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Gluing endo-permutation modules

Serge Bouc

Abstract : In this paper, I show that if p is an odd prime, and if P is a finite p -group, then there exists an exact sequence of abelian groups

$$0 \rightarrow T(P) \rightarrow D(P) \rightarrow \varprojlim_{1 < Q \leq P} D(N_P(Q)/Q) \rightarrow H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)},$$

where $D(P)$ is the Dade group of P and $T(P)$ is the subgroup of endo-trivial modules.

Here $\varprojlim_{1 < Q \leq P} D(N_P(Q)/Q)$ is the group of sequences of compatible elements in the Dade groups $D(N_P(Q)/Q)$ for non trivial subgroups Q of P . The poset $\mathcal{A}_{\geq 2}(P)$ is the set of elementary abelian subgroups of rank at least 2 of P , ordered by inclusion. The group $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$ is the subgroup of $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})$ consisting of classes of P -invariant 1-cocycles.

A key result to prove that the above sequence is exact is a characterization of elements of $2D(P)$ by sequences of integers, indexed by sections (T, S) of P such that $T/S \cong (\mathbb{Z}/p\mathbb{Z})^2$, fulfilling certain conditions associated to subquotients of P which are either elementary abelian of rank 3, or extraspecial of order p^3 and exponent p .

AMS Subject classification : 20C20

Keywords : endo-permutation module, Dade group, gluing.

1. Introduction

The classification of all endo-permutation modules for finite p -groups has been completed recently, thanks to the work of several authors (see in particular [1], [11], [12], [5], [3]). This paper addresses the question of *gluing arbitrary endo-permutation modules*, and it is intended to be a complement to our previous joint work with Jacques Thévenaz ([6]), where the case of torsion endo-permutation modules was handled.

The gluing problem is the following : let p be an odd prime, let P be a finite p -group, and let k be a field of characteristic p . If v is an element of the Dade group $D(P)$ of endo-permutation kP -modules, and if Q is a non trivial subgroup of P , denote by v_Q the image of v by the deflation-restriction map $\text{Defres}_{N_P(Q)/Q}^P$. Then the v_Q 's are subject to some obvious compatibility conditions. Conversely, if Q is a non-trivial subgroup of P , let u_Q be an element of the Dade group $D_k(N_P(Q)/Q)$, and assume that these compatibility conditions between the u_Q 's are fulfilled. Is there an element $u \in D(P)$ such that for any non trivial subgroup Q of P

$$\text{Defres}_{N_P(Q)/Q}^P(u) = u_Q \quad ?$$

Such an element u is called a *solution to the gluing problem* for the *gluing data* $(u_Q)_{1 < Q \leq P}$.

When P is abelian, the gluing problem was completely solved by Puig [16] (see also Lemma 2.3 below), and he used the result to construct suitable stable equivalences between blocks.

The main result of the present paper is that if p is an odd prime, and if P is a finite p -group, then there exists an exact sequence of abelian groups

$$0 \longrightarrow T(P) \longrightarrow D(P) \longrightarrow \varprojlim_{1 < Q \leq P} D(N_P(Q)/Q) \xrightarrow{h_P} H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)},$$

where $D(P)$ is the Dade group of P and $T(P)$ is the subgroup of endo-trivial modules. Here $\varprojlim_{1 < Q \leq P} D(N_P(Q)/Q)$ is the group of gluing data for P , i.e. the group of sequences of compatible elements in the Dade groups $D(N_P(Q)/Q)$ for non trivial subgroups Q of P . The poset $\mathcal{A}_{\geq 2}(P)$ is the set of elementary abelian subgroups of rank at least 2 of P , ordered by inclusion. The group $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$ is the subgroup of $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})$ consisting of classes of P -invariant 1-cocycles.

The main consequence of this result is that if $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z}) = \{0\}$, then the gluing problem always has a solution. Unfortunately, the map h_P is not surjective in general, so when $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z}) \neq \{0\}$, not much can be said at the time for the gluing problem. In Section 6, the example of the extraspecial group of order p^5 and exponent p is described in details. In this case, the group $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$ is a free group of rank p^4 , and the image of h_P has finite index in this group. In particular it is non zero, and the gluing problem does not always have a solution.

It could be true in general that h_P always has finite cokernel, and this would be enough to show that if $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)} \neq \{0\}$, then the image of h_P is non zero, hence that the gluing problem does not always have a solution : it is known indeed that the group $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$ is a free abelian group, since the poset $\mathcal{A}_{\geq 2}(P)$ has the homotopy type of a wedge of spheres (see [8]).

1.1. Notation. Throughout this paper, the symbol p denotes an odd prime number, and P denotes a finite p -group. Inclusion of subgroups will be denoted by \leq , and strict inclusion by $<$. Inclusion up to P -conjugation will be denoted by \leq_P .

A *section* (T, S) of P is a pair of subgroups of P with $S \trianglelefteq T$. The factor group T/S is the corresponding *subquotient* of P . If (T, S) is a section of P , then $N_P(T, S)$ denotes $N_P(T) \cap N_P(S)$.

