in the null hypothesis. Silverman [16] extends these methods to the estimation of the density's derivatives.

In this work we generalize Bickel & Rosenblatt's result to kernel estimators of the density and of the derivative of the density, not only for the L^2 norm but also for L^p norms. The first work where this problem was studied was Csörgő y Horváth [7], also using the strong approximation method. From then on there has been much work done trying to simplify the hypothesis for the density and the kernel, as, for example, Horváth [12], Beirlant & Mason [3] and finally the work of Giné, Mason & Zaitsev [9], where a limit theorem uniform with respect to the regularization kernel is obtained. In the last three papers mentioned the technique of Poissonization is used, which proves to be stronger than strong approximation.

Our method is interesting because, apart from allowing us to study the derivative of the density, handles the asymptotic convergence of functionals of the type

$$\int_0^1 G(\sqrt{nh}(\hat{l}_n(u) - l(u))) du$$

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where \hat{l}_n is an estimator of the density or its derivatives. The p norm case corresponds to $G(x) = |x|^p$. Using this viewpoint we can look at the asymptotics for a variant of the Kullback deviation. The results are obtained passing to functionals of the Brownian Bridge via the strong approximation theorem and then studying the asymptotic behaviour of these functionals using methods developed in a previous article [4].

With these same methods we can also study the number of crossings of a level by the Smoothed Empirical Process, defined by Yukich [17], as we explain in the next section, where we describe more precisely our results.

2 Description of the Results

Let X_1, X_2, \dots, X_n be a random sample from the distribution F with density $s(u)$, F_n be the empirical distribution of the sample and $\beta_n^F(u) = \sqrt{n}(F_n(u) - F(u))$ be the empirical process. The kernel density estimator is

$$\hat{s}_n(u) = \hat{s}_{n,h}(u) = \frac{1}{h} \int_{-\infty}^{\infty} \varphi\left(\frac{u-v}{h}\right) dF_n(v)$$

where φ is the estimation kernel and h depends on n : $h = h(n)$.

1.- Define

$$\sqrt{n}[\hat{s}_n(u) - \mathbb{E}(\hat{s}_n(u))] = \frac{1}{h} \int_{-\infty}^{\infty} \varphi\left(\frac{u-v}{h}\right) d\beta_n^F(v)$$

Integrating by parts

$$\sqrt{nh}[\hat{s}_n(u) - \mathbb{E}(\hat{s}_n(u))] = \sqrt{h} \dot{\beta}_{n,h}(u)$$

where $\dot{\beta}_{n,h}$ is the derivative of the φ -regularization of the empirical process $\beta_{n,h}$. If s has two continuous and bounded derivatives it is known that the optimal window is $h(n) = O(n^{-1/5})$. The window is sub-optimal if $h = o(n^{-1/5})$, otherwise it is super-optimal. It is also known that

$$\lim_{n \rightarrow \infty} \sqrt{nh}[\mathbb{E}(\hat{s}_n(u)) - s(u)] = \alpha(u)$$

where $\alpha(u)$ is a known function of the unknown density for an optimal window and $\alpha(u) \equiv 0$ for sub-optimal windows.

Consider the function $g_p(x) = |x|^p - \mathbb{E}|N|^p$, $p \geq 1$ where N is a standard Gaussian r.v. The results that we will show imply for sub-optimal windows the following theorem of Csörgő & Horváth [7]:

$$\frac{1}{\sqrt{h}} \int_0^1 \left[|\sqrt{nh}(\hat{s}_n(u) - s(u))|^p - \mathbb{E}[|\sqrt{nh}(\hat{s}_n(u) - s(u))|^p] \right] du \rightarrow N(0, v_p^2)$$

Indeed the integral above is equivalent to

$$\frac{1}{\sqrt{h}} \int_0^1 g_p\left(\frac{\sqrt{h}\dot{\beta}_{n,h}(u)}{[\text{Var}(\sqrt{h}\dot{\beta}_{n,h}(u))]^{1/2}}\right) |\text{Var}(\sqrt{h}\dot{\beta}_{n,h}(u))|^{p/2} du.$$

Using now the KMT theorem we can replace $\dot{\beta}_{n,h}(u)$ by the derivative of the regularization of the F-B.B. and get that the last integral converges to $N(0, v_p^2)$.

In section 6 we obtain results along this line for the derivatives of the density and for a general function in place of $|x|^p$ for the sub-optimal and optimal cases.

Let $\beta_n^F(u) = \sqrt{n}(F_n(u) - F(u))$ be the empirical process and define

$$\nu_{n,h}(u) = \hat{s}_{n,h}(u) - s(u).$$

If $\varphi = \mathbf{1}_{[-1,1]}/2$ then

$$\sqrt{nh}\nu_{n,h}(u) = \frac{\beta_n^F(u+h) - \beta_n^F(u-h)}{2\sqrt{h}} + o(1)$$

as long as the density is sufficiently regular and h is chosen adequately. We want to study the asymptotic behaviour of the sums

$$\frac{1}{n} \sum_{i=1}^{[nt]} g_n\left(\frac{i}{n}, \sqrt{nh}\nu_{n,h}\left(\frac{i}{n}\right)\right) \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{[nt]} f\left(\beta_n^F\left(\frac{i}{n}\right)\right) g_n\left(\frac{i}{n}, \sqrt{nh}\nu_{n,h}\left(\frac{i}{n}\right)\right)$$

where g_n is a sequence of functions that satisfy certain properties that will be made explicit later on. These sums are generalizations of the so-called divisible statistics studied by Khmaladze [15] and they correspond to the following integrals

$$\int_0^t g_n(u, \sqrt{nh}\nu_{n,h}(u)) du \quad \text{and} \quad \int_0^t f(\beta_n^F(u)) g_n(u, \sqrt{nh}\nu_{n,h}(u)) du. \quad (1)$$

Using the Komlós, Major & Tusnády (KMT) strong approximation theorem ([8], see section 9 for the statement) we show that if $g_n \rightarrow g$ then

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^t g_n(u, \sqrt{nh}\nu_{n,h}(u)) du &= \int_0^t E\left[g(u, \sqrt{\frac{s(u)}{2}} N)\right] du \\ \lim_{n \rightarrow \infty} \int_0^t f(\beta_n^F(u)) g_n(u, \sqrt{nh}\nu_{n,h}(u)) du &= \int_0^t f(b^F(u)) E\left[g(u, \sqrt{\frac{s(u)}{2}} N)\right] du \end{aligned}$$

where $b^F(u)$ is the F-Brownian bridge (F-B.B.), N is a standard Gaussian variable and the convergence is in $L^2(\Omega)$ for the first limit and in law for the second. We also study the speed of convergence in both cases. Using a similar result for the F-B.B. shown in a previous work [4], we get that there exists a Brownian motion (B.M.) \widetilde{W} independent of b^F and a positive function $\sigma_g(\cdot)$, which depends only on g such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{h}} \int_0^t \{f(\beta_n^F(u)) g_n(u, \sqrt{nh}\nu_{n,h}(u)) - f(b^F(u)) E[g(u, \sqrt{\frac{s(u)}{2}} N)]\} du \\ = \int_0^t f(b^F(u)) d\widetilde{W}(\sigma_g(u)). \end{aligned}$$

with convergence in law. Making $f \equiv 1$ one gets the same result for the first kind of integrals in (1). We consider these problems for general kernel estimators of the density and also several applications of these results which we now describe briefly.

2.- We consider in section 7 the modified Kullback deviation between the kernel estimator of the density \hat{s}_n and s . Let

$$\mathcal{K}(\hat{s}_n, s) = \int_h^{1-h} \hat{s}_n(u) \ln\left(\frac{\hat{s}_n(u)}{s(u)}\right) du.$$

We show that

$$n\sqrt{h}[\mathcal{K}(\hat{s}_n, s) - \frac{1}{2} \int_h^{1-h} \frac{1}{s(u)} E[\hat{s}_n(u) - s(u)]^2 du - \int_h^{1-h} (\hat{s}_n(u) - s(u)) du]$$

is equivalent, in the sub-optimal and optimal cases, to

$$\frac{1}{2\sqrt{h}} \int_0^1 s^{-1}(u) \left[[\sqrt{nh}(\hat{s}_n(u) - s(u))]^2 - E[\sqrt{nh}(\hat{s}_n(u) - s(u))]^2 \right] du$$

which is similar to the previous result for $p = 2$ and converges to a normal distribution. We show a similar result for the super-optimal case but with a different normalization.

3.- Define as in Yukich [17] the Smoothed Empirical Process:

$$\tilde{\beta}_{n,h}(u) = \sqrt{n}(\hat{F}_n(u) - F(u))$$

where $\hat{F}_n(u) = \int_{-\infty}^u \hat{s}_n(v) dv$.

Yukich proved that if $\sqrt{nh^2} \rightarrow 0$ then $\tilde{\beta}_{n,h}(u)$ converges in law (in notation $\xrightarrow{\mathcal{D}}$) to the F-B.B. $b^F(u)$. Moreover, using a Taylor expansion for $\sqrt{n}(\tilde{F}_n(u) - F(u))$, where

$$\tilde{F}_n(u) = \int_{-\infty}^{\infty} \varphi(v)F(u - hv) dv,$$

he shows that, if $\sqrt{nh^2} \rightarrow a$, then $\tilde{\beta}_{n,h}(u) \xrightarrow{\mathcal{D}} b^F(u) + \frac{a}{2}K_2\dot{s}(u)$, where $K_2 = \int_{-\infty}^{\infty} \varphi(v)v^2 dv$.

