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Magnetic bottles on geometrically finite hyperbolic surfaces

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Abstract

We consider a magnetic Laplacian $-\Delta_A = (id + A)^*(id + A)$ on a hyperbolic surface \mathbf{M} , when the magnetic field dA is infinite at the boundary at infinity. We prove that the counting function of the eigenvalues has a particular asymptotic behavior when \mathbf{M} has an infinite area. ¹

1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface (\mathbf{M}, g) and a smooth, real one-form A on \mathbf{M} . We define the magnetic Laplacian

$$\begin{aligned} -\Delta_A &= (i d + A)^*(i d + A), \\ ((i d + A)u &= i du + uA, \forall u \in C_0^\infty(\mathbf{M}; \mathbb{C})) . \end{aligned} \tag{1.1}$$

The magnetic field is the exact two-form $\rho_B = dA$.

If dm is the Riemannian measure on \mathbf{M} , then

$$\rho_B = \tilde{\mathbf{b}} dm, \quad \text{with } \tilde{\mathbf{b}} \in C^\infty(\mathbf{M}; \mathbb{R}). \tag{1.2}$$

The magnetic intensity is $\mathbf{b} = |\tilde{\mathbf{b}}|$.

¹ *Keywords* : spectral asymptotics, magnetic bottles, hyperbolic surface.

It is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(\mathbf{M})$, containing in its domain $C_0^\infty(\mathbf{M}; \mathbb{C})$, the space of smooth and compactly supported functions.

When \mathbf{b} is infinite at the infinity, (with some additional assumption), the spectrum of $-\Delta_A$ is discrete, and we denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda) = \sum_{\lambda_j < \lambda} 1 . \quad (1.3)$$

We are interested by the hyperbolic surfaces \mathbf{M} , when the curvature of \mathbf{M} is constant and negative.

In this case, when \mathbf{M} has finite area, the asymptotic behavior of $N(\lambda)$ seems to be the Weyl formula : $N(\lambda) \sim_{+\infty} \frac{\lambda}{4\pi} |\mathbf{M}|$.

S. Golénia and S. Moroianu in [Go-Mo] have such examples.

In the case of the Poincaré half-plane, $\mathbf{M} = \mathbb{H}$, we prove in [Mo-Tr] that the Weyl formula is not valid : $\lim_{\lambda \rightarrow +\infty} \lambda^{-1} N(\lambda) = +\infty$.

For example when $\mathbf{b}(z) = a_0^2(x/y)^{2m_0} + a_1^2 y^{m_1} + a_2^2/y^{m_2}$, $a_j > 0$ and $m_j \in \mathbb{N}^*$, then

$$N(\lambda) \sim_{+\infty} \lambda^{1+1/(2m_0)} \ln(\lambda) \alpha(m_0, m_1, m_2) .$$

In this paper, we are interested by the hyperbolic surfaces with infinite area. When \mathbf{M} is a geometrically finite hyperbolic surface of infinite area and when the above example is arranged for this new situation, (m_0 is absent, m_1 appears in the cusps and m_2 in the funnels), we get

$$N(\lambda) \sim_{+\infty} \lambda^{1+1/m_2} \alpha(m_2) :$$

the cusps do not contribute to the leading part of $N(\lambda)$.

2 Main result

We assume that (\mathbf{M}, g) is a smooth connected Riemannian manifold of dimension two, which is a geometrically finite hyperbolic surface of infinite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$\mathbf{M} = \left(\bigcup_{j=0}^{J_1} M_j \right) \cup \left(\bigcup_{k=1}^{J_2} F_k \right) ; \quad (2.1)$$

where the M_j and the F_k are open sets of \mathbf{M} , such that the closure of M_0 is compact, and if $J_1 > 0$, the other M_j are cuspidal ends of \mathbf{M} , and the F_k are funnel ends of \mathbf{M} .