A class \mathcal{Y} of p -groups is said to be *closed under taking subquotients* if for any $Y \in \mathcal{Y}$ and any section (T, S) of Y , any group isomorphic to T/S belongs to \mathcal{Y} . If \mathcal{Y} is such a class, and P is a finite p -group, let $\mathcal{Y}(P)$ be the set of sections (T, S) of P such that $T/S \in \mathcal{Y}$.

The symbol X_{p^3} denotes an extraspecial p -group of order p^3 and exponent p . The symbol \mathcal{X}_3 denotes the class of p -groups which are either elementary abelian of rank at most 3, or isomorphic to X_{p^3} . Thus, the symbol $\mathcal{X}_3(P)$ denotes the set of sections (T, S) of P such that T/S is elementary abelian of rank at most 3, or isomorphic to X_{p^3} . Let moreover $\mathcal{E}_2^\sharp(P)$ denote the set of sections (T, S) of P such that $T/S \cong (\mathbb{Z}/p\mathbb{Z})^2$.

If P is a finite p -group, and k is a field of characteristic p , let $D(P)$ denote the Dade group of endo-permutation kP -modules. The field k does not appear in this notation, because it turns out that $D(P)$ is independent of k , at least when p is odd (see [3] Theorem 9.5 for details).

When (T, S) is a section of P , there is a *deflation-restriction map* $\text{Defres}_{T/S}^P : D(P) \rightarrow D(T/S)$, which is the group homomorphism obtained by composing the *restriction map* $\text{Res}_T^P : D(P) \rightarrow D(T)$, followed by the *deflation map* $\text{Def}_T^T : D(T) \rightarrow D(T/S)$.

Recall that if X is a finite P -set, there is a corresponding element Ω_X of the Dade group of P , called the syzygy of the trivial module relative to X (or the *X -relative syzygy* for short) : it is defined as the class of the kernel of the augmentation map $kX \rightarrow k$ when this does make sense, and by 0 otherwise (see e.g. [2] for details). When X is the set P itself, on which P acts by multiplication, the corresponding element will be denoted by $\Omega_{P/1}$ or Ω_P .

1.2. Contents. This paper is organized as follows :

- In Section 2, I state the main theorem (Theorem 2.15), and this requires in particular the definition of some objects and maps between them.
- Section 3 recalls some notation on biset functors, forgetful functors between categories of biset functors, and corresponding adjoint functors.
- Section 4 is devoted to the main tool (Theorem 4.5) used in the proof of Theorem 2.15, namely a characterization by linear equations of the image of the group $2D(P)$ by the deflation-restriction maps to all subquotients T/S of P which are elementary abelian of rank 2. This characterization may be a result of independent interest.
- Section 5 exposes the proof of Theorem 2.15.
- Finally, Section 6 focuses on the example of the extraspecial p -group of order p^5 and exponent p : the reason for choosing this particular group is twofold : it is one of the smallest p -groups P for which $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z}) \neq \{0\}$, and moreover the Dade group of this p -group is rather well known, thanks to our joint work with Nadia Mazza ([5]).

2. Statement of the main theorem

2.1. Notation. If P is a finite p -group, then $\mathcal{A}_{\geq 2}(P)$ denotes the poset of elementary abelian subgroups of P of rank at least 2. Let $\underline{\mathcal{A}}_{=2}(P)$ denote the set of elementary abelian subgroups of rank 2 of P .

Recall that if the p -rank of P is at least equal to 3, then all the elementary abelian subgroup of P of rank at least 3 are in the same connected component of $\mathcal{A}_{\geq 2}(P)$. This component is called *the big component*. It is obviously invariant under P -conjugation. Each of the other connected components, if there are any, consists of a single maximal elementary abelian subgroup of rank 2.

2.2. Notation. Denote by $\varprojlim_{\mathbf{1} < Q \leq P} D(N_P(Q)/Q)$ the set of sequences $(u_Q)_{\mathbf{1} < Q \leq P}$, indexed by non trivial subgroups of P , where $u_Q \in D(N_P(Q)/Q)$, such that :

- If $x \in P$, then ${}^x u_Q = u_{xQ}$.
- If $Q \trianglelefteq R$, then $\text{Defres}_{N_P(Q,R)/R}^{N_P(Q)/Q} u_Q = \text{Res}_{N_P(Q,R)/R}^{N_P(R)/R} u_R$.

Denote by $r_P : D(P) \rightarrow \varprojlim_{\mathbf{1} < Q \leq P} D(N_P(Q)/Q)$ the map sending $v \in D(P)$ to the sequence $(\text{Defres}_{N_P(Q)/Q}^P v)_{\mathbf{1} < Q \leq P}$.

If E is an abelian p -group, denote by σ_E the map $\varprojlim_{\mathbf{1} < F \leq E} D(E/F) \rightarrow D(E)$ defined by

$$\sigma_E(u) = - \sum_{\mathbf{1} < F \leq E} \mu(\mathbf{1}, F) \text{Inf}_{E/F}^E u_F,$$

where μ is the Möbius function of the poset of subgroups of P .

