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POSITIVE LEGENDRIAN REGULAR HOMOTOPIES

FRANÇOIS LAUDENBACH

ABSTRACT. In contrast with what happens for Legendrian embeddings, there always exist positive loops of Legendrian immersions.

1. INTRODUCTION

In this note, the following is proved:

Theorem 1.1. *Let L be an n -dimensional closed manifold, $J^1(L, \mathbb{R})$ be its space of 1-jets, $\alpha = dz - pdq$ be its canonical contact form and $\xi = \ker \alpha$ be its associated contact structure. There exists a loop of Legendrian immersions $\varphi_t : L \rightarrow J^1(L, \mathbb{R})$, $t \in S^1$, starting from the 0-section and positive in the following sense: $\frac{\partial \varphi}{\partial t}$ is transverse to ξ at every point. Moreover, φ_t may be chosen C^0 -close to the stationary loop.¹*

In [4] Y. Eliashberg and L. Polterovich emphasized that the existence of positive contractible loops of Hamiltonian contact diffeomorphisms on a contact manifold should have an important topological significance. In [2] with S. S. Kim, they proved that such loops exist on the standard spheres S^{2n+1} , $n > 0$, but not on $J^1(L, \mathbb{R})$. By using invariants defined by C. Viterbo in [7], V. Colin, E. Ferrand and P. Pushkar proved that there do not exist any positive loops of Legendrian embeddings of L into $J^1(L, \mathbb{R})$ starting from the zero-section (see [1]). Thereafter, Emmanuel Ferrand asked me about the same question in replacing embeddings by immersions; I am grateful to him for that question. I thought that methods which are developed in the marvellous book by Y. Eliashberg and N. Mishachev [3] could apply. This is the case as is explained below; actually we do not apply a known h -principle, but we prove the h -principle for the problem at hand. I thank Vincent Colin for interesting discussions on related topological questions.

I am glad to have here the opportunity to express my deep admiration to Yasha.

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¹As a consequence, the theorem holds for any contact manifold (M, ξ) and any Legendrian immersed submanifold L (apply the *tubular neighborhood theorem* for Legendrian immersions in a contact manifold). I am grateful to the referee for this remark and, more generally, for his very careful reading of the first version.

2. FIELD OF TRANSVERSE LOOPS

Let X be the Reeb vector field on $J^1(L, \mathbb{R})$ associated to α , that is the unique contact vector field such that $\alpha(\xi) = 1$. A *transverse loop* will be an immersed closed curve which is transverse to ξ .

Proposition 2.1. *There exists a smooth family of transverse loops passing through every point in L , identified with the 0-section of $J^1(L, \mathbb{R})$. More precisely, there exists $\varphi : L \times S^1 \rightarrow J^1(L, \mathbb{R})$ such that:*

- 1) $\varphi(x, 0) = x$ for every $x \in L$;
- 2) $\frac{\partial \varphi}{\partial t}(x, t) = X(\varphi(x, t))$ when t is close to 0 in S^1 (here t stands for the variable in S^1);
- 3) $\frac{\partial \varphi}{\partial t}(x, t)$ is transverse to ξ for every $(x, t) \in L \times S^1$.

Proof. Here we think of S^1 as $[-1, +1]$ with identified end points. If X^t denotes the flow of X , we define the required family by $\varphi(x, t) = X^t(x)$ when $t \in [-1/2, +1/2]$. For extending φ to $L \times S^1$, we will use a suitable family of contact Hamiltonians.

We choose a triangulation \mathcal{T} of L and a finite open covering of L by charts $\{\mathcal{O}_i\}_{i \in I}$ such that each simplex σ is contained in some $\mathcal{O}_{i(\sigma)}$ and, when τ is a face of σ , $\mathcal{O}_{i(\tau)} \subset \mathcal{O}_{i(\sigma)}$. Let $U_i = J^1(\mathcal{O}_i, \mathbb{R})$. We are going to construct φ so that $\varphi(\sigma \times S^1) \subset U_{i(\sigma)}$ for each $\sigma \in \mathcal{T}$.