This means that, for any j , $1 \leq j \leq J_1$, there exist strictly positive constants a_j and L_j such that M_j is isometric to $\mathbb{S} \times]a_j^2, +\infty[$, equipped with the metric

$$ds_j^2 = y^{-2}(L_j^2 d\theta^2 + dy^2); \quad (2.2)$$

($\mathbb{S} = \mathbb{S}^1$ is the unit circle.)

In the same way, for any k , $1 \leq k \leq J_2$, there exist strictly positive constants α_k and τ_k such that F_k is isometric to $\mathbb{S} \times]\alpha_k^2, +\infty[$, equipped with the metric

$$ds_k^2 = \tau_k^2 \cosh^2(t) d\theta^2 + dt^2; \quad (2.3)$$

moreover, for any two integers $j, k > 0$, we have $M_j \cap F_k = \emptyset$ and $M_j \cap M_k = F_j \cap F_k = \emptyset$ if $j \neq k$.

Let us choose some $z_0 \in M_0$ and let us define

$$d : \mathbf{M} \rightarrow \mathbb{R}_+; \quad d(z) = d_g(z, z_0); \quad (2.4)$$

$d_g(\cdot, \cdot)$ denotes the distance with respect to the metric g .

We assume the smooth one-form A to be given such that the magnetic field $\tilde{\mathbf{b}}$ satisfies

$$\lim_{d(z) \rightarrow \infty} \mathbf{b}(z) = +\infty. \quad (2.5)$$

If $J_1 > 0$, there exists a constant $C_1 > 0$ such

$$|X\tilde{\mathbf{b}}(z)| \leq C_1(\mathbf{b}(z) + 1)e^{d(z)}|X|_g; \quad (2.6)$$

$$\forall z \in M_j, \forall X \in T_z\mathbf{M} \text{ and } \forall j = 1, \dots, J_1.$$

There exists a constant $C_2 > 0$ such

$$|X\tilde{\mathbf{b}}(z)| \leq C_2(\mathbf{b}(z) + 1)|X|_g; \quad (2.7)$$

$$\forall z \in F_k, \forall X \in T_z\mathbf{M} \text{ and } \forall k = 1, \dots, J_2.$$

For any self-adjoint operator P , and for any real λ , we will denote by $E_\lambda(P)$ its spectral projection, and when its trace is finite we will denote it by

$$N(\lambda; P) = \text{Tr}(E_\lambda(P)).$$

$N(\lambda; P)$ is the number of eigenvalues of P , (counted with their multiplicity), which are in $] -\infty, \lambda[$.

Theorem 2.1 *Under the above assumptions, $-\Delta_A$ has a compact resolvent and for any $\delta \in]\frac{1}{3}, \frac{2}{5}[$, there exists a constant $C > 0$ such that*

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbf{M}} \left(1 - \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathcal{N}\left(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m)\right) dm \\ \leq N(\lambda, -\Delta_A) \leq \\ \frac{1}{2\pi} \int_{\mathbf{M}} \left(1 + \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathcal{N}\left(\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m)\right) dm \end{aligned} \quad (2.8)$$

where

$$\mathcal{N}(\mu, \mathbf{b}(m)) = \mathbf{b}(m) \sum_{k=0}^{+\infty} [\mu - (2k+1)\mathbf{b}(m)]_+^0 \quad \text{if } \mathbf{b}(m) > 0,$$

and

$$\mathcal{N}(\mu, \mathbf{b}(m)) = \mu/2 \quad \text{if } \mathbf{b}(m) = 0.$$

$[\rho]_+^0$ is the Heaviside function:

$$[\rho]_+^0 = \begin{cases} 1, & \text{if } \rho > 0 \\ 0, & \text{if } \rho \leq 0. \end{cases}$$

The Theorem remains true if we replace $\int_{\mathbf{M}}$ by $\sum_{k=1}^{J_2} \int_{F_k}$, due to the fact that the other parts are bounded by $C\lambda$.