It has been shown by Puig ([15] 2.1.2) that the kernel of r_P is equal to the group $T(P)$ of endo-trivial modules. Moreover, when E is an abelian group, the map r_E is surjective ([16] Proposition 3.6). More precisely :

2.3. Lemma. *Let E be an abelian p -group. Then σ_E is a section of r_E , i.e. $r_E\sigma_E$ is equal to the identity map of $\varprojlim_{\mathbf{1} < F \leq E} D(E/F)$.*

Proof. Let $\mathbf{1} < G \leq E$. Then

$$\begin{aligned} \text{Def}_{E/G}^E \sigma_E(u) &= - \sum_{\mathbf{1} < F \leq E} \mu(\mathbf{1}, F) \text{Def}_{E/G}^E \text{Inf}_{E/F}^G u_F \\ &= - \sum_{\mathbf{1} < F \leq E} \mu(\mathbf{1}, F) \text{Inf}_{E/FG}^{E/G} \text{Def}_{E/FG}^{E/F} u_F \\ &= - \sum_{\mathbf{1} < F \leq E} \mu(\mathbf{1}, F) \text{Inf}_{E/FG}^{E/G} u_{FG} \\ &= - \sum_{G \leq R \leq E} \left(\sum_{\substack{\mathbf{1} < F \leq R \\ FG=R}} \mu(\mathbf{1}, F) \right) \text{Inf}_{E/R}^{E/G} u_R. \end{aligned}$$

Now if $G < R$

$$\sum_{\substack{\mathbf{1} < F \leq R \\ FG=R}} \mu(\mathbf{1}, F) = \sum_{\substack{\mathbf{1} \leq F \leq R \\ FG=R}} \mu(\mathbf{1}, F),$$

and this is equal to zero, by a classical combinatorial lemma, since $G \neq \mathbf{1}$. And if $G = R$

$$\sum_{\substack{\mathbf{1} < F \leq R \\ FG=R}} \mu(\mathbf{1}, F) = -\mu(\mathbf{1}, \mathbf{1}) + \sum_{\mathbf{1} \leq F \leq R} \mu(\mathbf{1}, F) = -1.$$

Thus $\text{Def}_{E/G}^E \sigma_E(u) = \text{Inf}_{E/G}^{E/G} u_G = u_G$, as was to be shown. \square

2.4. Lemma. *Let E be an elementary abelian group of rank at least 2. Then the map r_E is surjective, and its kernel is the free abelian group of rank one generated by $\Omega_{E/\mathbf{1}}$.*

Proof. The kernel of r_E is the group $T(E)$ of endo-trivial modules. Since E is elementary abelian, this group is free of rank one, generated by $\Omega_{E/\mathbf{1}}$, by Dade's Theorem ([13] [14]). The surjectivity of r_E follows from Lemma 2.3. \square

2.5. Restriction and conjugation. The following construction has been introduced in [6], for the torsion subgroup of the Dade group, but it works as well for the whole Dade group: let P be a finite p -group, and H be a subgroup of P . If $u \in \varprojlim_{\mathbf{1} < Q \leq P} D(N_P(Q)/Q)$,

then the sequence $(v_Q)_{\mathbf{1} < Q \leq H}$ defined by

$$v_Q = \text{Res}_{N_H(Q)/Q}^{N_P(Q)/Q} u_Q$$

is an element of $\varprojlim_{\mathbf{1} < Q \leq H} D(N_H(Q)/Q)$, denoted by $\text{Res}_H^P u$. The map $u \mapsto \text{Res}_H^P u$

is a linear map $\varprojlim_{\mathbf{1} < Q \leq P} D(N_P(Q)/Q)$ to $\varprojlim_{\mathbf{1} < Q \leq H} D(N_H(Q)/Q)$. The following is the analogue of Lemma 2.4 of [6]:

2.6. Lemma. *Let H be a subgroup of P . The diagram*

$$\begin{array}{ccc} D(P) & \xrightarrow{r_P} & \varprojlim_{1 < Q \leq P} D(N_P(Q)/Q) \\ \text{Res}_H^P \downarrow & & \downarrow \text{Res}_H^P \\ D(H) & \xrightarrow{r_H} & \varprojlim_{1 < Q \leq H} D(N_H(Q)/Q) \end{array}$$

is commutative.

