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# Magnetic bottles on geometrically finite hyperbolic surfaces

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## Abstract

We consider a magnetic Laplacian  $-\Delta_A = (id + A)^*(id + A)$  on a hyperbolic surface  $\mathbf{M}$ , when the magnetic field  $dA$  is infinite at the boundary at infinity. We prove that the counting function of the eigenvalues has a particular asymptotic behavior when  $\mathbf{M}$  has an infinite area. <sup>1</sup>

## 1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface  $(\mathbf{M}, g)$  and a smooth, real one-form  $A$  on  $\mathbf{M}$ . We define the magnetic Laplacian

$$\begin{aligned} -\Delta_A &= (i d + A)^*(i d + A) , \\ ((i d + A)u &= i du + uA , \forall u \in C_0^\infty(\mathbf{M}; \mathbb{C}) ) . \end{aligned} \quad (1.1)$$

The magnetic field is the exact two-form  $\rho_B = dA$ .

If  $dm$  is the Riemannian measure on  $\mathbf{M}$ , then

$$\rho_B = \tilde{\mathbf{b}} dm , \quad \text{with} \quad \tilde{\mathbf{b}} \in C^\infty(\mathbf{M}; \mathbb{R}) . \quad (1.2)$$

The magnetic intensity is  $\mathbf{b} = |\tilde{\mathbf{b}}|$ .

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It is well known, (see [Shu] ), that  $-\Delta_A$  has a unique self-adjoint extension on  $L^2(\mathbf{M})$  , containing in its domain  $C_0^\infty(\mathbf{M}; \mathbb{C})$  , the space of smooth and compactly supported functions.

When  $\mathbf{b}$  is infinite at the infinity, (with some additional assumption), the spectrum of  $-\Delta_A$  is discrete, and we denote by  $(\lambda_j)_j$  the increasing sequence of eigenvalues of  $-\Delta_A$  , (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda) = \sum_{\lambda_j < \lambda} 1 . \quad (1.3)$$

We are interested by the hyperbolic surfaces  $\mathbf{M}$ , when the curvature of  $\mathbf{M}$  is constant and negative.

In this case, when  $\mathbf{M}$  has finite area, the asymptotic behavior of  $N(\lambda)$  seems to be the Weyl formula :  $N(\lambda) \sim_{+\infty} \frac{\lambda}{4\pi} |\mathbf{M}|$  .

S. Golénia and S. Moroianu in [Go-Mo] have such examples.

In the case of the Poincaré half-plane,  $\mathbf{M} = \mathbb{H}$  , we prove in [Mo-Tr] that the Weyl formula is not valid :  $\lim_{\lambda \rightarrow +\infty} \lambda^{-1} N(\lambda) = +\infty$  .

For example when  $\mathbf{b}(z) = a_0^2(x/y)^{2m_0} + a_1^2 y^{m_1} + a_2^2/y^{m_2}$  ,  $a_j > 0$  and  $m_j \in \mathbb{N}^*$  , then

$$N(\lambda) \sim_{+\infty} \lambda^{1+1/(2m_0)} \ln(\lambda) \alpha(m_0, m_1, m_2) .$$

In this paper, we are interested by the hyperbolic surfaces with infinite area. When  $\mathbf{M}$  is a geometrically finite hyperbolic surface of infinite area and when the above example is arranged for this new situation, ( $m_0$  is absent,  $m_1$  appears in the cusps and  $m_2$  in the funnels), we get

$$N(\lambda) \sim_{+\infty} \lambda^{1+1/m_2} \alpha(m_2) :$$

the cusps do not contribute to the leading part of  $N(\lambda)$  .

## 2 Main result

We assume that  $(\mathbf{M}, g)$  is a smooth connected Riemannian manifold of dimension two, which is a geometrically finite hyperbolic surface of infinite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$\mathbf{M} = \left( \bigcup_{j=0}^{J_1} M_j \right) \cup \left( \bigcup_{k=1}^{J_2} F_k \right) ; \quad (2.1)$$

where the  $M_j$  and the  $F_k$  are open sets of  $\mathbf{M}$ , such that the closure of  $M_0$  is compact, and if  $J_1 > 0$ , the other  $M_j$  are cuspidal ends of  $\mathbf{M}$ , and the  $F_k$  are funnel ends of  $\mathbf{M}$ .