Corollary 2.2 *Under the assumptions of Theorem 2.1 and if the function*

$$\omega(\mu) = \int_{\mathbf{M}} [\mu - \mathbf{b}(m)]_+^0 dm$$

satisfies, $\exists C_1 > 0$ s.t. $\forall \mu > C_1, \forall \tau \in]0, 1[$,

$$\omega((1+\tau)\mu) - \omega(\mu) \leq C_1 \tau \omega(\mu), \quad (2.9)$$

then

$$N(\lambda; -\Delta_A) \sim \frac{1}{2\pi} \int_{\mathbf{M}} \mathcal{N}\left(\lambda - \frac{1}{4}, \mathbf{b}(m)\right) dm. \quad (2.10)$$

For example this allows us to consider magnetic fields of the following type:

$$\begin{aligned} & \text{on } F_k, \mathbf{b}(\theta, t) = p_k(1/\cosh(t)), \\ & \text{and on } M_j, j > 0, \mathbf{b}(\theta, y) = q_j(y), \end{aligned}$$

where the $p_k(s)$ and the $q_j(s)$ are, for large s , polynomial functions of order ≥ 1 . In this case, if d is the largest order of the $p_k(s)$, then

$$N(\lambda; -\Delta_A) \sim \alpha \lambda^{1+1/d},$$

for some constant $\alpha > 0$, depending only on the funnels F_k where the order of $p_k(s)$ is d .

3 Estimate for Dirichlet operators

3.1 The main propositions

In this section, we consider some particular open set Ω of \mathbf{M} with smooth boundary. To Ω and $-\Delta_A$, we associate the Dirichlet operator $-\Delta_A^\Omega$, and we estimate $N(\lambda; -\Delta_A^\Omega)$.

Proposition 3.1 *Let Ω an open set of M_0 with smooth boundary. Then there exists a constant $C_\Omega > 0$ s.t.*

$$\left| N(\lambda; -\Delta_A^\Omega) - \frac{|\Omega|}{4\pi} \lambda \right| \leq C_\Omega \sqrt{\lambda}; \quad \forall \lambda > 1.$$

As $\bar{\Omega}$ is compact, the above estimate is well known. See for example Theorem 29.3.3 in [Hor].

Proposition 3.2 *Let $j > 0$ and Ω an open set of the cusp M_j , isometric to $\mathbb{S} \times]a^2, +\infty[$, equipped with the metric*

$$ds^2 = y^{-2}(L^2 d\theta^2 + dy^2); \quad (a \text{ and } L \text{ are strictly positive constants}).$$

Then $-\Delta_A^\Omega$ has a compact resolvent and

$$N(\lambda; -\Delta_A^\Omega) \sim \frac{|\Omega|}{4\pi} \lambda; \quad \text{as } \lambda \rightarrow +\infty.$$

We will prove it in the next subsection.

Proposition 3.3 *Let Ω an open set of a funnel F_k , isometric to $\mathbb{S} \times]a^2, +\infty[$, equipped with the metric*

$$ds^2 = L^2 \cosh^2(t) d\theta^2 + dt^2 ; \quad (a \text{ and } L \text{ are strictly positive constants}) .$$

Then $-\Delta_A^\Omega$ has a compact resolvent and for any $\delta \in]\frac{1}{3}, \frac{2}{5}[$, there exists a constant $C > 0$ such that

$$\begin{aligned} \frac{1}{2\pi} \int_{\Omega} \left(1 - \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathcal{N}(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m)) dm \\ \leq N(\lambda, -\Delta_A^\Omega) \leq \\ \frac{1}{2\pi} \int_{\Omega} \left(1 + \frac{C}{(\mathbf{b}(m) + 1)^{(2-5\delta)/2}}\right) \mathcal{N}(\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m)) dm \end{aligned}$$

The proof comes easily following the ones in the Poincaré half-plane of [Mo-Tr], using the method of [Col], in the neighbourhood of the boundary at infinity. It corresponds to a context where the partitions of unity were fine, so they can be performed on $\mathbb{S} \times]a^2, +\infty[$, (instead of $\mathbb{R} \times]-\infty, 0[$).