Proof. This is straightforward. \square

Similarly, if $x \in P$, denote by $c_{x,H} : D(H) \rightarrow D({}^xH)$ the conjugation by x , sending v to xv . If $u \in \varprojlim_{1 < Q \leq H} D(N_H(Q)/Q)$, then the sequence $(v_R)_{1 < R \leq {}^xH}$ defined by $v_R = c_{x,R^x}(u_{R^x})$ is an element of $\varprojlim_{1 < R \leq {}^xH} D(N_{{}^xH}(R)/R)$, that will be denoted by xu . The assignment $u \mapsto {}^xu$ is a linear map from $\varprojlim_{1 < Q \leq H} D(N_H(Q)/Q)$ to

$\varprojlim_{1 < R \leq {}^xH} D(N_{{}^xH}(R)/R)$, also denoted by $c_{x,H}$. Then :

2.7. Lemma. *Let H be a subgroup of P , and let E be an abelian subgroup of P . The following diagrams are commutative :*

$$\begin{array}{ccc} D(H) \xrightarrow{r_H} \varprojlim_{1 < Q \leq H} D(N_H(Q)/Q) & & D(E) \xleftarrow{\sigma_E} \varprojlim_{1 < R \leq E} D(E/R) \\ c_{x,H} \downarrow & & c_{x,E} \downarrow \\ D({}^xH) \xrightarrow{r_{{}^xH}} \varprojlim_{1 < R \leq {}^xH} D(N_{{}^xH}(R)/R) & & D({}^xE) \xleftarrow{\sigma_{{}^xE}} \varprojlim_{1 < R \leq {}^xE} D(E/R) \end{array}$$

Proof. This is also straightforward. \square

2.8. Construction of a map. Let E and F be elements of $\mathcal{A}_{\geq 2}(P)$ such that $E < F$. If $v \in \varprojlim_{1 < Q \leq P} D(N_P(Q)/Q)$, consider the element

$$d_{E,F} = \text{Res}_E^F \sigma_F \text{Res}_F^P v - \sigma_E \text{Res}_E^P v$$

of $D(E)$. Then by Lemma 2.4 and Lemma 2.3

$$\begin{aligned} r_E(d_{E,F}) &= r_E \text{Res}_E^F \sigma_F \text{Res}_F^P v - r_E \sigma_E \text{Res}_E^P v \\ &= \text{Res}_E^F r_F \sigma_F \text{Res}_F^P v - r_E \sigma_E \text{Res}_E^P v \\ &= \text{Res}_E^F \text{Res}_F^P v - \text{Res}_E^P v = 0. \end{aligned}$$

By Lemma 2.4, there exists a unique integer $w_{E,F}$ such that

$$d_{E,F} = w_{E,F} \cdot \Omega_{E/1}.$$

If $x \in P$, then it is clear from Lemma 2.7 that ${}^x d_{E,F} = d_{{}^x E, {}^x F}$, and it follows that $w_{{}^x E, {}^x F} = w_{E,F}$. Moreover, if $E, F, G \in \mathcal{A}_{\geq 2}(P)$ with $E < F < G$, then

$$d_{E,F} + \text{Res}_E^F d_{F,G} = d_{E,G} ,$$

hence $w_{E,F} + w_{F,G} = w_{E,G}$. In other words the function sending the pair (E, F) of elements of $\mathcal{A}_{\geq 2}(P)$, with $E < F$, to $w_{E,F}$, is a P -invariant 1-cocycle on $\mathcal{A}_{\geq 2}(P)$, with values in \mathbb{Z} :

2.9. Notation. *Let P be a finite p -group. A P -invariant 1-cocycle on $\mathcal{A}_{\geq 2}(P)$, with values in \mathbb{Z} , is a function sending a pair (E, F) of elements of $\mathcal{A}_{\geq 2}(P)$, with $E < F$, to an integer $w_{E,F}$, with the following two properties :*

1. *If $x \in P$ and $E < F$ in $\mathcal{A}_{\geq 2}(P)$, then $w_{{}^x E, {}^x F} = w_{E,F}$.*
2. *If $E < F < G$ in $\mathcal{A}_{\geq 2}(P)$, then $w_{E,F} + w_{F,G} = w_{E,G}$.*

The set $\left(Z^1(\mathcal{A}_{\geq 2}(P)) \right)^P$ of P -invariants 1-cocycles is a group for addition of functions.

Denote by $\left(B^1(\mathcal{A}_{\geq 2}(P)) \right)^P$ the subgroup of $\left(Z^1(\mathcal{A}_{\geq 2}(P)) \right)^P$ consisting of cocycles w for which there exists a P -invariant function $E \mapsto m_E$ from $\mathcal{A}_{\geq 2}(P)$ to \mathbb{Z} such that

$$\forall E < F \in \mathcal{A}_{\geq 2}(P), w_{E,F} = m_F - m_E .$$

Denote by $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$ the factor group $\left(Z^1(\mathcal{A}_{\geq 2}(P)) \right)^P / \left(B^1(\mathcal{A}_{\geq 2}(P)) \right)^P$

2.10. Remark: One can show that the group $\left(B^1(\mathcal{A}_{\geq 2}(P)) \right)^P$ is also equal to the set of elements w of $\left(Z^1(\mathcal{A}_{\geq 2}(P)) \right)^P$ for which there exists a (not necessarily P -invariant) function $E \mapsto m_E$ such that $w_{E,F} = m_F - m_E$ for any $E < F$ in $\mathcal{A}_{\geq 2}(P)$. This is because if $E < F$ in $\mathcal{A}_{\geq 2}(P)$, then E and F are the ‘‘big component’’, which is P -invariant. Since w is P -invariant, it follows that the function

$$x \in P \mapsto m_{{}^x E} - m_E$$

does not depend on the choice of E , and that it is a group homomorphism from P to \mathbb{Z} (i.e. an element of $H^1(P, \mathbb{Z})$). There are no non zero such homomorphisms, so m is actually P -invariant.