Let $\tilde{N}_n^h(x)$ be the number of times that the process $\tilde{\beta}_{n,h}(\cdot)$ crosses level x before time 1, and define $\tilde{\ell}_n^h(\cdot)$ as a modification of the occupation measure for $\tilde{\beta}_{n,h}$ (see Corollary 2 in section 9 for an exact definition). By a formula of Banach-Kac [2, 14]

$$\int_{-\infty}^{\infty} f(x) \tilde{N}_n^h(x) dx = \int_0^1 f(\tilde{\beta}_{n,h}(u)) |\dot{\tilde{\beta}}_{n,h}(u)| du.$$

Let $\Lambda_h = (\frac{\pi h}{2\|\varphi\|_2^2})^{\frac{1}{2}}$, using results included in a previous work [4] we obtain in section 9 the following:

a) If $\sqrt{nh^2} \rightarrow 0$ then

$$\tilde{\xi}_n^h(f) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} f(x)(\Lambda_h \tilde{N}_n(x) - \tilde{\ell}_n^h(x)) dx \xrightarrow{\mathcal{D}} V$$

where V is a r.v. which, conditional on \mathcal{F}_∞ , the σ -field generated by $\{b^F(u), 0 \leq u \leq 1\}$, has the same law as $\sigma_0 \int_0^1 f(b^F(u)) \sqrt{s(u)} d\tilde{W}(u)$, $\tilde{W}(\cdot)$ is a B.M. independent of $b^F(\cdot)$ (see Theorem 3 in [4]) and σ_0 is defined in the next section.

b) On the other hand if $\sqrt{nh^2} \rightarrow a$ then

$$\Lambda_h \int_{-\infty}^{\infty} f(x) \tilde{N}_n^h(x) dx \xrightarrow{\mathcal{D}} \int_0^1 f(b^F(u) + \frac{a}{2}K_2\dot{s}(u)) \sqrt{s(u)} du.$$

Moreover, we show that $\tilde{\xi}_n^h(f)$ converges weakly to a r.v. \tilde{V} such that

$$\mathcal{L}(\tilde{V}/\mathcal{F}_\infty) = \sigma_0 \int_0^1 f(b^F(u) + \frac{a}{2}K_2\dot{s}(u)) \sqrt{s(u)} d\tilde{W}(u).$$

c) In section 9 we also treat the case $\sqrt{nh^2} \rightarrow +\infty$. In this situation we need a different normalization sequence for the process $\tilde{\beta}_{n,h}$.

3 Hypothesis and notation

Let F be a distribution function with bounded support and density s . To simplify the notation we shall suppose that its support is $[0, 1]$ i.e. $F(0) = 0$ and $F(1) = 1$. The F-Brownian Motion is defined as $W^F(t) = W(F(t))$, where W is a standard B. M. With this definition one has $E(W^F(u)W^F(v)) = F(u \wedge v)$. The F-B.B. is defined as $b^F(t) = W^F(t) - F(t)W(1)$ and then $E(b^F(u)b^F(v)) = F(u \wedge v)[1 - F(u \vee v)]$.

For each t and $h > 0$ we define the regularized processes $b_h^F(t) = \varphi_h * b^F(t)$ and $W_h^F(t) = \varphi_h * W^F(t)$ where $\varphi_h(t) = \frac{1}{h}\varphi(\frac{t}{h})$ and $*$ denotes the convolution. We use h instead of ε (as in [4]) for the regularization parameter to be consistent with the usual notation in density estimation.

We shall use the Hermite polynomials, which can be defined by $\exp(tx - \frac{t^2}{2}) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$. They form an orthogonal system for the standard Gaussian measure $\phi(x)dx$ and, if $h \in L^2(\phi(x) dx)$, $h(x) = \sum_{n=0}^{\infty} \hat{h}_n H_n(x)$ and $\|h\|_{2,\phi}^2 = \sum_{n=0}^{\infty} n! \hat{h}_n^2$. Mehler's formula [6] gives a simple form to compute the covariance between two L^2 functions of Gaussian r.v.'s: If (X, Y) is a Gaussian random vector having correlation ρ then

$$E[h(X)k(Y)] = \sum_{n=0}^{\infty} \hat{h}_n \hat{k}_n n! \rho^n. \quad (2)$$

We have the following hypothesis

(H1) For the kernel φ : $\int_{-1}^1 \varphi(t) dt = 1$, $\varphi \geq 0$, φ is absolutely continuous with support in $[-1, 1]$. Define $\psi(u) = \varphi * \bar{\varphi}(u)$ where $\bar{\varphi}(u) = \varphi(-u)$ and $\theta(u) = \psi(u) \|\varphi\|_2^{-2}$, $u \in \mathbb{R}$. Let $K_2 = \int_{-\infty}^{\infty} \varphi(v) v^2 dv$. Further conditions will be required for some applications.

(H2) For the function s : $s \in C^2[0, 1]$, $0 < s(x)$ for all $x \in [0, 1]$.

Let $g(x, y)$ be a function in $L^2(\phi(y)dy)$ a.s. continuous in the first variable and with polynomial growth in the second variable ($|g(x, y)| \leq K P(|y|)$ for x in a set compact) that satisfies the following two conditions:

i) $Eg(x, N) = 0$, $0 \leq x$.

ii) $E[Ng(x, N)] = 0$, $0 \leq x$.

Define $g(x, y) = \sum_{k=2}^{\infty} c_k(x) H_k(y)$ the Hermite expansion of g , where we suppose that $\forall x > 0$, $\exists k \geq 2$ such that $c_k(x) > 0$, $(\dot{\sigma}_h^b(u))^2 = \text{Var}(\dot{b}_h^F(u))$ and

$$g_h(u, y) = g(\sqrt{h} \dot{\sigma}_h^b(u), y), \quad (3)$$

and for $0 \leq t \leq 1$

$$S_h^{b^F}(t) = \frac{1}{\sqrt{h}} \int_0^t g_h(u, \xi_h^b(u)) du \quad \text{with} \quad \xi_h^b(u) = \frac{\dot{b}_h^F(u)}{\dot{\sigma}_h^b(u)} \quad (4)$$

Define also

$$\sigma^2(t) = \int_0^t \kappa(u) du \quad (5)$$

where

$$\begin{aligned} \kappa(u) &= \sum_{k=2}^{\infty} k! c_k^2(\sqrt{s(u)} \|\varphi\|_2) \left[\int_{-2}^2 \theta^k(w) dw \right] \\ &= \int_{-2}^2 E[g(\sqrt{s(u)} \|\varphi\|_2, X) g(\sqrt{s(u)} \|\varphi\|_2, \theta(w)X + \sqrt{1 - \theta^2(w)}Y)] dw \end{aligned}$$

and X, Y are independent, standard Gaussian variables. We shall write

$$l_q(u, x) = \sqrt{\frac{\pi}{2}} \left| x + \frac{qK_2 \ddot{s}(u)}{2\sqrt{s(u)} \|\varphi\|_2} \right| = \sum_{k=0}^{+\infty} d_{k,q}(u) H_k(x)$$

and

$$\sigma_q^2(u) = \int_{-2}^2 \sum_{n=2}^{\infty} d_{n,q}^2(u) n! \theta^n(w) dw,$$

for $u \in [0, 1]$, $x \in \mathbb{R}$ and $q \geq 0$ (note that $\sigma_0^2(u)$ doesn't depend on u so we will note $\sigma_0^2(u) = \sigma_0^2$).

Also define $c_n = \frac{2}{K_2 \sqrt{n} h^2}$.

We have $\dot{b}_h^F(t) = \frac{1}{h} \int_{-\infty}^{t/h} b^F(t - hy) d\varphi(y)$. We introduce

$$Z_h^{b^F}(f) = h^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x) [\Lambda_h N_h^{b^F}(x) - \ell^{b^F}(x)] dx \quad \text{with} \quad \Lambda_h = \sqrt{\frac{\pi h}{2}} \|\varphi\|_2^{-1},$$

where $N_h^{b^F}(x)$ is the number of times that the process $b_h^F(\cdot)$ crosses level x before time 1 and $\ell^{b^F}(\cdot)$ is a modification of the local time for the F-B.B. on $[0, 1]$ that satisfies, for any continuous function f ,

$$\int_{-\infty}^{\infty} f(x) \ell^{b^F}(x) dx = \int_0^1 f(b^F(u)) \sqrt{s(u)} du$$

In what follows we shall drop out the indices F when no confusion is possible. Throughout the paper, Const shall stand for a generic constant, whose value may change during a proof, N denotes a standard Gaussian r.v.

3.1 Stable Convergence

We shall use the notion of stable convergence, which we describe now following [1, 11] and [13].

Let X_n be a sequence of r. v.'s defined over (Ω, \mathcal{F}, P) and taking values in $C[0, 1]$, and let $\mathcal{G} \subset \mathcal{F}$ be a sub- σ -field. Let X be another r. v. defined over an extension $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ of the original probability space, with values in $C[0, 1]$. The sequence X_n converges \mathcal{G} -stably to X if

$$\lim_n \mathbb{E}(Yh(X_n)) = \bar{\mathbb{E}}(Yh(X)),$$

for all bounded and continuous functions $h : C[0, 1] \rightarrow \mathbb{R}$ and all \mathcal{G} -measurable and bounded r.v. Y .

Stable convergence is invariant under absolutely continuous changes of the measure.

4 Preliminary Results

We shall say that a function f is locally Lipschitz if

$$|f(x) - f(y)| \leq |P(x, y)||x - y| \quad (6)$$

where P is a polynomial. .

We need two results from [4] which we state for completeness.

Theorem 1 *Under H1, H2, i) and ii)*

$$S_h^b(t) \rightarrow \widetilde{W}(\sigma^2(t))$$

stably in $C[0, 1]$ where \widetilde{W} is a B.M. and $\sigma^2(t)$ is given by (5). Furthermore the vector process $(b_h^F(\cdot), S_h^b(\cdot))$ converges weakly in $C[0, 1] \times C[0, 1]$ towards $(b^F(\cdot), \widetilde{W}(\sigma^2(\cdot)))$ and the processes $b^F(\cdot)$ and $\widetilde{W}(\sigma^2(\cdot))$ are independent.

Corollary 1 *Under H1, H2 and if $f \in C^2$ with \ddot{f} bounded, $Z_h^b(f)$ converges stably as $h \rightarrow 0$ towards a r.v. $V \in L^2(\Omega)$ and the conditional distribution $\mathcal{L}(V|\mathcal{F}_\infty)$ is Gaussian with zero mean and random variance equal to $\sigma_0^2 \int_0^1 f^2(b^F(u))s(u)du$.*

(σ_0^2 has been defined in previous section 3).

Remark 1. Conditionally on \mathcal{F}_∞ , V has the same distribution as $\sigma_0 \int_0^1 f(b^F(u)) \sqrt{s(u)} d\widetilde{W}(u)$ (remember that \mathcal{F}_∞ is the σ -field generated by $\{b^F(u), 0 \leq u \leq 1\}$).

Remark 2. Although the theorem has been stated assuming that in the definition of $S_h^b(t)$, $g_h(u, y)$ is of the form $g(\sqrt{h}\dot{\sigma}_h^b(u), y)$, (see (3) and (4)), a slight modification of the proof shows that the result is true for other forms of the function $g_h(u, y)$ that will appear in sections 6 and 9.