3.2 Proof of Proposition 3.2

For simplicity we change the unit circle $\mathbb{S} = \mathbb{S}_1$ into the circle \mathbb{S}_L , of radius L , so

$$\begin{aligned} \Omega = \mathbb{S}_L \times]a^2, +\infty[, \quad ds^2 = y^{-2}(dx^2 + dy^2) , \quad \text{and} \quad (3.1) \\ -\Delta_A^\Omega u(z) = y^2[(D_x - A_1)^2 u(z) + (D_y - A_2)^2 u(z)] ; \end{aligned}$$

moreover $d(z, z') = \arg \cosh \frac{y^2 + y'^2 + d_{\mathbb{S}_L}^2(x, x')}{2yy'}$.

We begin by proving the compactness of the resolvent of $-\Delta_A^\Omega$.

Lemma 3.4 *There exists $C_0 > 1$ such that*

$$\int_{\Omega} (\mathbf{b}(z) - C_0) |u(z)|^2 dm \leq \int_{\Omega} -\Delta_A^\Omega u(z) \overline{u(z)} dm ; \quad \forall u \in C_0^\infty(\Omega) .$$

Proof. Let us denote the quadratic form

$$q_A^\Omega(u) = \int_{\Omega} -\Delta_A^\Omega u(z) \overline{u(z)} dm \quad \forall u \in C_0^\infty(\Omega). \quad (3.2)$$

Then $q_A^\Omega(u) = \int_{\Omega} [|(D_x - A_1)u|^2 + (D_y - A_2)u|^2] dx dy$,

$$\text{and } \left| \int_{\Omega} \tilde{\mathbf{b}}(z) |u(z)|^2 dm \right| \\ = \left| \int_{\Omega} [(D_x - A_1)u(z) \overline{(D_y - A_2)u(z)} - (D_y - A_2)u(z) \overline{(D_x - A_1)u(z)}] dx dy \right|.$$

Therefore we get that $\left| \int_{\Omega} \tilde{\mathbf{b}}(z) |u(z)|^2 dm \right| \leq q_A^\Omega(u)$.

As $\mathbf{b}(z) = |\tilde{\mathbf{b}}(z)| \rightarrow +\infty$ at the infinity, the Lemma comes easily.

The Lemma 3.4 and the assumption (2.5) prove that $-\Delta_A^\Omega$ has compact resolvent.

Later on, we will need that the assumptions (2.5) and (2.6) ensure that there exists $C > 1$ such that $\forall z = (x, y), z' = (x', y') \in \Omega$,

$$\mathbf{b}(z)/C \leq \mathbf{b}(z') \leq C\mathbf{b}(z), \quad \text{if } |y - y'| \leq 1 \text{ and } y > C. \quad (3.3)$$

This comes from the fact that $d(z)$ is equivalent to $\ln(y)$ for $y(> 1)$ large enough, so the assumption (2.6) ensures that $|\partial_x b(z)| + |\partial_y b(z)| \leq C(|b(z)| + 1)$.

Lemma 3.5 *There exists a constant $C_0 > 1$ such that, for any $\lambda > 1$ and for any $K \subset \Omega$ isometric to $I_1 \times I_2$, endowed with the metric in (3.1), with*

$$I_1 =]x_0 - \epsilon_1, x_0 + \epsilon_1[, \quad I_2 =]y_0 - \epsilon_2, y_0 + \epsilon_2[,$$

$$\epsilon_1 \in]C_0^{-1}, 1[, \quad \epsilon_2 = \sqrt{y_0}/\sqrt{\mathbf{b}(z_0)}, \quad (y_0 > C_0);$$

the following estimates hold:

$$[\lambda(1 - \frac{1}{\sqrt{y_0}}) - C_0] \frac{|K|_g}{4\pi} \leq N(\lambda; -\Delta_A^K) \leq [\lambda(1 + \frac{1}{\sqrt{y_0}}) + C_0] \frac{|K|_g}{4\pi}. \quad (3.4)$$

Proof. If $\mathbf{b}(z_0) > C\lambda$, then, according to the estimate of Lemma 3.4 with K instead of Ω , $N(\lambda; -\Delta_A^K) = 0$.