2.11. Remark: On the other hand, one can consider the ordinary first cohomology group $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})$ of $\mathcal{A}_{\geq 2}(P)$ over \mathbb{Z} , which is defined similarly to $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$, but forgetting all conditions of P -invariance. Then the group P acts on $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})$, and it follows from Remark 2.10 that $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$ is a subgroup of the group $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^P$ of P -invariant elements in $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})$. It might happen however that this inclusion is proper : an argument similar to the one used in Remark 2.10 yields an element in $H^2(P, \mathbb{Z})$, and this group need not be zero.

2.12. Notation. *Let P be a finite p -group. Denote by*

$$h_P : \varprojlim_{1 < Q \leq P} D(N_P(Q)/Q) \rightarrow H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$$

the map sending $v \in \varprojlim_{1 < Q \leq P} D(N_P(Q)/Q)$ to the class of the 1-cocycle w defined by the following equality, for $E < F$ in $\mathcal{A}_{\geq 2}(P)$:

$$(2.13) \quad w_{E,F} \cdot \Omega_{E/1} = \text{Res}_E^F \sigma_F \text{Res}_F^P v - \sigma_E \text{Res}_E^P v .$$

2.14. Proposition. *Let P be a finite p -group. Then h_P is a group homomorphism, and the composition*

$$D(P) \xrightarrow{r_P} \varprojlim_{1 < Q \leq P} D(N_P(Q)/Q) \xrightarrow{h_P} H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$$

is equal to 0.

Proof. Clearly, the definition of h_P implies that it is a group homomorphism. Observe next that for any $E \in \mathcal{A}_{\geq 2}(P)$, since $r_E \sigma_E$ is the identity map, the image of the map $\sigma_E r_E - \text{Id}_{D(E)}$ is contained in the kernel of r_E . By Lemma 2.4, it follows that there is a unique linear form s_E on $D(E)$, with values in \mathbb{Z} , such that

$$\sigma_E r_E(u) = u + s_E(u) \cdot \Omega_{E/1} ,$$

for any $u \in D(E)$. By Lemma 2.7, this definition clearly implies that if $x \in P$, then $s_{xE}(x u) = s_E(u)$, for any $u \in D(E)$.

Now if $E < F$ in $\mathcal{A}_{\geq 2}(P)$, and if $v = r_P(t)$, for $t \in D(P)$, Equation 2.13 becomes

$$\begin{aligned} w_{E,F} \cdot \Omega_{E/1} &= \text{Res}_E^F \sigma_F \text{Res}_F^P r_P(t) - \sigma_E \text{Res}_E^P r_P(t) \\ &= \text{Res}_E^F \sigma_F r_F \text{Res}_F^P t - \sigma_E r_E \text{Res}_E^P t \quad (\text{by Lemma 2.6}) \\ &= \text{Res}_E^F (\text{Res}_F^P t + s_F(\text{Res}_F^P t) \cdot \Omega_{F/1}) - (\text{Res}_E^P t + s_E(\text{Res}_E^P t) \cdot \Omega_{E/1}) \\ &= (s_F(\text{Res}_F^P t) - s_E(\text{Res}_E^P t)) \cdot \Omega_{E/1} . \end{aligned}$$

Setting $m_E = s_E(\text{Res}_E^P t)$, for $E \in \mathcal{A}_{\geq 2}(P)$, yields

$$w_{E,F} = m_F - m_E ,$$

hence $w \in \left(B^1(\mathcal{A}_{\geq 2}(P)) \right)^P$ (the P -invariance of m follows easily from the above remark, or from Remark 2.10). Thus $h_P r_P(u) = 0$, as was to be shown. \square

The main theorem of this paper is the following :

2.15. Theorem. *Let P be a finite p -group. Then the sequence of abelian groups*

$$0 \longrightarrow T(P) \longrightarrow D(P) \xrightarrow{r_P} \varprojlim_{1 < Q \leq P} D(N_P(Q)/Q) \xrightarrow{h_P} H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$$

is exact.

The key point in this theorem is to show that the kernel of h_P is equal to the image of r_P . This will be done in two steps : first take an element $u \in \text{Ker } h_P$, and show that $2u \in r_P(2D(P))$. This amounts to replacing D by $2D$, which is easier to handle, since it is torsion free. Next, write $2u = r_P(2v)$, for some $v \in D(P)$. Then $u - r_P(v)$ is an element in $\varprojlim_{1 < Q \leq P} D_t(N_P(Q)/Q)$, and it has been shown in [6] that such a sequence of compatible torsion elements can always be glued (i.e. it always lies in $r_P(D(P))$, though possibly not in $r_P(D_t(P))$).