We first describe a transverse loop passing through a vertex A . Let $A^\pm = X^{\pm 3/4}(A)$. We choose a smooth path γ from A^+ to A^- in $U_{i(A)}$ avoiding $X^t(A)$, $t \in [-1/2, +1/2]$; it is not required to be transverse to ξ . It may be viewed as the trajectory of A^+ during a contact isotopy ψ_s , $s \in [0, 1]$, generated by some time dependent contact Hamiltonian h_s . We assume that h_s is constant on a small interval of the Reeb orbit $[X^{-\varepsilon}(\gamma(s)), X^{+\varepsilon}(\gamma(s))]$. The Hamiltonian is extended to $J^1(L, \mathbb{R})$ with compact support in $U_{i(A)}$ avoiding $X^t(A)$, $t \in [-1/2, +1/2]$. The arc $\psi_1(X^t(A))$, $t \in [-1/2, +3/4]$ is transverse to ξ , starts at $X^{-1/2}(A)$ and ends at $X^{-3/4}(A)$.

Moreover, it is vertical, *i.e.* tangent to $\frac{\partial}{\partial z}$, near its end points. Therefore this arc can be smoothly closed by adding the vertical interval $J(A) = \{X^t(A) \mid t \in [-3/4, -1/2]\}$. In general the resulting loop is only immersed with possible double points in $J(A)$.

Now, assume recursively that φ is already constructed on $\partial\sigma \times S^1$, where σ is a simplex of \mathcal{T} . We perform a similar construction where A is replaced by x running in σ . We have to find a family of paths γ_x from x^+ to x^- in $U_{i(\sigma)}$, avoiding $X^t(x)$, $t \in [-1/2, +1/2]$, depending smoothly on x and extending the given family on $\partial\sigma$. As the dimension of $U_{i(\sigma)}$ is much bigger than the dimension of σ , there is no obstruction for solving this homotopy problem. \square

Remark. By rescaling it is easy to make the diameter of each loop in the family φ less than any positive ε .

In the next step, we will thicken each transverse loop φ_x , $x \in L$, from the family φ yielded by proposition 2.1, into an immersed tube equipped with a contact action of S^1 , depending smoothly on x . Even forgetting that the action has to be contact, such a family of thickenings is not so obvious to get, except when ξ is a trivial bundle. Of course one transverse loop may

be thickened into a tube with a S^1 -action, but a problem appears when one tries to do it in a family. For that, we will take care of the concept of trivialization. For $x \in L$, we use the same notation φ_x for denoting the immersed curve $\varphi(x, S^1)$ or its parametrization $t \in S^1 \mapsto \varphi(x, t)$. We think of $\nu_x := \xi|_{\varphi_x}$ as its normal bundle, which is a trivial bundle since the loop lies in a chart. Classically, a trivialization of ν_x is a framing. But, if ξ is not a trivial bundle, such a framing does not exist if it is required to depend continuously on x . We use another definition of trivialization.

Definition 2.2. *Let A be a submanifold of $J^1(L, \mathbb{R})$, with a base point a . A trivialization of $\xi|_A$ is a smooth field of linear isomorphisms $\pi_z : \xi(z) \rightarrow \xi(a)$, $z \in A$.*

For instance, if A is a convex Darboux chart, we have a canonical trivialization obtained by parallelism and projection along $\frac{\partial}{\partial z}$. Therefore, if σ is a simplex in \mathcal{T} and $x \in \sigma$, the Darboux chart $U_{i(\sigma)}$ induces a canonical trivialization on ν_x using $\varphi(x, 0) = x$ as a base point of φ_x . When τ is a face of σ , for $x \in \tau$ we have two trivializations of ν_x : one when φ_x is viewed as a loop in $U_{i(\sigma)}$ and the other one when it is viewed as a loop in $U_{i(\tau)}$. But, if the triangulation is fine enough and the diameter of the loops φ_x is small enough, both trivializations are close to each other. Arguing in this way, we have the following.

Lemma 2.3. *There exists a smooth family of trivializations $\pi_{x,t} : \xi(\varphi(x, t)) \rightarrow \xi(x)$, $(x, t) \in L \times S^1$. This family may be viewed as a family of linear actions of S^1 on ν_x , depending smoothly on $x \in L$.*

Proof. The question is to find a global section of some bundle \mathcal{E} over $L \times S^1$ whose fiber at (x, t) is the space of linear isomorphisms $\pi_{x,t} : \xi(\varphi(x, t)) \rightarrow \xi(x)$. Two such isomorphisms differ by an automorphism of $\xi(x)$ depending on t . Therefore we introduce the group bundle \mathcal{A} over $L \times S^1$ whose fiber $A_{x,t}$ over (x, t) is the group of linear automorphisms of $\xi(x)$. Around the identity section in \mathcal{A} we choose n disk sub-bundles $V_1 \subset V_2 \dots \subset V_n$ such that, if f and g are in $V_j \cap A_{x,t}$, then f^{-1} is in V_j and $f \circ g$ is in V_{j+1} . As above, we choose the triangulation \mathcal{T} fine enough and the diameters of the loops φ_x small enough so that the following condition is fulfilled:

for each simplex $\sigma \in \mathcal{T}$ and any face τ , and for any $x \in \tau$, the canonical trivializations of ν_x associated to the Darboux charts $U_{i(\sigma)}$ and $U_{i(\tau)}$ differ by a section of V_1 over $\{x\} \times S^1$.