So we can assume that $\mathbf{b}(z_0) \leq C\lambda$.

We use that the spectrum of $-\Delta_A^K$ is gauge-invariant, so we can suppose that in K

$$A_2 = 0 \quad \text{and} \quad A_1(x, y) = - \int_{y_0}^y \frac{\tilde{\mathbf{b}}(x, \rho)}{\rho^2} d\rho .$$

Then $|A_1(x, y)| \leq C\epsilon_2 \frac{\mathbf{b}(z_0)}{y_0^2}$.

From this estimate, we get that for any $\epsilon \in]0, 1[$,

$$-(1 - \epsilon)\Delta_0^K - C\epsilon_2^2 \frac{\mathbf{b}^2(z_0)}{\epsilon y_0^2} \leq -\Delta_A^K \leq -(1 + \epsilon)\Delta_0^K + C\epsilon_2^2 \frac{\mathbf{b}^2(z_0)}{\epsilon y_0^2} .$$

We take $\epsilon = 1/\sqrt{y_0}$, to get

$$-(1 - \frac{1}{\sqrt{y_0}})\Delta_0^K - C \frac{\mathbf{b}(z_0)}{\sqrt{y_0}} \leq -\Delta_A^K \leq -(1 + \frac{1}{\sqrt{y_0}})\Delta_0^K + C \frac{\mathbf{b}(z_0)}{\sqrt{y_0}} .$$

As $\mathbf{b}(z_0) \leq C\lambda$, the Lemma follows easily from the min-max principle and the well-known estimate for $N(\lambda; -\Delta_0^K)$.

Proof of Proposition 3.2.

It follows easily from Lemma 3.5, (for large y), using the same tricks as in [Mo-Tr].

4 Proof of the main Theorem 2.1

The proof comes easily from the three propositions 3.1 - - 3.3, following the method developped in [Mo-Tr].

5 Remark on the case of constant magnetic field

It is not always possible to have a constant magnetic field on \mathbf{M} , (for topological reason), but for any $(b, \beta) \in \mathbb{R}^{J_1} \times \mathbb{R}^{J_2}$, there exists a one-form A , such that the corresponding magnetic field dA satisfies

$$dA = \tilde{\mathbf{b}}(z)dm \quad \begin{cases} \tilde{\mathbf{b}}(z) = b_j \forall z \in M_j \\ \tilde{\mathbf{b}}(z) = \beta_k \forall z \in F_k \end{cases} \quad (5.1)$$

Theorem 5.1 *Assume (2.1) and (5.1).*

If $J_1 = 0$ and $J_2 > 0$, then the essential spectrum of $-\Delta_A$ is

$$\mathrm{sp}_{\mathrm{ess}}(-\Delta_A) = \left[\frac{1}{4} + \inf_k \beta_k^2, +\infty \left[\bigcup_{k=1}^{J_2} S(\beta_k) \right) \right) \quad (5.2)$$

with $S(\beta_k) = \emptyset$ when $|\beta_k| \leq 1/2$ and when $|\beta_k| > 1/2$

$S(\beta_k) = \{(2j+1)|\beta_k| - j(j+1); j \in \mathbb{N}, j < |\beta_k| - 1/2\}$.