3. Biset functors

The main ingredient in the proof of Theorem 2.15 is the formalism of *biset functors*. A short exposition of the notation and main results on this subject can be found in Section 2 of [9], Section 3 of [4], or Section 3 of [3].

Recall in particular that if (T, S) is a section of the group P , and if M is a biset functor, then the set P/S is a $(P, T/S)$ -biset, and the corresponding *induction-inflation* morphism

$$M(P/S) : M(T/S) \rightarrow M(P)$$

is denoted by $\text{Indinf}_{T/S}^P$. Similarly, the set $S \setminus P$ is a $(T/S, P)$ -biset, and the corresponding *deflation-restriction* map

$$M(S \setminus P) : M(P) \rightarrow M(T/S)$$

is denoted by $\text{Defres}_{T/S}^P$.

This notation was already used for the Dade group, and it is coherent : it was shown more generally in [7] that, if P and Q are finite p -groups, and U is a finite (Q, P) -bisets, one can define a natural group homomorphism $D(U) : D(P) \rightarrow D(Q)$. For an arbitrary prime number p , this construction does not yield a structure of biset functor for the whole Dade group, but only on the correspondence sending a p -group P to the subgroup $D^\Omega(P)$ of $D(P)$ generated by all the relative syzygies Ω_X obtained for various P -sets X . However, in the case $p = 2$, it was shown in [3] that $D = D^\Omega$, so D is indeed a biset functor in this case.

The following additional notation was introduced in [9] : a class \mathcal{Y} of p -groups is said to be *closed under taking subquotients* if for any $Y \in \mathcal{Y}$ and any section (T, S) of Y , the corresponding subquotient T/S belongs to \mathcal{Y} . If \mathcal{Y} is such a class, and P is a finite p -group, denote by $\mathcal{Y}(P)$ the set of sections (T, S) of P such that $T/S \in \mathcal{Y}$. One can consider biset functors defined only on \mathcal{Y} , with values in abelian groups. Let $\mathcal{F}_{\mathcal{Y}}$ be the category of all such functors, and let \mathcal{F} denote the category of functors defined on all finite p -groups. These categories are abelian categories.

The obvious forgetful functor

$$\mathcal{O}_{\mathcal{Y}} : \mathcal{F} \rightarrow \mathcal{F}_{\mathcal{Y}}$$

admits left and right adjoints

$$\mathcal{L}_{\mathcal{Y}} : \mathcal{F}_{\mathcal{Y}} \rightarrow \mathcal{F} \quad \text{and} \quad \mathcal{R}_{\mathcal{Y}} : \mathcal{F}_{\mathcal{Y}} \rightarrow \mathcal{F},$$

whose evaluations can be computed as follows (cf. [9] Theorem 1.2 for details, in particular on the direct and inverse limits appearing in this statement) :

3.1. Theorem. *With the notation above, for any functor $M \in \mathcal{F}_{\mathcal{Y}}$, we have :*

$$\mathcal{L}_{\mathcal{Y}}M(P) \cong \varinjlim_{(T,S) \in \mathcal{Y}(P)} M(T/S) \quad \text{and} \quad \mathcal{R}_{\mathcal{Y}}M(P) \cong \varprojlim_{(T,S) \in \mathcal{Y}(P)} M(T/S).$$

Moreover ([9] Corollary 6.17), for any biset functor M and any p -group P , the unit map

$$\eta_{M,P} : M(P) \rightarrow \mathcal{R}_{\mathcal{Y}}\mathcal{O}_{\mathcal{Y}}(P) = \varprojlim_{(T,S) \in \mathcal{Y}(P)} M(T/S)$$

is given by

$$\eta_{M,P}(u)_{T,S} = \text{Defres}_{T/S}^P u,$$

for any $u \in M(P)$ and any $(T, S) \in \mathcal{Y}(P)$.

4. Image in subquotients of rank two

4.1. Lemma. *Let \mathcal{X} be a non-empty class of finite p -groups, closed under taking subquotients. Let A be any abelian group, and denote by \hat{B} the functor $\text{Hom}_{\mathbb{Z}}(B(-), A)$, where B is the Burnside functor. Then the unit map*

$$\beta : \hat{B} \rightarrow \mathcal{R}_{\mathcal{X}} \mathcal{O}_{\mathcal{X}} \hat{B}$$

is an isomorphism.

Proof. Indeed if P is a p -group, if $\varphi \in \text{Hom}_{\mathbb{Z}}(B(P), A)$, if (T, S) is a section of P and X is a subgroup such that $S \leq X \leq T$, then

$$(\text{Defres}_{T/S}^P \varphi)((T/S)/(X/S)) = \varphi(\text{Indinf}_{T/S}^P (T/S)/(X/S)) = \varphi(P/X).$$

Suppose that $\varphi \in \text{Ker } \beta$. Since the class \mathcal{X} is non empty and closed under taking subquotients, then it contains the trivial group, and for any subgroup X of P , the section (X, X) is in $\mathcal{X}(P)$. It follows in particular that

$$0 = (\text{Defres}_{X/X}^P \varphi)((X/X)/(X/X)) = \varphi(P/X).$$

Thus $\varphi = 0$, and β is injective.