Now the wanted section of \mathcal{E} will be constructed recursively over the skeleta $\mathcal{T}^{[k]} \times S^1$. Let σ be a k -simplex of \mathcal{T} . Assume that we have a smooth family of trivializations $\pi_{x,t}$ of ν_x for $x \in \partial\sigma$ and assume that, for any $(k-1)$ -face τ of σ , it differs from the canonical trivialization associated to $U_{i(\tau)}$ by a section of V_{k-1} . Hence it differs from the canonical trivialization associated to $U_{i(\sigma)}$ by a section of $V_1 \circ V_{k-1} \subset V_k$. As the fiber of V_k is contractible, two sections of V_k are homotopic and the family $\{\pi_{x,t}\}$ extends over $\sigma \times S^1$. \square

Proposition 2.4. *There exists a family of closed immersed tubes $\theta_x : D_x \times S^1 \rightarrow J^1(L, \mathbb{R})$, $x \in L$, where D_x is a small $2n$ -dimensional disk centered at x and tangent to $\xi(x)$ at x , with the following properties:*

- 1) θ_x depends smoothly on $x \in L$;
- 2) $\theta_x(x, t) = \varphi(x, t)$ for every $(x, t) \in L \times S^1$;
- 3) for $z \in D_x$, $\theta_x(z, 0) = \theta_x(z, 1) = z$ and, for t close to $0 = 1$ in S^1 , $\theta_x(z, t) = X^t(z)$;
- 4) S^1 acts on the source of θ_x by contact diffeomorphisms with respect to the contact structure $\theta_x^*(\xi)$.

Proof. According to lemma 2.3, we have a family of linear S^1 -actions on the $2n$ -disk bundle ν_x^r of radius r about the 0-section in ν_x , $x \in L$; here an S^1 -invariant metric is used. By choosing an exponential map $\exp : \xi \rightarrow J^1(L, \mathbb{R})$ and taking r small enough, $\exp(\nu_x^r)$ is a family of immersed tubes $\theta_x : D_x \times S^1 \rightarrow J^1(L, \mathbb{R})$, $x \in L$, which meets the wanted conditions but the contactness; here D_x is the fiber of the tube at x . The contact form α induces on D_x a Liouville form λ_x for the symplectic structure induced by $d\alpha$ on D_x . Hence we get an S^1 -invariant contact form on the tube: $\tilde{\alpha}_x = dt + \lambda_x$. So we have two contact forms on $D_x \times S^1$, $\theta_x^*(\alpha)$ and $\tilde{\alpha}_x$. The underlying plane fields are both transverse to the S^1 -orbits and coincide along the core of the tube $\{x\} \times S^1$. Gray's stability theorem [5] applies relatively to the core curve and yields a conjugation of the germs of both contact structures along the core. Carrying the S^1 -action over this conjugation we fulfill condition 4) on a small tube around $\{x\} \times S^1$. \square

We thicken the disk D_x into a $2n+1$ -ball B_x contained in the cylinder $C_x := \cup_{s \in [-\varepsilon, +\varepsilon]} X^s(D_x)$. A point $z \in C_x$ reads $z = X^s(y)$, $y \in D_x$, $s \in [-\varepsilon, +\varepsilon]$. The immersion θ_x obviously extends as a map (not an immersion) $\Theta_x : B_x \times S^1 \rightarrow J^1(L, \mathbb{R})$ by the following formula:

$$\Theta_x(z, t) = \theta_x(y, t + s).$$

For a given $x \in L$, $t \mapsto \Theta_x(-, t)$ is a periodic positive contact regular homotopy² of B_x into $J^1(L, \mathbb{R})$ starting from $B_x \hookrightarrow J^1(L, \mathbb{R})$ at $t = 0$ (in fact, if B_x is small, it is an isotopy). The family of the germs of Θ_x along $\{x\} \times S^1$ may be thought of as a *formal solution* of the problem that theorem 1.1 solves. In the next section we follow the book by Eliashberg-Mishachev [3] for modifying this formal solution into a genuine solution.