5 Convergence in L^2 for the Empirical Process

Let $\beta_n^F(u) = \sqrt{n}(F_n(u) - F(u))$ be the empirical process associated to a n -sample X_1, \dots, X_n of the continuous distribution F , where F_n is the corresponding empirical d. f. and F satisfies the conditions of section 3. The transformation $U_i = F(X_i), i = 1, \dots, n$ gives a n -sample of the uniform distribution $\mathcal{U}[0, 1]$. If $E_n(v)$ is the associated empirical distribution function, the corresponding empirical process is $\alpha_n(v) = \sqrt{n}(E_n(v) - v)$, $0 \leq v \leq 1$. Note that $\alpha_n(F(u)) = \beta_n^F(u)$ and if $b(t)$ denotes the standard B.B.

$$\sup_{t \in [0, 1]} \sqrt{n} |\alpha_n(t) - b(t)| = \sqrt{n} \|\beta_n^F - b^F\|_\infty$$

The following approximation theorem is due to Komlós, Major and Tusnády (1975) [8].

Theorem [KMT] *For any integer n there exists some B.B. b_n such that for some absolute positive constants C, Λ, γ the following inequality holds:*

$$P\left(\sup_{t \in [0, 1]} \sqrt{n} |\alpha_n(t) - b_n(t)| \geq x + C \ln(n)\right) \leq \Lambda \exp(-\gamma x)$$

for any positive x . Consequently

$$\sup_{t \in [0,1]} |\alpha_n(t) - b_n(t)| = O\left(\frac{\ln(n)}{\sqrt{n}}\right) \quad a.s.$$

Let $\beta_{n,h}(u) = \frac{1}{h} \int_{-\infty}^{\infty} \varphi\left(\frac{u-v}{h}\right) \beta_n^F(v) dv$, with φ verifying **H1**, be the regularized empirical process, and denote by $\dot{\beta}_{n,h}(u)$ its derivative. Defining $b_n^F(\cdot) = b_n(F(\cdot))$, $b_{n,h} = b_n \star \varphi_h$ and using the KMT theorem, it is easy to see that

$$\|\beta_{n,h} - b_{n,h}\|_{\infty} \leq \text{Const} \frac{\ln(n)}{\sqrt{n}} \quad \text{and} \quad \|\dot{\beta}_{n,h} - \dot{b}_{n,h}\|_{\infty} \leq \text{Const} \frac{\ln(n)}{h\sqrt{n}},$$

where the constant is random.

Lemma 1 For $p \geq 1$,

$$\mathbb{E}(\sqrt{n} \|\beta_n^F - b_n^F\|_{\infty})^p \leq \text{Const} (\ln(n))^p \quad (7)$$

$$\mathbb{E}(\sqrt{n} \|\beta_{n,h} - b_{n,h}\|_{\infty})^p \leq \text{Const} (\ln(n))^p \quad (8)$$

$$\mathbb{E}(\sqrt{n} \|\dot{\beta}_{n,h} - \dot{b}_{n,h}\|_{\infty})^p \leq \text{Const} h^{-p} (\ln(n))^p \quad (9)$$

Proof. We only consider the case $p = 2$, the proof is similar for general p . Define $Y_n(u) = \sqrt{n} |\beta_n^F(u) - b_n^F(u)|$ and let $A_{\lambda} = \{\|Y_n\|_{\infty} \geq \lambda + C \ln(n)\}$, where $\lambda \geq 0$ and C is the constant in the KMT theorem. We have

$$\mathbb{E}(\|Y_n\|_{\infty}^2) = \mathbb{E}(\|Y_n\|_{\infty}^2 \mathbf{1}_{A_{\lambda}^c}) + \mathbb{E}(\|Y_n\|_{\infty}^2 \mathbf{1}_{A_{\lambda}}) \leq (\lambda + C \ln(n))^2 + \mathbb{E}(\|Y_n\|_{\infty}^2 \mathbf{1}_{A_{\lambda}}) \quad (10)$$

Call G_n the distribution function of $\|Y_n\|_{\infty}$, then, integrating by parts and using KMT

$$\begin{aligned} \mathbb{E}(\|Y_n\|_{\infty}^2 \mathbf{1}_{A_{\lambda}}) &= \int_{\lambda + C \ln(n)}^{\infty} u^2 dG_n(u) \\ &= (\lambda + C \ln(n))^2 (1 - G_n(\lambda + C \ln(n))) + \int_{\lambda + C \ln(n)}^{\infty} 2u(1 - G_n(u)) du \end{aligned} \quad (11)$$

but by the inequality in the KMT theorem

$$1 - G_n(\lambda + C \ln(n)) \leq \Lambda e^{-\gamma \lambda}; \quad 1 - G_n(u) \leq \Lambda e^{-\gamma(u - C \ln(n))}$$

thus

$$\begin{aligned} (11) &\leq \Lambda (\lambda + C \ln(n))^2 e^{-\gamma \lambda} + 2\Lambda \int_{\lambda + C \ln(n)}^{\infty} u e^{-\gamma(u - C \ln(n))} du \\ &\leq \text{Const} (\ln(n))^2 + \text{Const} n^{\gamma C} \left((\lambda + C \ln(n)) e^{-\gamma(\lambda - C \ln(n))} + e^{-\gamma(\lambda - C \ln(n))} \right) \\ &\leq \text{Const} (\ln(n))^2 \end{aligned} \quad (12)$$

Using (12) in (10) we get (7). We shall now prove (9), the proof of (8) is similar but simpler.

$$\begin{aligned} \mathbb{E}(n \|\dot{\beta}_{n,h} - \dot{b}_{n,h}\|_{\infty}^2) &= \mathbb{E}\left(n \sup_x \left| \int_{-\infty}^{\infty} \frac{1}{h^2} \dot{\varphi}\left(\frac{x-y}{h}\right) (\beta_n^F(y) - b_n^F(y)) dy \right|^2\right) \\ &\leq \mathbb{E}\left(n \left(\sup_x \int_{-\infty}^{\infty} \frac{1}{h} |\dot{\varphi}(u)| |\beta_n^F(x-uh) - b_n^F(x-uh)| du\right)^2\right) \\ &\leq \text{Const} \frac{1}{h^2} \mathbb{E} \|Y_n\|_{\infty}^2 \leq \text{Const} \frac{(\ln(n))^2}{h^2}. \quad \blacksquare \end{aligned}$$

Theorem 2 Let g be an even function verifying (6), under **H1** and **H2** and if $\frac{\ln(n)}{\sqrt{hn}} \rightarrow 0$ as $h \rightarrow 0$,

$$\lim_{n \rightarrow \infty} \int_0^1 g(\sqrt{h}\dot{\beta}_{n,h}(u))du = \int_0^1 E\{g(\sqrt{s(u)}N\|\varphi\|_2)\}du \quad \text{in } L^2$$

Proof. We have

$$\begin{aligned} E\left[\int_0^1 g(\sqrt{h}\dot{\beta}_{n,h}(u))du - \int_0^1 E\{g(\sqrt{s(u)}\|\varphi\|_2N)\}du\right]^2 &\leq \text{Const} E\left[\int_0^1 g(\sqrt{h}\dot{\beta}_{n,h}(u))du - \int_0^1 g(\sqrt{h}\dot{b}_{n,h}(u))du\right]^2 \\ &+ \text{Const} E\left[\int_0^1 g(\sqrt{h}\dot{b}_{n,h}(u))du - \int_0^1 E\{g(\sqrt{s(u)}\|\varphi\|_2N)\}du\right]^2 \end{aligned} \quad (13)$$

Using the Lipschitz property for g , Hölder's inequality and lemma 1 the first term is bounded by

$$\text{Const} \frac{(\ln(n))^2}{nh}.$$

Up to a constant the second term is equal to

$$\begin{aligned} E\left[\int_0^1 g(\sqrt{h}\dot{b}_h^F(u)) - E(g(\sqrt{s(u)}\|\varphi\|_2N))^2 du\right] &\leq \text{Const} E\left[\int_0^1 (g(\xi_h^b(u)\sqrt{h}\dot{\sigma}_h^b(u)) - g(\xi_h^b(u)\|\varphi\|_2\sqrt{s(u)})) du\right]^2 \\ &+ \text{Const} E\left[\int_0^1 (g(\xi_h^b(u)\|\varphi\|_2\sqrt{s(u)}) - E[g(\|\varphi\|_2\sqrt{s(u)}N)]) du\right]^2. \end{aligned}$$

For the first of these two terms, use the fact that

$$|\sqrt{h}\dot{\sigma}_h^b(u) - \|\varphi\|_2\sqrt{s(u)}| \leq \text{Const} h$$

for $h \leq u \leq 1-h$, the Lipschitz property and Hölder's inequality to obtain that it is bounded by $\text{Const} h^2$. For the second term we consider the function

$$g(u, y) = g(y\|\varphi\|_2\sqrt{s(u)}) - E[g(\sqrt{s(u)}\|\varphi\|_2N)].$$

It verifies i) and ii), hence by Theorem 1 the second term is $O(h)$. We thus obtain that

$$(13) \leq \text{Const} \frac{(\ln(n))^2}{hn} + O(h)$$

and the theorem holds. ■

6 Estimation of a Density's Derivatives

In [7] Csörgő and Horváth consider the asymptotic behaviour of the p -th order risk for the kernel estimator of a density. In previous works the case $p = 2$ was established for sub-optimal windows by Bickel & Rosenblatt [5] and in all generality by Hall [10].

With our method we can consider the same problem as Csörgő and Horváth but with a general G instead of $G(u) = |u|^p$ and for the estimation of the derivatives of the density.