If J_1 and J_2 are > 0 , then for any j , $1 \leq j \leq J_1$ and for any $z \in M_j$ there exists a unique closed curve through z , $\mathcal{C}_{j,z}$ in (M_j, g) , not contractible and with zero g -curvature. The following limit exists and is finite:

$$[A]_{M_j} = \lim_{d(z) \rightarrow +\infty} \int_{\mathcal{C}_{j,z}} A. \quad (5.3)$$

If $J_1^A = \{j \in \mathbb{N}, 1 \leq j \leq J_1 \text{ s.t. } [A]_{M_j} \in 2\pi\mathbb{Z}\}$, then

$$\mathrm{sp}_{\mathrm{ess}}(-\Delta_A) = \left[\frac{1}{4} + \min \left\{ \inf_{j \in J_1^A} b_j^2, \inf_{1 \leq k \leq J_2} \beta_k^2 \right\}, +\infty \left[\bigcup_{k=1}^{J_2} S(\beta_k) \right) \right) . \quad (5.4)$$

If $J_2 = 0$ and $J_1^A = \emptyset$, then $\mathrm{sp}_{\mathrm{ess}}(-\Delta_A) = \emptyset$: $-\Delta_A$ has purely discrete spectrum, (its resolvent is compact).

Remark 5.2 *In Theorem 5.1, one can change $\mathcal{C}_{j,z}$ into $\mathcal{S}_{j,z}$, the unique closed curve through z , not contractible and with minimal g -length.*

$\mathcal{S}_{j,z}$ is not smooth at z , $\mathcal{S}_{j,z}$ is part of two geodesics through z , so there is an out-going tangent and an incoming tangent at z . It is easy to see that $\mathcal{C}_{j,z} \cap \mathcal{S}_{j,z} = \{z\}$, so by Stokes formula

$$\int_{\mathcal{S}_{j,z}} (A - A^0) = \int_{\mathcal{C}_{j,z}} (A - A^0),$$

where A^0 is a one-form on M , such that

$$dA = dA^0 \quad \text{on } M_j \quad \text{and} \quad [A^0]_{M_j} = 0; \quad \forall j.$$

The orientation in both cases $\mathcal{C}_{j,z}$ and $\mathcal{S}_{j,z}$, is chosen such that, if $u_z, v_z \in T_z M_j$, $g_z(u_z, v_z) = 0$, $dm(u_z, v_z) > 0$, and u_z is tangent to the curve (in the positive direction), then v_z points to boundary at infinity; (for $\mathcal{S}_{j,z}$, one can take as u_z the out-going tangent, or the incoming tangent).

Proof of Theorem 5.1. It is clear that

$$\text{sp}_{\text{ess}}(-\Delta_A) = \left(\bigcup_{j=1}^{J_1} \text{sp}_{\text{ess}}(-\Delta_A^{M_j}) \right) \cup \left(\bigcup_{k=1}^{J_2} \text{sp}_{\text{ess}}(-\Delta_A^{F_k}) \right) ; \quad (5.5)$$

so the proof will result on the two lemmas below.

Lemma 5.3

$$\text{sp}_{\text{ess}}(-\Delta_A^{F_k}) = \left[\frac{1}{4} + \beta_k^2, +\infty[\cup S(\beta_k) \right] .$$

Proof. We have $-\Delta_A^{F_k} = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + \cosh^{-1}(t)(D_t - A_2) [\cosh(t)(D_t - A_2)]$.

Since $\tilde{\mathbf{b}} = \beta_k = \tau_k^{-1} \cosh^{-1}(t)(\partial_\theta A_2 - \partial_t A_1)$, there exists a function φ such that $A - \tilde{A} = d\varphi$ if $\tilde{A} = (\xi - \beta_k \tau_k \sinh(t))d\theta$, (for some constant ξ) .

So we can assume that $A = \tilde{A}$.