3. TOWARDS A GENUINE SOLUTION

In the sequel, it is more convenient to work with a cubication \mathcal{C} (cell decomposition made of cubes) instead of a triangulation of L . By adding the barycenter of each simplex of a triangulation one easily gets a cubication. In the sequel the germ at x of some periodic homotopy $\Theta : B \times S^1 \rightarrow J^1(L, \mathbb{R})$ will mean the germ of Θ along $\{x\} \times S^1$.

Proposition 3.1. *There exist the following families:*

- 1) For each edge $\sigma \in \mathcal{C}$ whose end points are x_0 and x_1 , there exist a small neighborhood B_σ of σ in $J^1(L, \mathbb{R})$ and a periodic positive contact regular homotopy $\Theta_\sigma : B_\sigma \times S^1 \rightarrow J^1(L, \mathbb{R})$ starting from $B_\sigma \hookrightarrow J^1(L, \mathbb{R})$ at $t = 0$ and whose germ at $\{x_i\}$ is the one of Θ_{x_i} for $i = 0, 1$.

²We recall that a regular homotopy is a homotopy through immersions.

2) There exists a smooth family Θ_x^1 , $x \in L$, of periodic positive contact regular homotopies, defined near x , whose germ at x is the one of Θ_σ when $x \in \sigma$.

Proof. Take the barycentric parametrization of σ , $\gamma(u)$ with $u \in [0, 1]$, and a fine subdivision $u_0 = 0, u_1 = 1/N, \dots, u_N = 1$. Let $x_k = \gamma(u_k)$, $u'_k = \frac{u_k + u_{k+1}}{2}$ and $x'_k = \gamma(u'_k)$.

LEMMA. If N is large enough, there exists a periodic Hamiltonian contact isotopy F^t , $t \in S^1$, of B_{x_k} with support in $2/3B_{x_k}$, being Identity at $t = 0$, such that:

- i) $\Theta_{x_k} \circ F$ is a positive regular homotopy, where the composition is meant at each time $t \in S^1$.
- ii) x'_k belongs to $1/3B_{x_k}$ and the germ of $\Theta_{x_{k+1}}$ at x'_k is the one of $\Theta_{x_k} \circ F$.

PROOF OF THE LEMMA. If N is large enough, $\Theta_{x_{k+1}}(x'_k, t)$ belongs to the ball $\Theta_{x_k}(1/3B_{x_k}, t)$ for every $t \in [0, 1]$. Therefore, near $\{x'_k\} \times S^1$, $\Theta_{x_{k+1}}$ can be lifted to the source of Θ_{x_k} ; in other words, the germ of $\Theta_{x_{k+1}}$ along $\{x'_k\} \times S^1$ has a time preserving factorization through Θ_{x_k} . This lift is a periodic contact Hamiltonian isotopy of embeddings of a small ball centered at x'_k into $1/3B_{x_k}$. Moreover, if N is large, it can be chosen ε -close to Identity in the C^1 -topology. It extends as a Hamiltonian isotopy F of B_{x_k} supported in $2/3B_{x_k}$ and 2ε -close to Identity. In general, such an F is not a periodic isotopy; F^1 is Identity only on a neighborhood $N(x'_1)$ of x'_1 and outside $2/3B_{x_k}$. But being C^1 -close to Identity, F^1 is isotopic to Identity by a contact Hamiltonian isotopy supported in $W := 2/3B_{x_k} \setminus N(x'_1)$; indeed, the group $Diff_{cont}(W, \partial W)$ is locally contractible³. This allows us to modify F so that it becomes a periodic isotopy. As Θ_{x_k} is a positive regular homotopy, if F is close enough to Identity, $\Theta_{x_k} \circ F$ is still positive. \square

For $0 < k < N$, let $B_k := B_{x_k}$ and let Θ'_k denote the modified regular homotopy described above. We also choose a contact Hamiltonian diffeomorphism ψ_k with support in a small neighborhood of some ray R_k in B_k , leaving a neighborhood of x'_{k-1} and x'_k fixed, and moving x_k into $B_k \setminus (2/3B_k)$. We look at the path γ' defined by:

$$\begin{aligned} \gamma'(u) &= \psi_k(\gamma(u)) \text{ when } 0 < k < N \text{ and } u \in [u'_{k-1}, u'_k], \\ \gamma'(u) &= \gamma(u) \text{ when } u \in [0, u'_0] \text{ or } u \in [u'_{N-1}, 1]. \end{aligned}$$

We think of the process changing γ to γ' as a *making waves* process on σ , according to the terminology of Bill Thurston in [6].