This means that, for any  $j$ ,  $1 \leq j \leq J_1$ , there exist strictly positive constants  $a_j$  and  $L_j$  such that  $M_j$  is isometric to  $\mathbb{S} \times ]a_j^2, +\infty[$ , equipped with the metric

$$ds_j^2 = y^{-2} (L_j^2 d\theta^2 + dy^2); \quad (2.2)$$

( $\mathbb{S} = \mathbb{S}^1$  is the unit circle.)

In the same way, for any  $k$ ,  $1 \leq k \leq J_2$ , there exist strictly positive constants  $\alpha_k$  and  $\tau_k$  such that  $F_k$  is isometric to  $\mathbb{S} \times ]\alpha_k^2, +\infty[$ , equipped with the metric

$$ds_k^2 = \tau_k^2 \cosh^2(t) dt^2 + d\theta^2; \quad (2.3)$$

moreover, for any two integers  $j, k > 0$ , we have  $M_j \cap F_k = \emptyset$  and  $M_j \cap M_k = F_j \cap F_k = \emptyset$  if  $j \neq k$ .

Let us choose some  $z_0 \in M_0$  and let us define

$$d : \mathbf{M} \rightarrow \mathbb{R}_+; \quad d(z) = d_g(z, z_0); \quad (2.4)$$

$d_g(\cdot, \cdot)$  denotes the distance with respect to the metric  $g$ .

We assume the smooth one-form  $A$  to be given such that the magnetic field  $\tilde{\mathbf{b}}$  satisfies

$$\lim_{d(z) \rightarrow \infty} \mathbf{b}(z) = +\infty. \quad (2.5)$$

If  $J_1 > 0$ , there exists a constant  $C_1 > 0$  such

$$|X\tilde{\mathbf{b}}(z)| \leq C_1(\mathbf{b}(z) + 1)e^{d(z)}|X|_g; \quad (2.6)$$

$$\forall z \in M_j, \forall X \in T_z\mathbf{M} \text{ and } \forall j = 1, \dots, J_1.$$

There exists a constant  $C_2 > 0$  such

$$|X\tilde{\mathbf{b}}(z)| \leq C_2(\mathbf{b}(z) + 1)|X|_g; \quad (2.7)$$

$$\forall z \in F_k, \forall X \in T_z\mathbf{M} \text{ and } \forall k = 1, \dots, J_2.$$

For any self-adjoint operator  $P$ , and for any real  $\lambda$ , we will denote by  $E_\lambda(P)$  its spectral projection, and when its trace is finite we will denote it by

$$N(\lambda; P) = \text{Tr}(E_\lambda(P)).$$

$N(\lambda; P)$  is the number of eigenvalues of  $P$ , (counted with their multiplicity), which are in  $] -\infty, \lambda[$ .

**Theorem 2.1** *Under the above assumptions,  $-\Delta_A$  has a compact resolvent and for any  $\delta \in ]\frac{1}{3}, \frac{2}{5}[$ , there exists a constant  $C > 0$  such that*

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbf{M}} \left(1 - \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathcal{N}(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m)) \, dm \\ \leq N(\lambda, -\Delta_A) \leq \\ \frac{1}{2\pi} \int_{\mathbf{M}} \left(1 + \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathcal{N}(\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m)) \, dm \end{aligned} \quad (2.8)$$

where

$$\mathcal{N}(\mu, \mathbf{b}(m)) = \mathbf{b}(m) \sum_{k=0}^{+\infty} [\mu - (2k+1)\mathbf{b}(m)]_+^0 \quad \text{if } \mathbf{b}(m) > 0 ,$$

and

$$\mathcal{N}(\mu, \mathbf{b}(m)) = \mu/2 \quad \text{if } \mathbf{b}(m) = 0 .$$

$[\rho]_+^0$  is the Heaviside function:

$$[\rho]_+^0 = \begin{cases} 1 , & \text{if } \rho > 0 \\ 0 , & \text{if } \rho \leq 0 . \end{cases}$$

The Theorem remains true if we replace  $\int_{\mathbf{M}}$  by  $\sum_{k=1}^{J_2} \int_{F_k}$ , due to the fact that the other parts are bounded by  $C\lambda$ .