Assume that s is in $C^{m+2}[0, 1]$ and satisfies the conditions of section 3. Define the estimator for the m th-derivative as in [16] by

$$\hat{s}_n^{(m)}(u) = \frac{1}{nh^{m+1}} \sum_{k=1}^n \varphi^{(m)}\left(\frac{u - X_k}{h}\right)$$

with φ even and in C^{m+1} . We shall use the notation $s_n^{(m)}(u) = E[\hat{s}_n^{(m)}(u)]$. It is easy to show, using a Taylor expansion of order $m + 2$ for s , that for $0 < u < 1$ the bias is

$$E[\hat{s}_n^{(m)}(u) - s^{(m)}(u)] = \frac{(-1)^{m+2}}{(m+2)!} h^2 \int_{-\infty}^{\infty} \varphi^{(m)}(v) u^{m+2} s^{(m+2)}(\theta_1) dv,$$

where θ_1 depends on u, v and n . Thus $h^{-2}(\mathbb{E}[\hat{s}_n^{(m)}(u) - s^{(m)}(u)]) \rightarrow K_2 s^{(m+2)}(u)/2$ as n goes to infinity (remember that $K_2 = \int_{-\infty}^{\infty} v^2 \varphi(v) dv$). The variance term can also be calculated, giving

$$\begin{aligned} nh^{2m+1} \mathbb{E}(\hat{s}_n^{(m)}(u) - s_n^{(m)}(u))^2 &= \int_{-\infty}^{\infty} [\varphi^{(m)}(v)]^2 s(u - hv) dv - h \left[\int_{-\infty}^{\infty} \varphi^{(m)}(v) s(u - hv) dv \right]^2 \\ &\rightarrow \int_{-\infty}^{\infty} (\varphi^{(m)}(v))^2 dv s(u) = \|\varphi^{(m)}\|_2^2 s(u) \\ \text{i.e. } nh^{2m+1} \sigma_{n,m}^2(u) &\rightarrow \|\varphi^{(m)}\|_2^2 s(u) \text{ as } n \rightarrow \infty. \end{aligned}$$

Let now G be a locally Lipschitz function (see (6) for the definition), we want to study

$$I_n^{(m)} = \frac{1}{\sqrt{h}} \int_h^{1-h} \left\{ G\left(\frac{\hat{s}_n^{(m)}(u) - s^{(m)}(u)}{\sigma_{n,m}(u)}\right) - \mathbb{E} G\left(\frac{\hat{s}_n^{(m)}(u) - s^{(m)}(u)}{\sigma_{n,m}(u)}\right) \right\} du.$$

It is well known, and can also be shown using the previous expressions for the bias and variance, that the optimal estimation window for the mean square error is $h = O(n^{-1/2m+5})$ i.e. $\sqrt{nh}^{m+\frac{5}{2}} \rightarrow C_m$ and in this case

$$\frac{\sqrt{nh}^{m+\frac{1}{2}}(s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{nh}^{m+\frac{1}{2}}\sigma_{n,m}(u)} \rightarrow f_m(u) = \frac{K_2 C_m s^{(m+2)}(u)}{2\|\varphi^{(m)}\|_2 \sqrt{s(u)}} \quad (14)$$

as $n \rightarrow \infty$. The window is sub-optimal if $h = o(n^{-1/2m+5})$.

Let us introduce some notation before stating the next result.

$$\sigma_{so,m}^2 = \sum_{k=2}^{\infty} k! \left[\frac{1}{k!} \int_{-\infty}^{+\infty} H_k(y) \phi(y) G(y) dy \right]^2 \left[\int_{-2}^2 \theta_m^k(\omega) d\omega \right]$$

with

$$\begin{aligned} \theta_m(\omega) &= \left[\int_{-\infty}^{\infty} \varphi^{(m)}(y) \varphi^{(m)}(y + \omega) dy \right] \|\varphi^{(m)}\|_2^{-2}, \\ \sigma_{o,m}^2 &= \int_0^1 \sum_{k=2}^{\infty} k! \left[\frac{1}{k!} \int_{-\infty}^{+\infty} H_k(y) \phi(y) G(y + f_m(u)) dy \right]^2 \left[\int_{-2}^2 \theta_m^k(\omega) d\omega \right] du, \\ \zeta_{so}(u) &= \left[\int_{-\infty}^{+\infty} -z G(z) \phi(z) dz \right] \frac{\dot{s}(u)}{2\sqrt{s(u)}\|\varphi\|_2 s(u)}, \\ \zeta_o(u) &= \int_{-\infty}^{+\infty} G(y) \phi(y - f_0(u)) \frac{1}{\eta^2(u)} \left[(y - f_0(u))^2 \dot{f}_0(u) \eta(u) - \dot{f}_0(u) \eta(u) - (y - f_0(u)) \dot{\eta}(u) \right] dy \end{aligned}$$

where

$$\eta^2(u) = \|\varphi\|_2^2 s(u) \quad (15)$$

(see (14) for the definition of f_m). We also define the following integral operator

$$\langle Ka, a \rangle = \int_0^1 \int_0^1 F(u \wedge v) (1 - F(u \vee v)) a(u) a(v) du dv \quad (16)$$

Theorem 3 *Let G be a function verifying (6), under **H1**, **H2**, if $s \in C^{(m+3)}[0, 1]$, φ is an even function and $\frac{\ln(n)}{\sqrt{nh}} \rightarrow 0$ as $h \rightarrow 0$, then*

$$I_n^{(m)} \xrightarrow{\mathcal{D}} N(0, \sigma_{G,m}^2)$$

where if $m \neq 0$

$$\sigma_{G,m}^2 = \begin{cases} \sigma_{so,m}^2 & \text{in the sub-optimal case} \\ \sigma_{o,m}^2 & \text{in the optimal case} \end{cases}$$

and if $m = 0$

$$\sigma_{G,m}^2 = \begin{cases} \sigma_{so,m}^2 + \langle K\zeta_{so}, \zeta_{so} \rangle & \text{in the sub-optimal case} \\ \sigma_{o,m}^2 + \langle K\zeta_o, \zeta_o \rangle & \text{in the optimal case} \end{cases}$$

Remark 1: For $m = 0$ and $G(u) = |u|^p$ the result was proved by Csörgő & Horváth in [7]. Note that, if G is even, $\zeta_{so}(u) = 0$ and then $\langle K\zeta_{so}, \zeta_{so} \rangle = 0$.

Remark 2: In the sub-optimal case, if $m = 0$ and G is even we can drop the hypothesis that $s \in C^{m+3} \in [0, 1]$ and use the less restrictive condition $s \in C^{m+2} \in [0, 1]$.

The same conclusion holds if $m \neq 0$.

Proof.

$$\begin{aligned} I_n^{(m)} &= \frac{1}{\sqrt{h}} \int_h^{1-h} \left\{ G \left[\frac{h^{m+\frac{1}{2}} \beta_{n,h}^{(m+1)}(u)}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} + \frac{\sqrt{nh} h^{m+\frac{1}{2}} (s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} \right] \right. \\ &\quad \left. - \mathbb{E} \left\{ G \left[\frac{h^{m+\frac{1}{2}} \beta_{n,h}^{(m+1)}(u)}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} + \frac{\sqrt{nh} h^{m+\frac{1}{2}} (s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} \right] \right\} \right\} du \end{aligned}$$

where

$$\beta_{n,h}^{(m+1)}(u) = \frac{1}{h^{m+2}} \int_{-\infty}^{\infty} \varphi^{(m+1)}\left(\frac{u-v}{h}\right) \beta_n^F(v) dv$$

thus

$$h^{m+\frac{1}{2}} \beta_{n,h}^{(m+1)}(u) = \frac{1}{h\sqrt{h}} \int_{-\infty}^{\infty} \varphi^{(m+1)}\left(\frac{u-v}{h}\right) \beta_n^F(v) dv = \sqrt{h} \frac{1}{h} \int_{-\infty}^{\infty} \varphi^{(m+1)}(v) \beta_n^F(u-hv) dv$$

and then $h^{m+\frac{1}{2}} \beta_{n,h}^{(m+1)}(u)$ has the same form as $\sqrt{h} \dot{\beta}_{n,h}^F(u)$ replacing $\dot{\varphi}$ by $\varphi^{(m+1)}$. We have:

$$\begin{aligned} I_n^{(m)} &= \frac{1}{\sqrt{h}} \int_h^{1-h} \left\{ G \left[\frac{h^{m+\frac{1}{2}} b_{n,h}^{(m+1)}(u)}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} + \frac{\sqrt{nh} h^{m+\frac{1}{2}} (s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} \right] \right. \\ &\quad \left. - \mathbb{E} \left\{ G \left[\frac{h^{m+\frac{1}{2}} b_{n,h}^{(m+1)}(u)}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} + \frac{\sqrt{nh} h^{m+\frac{1}{2}} (s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} \right] \right\} \right\} du \\ &+ \frac{1}{\sqrt{h}} \int_h^{1-h} \left\{ G \left[\frac{h^{m+\frac{1}{2}} \beta_{n,h}^{(m+1)}(u)}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} + \frac{\sqrt{nh} h^{m+\frac{1}{2}} (s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} \right] \right. \\ &\quad \left. - G \left[\frac{h^{m+\frac{1}{2}} b_{n,h}^{(m+1)}(u)}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} + \frac{\sqrt{nh} h^{m+\frac{1}{2}} (s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} \right] \right\} du \\ &+ \frac{1}{\sqrt{h}} \int_h^{1-h} \left\{ \mathbb{E} \left\{ G \left[\frac{h^{m+\frac{1}{2}} b_{n,h}^{(m+1)}(u)}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} + \frac{\sqrt{nh} h^{m+\frac{1}{2}} (s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} \right] \right\} \right. \\ &\quad \left. - \mathbb{E} \left\{ G \left[\frac{h^{m+\frac{1}{2}} \beta_{n,h}^{(m+1)}(u)}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} + \frac{\sqrt{nh} h^{m+\frac{1}{2}} (s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n} \sigma_{n,m}(u) h^{m+\frac{1}{2}}} \right] \right\} \right\} du \\ &= (I) + (II) + (III) \end{aligned}$$

We need the following lemma, which is similar to Lemma 1.

Lemma 2 For $p \geq 1$,

$$\mathbb{E}(\sqrt{n} \|\beta_{n,h}^{(m+1)} - b_{n,h}^{(m+1)}\|_{\infty})^p \leq \text{Const} \left(\frac{\ln(n)}{h^{m+1}} \right)^p$$

Proof. The proof is similar to that of Lemma 1, putting $\varphi^{(m+1)}$ instead of $\dot{\varphi}$. ■

Using the Lipschitz property (6) for G and Lemma 2, we can show that $|(III)| \leq \text{Const} \frac{\ln(n)}{\sqrt{nh}}$. In the same manner we obtain $\mathbb{E}[(II)^2] \leq \text{Const} \left(\frac{\ln(n)}{\sqrt{nh}} \right)^2$.