Conversely, let $\psi = (\psi_{T,S})_{(T,S) \in \mathcal{X}(P)}$ be an element of $\varinjlim_{\mathcal{X}(P)} \text{Hom}_{\mathbb{Z}}(B(T/S), A)$.

Equivalently, for each section $(T, S) \in \mathcal{X}(P)$ and each subgroup X with $S \leq X \leq T$, we have an element $\psi_{T,S}((T/S)/(X/S))$ of A , fulfilling the following two conditions :

- i)* If $x \in P$, then $\psi_{xT, xS}((xT/xS)/(xX/xS)) = \psi_{T,S}((T/S)/(X/S))$.
- ii)* If $(T, S) \in \mathcal{X}(P)$ and $(T', S') \in \mathcal{X}(P)$, and if X is a subgroup of P such that $S \leq S' \leq X \leq T' \leq T$, then $\psi_{T,S}((T/S)/(X/S)) = \psi_{T',S'}((T'/S')/(X/S'))$.

Condition *i)* implies in particular that the element $\psi_{X,X}((X/X)/(X/X))$ is constant on the conjugacy class of X in P . Hence we can define an element $\varphi \in \text{Hom}_{\mathbb{Z}}(B(P), A)$ by setting

$$\varphi(P/X) = \psi_{X,X}((X/X)/(X/X)),$$

for any subgroup X of P . Now if $(T, S) \in \mathcal{X}(P)$ and if $S \leq X \leq T$

$$\begin{aligned} (\text{Defres}_{T/S}^P \varphi)((T/S)/(X/S)) &= \varphi(P/X) = \psi_{X,X}((X/X)/(X/X)) \\ &= (\text{Defres}_{X/X}^{T/S} \psi_{T,S})((X/X)/(X/X)) \\ &= \psi_{T,S}(\text{Indinf}_{(T/S)/(X/X)}^{T/S} (X/X)/(X/X)) \\ &= \psi_{T,S}((T/S)/(X/S)). \end{aligned}$$

Thus $\text{Defres}_{T/S}^P \varphi = \psi_{T,S}$ for any $(T, S) \in \mathcal{X}(P)$. Equivalently $\beta(\varphi) = \psi$, so β is surjective, hence it is an isomorphism. \square

4.2. Lemma. *When p is odd, the map*

$$\varepsilon = \prod_{(T,S) \in \mathcal{X}_3(P)} \text{Defres}_{T/S}^P : 2D(P) \longrightarrow \varinjlim_{(T,S) \in \mathcal{X}_3(P)} 2D(T/S)$$

is an isomorphism.

Proof. Consider the short exact sequence of p -biset functors

$$0 \rightarrow 2D \rightarrow D \rightarrow \mathbb{F}_2 D \rightarrow 0 .$$

Applying the functor $\varprojlim_{\mathcal{X}_3(P)}$ yields the bottom line of the following commutative diagram with exact lines

$$(4.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & 2D(P) & \longrightarrow & D(P) & \longrightarrow & \mathbb{F}_2 D(P) \longrightarrow 0 \\ & & \downarrow \varepsilon & & \downarrow \delta & & \downarrow \gamma \\ 0 & \longrightarrow & \varprojlim_{\mathcal{X}_3(P)} 2D & \longrightarrow & \varprojlim_{\mathcal{X}_3(P)} D & \longrightarrow & \varprojlim_{\mathcal{X}_3(P)} \mathbb{F}_2 D \end{array}$$

where the map δ is an isomorphism, by Theorem 1.1 of [9]. Moreover, by Corollary 1.5 of [10], there is an exact sequence of p -biset functors

$$0 \rightarrow B^\times \rightarrow \mathbb{F}_2 B^* \rightarrow \mathbb{F}_2 D^\Omega \rightarrow 0 ,$$

where B^\times is the functor of units of the Burnside ring, which is isomorphic to the constant functor $\Gamma_{\mathbb{F}_2}$ for p odd. Moreover $D^\Omega = D$ in this case. Applying the functor $\varprojlim_{\mathcal{X}_3(P)}$ to this sequence yields the bottom line of the following commutative diagram with exact lines

$$(4.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 B^*(P) & \longrightarrow & \mathbb{F}_2 D(P) \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & \varprojlim_{\mathcal{X}_3(P)} \Gamma_{\mathbb{F}_2} & \longrightarrow & \varprojlim_{\mathcal{X}_3(P)} \mathbb{F}_2 B^* & \longrightarrow & \varprojlim_{\mathcal{X}_3(P)} \mathbb{F}_2 D \end{array}$$