**Corollary 2.2** *Under the assumptions of Theorem 2.1 and if the function*

$$\omega(\mu) = \int_{\mathbf{M}} [\mu - \mathbf{b}(m)]_+^0 dm$$

satisfies,  $\exists C_1 > 0$  s.t.  $\forall \mu > C_1, \forall \tau \in ]0, 1[$ ,

$$\omega((1+\tau)\mu) - \omega(\mu) \leq C_1 \tau \omega(\mu) , \quad (2.9)$$

then

$$N(\lambda; -\Delta_A) \sim \frac{1}{2\pi} \int_{\mathbf{M}} \mathcal{N}(\lambda - \frac{1}{4}, \mathbf{b}(m)) \, dm . \quad (2.10)$$

For example this allows us to consider magnetic fields of the following type:

$$\begin{aligned} & \text{on } F_k, \quad \mathbf{b}(\theta, t) = p_k(1/\cosh(t)) , \\ & \text{and on } M_j, \quad j > 0, \quad \mathbf{b}(\theta, y) = q_j(y) , \end{aligned}$$

where the  $p_k(s)$  and the  $q_j(s)$  are, for large  $s$ , polynomial functions of order  $\geq 1$ . In this case, if  $d$  is the largest order of the  $p_k(s)$ , then

$$N(\lambda; -\Delta_A) \sim \alpha \lambda^{1+1/d} ,$$

for some constant  $\alpha > 0$ , depending only on the funnels  $F_k$  where the order of  $p_k(s)$  is  $d$ .

### 3 Estimate for Dirichlet operators

#### 3.1 The main propositions

In this section, we consider some particular open set  $\Omega$  of  $\mathbf{M}$  with smooth boundary. To  $\Omega$  and  $-\Delta_A$ , we associate the Dirichlet operator  $-\Delta_A^\Omega$ , and we estimate  $N(\lambda; -\Delta_A^\Omega)$ .

**Proposition 3.1** *Let  $\Omega$  an open set of  $M_0$  with smooth boundary. Then there exists a constant  $C_\Omega > 0$  s.t.*

$$\left| N(\lambda; -\Delta_A^\Omega) - \frac{|\Omega|}{4\pi} \lambda \right| \leq C_\Omega \sqrt{\lambda}; \quad \forall \lambda > 1 .$$

As  $\overline{\Omega}$  is compact, the above estimate is well known. See for example Theorem 29.3.3 in [Hor].

**Proposition 3.2** *Let  $j > 0$  and  $\Omega$  an open set of the cusp  $M_j$ , isometric to  $\mathbb{S} \times ]a^2, +\infty[$ , equipped with the metric*

$$ds^2 = y^{-2} ( L^2 d\theta^2 + dy^2 ); \quad (a \text{ and } L \text{ are strictly positive constants}) .$$

*Then  $-\Delta_A^\Omega$  has a compact resolvent and*

$$N(\lambda; -\Delta_A^\Omega) \sim \frac{|\Omega|}{4\pi} \lambda; \quad \text{as } \lambda \rightarrow +\infty .$$

We will prove it in the next subsection.

**Proposition 3.3** *Let  $\Omega$  an open set of a funnel  $F_k$ , isometric to  $\mathbb{S} \times ]a^2, +\infty[$ , equipped with the metric*

$$ds^2 = L^2 \cosh^2(t) d\theta^2 + dt^2 ; \quad (a \text{ and } L \text{ are strictly positive constants}) .$$

*Then  $-\Delta_A^\Omega$  has a compact resolvent and for any  $\delta \in ]\frac{1}{3}, \frac{2}{5}[$ , there exists a constant  $C > 0$  such that*

$$\begin{aligned} \frac{1}{2\pi} \int_{\Omega} \left(1 - \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathcal{N}(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m)) dm \\ \leq N(\lambda, -\Delta_A^\Omega) \leq \\ \frac{1}{2\pi} \int_{\Omega} \left(1 + \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathcal{N}(\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m)) dm \end{aligned}$$

The proof comes easily following the ones in the Poincaré half-plane of [Mo-Tr], using the method of [Col], in the neighbourhood of the boundary at infinity. It corresponds to a context where the partitions of unity were fine, so they can be performed on  $\mathbb{S} \times ]a^2, +\infty[$ , (instead of  $\mathbb{R} \times ]-\infty, 0[$ ) .