We have then proved that asymptotically $I_n^{(m)} \sim (I)$ and also

$$\begin{aligned}
(I) &\stackrel{\mathcal{D}}{=} \frac{1}{\sqrt{h}} \int_h^{1-h} \left\{ G \left[\xi_h^{b,(m)}(u) + \frac{\sqrt{nh}^{m+\frac{1}{2}}(s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n}\sigma_{n,m}(u)h^{m+\frac{1}{2}}} \right] \right. \\
&- \left. \mathbb{E} \left\{ G \left[\xi_h^{b,(m)}(u) + \frac{\sqrt{nh}^{m+\frac{1}{2}}(s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n}\sigma_{n,m}(u)h^{m+\frac{1}{2}}} \right] \right\} \right\} du \\
&= \frac{1}{\sqrt{h}} \int_h^{1-h} G_h(u, \xi_h^{b,(m)}(u)) du + E_{n,m}
\end{aligned}$$

where

$$\begin{aligned}
\xi_h^{b,(m)}(u) &= \frac{h^{m+\frac{1}{2}} b_h^{F,(m+1)}(u)}{\sqrt{n}\sigma_{n,m}(u)h^{m+\frac{1}{2}}} \\
b_h^{F,(m+1)}(u) &= \frac{1}{h^{m+2}} \int_{-\infty}^{\infty} \varphi^{(m+1)}\left(\frac{u-v}{h}\right) b^F(v) dv.
\end{aligned}$$

Then

$$\begin{aligned}
G_h(u, y) &= G\left(y + \frac{\sqrt{nh}^{m+\frac{1}{2}}(s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n}\sigma_{n,m}(u)h^{m+\frac{1}{2}}}\right) \\
&- \mathbb{E} \left[G\left(N + \frac{\sqrt{nh}^{m+\frac{1}{2}}(s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n}\sigma_{n,m}(u)h^{m+\frac{1}{2}}}\right) \right] - y \mathbb{E} \left[NG\left(N + \frac{\sqrt{nh}^{m+\frac{1}{2}}(s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n}\sigma_{n,m}(u)h^{m+\frac{1}{2}}}\right) \right]
\end{aligned}$$

and

$$E_{n,m} = \frac{1}{\sqrt{h}} \int_h^{1-h} \xi_h^{b,(m)}(u) \mathbb{E} \left[NG\left(N + \frac{\sqrt{nh}^{m+\frac{1}{2}}(s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n}\sigma_{n,m}(u)h^{m+\frac{1}{2}}}\right) \right] du.$$

Note that

$$h^{m+\frac{1}{2}} b_h^{F,(m+1)}(u) = \sqrt{h} \frac{1}{h} \int_{-\infty}^{\infty} \varphi^{(m+1)}(v) b^F(u-hv) dv = \sqrt{h} \dot{b}_h^F(u)$$

and then $h^{m+\frac{1}{2}} b_h^{F,(m+1)}(u)$ is of the same form as $\sqrt{h} \dot{b}_h^F(u)$, using $\varphi^{(m)}$ instead of φ , and

$$\xi_h^{b,(m)}(u) = \frac{\dot{b}_h^F(u)}{\dot{\sigma}_h^b(u)} \stackrel{\mathcal{D}}{=} N(0, 1)$$

Observe that $G_h(u, y)$ is not of the form $g(\sqrt{h}\dot{\sigma}_h^b(u), y)$ but as we remarked at the end of section 4, Theorem 1 is still true and

$$\frac{1}{\sqrt{h}} \int_h^{1-h} G_h(u, \xi_h^{b,(m)}(u)) du \rightarrow N(0, \sigma_{G,m}^2)$$

where

$$\sigma_{G,m}^2 = \begin{cases} \sigma_{so,m}^2 & \text{in the sub-optimal case} \\ \sigma_{o,m}^2 & \text{in the optimal case} \end{cases}$$

On the other hand

$$E_{n,m} = \int_h^{1-h} q_{n,m}(u) \frac{h^m b_h^{F,(m+1)}(u)}{\sqrt{n}\sigma_{n,m}(u)h^{m+\frac{1}{2}}} du$$

and

$$q_{n,m}(u) = \int_{-\infty}^{\infty} y \phi(y) G\left(y + \frac{\sqrt{nh}^{m+\frac{1}{2}}(s_n^{(m)}(u) - s^{(m)}(u))}{\sqrt{n}\sigma_{n,m}(u)h^{m+\frac{1}{2}}}\right) dy.$$

Integrating by parts we can see that when $m \neq 0$, this expression converges a.s. to zero because $h^m b_h^{F,(m)}(u) \xrightarrow{a.s.} b^F(u) \int_{-\infty}^{\infty} \varphi^{(m)}(v) dv = 0$ and

$$\left| \left[\frac{q_{n,m}(u)}{\sqrt{n}\sigma_{n,m}(u)h^{m+\frac{1}{2}}} \right]' \right| \leq \text{Const}$$

so we only have to consider the case when $m = 0$ and then

$$E_{n,0} = \int_h^{1-h} q_{n,0}(u) \frac{\dot{b}_h^F(u)}{\sqrt{nh}\sigma_{n,0}(u)} du$$

with

$$q_{n,0}(u) = \int_{-\infty}^{\infty} y \phi(y) G\left(y + \frac{\sqrt{nh}(s_n(u) - s(u))}{\sqrt{nh}\sigma_{n,0}(u)}\right) dy.$$

Integrating by parts the last expression one obtains that $E_{n,0}$ converges in law to $-\int_0^1 b^F(u)\zeta_{so}(u)du$ in the sub-optimal case (resp. to $-\int_0^1 b^F(u)\zeta_o(u)du$ in the optimal one) and the theorem plus the remarks follow. ■

7 Kullback Deviation

Consider the "modified" Kullback deviation between the kernel estimator of the density \hat{s}_n and the true density s verifying the conditions of section 3 and **H2**. The deviation is 'modified' since we consider the integration over the interval $[h, 1-h]$ for $h > 0$ and not over $[0, 1]$. Let

$$\mathcal{K}(\hat{s}_n, s) = \int_h^{1-h} \hat{s}_n(u) \ln\left(\frac{\hat{s}_n(u)}{s(u)}\right) du.$$

We suppose in this section that φ is even.

Let us show that this definition makes sense under the condition $nh^{1+a} \rightarrow +\infty$ as $h \rightarrow 0$, for some $a > 0$.

For $u \in [h, 1-h]$ we can decompose $\frac{\hat{s}_n(u)}{s(u)}$ as

$$\frac{\hat{s}_n(u)}{s(u)} = 1 + \frac{1}{\sqrt{n}} \left[\frac{\dot{\beta}_{n,h}(u) - \dot{b}_{n,h}(u)}{s(u)} \right] + \frac{1}{\sqrt{n}} \frac{\dot{b}_{n,h}(u)}{s(u)} + \frac{s_n(u) - s(u)}{s(u)},$$

by Lemma 1 and since s is bounded below we know that

$$\sup\left\{ \left| \frac{1}{\sqrt{n}} \left[\frac{\dot{\beta}_{n,h}(u) - \dot{b}_{n,h}(u)}{s(u)} \right] \right|, u \in [h, 1-h] \right\} \leq C_1(\omega) \frac{\ln(n)}{hn}.$$

On the other hand we have the following inequality

$$\sup\left\{ \left| \frac{s_n(u) - s(u)}{s(u)} \right|, u \in [h, 1-h] \right\} \leq \text{Const } h^2$$

and since $\int_{-\infty}^{\infty} \dot{\varphi}(v) dv = 0$ we get

$$\dot{b}_{n,h}(u) = \frac{1}{h} \int_{-\infty}^{\infty} [b_n^F(u-hv) - b_n^F(u)] \dot{\varphi}(v) dv.$$

Using the modulus of continuity for the B.B. we finally obtain that

$$\sup\left\{ \left| \frac{1}{\sqrt{n}} \frac{\dot{b}_{n,h}(u)}{s(u)} \right|, u \in [h, 1-h] \right\} \leq C_2(\omega) \frac{1}{\sqrt{nh}h^{\frac{1}{2}+\delta}}.$$

We have then shown that for $u \in [h, 1-h]$,

$$\frac{\hat{s}_n(u)}{s(u)} = 1 + H_n(u)$$

and

$$\sup\{|H_n(u)|, u \in [h, 1-h]\} \leq C(\omega) \left[\frac{\ln(n)}{hn} + h^2 + \frac{1}{\sqrt{nh}^{\frac{1}{2}+\delta}} \right]$$

which gives the required result if δ is small enough.

Let us now introduce some notation before proving the next theorem. Let

$$w_k^2 = \frac{\|\varphi\|_2^4}{2} \left[\int_{-2}^2 \theta^2(\omega) d\omega \right], \quad c(u) = \left(\frac{s'(u)}{s(u)} \right)' \quad \text{and} \quad \tilde{a}(u) = \frac{1}{2} C_0 K_2 c(u)$$

(' denotes the derivative)

Theorem 4 Under **H1**, **H2** and the assumptions, $nh^{2+a} \rightarrow +\infty$ as $h \rightarrow 0$, for some $a > 0$, and φ is an even function,

$$\begin{aligned} n\sqrt{h} \left[\mathcal{K}(\hat{s}_n, s) - \frac{1}{2} \int_h^{1-h} \frac{1}{s(u)} \mathbb{E}[\hat{s}_n(u) - s(u)]^2 du - \int_h^{1-h} (\hat{s}_n(u) - s(u)) du \right] \\ \xrightarrow{\mathcal{D}} N(0, \sigma_2'^2) \end{aligned}$$

where

$$\sigma_2'^2 = \begin{cases} w_k^2 & \text{in the sub-optimal case} \\ w_k^2 + \langle K\tilde{a}, \tilde{a} \rangle & \text{in the optimal case} \end{cases}$$

(see (16) for the definition of $\langle K\tilde{a}, \tilde{a} \rangle$)

Remark: The term $\int_h^{1-h} (\hat{s}_n(u) - s(u)) du$ tends to zero more slowly than the required normalization $n\sqrt{h}$ so it cannot be dropped.

Proof.

$$\mathcal{K}(\hat{s}_n, s) = \int_h^{1-h} \hat{s}_n(u) \ln\left(1 + \frac{\hat{s}_n(u) - s(u)}{s(u)}\right) du$$

It is easy to show, using a Taylor expansion of order three for the logarithm, that

$$\begin{aligned} \mathcal{K}(\hat{s}_n, s) &= \frac{1}{2} \int_h^{1-h} \frac{(\hat{s}_n(u) - s(u))^2}{s(u)} du + \int_h^{1-h} (\hat{s}_n(u) - s(u)) du + \\ &- \frac{1}{2} \int_h^{1-h} \frac{(\hat{s}_n(u) - s(u))^3}{s^2(u)} du + \frac{1}{3} \int_h^{1-h} \hat{s}_n(u) \left(\frac{\hat{s}_n(u) - s(u)}{s(u)} \right)^3 \frac{1}{(1 + \theta(u, n, \omega))^3} du \end{aligned}$$

where $\theta(u, n, \omega)$ is a point between 0 and $\frac{\hat{s}_n(u) - s(u)}{s(u)}$, and then

$$\begin{aligned} &n\sqrt{h} \left[\mathcal{K}(\hat{s}_n, s) - \frac{1}{2} \int_h^{1-h} \frac{1}{s(u)} \mathbb{E}[\hat{s}_n(u) - s(u)]^2 du - \int_h^{1-h} (\hat{s}_n(u) - s(u)) du \right] \tag{17} \\ &= \frac{1}{2\sqrt{h}} \int_h^{1-h} \frac{1}{s(u)} \left\{ \left[\sqrt{nh}(\hat{s}_n(u) - s(u)) \right]^2 - \mathbb{E} \left[\left[\sqrt{nh}(\hat{s}_n(u) - s(u)) \right]^2 \right] \right\} du \\ &- \frac{n\sqrt{h}}{2} \int_h^{1-h} \frac{(\hat{s}_n(u) - s(u))^3}{s^2(u)} du + \frac{n\sqrt{h}}{3} \int_h^{1-h} \hat{s}_n(u) \left(\frac{\hat{s}_n(u) - s(u)}{s(u)} \right)^3 \frac{1}{(1 + \theta(u, n, \omega))^3} du \\ &= (I) + (II) + (III). \end{aligned}$$

By the results of the previous section we know that $(I) \rightarrow N(0, \sigma_2'^2)$ in law when $n \rightarrow \infty$ (In fact, this is not a straightforward application of Theorem 5 with $G(x) = x^2$, because we don't consider here the

normalization $\sqrt{nh}\sigma_{n,0}(u)$ and furthermore, we have the term $\frac{1}{s(u)}$ which does not appear in Theorem 5. However, the idea of the proof is the same and using Mehler's formula (2) the variance is easily computable because of the very simple form of G . Thus to finish it is sufficient to prove that (II) and (III) tend to zero in probability when $n \rightarrow \infty$. For this it is enough to prove that $E[(II)^2] \rightarrow 0$ and that (III) tends to zero a.s., but

$$E[(II)^2] \leq \text{Const} \frac{1}{nh^2} \int_h^{1-h} E[\sqrt{nh}(\hat{s}_n(u) - s(u))]^6 du$$

and $E[\sqrt{nh}(\hat{s}_n(u) - s(u))]^6 \leq \text{Const}$ then

$$E[(II)^2] \leq \text{Const} \frac{1}{nh^2}$$

and the last term tends to zero because of the hypothesis.