We change the density $dm = \tau_k \cosh(t)d\theta dt$ for $d\theta dt$, using the unitary operator $Uf = (\tau_k \cosh(t))^{1/2}f$, so

$$P = -U\Delta_A^{F_k}U^* = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}(1 + \cosh^{-2}(t)) .$$

We remind that $\lambda \in \text{sp}_{\text{ess}}(-\Delta_A^{F_k})$ iff there exists a sequence $(u_j)_j \in \text{Dom}(-\Delta_A^{F_k})$ converging weakly in $L^2(F_k)$ to zero, $\|u_j\|_{L^2(F_k)} = 1$ and such that the sequence $(-\Delta_A^{F_k}u_k - \lambda u_k)_k$ converges strongly to zero.

It is clear that $\text{sp}(-\Delta_A^{F_k}) = \text{sp}\left(\bigoplus_{\ell \in \mathbb{Z}} P_\ell\right)$,

$$P_\ell = D_t^2 + \tau_k^{-2} \cosh^{-2}(t)(\ell + \beta_k \tau_k \sinh(t) - \xi)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)) ,$$

for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha_k^2, +\infty[$.

So $\text{sp}(-\Delta_A^{F_k}) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell)$.

Writing that $P_\ell = D_t^2 + \left(\frac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t) \right)^2 + \frac{1}{4}(1 + \cosh^{-2}(t))$,

we get easily that $\text{sp}_{\text{ess}}(P_\ell) = \left[\frac{1}{4} + \beta_k^2, +\infty[$, and that the number of eigenvalues $< \frac{1}{4} + \beta_k^2$ is finite for all $\ell < \xi$ and equal to zero for all $\ell \geq \xi$. Here

we assume $\beta_k > 0$. So $[\frac{1}{4} + \beta_k^2, +\infty[\subset \text{sp}_{ess}(-\Delta_A^{F_k})$ and the other part of $\text{sp}_{ess}(-\Delta_A^{F_k})$ is $S_\infty = \{\lambda; \lambda = \lim_{j \rightarrow +\infty} \lambda_{\ell(j)}, \lambda_{\ell(j)} \in \text{sp}_d(P_{\ell(j)})\}$, where $(\ell(j))_j$ denotes any decreasing sequence of negative integers.

Now we use again the formula

$$P_\ell = D_t^2 + \left(\frac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t) \right)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)).$$

Assuming $\ell - \xi < 0$, we set $\rho = |\ell - \xi|/\tau_k$ and we introduce the new variable $y = 2\rho e^{-t}$. We get that P_ℓ is unitarily equivalent to \tilde{P}_ρ defined as a Dirichlet type operator in $L^2(]0, 2\rho e^{-\alpha_k^2}]; dy)$, (zero boundary condition is only required on the right boundary):

$$\tilde{P}_\rho = D_y(y^2 D_y) + W_\rho(y), \quad \text{with}$$

$$W_\rho(y) = \left(\beta_k \frac{(1 - y^2/(4\rho^2))}{1 + y^2/(4\rho^2)} - \frac{y}{1 + y^2/(4\rho^2)} \right)^2 + \left(\frac{y/(2\rho)}{1 + y^2/(4\rho^2)} \right)^2.$$

So we have $\lim_{\rho \rightarrow +\infty} W_\rho(y) = W_\infty(y) = (\beta_k - y)^2$, and the operator

$\tilde{P}_\infty = D_y(y^2 D_y) + W_\infty(y)$ on $L^2(]0, +\infty[; dy)$ satisfies, (see [Mo-Tr]), $\text{sp}(\tilde{P}_\infty) = \text{sp}_{ess}(\tilde{P}_\infty) \cup \text{sp}_d(\tilde{P}_\infty)$ with

$$\text{sp}_{ess}(\tilde{P}_\infty) = [\frac{1}{4} + \beta_k^2, +\infty[; \quad \text{sp}_d(\tilde{P}_\infty) = S(\beta_k).$$

We remind that the eigenfunctions associated to the eigenvalues in $S(\beta_k)$ of \tilde{P}_∞ are exponentially decreasing, so if $\lambda_0(\rho) \leq \dots \leq \lambda_j(\rho) \leq \lambda_{j+1}(\rho) \dots$

are the eigenvalues of \tilde{P}_ρ then for any j ,

$\lim_{\rho \rightarrow +\infty} \lambda_j(\rho) = \lambda_j(\infty) = (2j + 1)\beta_k - j(j + 1)$, if $\beta_k > 1/2$ and $j < \beta_k - 1/2$,

otherwise $\lim_{\rho \rightarrow +\infty} \lambda_j(\rho) = \frac{1}{4} + \beta_k^2$.