### 3.2 Proof of Proposition 3.2

For simplicity we change the unit circle  $\mathbb{S} = \mathbb{S}_1$  into the circle  $\mathbb{S}_L$ , of radius  $L$ , so

$$\begin{aligned} \Omega = \mathbb{S}_L \times ]a^2, +\infty[ , \quad ds^2 = y^{-2}(dx^2 + dy^2) , \quad \text{and} \quad (3.1) \\ -\Delta_A^\Omega u(z) = y^2[(D_x - A_1)^2 u(z) + (D_y - A_2)^2 u(z)] ; \end{aligned}$$

$$\text{moreover } d(z, z') = \arg \cosh \frac{y^2 + y'^2 + d_{\mathbb{S}_L}^2(x, x')}{2yy'} .$$

We begin by proving the compactness of the resolvent of  $-\Delta_A^\Omega$ .

**Lemma 3.4** *There exists  $C_0 > 1$  such that*

$$\int_{\Omega} (\mathbf{b}(z) - C_0) |u(z)|^2 dm \leq \int_{\Omega} -\Delta_A^\Omega u(z) \overline{u(z)} dm ; \quad \forall u \in C_0^\infty(\Omega) .$$

**Proof.** Let us denote the quadratic form

$$q_A^\Omega(u) = \int_{\Omega} -\Delta_A^\Omega u(z) \overline{u(z)} dm \quad \forall u \in C_0^\infty(\Omega). \quad (3.2)$$

Then  $q_A^\Omega(u) = \int_{\Omega} [| (D_x - A_1)u|^2 + (D_y - A_2)u|^2] dx dy$ ,  
and  $\left| \int_{\Omega} \tilde{\mathbf{b}}(z) |u(z)|^2 dm \right|$   
 $= \left| \int_{\Omega} [(D_x - A_1)u(z) \overline{(D_y - A_2)u(z)} - (D_y - A_2)u(z) \overline{(D_x - A_1)u(z)}] dx dy \right|.$

Therefore we get that  $\left| \int_{\Omega} \tilde{\mathbf{b}}(z) |u(z)|^2 dm \right| \leq q_A^\Omega(u).$

As  $\mathbf{b}(z) = |\tilde{\mathbf{b}}(z)| \rightarrow +\infty$  at the infinity, the Lemma comes easily.

The Lemma 3.4 and the assumption (2.5) prove that  $-\Delta_A^\Omega$  has compact resolvent.

Later on, we will need that the assumptions (2.5) and (2.6) ensure that there exists  $C > 1$  such that  $\forall z = (x, y), z' = (x', y') \in \Omega$ ,

$$\mathbf{b}(z)/C \leq \mathbf{b}(z') \leq C\mathbf{b}(z), \quad \text{if } |y - y'| \leq 1 \text{ and } y > C. \quad (3.3)$$

This comes from the fact that  $d(z)$  is equivalent to  $\ln(y)$  for  $y(> 1)$  large enough, so the assumption (2.6) ensures that  $|\partial_x b(z)| + |\partial_y b(z)| \leq C(|b(z)| + 1).$

**Lemma 3.5** *There exists a constant  $C_0 > 1$  such that, for any  $\lambda > 1$  and for any  $K \subset \Omega$  isometric to  $I_1 \times I_2$ , endowed with the metric in (3.1), with*

$$I_1 = ]x_0 - \epsilon_1, x_0 + \epsilon_1[, \quad I_2 = ]y_0 - \epsilon_2, y_0 + \epsilon_2[,$$

$$\epsilon_1 \in ]C_0^{-1}, 1[, \quad \epsilon_2 = \sqrt{y_0}/\sqrt{\mathbf{b}(z_0)}, \quad (y_0 > C_0);$$

*the following estimates hold:*

$$[\lambda(1 - \frac{1}{\sqrt{y_0}}) - C_0] \frac{|K|_g}{4\pi} \leq N(\lambda; -\Delta_A^K) \leq [\lambda(1 + \frac{1}{\sqrt{y_0}}) + C_0] \frac{|K|_g}{4\pi}. \quad (3.4)$$

**Proof.** If  $\mathbf{b}(z_0) > C\lambda$ , then, according to the estimate of Lemma 3.4 with  $K$  instead of  $\Omega$ ,  $N(\lambda; -\Delta_A^K) = 0$ .  
So we can assume that  $\mathbf{b}(z_0) \leq C\lambda$ .