Let us look more closely at the last term of (17). We have already seen that

$$|\hat{s}_n(u) - s(u)| \leq C_3(\omega) \left[\frac{\ln(n)}{hn} + h^2 + \frac{1}{\sqrt{nh}^{\frac{1}{2}+\delta}} \right] \leq C_3(\omega) a(h)$$

with $a(h) \rightarrow 0$ when $h \rightarrow 0$, then

$$|\theta(u, n, \omega)| \leq C_3(\omega) a(h)$$

and

$$|\hat{s}_n(u)| \leq d(\omega).$$

Finally

$$n\sqrt{h} |\hat{s}_n(u)| \left| \frac{\hat{s}_n(u) - s(u)}{s(u)} \right|^3 \leq C'(\omega) \left[\frac{\ln^3(n)}{n^2 h^{\frac{3}{2}}} + nh^{\frac{13}{2}} + \frac{1}{\sqrt{nh}^{1+3\delta}} \right]$$

and then under the hypothesis and for δ small enough, (III) $\rightarrow 0$ a.s. Hence the result follows. ■

We consider also the super-optimal case; following the same arguments as before we get the following

Theorem 5 *Under H1, H2, if $s \in C^3[0, 1]$, φ is an even function, $nh^5 \rightarrow +\infty$ as $h \rightarrow 0$ and $nh^8 \rightarrow 0$, then*

$$\begin{aligned} \frac{2\sqrt{n}}{K_2 h^2} \left[\mathcal{K}(\hat{s}_n, s) - \frac{1}{2} \int_h^{1-h} \frac{1}{s(u)} E[\hat{s}_n(u) - s(u)]^2 du - \int_h^{1-h} (\hat{s}_n(u) - s(u)) du \right] \\ \xrightarrow{\mathcal{D}} = - \int_0^1 b^F(u) c(u) du \equiv N(0, \langle Kc, c \rangle) \end{aligned}$$

(Remember that $c(u) = \left(\frac{s''(u)}{s(u)} \right)'$).

8 Crossings of the Empirical Process

Denote by $N_n^h(x)$ the number of crossings before time 1 of the process $\beta_{n,h}$ at level x . Having in mind the result for the F-B.B. (see Corollary 1) we have the following

Theorem 6 *Let f be a function in C^2 with \ddot{f} bounded. Under H1, H2 and if $\frac{\ln(n)}{\sqrt{nh}} \rightarrow 0$ as $h \rightarrow 0$ then*

$$V_h(f) = \Lambda_h \int_{-\infty}^{\infty} f(x) N_n^h(x) dx - \int_{-\infty}^{\infty} f(x) \ell_n^h(x) dx \rightarrow 0 \quad \text{in } L^2$$

where we define, $\int_{-\infty}^{\infty} f(x) \ell_n^h(x) dx$ as the 'modified occupation measure'

$$\int_{-\infty}^{\infty} f(x) \ell_n^h(x) dx = \int_0^1 f(\beta_{n,h}(u)) \sqrt{s(u)} du$$

(Remember that $\Lambda_h = \sqrt{\frac{\pi h}{2}} \|\varphi\|_2^{-1}$)

Proof.

The Banach-Kac [2] [14] formula gives

$$\int_{-\infty}^{\infty} f(x) N_n^h(x) dx = \int_0^1 f(\beta_{n,h}(u)) |\dot{\beta}_{n,h}(u)| du.$$

To study the behaviour of this r.v. we apply the previous results to the function $g(x) = |x|$, to get that if $h \rightarrow 0$

$$\sqrt{h} \int_0^1 |\dot{\beta}_{n,h}(u)| du \rightarrow \|\varphi\|_2 \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{s(u)} du \quad \text{in } L^2$$

and then

$$\int_{-\infty}^{\infty} \sqrt{h} N_n^h(x) dx \rightarrow \|\varphi\|_2 \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \ell^b(x) dx \quad \text{in } L^2.$$

We consider, since f is continuous, the following normalization for the integrated number of crossings

$$\Lambda_h \int_{-\infty}^{\infty} f(x) N_n^h(x) dx = \Lambda_h \int_0^1 f(\beta_{n,h}(u)) |\dot{\beta}_{n,h}(u)| du.$$

Thus

$$\begin{aligned} \mathbb{E}[V_h(f)]^2 &= \mathbb{E} \left[\int_0^1 f(\beta_{n,h}(u)) \Lambda_h |\dot{\beta}_{n,h}(u)| du - \int_0^1 f(\beta_{n,h}(u)) \sqrt{s(u)} du \right]^2 \\ &\leq \text{Const} \left\{ \mathbb{E} \left[\int_0^1 [f(\beta_{n,h}(u)) - f(b_{n,h}(u))] \Lambda_h |\dot{\beta}_{n,h}(u)| du \right]^2 + \mathbb{E} \left[\int_0^1 f(b_{n,h}(u)) \Lambda_h \{ |\dot{\beta}_{n,h}(u)| - |\dot{b}_{n,h}(u)| \} du \right]^2 \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^1 f(b_{n,h}(u)) (\Lambda_h |\dot{b}_{n,h}(u)| - \sqrt{s(u)}) du \right]^2 + \mathbb{E} \left[\int_0^1 (f(b_{n,h}(u)) - f(\beta_{n,h}(u))) \sqrt{s(u)} du \right]^2 \right\} \end{aligned}$$

Using the Lipschitz property (6), Hölder's inequality for f and Lemma 1, the first and fourth term are $O((\frac{\ln(n)}{\sqrt{n}})^2)$ and the second term is $O((\frac{\ln(n)}{\sqrt{nh}})^2)$. The third term is

$$\mathbb{E} \left[\int_0^1 f(b_h^F(u)) (\Lambda_h |\dot{b}_h^F(u)| - \sqrt{s(u)}) du \right]^2 \leq h \mathbb{E}[Z_h^b(f)]^2 + \mathbb{E} \left[\int_0^1 (f(b^F(u)) - f(b_h^F(u))) \sqrt{s(u)} du \right]^2$$

where $Z_h^b(f)$ was defined in section 3; by Corollary 1, we know that $h \mathbb{E}[Z_h^b(f)]^2 = O(h)$. Moreover, working in the same manner as in that corollary for the second term we get that

$$\mathbb{E} \left[\int_0^1 (f(b^F(u)) - f(b_h^F(u))) \sqrt{s(u)} du \right]^2 = o(h),$$

thus

$$\mathbb{E}[V_h(f)]^2 \leq \text{Const} \left[\left(\frac{\ln(n)}{\sqrt{n}} \right)^2 + \left(\frac{\ln(n)}{\sqrt{nh}} \right)^2 + h + o(h) \right].$$

Hence the theorem holds. ■

Remark: Using this result one has the following convergence

$$\Lambda_h \int_{-\infty}^{\infty} f(x) N_n^h(x) dx \xrightarrow{\mathcal{D}} \int_{-\infty}^{\infty} f(x) \ell^b(x) dx.$$

We only sketch the proof: Let be $a_h(f) = \Lambda_h \int_{-\infty}^{\infty} f(x) N_n^h(x) dx$, $\zeta_h(f) = \int_{-\infty}^{\infty} f(x) \ell_n^h(x) dx$, $Y(f) = \int_{-\infty}^{\infty} f(x) \ell^b(x) dx$, $Y_n(f) = \int_{-\infty}^{\infty} f(x) \ell^{b_n}(x) dx$ and d a distance that metrizes the convergence in law.

We have the following inequality

$$d(a_h(f), Y(f)) \leq d(a_h(f), \zeta_h(f)) + d(\zeta_h(f), Y(f))$$

By the previous result we know that $d(a_h(f), \zeta_h(f)) \rightarrow 0$. It is easy to prove that $\|\zeta_h(f) - Y_n(f)\|_2 \rightarrow 0$ and since $d(\zeta_h(f), Y(f)) = d(\zeta_h(f), Y_n(f)) \leq \|\zeta_h(f) - Y_n(f)\|_2$, the remark follows.

Theorem 7 Under the hypothesis of Theorem 6,

$$\xi_n(f) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} f(x) (\Lambda_h N_n^h(x) - \ell_n^h(x)) dx \rightarrow V \quad \text{stably}$$

where V is as in Corollary 1.