We now define a positive contact regular homotopy Θ' on a neighborhood of γ' by the following formulas which are matching:

$$\begin{aligned} \Theta'(\gamma'(u), t) &= \Theta'_k(\gamma'(u), t) \text{ when } 0 < k < N \text{ and } u \in [u_k, u'_k], \\ \Theta'(\gamma'(u), t) &= \Theta_k(\gamma'(u), t) \text{ when } 0 < k \leq N, u \in [u'_{k-1}, u_k] \text{ or } k = 0, u \in [0, u'_0]. \end{aligned}$$

This regular homotopy has the property which is required in point 1), except that it is not defined near σ but near the path γ' .

If the rays R_k are chosen to be mutually disjoint and so that $R_k \cap L = \{x_k\}$ for every k , there is a Hamiltonian contact diffeomorphism ψ , with compact support in $J^1(L, \mathbb{R})$, such that

³ $Diff(W, \partial W)$ is locally contractible and there is a locally trivial fibration $Diff_{cont}(W, \partial W) \hookrightarrow Diff(W, \partial W) \rightarrow Cont(W, \partial W)$ whose base is the locally contractible space of contact structures coinciding with a given one along the boundary.

$\gamma' = \psi \circ \gamma$ and ψ leaves the other edges of \mathcal{C} fixed. So we define for every $t \in S^1$

$$\Theta_\sigma^t = \psi^{-1} \circ (\Theta')^t \circ \psi.$$

It is well-defined on a small neighborhood of σ and $t \mapsto \Theta_\sigma^t$ is still a positive homotopy as ψ is independent of $t \in S^1$. Hence point 1) is proved.

2) When $x \in \sigma$, it is not difficult to interpolate between Θ_x and the germ of Θ_σ at x (interpolate between ψ and Identity and between Θ_k and Θ'_k if $x = \gamma(u)$, $u \in [u_k, u'_k]$). This interpolation extends when x is close to σ . Using a partition of unity, one easily finds a family Θ_x^1 with the desired property. \square

Applying proposition 3.1 above simultaneously to each 1-cell of \mathcal{C} yields a positive contact regular homotopy defined near the 1-skeleton. So we have a periodic contact positive regular homotopy Θ^1 defined near the 1-skeleton together with a family of Θ_x defined near each point $x \in L$. For going further, as in [3], we need a parametric version of 3.1.

Proposition 3.2. *Let σ be a 2-cell in \mathcal{C} , τ be a 1-face of σ and τ^* be a non-parallel 1-face of σ ; so σ is foliated by intervals $\tau(y)$ parallel to τ , starting at $y \in \tau^*$ and ending at a point of the edge opposite to τ^* . Then there exist periodic positive contact regular homotopies $\Theta_{\tau(y)}$, $y \in \tau^*$, defined near $\tau(y)$ and depending smoothly on y . Moreover its germ at any point of $\partial\sigma$ is the one of Θ^1 .*

Proof. Note that σ is a square. Clearly the proof we have done for one edge in 3.1 can be performed with parameters. When $y \in \partial\tau^*$ ($y = 0$ or 1), as Θ^1 is already defined, it is not necessary to replace $\tau(y)$ by a very much oscillating C^0 -approximation of $\tau(y)$. In other words, the contact diffeomorphism ψ from the proof of 3.1 can be interpolated with Identity when y approaches one end point of τ^* . \square

Proof of theorem 1.1. Here we explain how to construct a periodic positive contact regular homotopy defined near a 2-cell σ of \mathcal{C} extending Θ^1 , the regular homotopy we have near the 1-skeleton $L^{[1]}$. When doing it for all 2-cells simultaneously, we get the desired regular homotopy Θ^2 defined near the 2-skeleton $L^{[2]}$. And one goes on recursively until Θ^n which is the desired regular homotopy.

For constructing Θ^2 near σ , we use proposition 3.2 which yields a 1-parameter family $\Theta^1(y)$ of periodic contact regular homotopy defined near $\tau(y)$. As in the case of a 1-simplex (proposition 3.1), we discretize the y -interval, deform slightly $\Theta^1(\tau(y_k))$ so that they glue together and yield a periodic contact regular homotopy near $\psi(\sigma)$, where ψ is a contact diffeomorphism of $J^1(L, \mathbb{R})$, *making waves* on σ . The process changing σ to $\psi(\sigma)$ is the 2-dimensional analogue of the one that we have described very precisely for a 1-cell; it is a universal process once we know $\Theta^1(\tau(y))$ for every $y \in \tau^*$. \square

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