We use that the spectrum of  $-\Delta_A^K$  is gauge-invariant, so we can suppose that in  $K$

$$A_2 = 0 \quad \text{and} \quad A_1(x, y) = - \int_{y_0}^y \frac{\tilde{\mathbf{b}}(x, \rho)}{\rho^2} d\rho .$$

Then  $|A_1(x, y)| \leq C\epsilon_2 \frac{\mathbf{b}(z_0)}{y_0^2}$ .

From this estimate, we get that for any  $\epsilon \in ]0, 1[$ ,

$$-(1 - \epsilon)\Delta_0^K - C\epsilon_2^2 \frac{\mathbf{b}^2(z_0)}{\epsilon y_0^2} \leq -\Delta_A^K \leq -(1 + \epsilon)\Delta_0^K + C\epsilon_2^2 \frac{\mathbf{b}^2(z_0)}{\epsilon y_0^2} .$$

We take  $\epsilon = 1/\sqrt{y_0}$ , to get

$$-(1 - \frac{1}{\sqrt{y_0}})\Delta_0^K - C \frac{\mathbf{b}(z_0)}{\sqrt{y_0}} \leq -\Delta_A^K \leq -(1 + \frac{1}{\sqrt{y_0}})\Delta_0^K + C \frac{\mathbf{b}(z_0)}{\sqrt{y_0}} .$$

As  $\mathbf{b}(z_0) \leq C\lambda$ , the Lemma follows easily from the min-max principle and the well-known estimate for  $N(\lambda; -\Delta_0^K)$ .

**Proof of Proposition 3.2.**

It follows easily from Lemma 3.5, (for large  $y$ ), using the same tricks as in [Mo-Tr].

## 4 Proof of the main Theorem 2.1

The proof comes easily from the three propositions 3.1 - - 3.3, following the method developped in [Mo-Tr].

## 5 Remark on the case of constant magnetic field

It is not always possible to have a constant magnetic field on  $\mathbf{M}$ , (for topological reason), but for any  $(b, \beta) \in \mathbb{R}^{J_1} \times \mathbb{R}^{J_2}$ , there exists a one-form  $A$ , such that the corresponding magnetic field  $dA$  satisfies

$$dA = \tilde{\mathbf{b}}(z)dm \quad \left\{ \begin{array}{l} \tilde{\mathbf{b}}(z) = b_j \quad \forall z \in M_j \\ \tilde{\mathbf{b}}(z) = \beta_k \quad \forall z \in F_k \end{array} \right. \quad (5.1)$$

**Theorem 5.1** Assume (2.1) and (5.1).

If  $J_1 = 0$  and  $J_2 > 0$ , then the essential spectrum of  $-\Delta_A$  is

$$\text{sp}_{\text{ess}}(-\Delta_A) = [\frac{1}{4} + \inf_k \beta_k^2, +\infty[ \bigcup \left( \bigcup_{k=1}^{J_2} S(\beta_k) \right) \quad (5.2)$$

with  $S(\beta_k) = \emptyset$  when  $|\beta_k| \leq 1/2$  and when  $|\beta_k| > 1/2$

$S(\beta_k) = \{(2j+1)|\beta_k| - j(j+1); j \in \mathbb{N}, j < |\beta_k| - 1/2\}$ .