Proof. We decompose the expression for $\xi_n(f)$ as:

$$\begin{aligned} \xi_n(f) &= \frac{1}{\sqrt{h}} \left[\int_0^1 f(\beta_{n,h}(u)) \Lambda_h |\dot{\beta}_{n,h}(u)| du - \int_0^1 f(\beta_{n,h}(u)) \sqrt{s(u)} du \right] \\ &= \frac{1}{\sqrt{h}} \left\{ \int_0^1 [f(\beta_{n,h}(u)) - f(b_{n,h}(u))] \Lambda_h |\dot{\beta}_{n,h}(u)| du + \int_0^1 f(b_{n,h}(u)) \Lambda_h \{ |\dot{\beta}_{n,h}(u)| - |\dot{b}_{n,h}(u)| \} du \right. \\ &\quad \left. + \int_0^1 f(b_{n,h}(u)) (\Lambda_h |\dot{b}_{n,h}(u)| - \sqrt{s(u)}) du + \int_0^1 (f(b_{n,h}(u)) - f(\beta_{n,h}(u))) \sqrt{s(u)} du \right\} \\ &= (I) + (II) + (III) + (IV) \end{aligned}$$

We have already seen that

$$E(I)^2 + E(II)^2 + E(IV)^2 \leq \text{Const} \left[\left(\frac{\ln(n)}{\sqrt{nh}} \right)^2 + \left(\frac{\ln(n)}{\sqrt{nh}} \right)^2 \right]$$

Hence it only remains to consider (III), but

$$(III) = \frac{1}{\sqrt{h}} \int_0^1 f(b_{n,h}(u)) (\Lambda_h |\dot{b}_{n,h}(u)| - \sqrt{s(u)}) du \stackrel{\mathcal{D}}{=} \frac{1}{\sqrt{h}} \int_0^1 f(b_h^F(u)) (\Lambda_h |\dot{b}_h^F(u)| - \sqrt{s(u)}) du$$

Thus

$$\begin{aligned} (III) &\stackrel{\mathcal{D}}{=} \frac{1}{\sqrt{h}} \left[\int_0^1 f(b_h^F(u)) \Lambda_h |\dot{b}_h^F(u)| du - \int_0^1 f(b^F(u)) \sqrt{s(u)} du \right] \\ &\quad + \frac{1}{\sqrt{h}} \int_0^1 (f(b^F(u)) - f(b_h^F(u))) \sqrt{s(u)} du \\ &= (III)_1 + (III)_2 \end{aligned}$$

In the proof of the last theorem we showed that $E[(III)_2]^2 = o(1)$. On the other hand, the term $(III)_1$ is exactly $Z_h^b(f)$ and converges stably to V when h tends to zero. Thus the theorem follows. ■

9 Smoothed Empirical Process

We can obtain an extension of the previous result by considering the Smoothed Empirical Process:

$$\tilde{\beta}_{n,h}(u) = \beta_{n,h}(u) + \sqrt{n}[\tilde{F}_n(u) - F(u)]$$

where $\tilde{F}_n(u) = E[\hat{F}_n(u)]$, $\hat{F}_n(u) = \int_{-\infty}^u \hat{s}_n(v) dv$ with F satisfying the conditions of section 3, s verifying

H2 and φ even. Note that $\tilde{\beta}_{n,h}$ has a bias. It is obvious that

$$\tilde{\beta}_{n,h}(u) = \sqrt{n}[\hat{F}_n(u) - F(u)]$$

and hence this process is the same as that defined by Yukich in [17]. He proved that if $\sqrt{nh^2} \rightarrow 0$ then $\tilde{\beta}_{n,h}(u) \rightarrow b^F(u)$ in law. Moreover using Taylor's development for $\sqrt{n}(\tilde{F}_n(u) - F(u))$, he gets that, if $\sqrt{nh^2} \rightarrow a$, then $\tilde{\beta}_{n,h}(u) \xrightarrow{\mathcal{D}} b^F(u) + \frac{a}{2} K_2 \dot{s}(u)$, where $K_2 = \int_{-\infty}^{\infty} \varphi(v) v^2 dv$. Working as in the previous

section we can obtain the convergence results for $\tilde{N}_n^h(x)$, the number of crossings on $[h, 1-h]$ for the process $\tilde{\beta}_{n,h}$ at level x , i.e. when $\sqrt{nh^2} \rightarrow a \geq 0$,

$$\Lambda_h \int_{-\infty}^{\infty} f(x) \tilde{N}_n^h(x) dx \xrightarrow{\mathcal{D}} \int_0^1 f(b^F(u) + \frac{a}{2} K_2 \dot{s}(u)) \sqrt{s(u)} du.$$

When $\sqrt{nh^2} \rightarrow +\infty$, using a Taylor development for $\frac{2}{K_2 h^2} (\tilde{F}_n(u) - F(u))$ $0 < u < 1$, we get that this term tends to $\dot{s}(u)$. So it is natural to consider $c_n \tilde{\beta}_{n,h}$ (remember that $c_n = \frac{2}{K_2 \sqrt{nh^2}}$) and to study the convergence results for $N_{n,h}^*(x)$, the number of times in $[h, 1-h]$ that the process $c_n \tilde{\beta}_{n,h}$ crosses level x . We obtain the following.

Corollary 2 *Suppose that f satisfies (6), $s \in C^2[0, 1]$ (instead of **H2**), **H1** holds, φ is an even function and $\sqrt{nh^2} \rightarrow +\infty$ as $h \rightarrow 0$ then*

a) *If $nh^5 \rightarrow +\infty$,*

$$\int_{-\infty}^{\infty} f(x) N_{n,h}^*(x) dx \xrightarrow{\mathcal{P}} \int_0^1 f(\dot{s}(u)) |\ddot{s}(u)| du = \int_{-\infty}^{\infty} f(x) N(\dot{s}, x) dx$$

where $N(\dot{s}, x)$ stands for the number of crossings of level x by \dot{s} in $[0, 1]$.

b) *If $\sqrt{nh^5} \rightarrow b$ (b can take the value 0 and if it is the case we will suppose furthermore that **H2** holds and $s \in C^3[0, 1]$),*

$$\sqrt{nh^5} \int_{-\infty}^{\infty} f(x) N_{n,h}^*(x) dx \xrightarrow{\mathcal{D}} \frac{2}{K_2} \|\varphi\|_2 \int_0^1 f(\dot{s}(u)) \sqrt{\frac{2}{\pi}} d_{0,b}(u) \sqrt{s(u)} du.$$

(See section 3 for the definition of $d_{0,b}(u)$ and note that $d_{0,0}(u) = 1$).

Proof. The proof consists in applying the KMT theorem and Theorem 1 of [4]. ■

Let us define the following modification of the occupation measure

$$\int_{-\infty}^{\infty} f(x) \tilde{\ell}_n^h(x) dx = \int_h^{1-h} f(\tilde{\beta}_{n,h}(u)) \sqrt{s(u)} du$$

and

$$\tilde{\xi}_n(f) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} f(x) [\Lambda_h \tilde{N}_n^h(x) - \tilde{\ell}_n^h(x)] dx$$

(Remember that $\tilde{N}_n^h(x)$ is the number of times that the process $\tilde{\beta}_{n,h}$ crosses level x before time 1)

Corollary 3 *Assume that f is in C^2 and \ddot{f} is bounded, under **H1** and **H2**, if $\frac{\ln(n)}{\sqrt{nh}} \rightarrow 0$ as $h \rightarrow 0$,*

$s \in C^3[0, 1]$ and φ is an even function, then

a) *If $\sqrt{nh^2} \rightarrow 0$ we obtain that*

$$\tilde{\xi}_n(f) \rightarrow V \quad \text{stably}$$

where V is the same r.v. that we obtained in Theorem 4.

b) *On the other hand, if $\sqrt{nh^2} \rightarrow a$ then*

$$\tilde{\xi}_n(f) \rightarrow \tilde{V} \quad \text{stably}$$

where

$$\mathcal{L}(\tilde{V} | \mathcal{F}_\infty) = \sigma_0 \int_0^1 f(b^F(u) + \frac{a K_2}{2} \dot{s}(u)) \sqrt{s(u)} d\tilde{W}(u)$$

Proof. We recall that \mathcal{F}_∞ is the σ -algebra generated by $\{b^F(u), 0 \leq u \leq 1\}$.

$$\begin{aligned}\tilde{\xi}_n(f) &= \frac{1}{\sqrt{h}} \left[\int_h^{1-h} f(\tilde{\beta}_{n,h}(u)) \Lambda_h |\dot{\tilde{\beta}}_{n,h}(u)| du - \int_h^{1-h} f(\tilde{\beta}_{n,h}(u)) \sqrt{s(u)} du \right] \\ &= \frac{1}{\sqrt{h}} \left[\int_h^{1-h} f(\beta_{n,h}(u) + a_n(u)) \Lambda_h |\dot{\beta}_{n,h}(u) + \dot{a}_n(u)| du - \int_h^{1-h} f(\beta_{n,h}(u) + a_n(u)) \sqrt{s(u)} du \right]\end{aligned}$$

where $a_n(u) = \sqrt{n}[\tilde{F}_n(u) - F(u)] = \sqrt{n} \int_{-\infty}^{\infty} \varphi(v) [F(u - hv) - F(u)] dv$. We decompose $\tilde{\xi}_n(f)$ as

$$\begin{aligned}\tilde{\xi}_n(f) &= \frac{1}{\sqrt{h}} \left[\int_h^{1-h} f(b_{n,h}(u) + a_n(u)) \Lambda_h |\dot{b}_{n,h}(u) + \dot{a}_n(u)| du - \int_h^{1-h} f(b_{n,h}(u) + a_n(u)) \sqrt{s(u)} du \right] \\ &\quad + \frac{1}{\sqrt{h}} \int_h^{1-h} [f(\beta_{n,h}(u) + a_n(u)) - f(b_{n,h}(u) + a_n(u))] \Lambda_h |\dot{\beta}_{n,h}(u) + \dot{a}_n(u)| du \\ &\quad + \frac{1}{\sqrt{h}} \int_h^{1-h} f(b_{n,h}(u) + a_n(u)) \Lambda_h [|\dot{\beta}_{n,h}(u) + \dot{a}_n(u)| - |\dot{b}_{n,h}(u) + \dot{a}_n(u)|] du \\ &\quad + \frac{1}{\sqrt{h}} \int_h^{1-h} [f(b_{n,h}(u) + a_n(u)) - f(\beta_{n,h}(u) + a_n(u))] \sqrt{s(u)} du \\ &= (I) + (II) + (III) + (IV).\end{aligned}$$