Therefore we get that $S_\infty = S(\beta_k)$, or $S_\infty = S(\beta_k) \cup \{\frac{1}{4} + \beta_k^2\}$: the formula of Lemma 5.3 follows.

Lemma 5.4 *If $1 \leq j \leq J_1$ and $j \notin J_1^A$, then*

$$\text{sp}_{ess}(-\Delta_A^{M_j}) = \emptyset.$$

If $j \in J_1^A$, then

$$\text{sp}_{ess}(-\Delta_A^{M_j}) = [\frac{1}{4} + b_j^2, +\infty[.$$

Proof. Use the coordinate $t = \ln y$ instead of y , so

$$M_j = \mathbb{S} \times]\alpha_j^2, +\infty[\quad \text{and} \quad ds_j^2 = L_j^2 e^{-2t} d\theta^2 + dt^2; \quad (\alpha_j = e^{a_j}).$$

Then $-\Delta_A^{M_j} = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + e^t (D_t - A_2) (e^{-t} (D_t - A_2))$,
 $\tilde{\mathbf{b}} = L_j^{-1} e^t (\partial_\theta A_2 - \partial_t A_1)$ and $dm = L_j e^{-t} d\theta dt$. As in Lemma 5.3, we have

$$A - \tilde{A} = d\varphi \text{ if } \tilde{A} = (\xi + L_j b_j e^{-t}) d\theta, \text{ (for some constant } \xi \text{).}$$

So we can also assume that $A = \tilde{A}$.

We replace the density dm by $d\theta dt$, using the unitary operator $Uf = \sqrt{L_j} e^{-t/2} f$, so

$$P = -U \Delta_A^{M_j} U^* = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}.$$

Then we get also that

$$\text{sp}(-\Delta_A^{M_j}) = \text{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell); \quad P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L_j} + b_j \right)^2,$$

for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha_j^2, +\infty[$.

When $\ell + \xi \neq 0$, the spectrum of P_ℓ is discrete. More precisely

$$\text{sp}(P_\ell) = \text{sp}(P^\pm), \quad \text{where} \quad P^\pm = D_t^2 + \frac{1}{4} + (\pm e^t + b_j)^2$$

for the Dirichlet condition on $L^2(I_{j,\ell}; dt)$; $I_{j,\ell} =]\alpha_j^2 + \ln(|\ell + \xi|/L_j), +\infty[$,
and $\pm = \frac{\ell + \xi}{|\ell + \xi|}$.

So $\lim_{|\ell| \rightarrow \infty} \inf \text{sp}(P_\ell) = +\infty$, and then we get easily that the spectrum of $-\Delta_A^{M_j}$ is discrete, when $\xi = [A]_{M_j} / (2\pi) \notin \mathbb{Z}$.

If $\ell + \xi = 0$, the spectrum of P_ℓ is absolutely continuous:

$$\text{sp}(P_{-\xi}) = \text{sp}_{ess}(P_{-\xi}) = \text{sp}_{ac}(P_{-\xi}) = \left[\frac{1}{4} + b_j^2, +\infty[;$$

and then, when $[A]_{M_j} \in 2\pi\mathbb{Z}$, $\text{sp}_{ess}(-\Delta_A^{M_j}) = \left[\frac{1}{4} + b_j^2, +\infty[$.

This achieves the proof of Lemma 5.4.

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