If  $J_1$  and  $J_2$  are  $> 0$ , then for any  $j$ ,  $1 \leq j \leq J_1$  and for any  $z \in M_j$  there exists a unique closed curve through  $z$ ,  $\mathcal{C}_{j,z}$  in  $(M_j, g)$ , not contractible and with zero  $g$ -curvature. The following limit exists and is finite:

$$[A]_{M_j} = \lim_{d(z) \rightarrow +\infty} \int_{\mathcal{C}_{j,z}} A. \quad (5.3)$$

If  $J_1^A = \{j \in \mathbb{N}, 1 \leq j \leq J_1 \text{ s.t. } [A]_{M_j} \in 2\pi\mathbb{Z}\}$ , then

$$\text{sp}_{\text{ess}}(-\Delta_A) = [\frac{1}{4} + \min\{\inf_{j \in J_1^A} b_j^2, \inf_{1 \leq k \leq J_2} \beta_k^2\}, +\infty[ \bigcup \left( \bigcup_{k=1}^{J_2} S(\beta_k) \right). \quad (5.4)$$

If  $J_2 = 0$  and  $J_1^A = \emptyset$ , then  $\text{sp}_{\text{ess}}(-\Delta_A) = \emptyset$ :  
 $-\Delta_A$  has purely discrete spectrum, (its resolvent is compact).

**Remark 5.2** In Theorem 5.1, one can change  $\mathcal{C}_{j,z}$  into  $\mathcal{S}_{j,z}$ , the unique closed curve through  $z$ , not contractible and with minimal  $g$ -length.

$\mathcal{S}_{j,z}$  is not smooth at  $z$ ,  $\mathcal{S}_{j,z}$  is part of two geodesics through  $z$ , so there is an out-going tangent and an incoming tangent at  $z$ . It is easy to see that  $\mathcal{C}_{j,z} \cap \mathcal{S}_{j,z} = \{z\}$ , so by Stokes formula

$$\int_{\mathcal{S}_{j,z}} (A - A^0) = \int_{\mathcal{C}_{j,z}} (A - A^0),$$

where  $A^0$  is a one-form on  $M$ , such that

$$dA = dA^0 \text{ on } M_j \text{ and } [A^0]_{M_j} = 0; \forall j.$$

The orientation in both cases  $\mathcal{C}_{j,z}$  and  $\mathcal{S}_{j,z}$ , is chosen such that, if  $u_z, v_z \in T_z M_j$ ,  $g_z(u_z, v_z) = 0$ ,  $dm(u_z, v_z) > 0$ , and  $u_z$  is tangent to the curve (in the positive direction), then  $v_z$  points to boundary at infinity; (for  $\mathcal{S}_{j,z}$ , one can take as  $u_z$  the out-going tangent, or the incoming tangent).

**Proof of Theorem 5.1.** It is clear that

$$\mathrm{sp}_{ess}(-\Delta_A) = \left( \bigcup_{j=1}^{J_1} \mathrm{sp}_{ess}(-\Delta_A^{M_j}) \right) \cup \left( \bigcup_{k=1}^{J_2} \mathrm{sp}_{ess}(-\Delta_A^{F_k}) \right) ; \quad (5.5)$$

so the proof will result on the two lemmas below.

**Lemma 5.3**

$$\mathrm{sp}_{ess}(-\Delta_A^{F_k}) = \left[ \frac{1}{4} + \beta_k^2, +\infty[ \cup S(\beta_k) \right] .$$

**Proof.** We have  $-\Delta_A^{F_k} = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + \cosh^{-1}(t)(D_t - A_2) [\cosh(t)(D_t - A_2)]$  .

Since  $\tilde{\mathbf{b}} = \beta_k = \tau_k^{-1} \cosh^{-1}(t)(\partial_\theta A_2 - \partial_t A_1)$  , there exists a function  $\varphi$  such that  $A - \tilde{A} = d\varphi$  if  $\tilde{A} = (\xi - \beta_k \tau_k \sinh(t))d\theta$  , (for some constant  $\xi$ ) .

So we can assume that  $A = \tilde{A}$  .