Consider the last three terms. Using the Lipschitz property (6) for f , Hölder's inequality, Lemma 1 and the fact that, by using a Taylor development for F and s respectively, $a_n(u)$ and $\dot{a}_n(u)$ are bounded by \sqrt{nh}^2 on $[h, 1-h]$, it is easy to see that $E(II)^2$ and $E(IV)^2$ are $O((\frac{\ln(n)}{\sqrt{nh}})^2)$ and $E(III)^2$ is $O((\frac{\ln(n)}{\sqrt{nh}})^2)$. $\tilde{\xi}_n(f)$ is then equivalent to (I) and

$$\begin{aligned}(I) &\stackrel{D}{=} \frac{1}{\sqrt{h}} \int_h^{1-h} f(b_h^F(u) + a_n(u)) \{ \Lambda_h |\dot{b}_h^F(u) + \dot{a}_n(u)| - \sqrt{s(u)} \} du \\ &= \frac{1}{\sqrt{h}} \int_h^{1-h} f(b_h^F(u) + a_n(u)) G_h(u, \xi_h^b(u)) du \\ &\quad + \frac{1}{\sqrt{h}} \int_h^{1-h} f(b_h^F(u) + a_n(u)) [k_h(u) E |\sqrt{\frac{\pi}{2}} N + p_n(u)| - \sqrt{s(u)}] du \\ &\quad + \frac{1}{\sqrt{h}} \int_h^{1-h} f(b_h^F(u) + a_n(u)) k_h(u) \xi_h^b(u) E [N |\sqrt{\frac{\pi}{2}} N + p_n(u)|] du \\ &= (I)_1 + (I)_2 + (I)_3\end{aligned}$$

where $k_h(u) = \sqrt{h} \dot{\sigma}_h^b(u) \|\varphi\|_2^{-1}$, $p_n(u) = \sqrt{\frac{\pi}{2}} \frac{\sqrt{h} \dot{a}_n(u)}{\sqrt{h} \dot{\sigma}_h^b(u)}$ and

$$G_h(u, y) = k_h(u) [|\sqrt{\frac{\pi}{2}} y + p_n(u)| - E |\sqrt{\frac{\pi}{2}} N + p_n(u)| - y E [N |\sqrt{\frac{\pi}{2}} N + p_n(u)|]].$$

Consider the second term $(I)_2$.

$$|k_h(u) E |\sqrt{\frac{\pi}{2}} N + p_n(u)| - \sqrt{s(u)}| \leq |k_h(u) [E |\sqrt{\frac{\pi}{2}} N + p_n(u)| - 1]| + |k_h(u) - \sqrt{s(u)}|$$

We have already seen in section 5 that on $[h, 1-h]$, $|k_h(u) - \sqrt{s(u)}| \leq \text{Const } h$. Furthermore, a simple calculation shows that

$$\begin{aligned}E |\sqrt{\frac{\pi}{2}} N + p_n(u)| - 1 &= \sqrt{\frac{\pi}{2}} [E |N + d_n(u)| - \sqrt{\frac{2}{\pi}}] \\ &= 2 \sqrt{\frac{\pi}{2}} \{ d_n(u) [\Phi(0) - \Phi(-d_n(u))] + [\phi(-d_n(u)) - \phi(0)] \}\end{aligned}$$

where $d_n(u) = \sqrt{\frac{2}{\pi}}p_n(u)$, ϕ and Φ are the standard Gaussian density and distribution function respectively. So we get that $|\mathbb{E}[\sqrt{\frac{\pi}{2}}N + p_n(u)] - 1| \leq \text{Const } h$ and therefore $|I_2| \leq \text{Const } \sqrt{h} \int_h^{1-h} |f(b_h^F(u) + a_n(u))| du$. We thus obtain that $\mathbb{E}[(I_2)^2] \leq \text{Const } h$. Consider now the third term.

$$(I_3) = \frac{1}{\|\varphi\|_2} \int_h^{1-h} f(b_h^F(u) + a_n(u)) \dot{b}_h^F(u) \mathbb{E}[N | \sqrt{\frac{\pi}{2}}N + p_n(u)] du.$$

By an elementary calculation we can see that

$$\frac{1}{\sqrt{h}} \mathbb{E}[N | \sqrt{\frac{\pi}{2}}N + p_n(u)] = \sqrt{\frac{\pi}{2h}} \left[\int_{-d_n(u)}^{d_n(u)} z^2 \phi(z) dz + 2d_n(u) \int_{d_n(u)}^{+\infty} z \phi(z) dz \right]$$

this expression tends to zero in the sub-optimal case and tends towards $\frac{aK_2\ddot{s}(u)}{2\|\varphi\|_2\sqrt{s(u)}}$ in the optimal case.

Moreover, the difference between $\frac{1}{\sqrt{h}} \mathbb{E}[N | \sqrt{\frac{\pi}{2}}N + p_n(u)]$ and its limit is $O((n\sqrt{h})h^4 + \sqrt{nh^2})$ in the sub-optimal case and $O(\sqrt{h} + h + |\sqrt{nh^2} - a|)$ in the optimal one. The convergence to zero, in probability, for the term (I_3) has been proved for the sub-optimal case. Hence we must only consider the optimal case, and then (I_3) is asymptotically equivalent to

$$\int_h^{1-h} \frac{1}{\|\varphi\|_2} f(b_h^F(u) + a_n(u)) \frac{aK_2}{2\|\varphi\|_2} \frac{\ddot{s}(u)}{\sqrt{s(u)}} \sqrt{h} \dot{b}_h^F(u) du.$$

Using the Lipschitz property (6) for f and the fact that $\sup_{0 \leq u \leq 1} \|b_h^F(u) - b^F(u)\|_2 = O(h)$ and

$$\sup_{h \leq u \leq 1-h} |a_n(u) - \frac{a}{2}K_2\dot{s}(u)| \leq \text{Const}\{|\sqrt{nh^2} - a| + \sqrt{nh^3}\},$$

we thus obtain that (I_3) is asymptotically equivalent to

$$\int_h^{1-h} f(b^F(u) + \frac{a}{2}K_2\dot{s}(u)) \frac{\ddot{s}(u)}{\sqrt{s(u)}} \sqrt{h} \dot{b}_h^F(u) du.$$

By Theorem 1 in [4] this quantity tends to zero in probability when h tends to zero. Thus (I_3) tends to zero in probability and $\tilde{\xi}_n(f) \sim (I_1) = \frac{1}{\sqrt{h}} \int_h^{1-h} f(b_h^F(u) + a_n(u)) G_h(u, \xi_h^b(u)) du$. following the same arguments as in the proof of Corollary 1 (see remark 2 at the end of section 4), we finally get that the last term tends stably to the required limit. ■

We consider now the case where $\sqrt{nh^2} \rightarrow +\infty$ and we get, in the next corollary, the speed at which the convergence takes place.

Corollary 4 *Assume that f verifies (6), $s \in C^3[0, 1]$, **H1** holds, φ is an even function and $nh^4 \rightarrow +\infty$ as $h \rightarrow 0$, then*

a) *If $nh^6 \rightarrow +\infty$ and $|\ddot{s}|$ is bounded below on $[0, 1]$, then*

$$c_n^{-1} \left[\int_{-\infty}^{\infty} f(x) N_{n,h}^*(x) dx - \int_h^{1-h} f(c_n \tilde{\beta}_{n,h}(u)) |c_n \sqrt{n} (\dot{F}_n(u) - \dot{F}(u))| du \right] \xrightarrow{\mathcal{D}} \int_0^1 f(\dot{s}(u)) \text{sign}(\ddot{s}(u)) db^F(u)$$

b) *If $\frac{nh^5 - q^2}{\sqrt{h}} \rightarrow 0$ ($q \geq 0$), under **H2**, $s \in C^3[0, 1]$ and assuming that f is in C^2 with bounded second derivative,*

$$\frac{1}{\sqrt{h}} \left[\|\varphi\|_2^{-1} \sqrt{nh^5} \int_{-\infty}^{\infty} f(x) N_{n,h}^*(x) dx - \frac{2}{K_2} \int_h^{1-h} f(c_n \tilde{\beta}_{n,h}(u)) \sqrt{\frac{2}{\pi}} d_{0,q}(u) \sqrt{s(u)} du \right] \xrightarrow{\mathcal{D}} \\ \frac{2}{K_2} \|\varphi\|_2^{-1} \int_0^1 f(\dot{s}(u)) \sqrt{\frac{2}{\pi}} d_{1,q}(u) db^F(u) + \frac{2}{K_2} \sqrt{\frac{2}{\pi}} \int_0^1 \sigma_q(u) f(\dot{s}(u)) \sqrt{s(u)} d\tilde{W}(u)$$

where \tilde{W} is a B. M. independent of b^F .

Remark: In the special case where $q = 0$, $d_{1,0}(u) = 0$ and the first term of the limit vanishes.

Proof. The proof consists, as before, in an application of the KMT theorem, Theorem 1 of [4], Theorem 1 and arguments similar to those used to prove Corollary 1. ■

References

- [1] D.J. ALDOUS & G.K. EAGLESTON: *On mixing and stability of limit theorems*, Ann. Prob., 6, 325-331, 1978.
- [2] S. BANACH: *Sur les lignes rectifiables et les surfaces dont l'aire est finie*, Fund. Math., 7, 225-237, 1925.
- [3] J. BEIRLANT & D.M. MASON: *On the asymptotic normality of L_p -norms of empirical functionals*, Math. Meth. Stats., to appear.
- [4] C. BERZIN-JOSEPH & J.R. LEÓN & J. ORTEGA: *Non-linear functionals of the brownian bridge and their applications*, submitted.
- [5] P. BICKEL & M. ROSENBLATT: *On some global measures of the deviation of density function estimates*, Ann. Stat., 1, 1071-1095, 1973.
- [6] P. BREUER & P. MAJOR: *Central Limit Theorems for non-linear functionals of Gaussian fields*, J. Mult. Anal., 13, 425-441, 1983.
- [7] M. CSÖRGŐ & L. HORVÁTH: *Central limit theorems for L_p -norms of density estimators*, Prob. Th. Rel. Fields., 80, 269-291, 1988.
- [8] M. CSÖRGŐ & P. REVESZ: *Strong Approximations in Probability and Statistics*, Academic Press, New-York, 1981.
- [9] E. GINÉ & D.M. MASSON & A. Yu ZAITSEV: *The L_1 -norm Density Estimator Process*, to appear.
- [10] P. HALL: *Central limit theorem for integrated square error of multivariate nonparametric density estimators*, J. Mult. Anal., 14, 1-16, 1984.
- [11] P. HALL & C. C. HEYDE: *Martingale Limit Theory and its Applications*, Academic Press, New-York, 1980.
- [12] L. HORVÁTH: *On L_p -norms of multivariate density estimators*, Ann. Stat., 19, 1933-1949, 1991.
- [13] J. JACOD: *On continuous conditional Gaussian martingales and stable convergence in law*, Sémin. Prob. XXXI, L.N. Maths. Springer, 1655, 232-246, 1997.
- [14] M. KAC: *On the average number of real roots of a random algebraic equation*, Bull. Amer. Math. Soc., 49, 314-320, 1943.
- [15] E. V. KHMALADZE: *Martingale limit theorems for divisible statistics*, Theor. Prob. Appl., 28, 530-548, 1983.
- [16] B.W. SILVERMAN: *Weak and strong uniform consistency of the kernel estimate of a density and its derivatives*, Ann. Stat., 6, 177-184, 1978.
- [17] J.E. YUKICH: *A note on limit theorems for perturbed empirical processes*, Stoc. Proc. Appl., 33, 163-173, 1989.