We change the density  $dm = \tau_k \cosh(t)d\theta dt$  for  $d\theta dt$  , using the unitary operator  $Uf = (\tau_k \cosh(t))^{1/2}f$  , so

$$P = -U\Delta_A^{F_k}U^\star = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}(1 + \cosh^{-2}(t)) .$$

We remind that  $\lambda \in \mathrm{sp}_{ess}(-\Delta_A^{F_k})$  iff there exists a sequence  $(u_j)_j \in \mathrm{Dom}(-\Delta_A^{F_k})$  converging weekly in  $L^2(F_k)$  to zero,  $\|u_j\|_{L^2(F_k)} = 1$  and such that the sequence  $(-\Delta_A^{F_k}u_k - \lambda u_k)_k$  converges strongly to zero.

It is clear that  $\mathrm{sp}(-\Delta_A^{F_k}) = \mathrm{sp}(\bigoplus_{\ell \in \mathbb{Z}} P_\ell)$  ,

$$P_\ell = D_t^2 + \tau_k^{-2} \cosh^{-2}(t)(\ell + \beta_k \tau_k \sinh(t) - \xi)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)) ,$$

for the Dirichlet condition on  $L^2(I; dt)$  ;  $I = ]\alpha_k^2, +\infty[$  .

So  $\mathrm{sp}(-\Delta_A^{F_k}) = \bigcup_{\ell \in \mathbb{Z}} \mathrm{sp}(P_\ell)$  .

Writing that  $P_\ell = D_t^2 + \left( \frac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t) \right)^2 + \frac{1}{4}(1 + \cosh^{-2}(t))$  ,

we get easily that  $\mathrm{sp}_{ess}(P_\ell) = [\frac{1}{4} + \beta_k^2, +\infty[$  , and that the number of eigenvalues  $< \frac{1}{4} + \beta_k^2$  is finite for all  $\ell < \xi$  and equal to zero for all  $\ell \geq \xi$  . Here

we assume  $\beta_k > 0$ . So  $[\frac{1}{4} + \beta_k^2, +\infty[ \subset \text{sp}_{ess}(-\Delta_A^{F_k})$  and the other part of  $\text{sp}_{ess}(-\Delta_A^{F_k})$  is  $S_\infty = \{\lambda; \lambda = \lim_{j \rightarrow +\infty} \lambda_{\ell(j)}, \lambda_{\ell(j)} \in \text{sp}_d(P_{\ell(j)})\}$ , where  $(\ell(j))_j$  denotes any decreasing sequence of negative integers.

Now we use again the formula

$$P_\ell = D_t^2 + \left( \frac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t) \right)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)).$$

Assuming  $\ell - \xi < 0$ , we set  $\rho = |\ell - \xi|/\tau_k$  and we introduce the new variable  $y = 2\rho e^{-t}$ . We get that  $P_\ell$  is unitarily equivalent to  $\tilde{P}_\rho$  defined as a Dirichlet type operator in  $L^2([0, 2\rho e^{-\alpha_k^2}]; dy)$ , (zero boundary condition is only required on the right boundary):

$$\tilde{P}_\rho = D_y(y^2 D_y) + W_\rho(y), \quad \text{with}$$

$$W_\rho(y) = \left( \beta_k \frac{(1 - y^2/(4\rho^2))}{1 + y^2/(4\rho^2)} - \frac{y}{1 + y^2/(4\rho^2)} \right)^2 + \left( \frac{y/(2\rho)}{1 + y^2/(4\rho^2)} \right)^2.$$

So we have  $\lim_{\rho \rightarrow +\infty} W_\rho(y) = W_\infty(y) = (\beta_k - y)^2$ , and the operator

$\tilde{P}_\infty = D_y(y^2 D_y) + W_\infty(y)$  on  $L^2([0, +\infty[; dy)$  satisfies, (see [Mo-Tr]),  $\text{sp}(\tilde{P}_\infty) = \text{sp}_{ess}(\tilde{P}_\infty) \cup \text{sp}_d(\tilde{P}_\infty)$  with

$$\text{sp}_{ess}(\tilde{P}_\infty) = [\frac{1}{4} + \beta_k^2, +\infty[; \quad \text{sp}_d(\tilde{P}_\infty) = S(\beta_k).$$

We remind that the eigenfunctions associated to the eigenvalues in  $S(\beta_k)$  of  $\tilde{P}_\infty$  are exponentially decreasing, so if  $\lambda_0(\rho) \leq \dots \leq \lambda_j(\rho) \leq \lambda_{j+1}(\rho) \dots$

are the eigenvalues of  $\tilde{P}_\rho$  then for any  $j$ ,

$\lim_{\rho \rightarrow +\infty} \lambda_j(\rho) = \lambda_j(\infty) = (2j+1)\beta_k - j(j+1)$ , if  $\beta_k > 1/2$  and  $j < \beta_k - 1/2$ ,

otherwise  $\lim_{\rho \rightarrow +\infty} \lambda_j(\rho) = \frac{1}{4} + \beta_k^2$ .

Therefore we get that  $S_\infty = S(\beta_k)$ , or  $S_\infty = S(\beta_k) \cup \{\frac{1}{4} + \beta_k^2\}$ : the formula of Lemma 5.3 follows.

**Lemma 5.4** *If  $1 \leq j \leq J_1$  and  $j \notin J_1^A$ , then*

$$\text{sp}_{ess}(-\Delta_A^{M_j}) = \emptyset.$$

*If  $j \in J_1^A$ , then*

$$\text{sp}_{ess}(-\Delta_A^{M_j}) = [\frac{1}{4} + b_j^2, +\infty[.$$

**Proof.** Use the coordinate  $t = \ln y$  instead of  $y$ , so

$$M_j = \mathbb{S} \times ]\alpha_j^2, +\infty[ \quad \text{and} \quad ds_j^2 = L_j^2 e^{-2t} d\theta^2 + dt^2; \quad (\alpha_j = e^{a_j}).$$

Then  $-\Delta_A^{M_j} = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + e^t (D_t - A_2)(e^{-t}(D_t - A_2))$ ,  
 $\tilde{\mathbf{b}} = L_j^{-1} e^t (\partial_\theta A_2 - \partial_t A_1)$  and  $dm = L_j e^{-t} d\theta dt$ . As in Lemma 5.3, we have

$$A - \tilde{A} = d\varphi \text{ if } \tilde{A} = (\xi + L_j b_j e^{-t}) d\theta, \text{ (for some constant } \xi).$$

So we can also assume that  $A = \tilde{A}$ .

We replace the density  $dm$  by  $d\theta dt$ , using the unitary operator  $Uf = \sqrt{L_j} e^{-t/2} f$ , so

$$P = -U \Delta_A^{M_j} U^* = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}.$$

Then we get also that

$$\text{sp}(-\Delta_A^{M_j}) = \text{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell); \quad P_\ell = D_t^2 + \frac{1}{4} + \left( e^t \frac{(\ell + \xi)}{L_j} + b_j \right)^2,$$

for the Dirichlet condition on  $L^2(I; dt)$ ;  $I = ]\alpha_j^2, +\infty[$ .

When  $\ell + \xi \neq 0$ , the spectrum of  $P_\ell$  is discrete. More precisely

$$\text{sp}(P_\ell) = \text{sp}(P^\pm), \quad \text{where} \quad P^\pm = D_t^2 + \frac{1}{4} + (\pm e^t + b_j)^2$$

for the Dirichlet condition on  $L^2(I_{j,\ell}; dt)$ ;  $I_{j,\ell} = ]\alpha_j^2 + \ln(|\ell + \xi|/L_j), +\infty[$ ,  
and  $\pm = \frac{\ell + \xi}{|\ell + \xi|}$ .

So  $\lim_{|\ell| \rightarrow \infty} \inf \text{sp}(P_\ell) = +\infty$ , and then we get easily that the spectrum of  $-\Delta_A^{M_j}$  is discrete, when  $\xi = [A]_{M_j}/(2\pi) \notin \mathbb{Z}$ .

If  $\ell + \xi = 0$ , the spectrum of  $P_\ell$  is absolutely continuous :

$$\text{sp}(P_{-\xi}) = \text{sp}_{ess}(P_{-\xi}) = \text{sp}_{ac}(P_{-\xi}) = [\frac{1}{4} + b_j^2, +\infty[;$$

and then, when  $[A]_{M_j} \in 2\pi\mathbb{Z}$ ,  $\text{sp}_{ess}(-\Delta_A^{M_j}) = [\frac{1}{4} + b_j^2, +\infty[$ .

This achieves the proof of Lemma 5.4.

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