Since $\mathbb{F}_2 B^*$ is naturally isomorphic to $\text{Hom}_{\mathbb{Z}}(B(-), \mathbb{F}_2)$, the map β is an isomorphism, by Lemma 4.1. Now the group $\varprojlim_{\mathcal{X}_3(P)} \Gamma_{\mathbb{F}_2}$ is the set of sequences $(u_{T,S})_{(T,S) \in \mathcal{X}_3(P)}$

fulfilling the two following conditions :

- i)* If $(T, S) \in \mathcal{X}_3(P)$ and $x \in P$, then $u_{xT, xS} = u_{T,S}$.
- ii)* If $(T, S) \in \mathcal{X}_3(P)$ and $(T', S') \in \mathcal{X}_3(P)$ are such that $S \leq S' \leq T' \leq T$, then $u_{T,S} = u_{T',S'}$.

Applying this for the case $T' = S' = S$, and next for the case $T = T' = S'$, it follows that $u_{T,S} = u_{S,S} = u_{T,T}$ for any $(T, S) \in \mathcal{X}_3(P)$. Thus $u_{T,T} = u_{S,S}$ if $T/S \in \mathcal{X}_3$. Since \mathcal{X}_3 contains the cyclic group of order p , and since P is a p -group, it follows that $u_{T,T} = u_{1,1}$ for any subgroup T of P . Hence $u_{T,S}$ is constant, and the map α is an isomorphism.

Now the Snake's Lemma, applied to Diagram 4.4, shows that the map γ is injective. And another application of this Lemma to Diagram 4.3 shows that the map ε is an isomorphism. \square

Recall that if E is an elementary abelian group of rank 2, then $2D(E)$ is free of rank one, generated by $2\Omega_{E/1}$. Thus if $u \in 2D(P)$, and if $(T, S) \in \mathcal{E}_2^\sharp(P)$, then $\text{Defres}_{T/S}^P u$ is a multiple of $2\Omega_{T/S}$. The following theorem characterizes the sequences of integers $(v_{T,S})_{(T,S) \in \mathcal{E}_2^\sharp(P)}$ which can be obtained that way from an element of $2D(P)$:

4.5. Theorem. *Let P be a p -group (with $p > 2$). The map*

$$\mathcal{D}_P : 2D(P) \rightarrow \prod_{(T,S) \in \mathcal{E}_2^\#(P)} \mathbb{Z}$$

sending an element $u \in 2D(P)$ to the sequence $\mathcal{D}_P(u)_{T,S}$ of integers defined by

$$\text{Defres}_{T/S}^P(u) = \mathcal{D}_P(u)_{T,S} \cdot 2\Omega_{T/S},$$

is injective, and its image is equal to the set of sequences $(v_{T,S})_{(T,S) \in \mathcal{E}_2^\#(P)}$ fulfilling the following conditions :

1. *If $(T, S) \in \mathcal{E}_2^\#(P)$ and $x \in P$, then $v_{T,S} = v_{xT, xS}$.*
2. *If (T, S) and (T', S) are in $\mathcal{E}_2^\#(P)$, if $T' \leq N_P(T)$, and $TT'/S \cong (\mathbb{Z}/p\mathbb{Z})^3$, then*

$$v_{T,S} + \sum_{S < X < T} v_{TT',X} = v_{T',S} + \sum_{S < X < T'} v_{TT',X}.$$

3. *If (T, S) and (T', S) are in $\mathcal{E}_2^\#(P)$, if $T' \leq N_P(T)$, and $TT'/S \cong X_{p^3}$, then*

$$v_{T,S} \equiv v_{T',S} \pmod{p}.$$

Proof. Let $u \in \text{Ker } \mathcal{D}_P$. Then $\text{Defres}_{T/S}^P u = 0$ for any $(T, S) \in \mathcal{E}_2^\#(P)$. Moreover $\text{Defres}_{T/S}^P u \in 2D(\mathbb{Z}/p\mathbb{Z}) = \{0\}$ when (T, S) is a section of P with $T/S \cong \mathbb{Z}/p\mathbb{Z}$. The detection theorem of Carlson and Thévenaz ([11] Theorem 13.1) shows that $u = 0$. Hence \mathcal{D}_P is injective.

Now Condition 1 of Theorem 4.5 holds obviously for the elements of the image of \mathcal{D}_P . To check that Conditions 2 and 3 also hold, suppose that $(T, S), (T', S) \in \mathcal{E}_2^\#(P)$, with $T' \leq N_P(T)$ and $|TT' : P| = p^3$, and observe that setting $R = TT'$, the diagram

$$\begin{array}{ccc} D(P) & \xrightarrow{\mathcal{D}_P} & \prod_{(T'', S'') \in \mathcal{E}_2^\#(P)} \mathbb{Z} \\ \text{Defres}_{R/S}^P \downarrow & & \downarrow \pi_{R,S} \\ D(R/S) & \xrightarrow{\mathcal{D}_P} & \prod_{(T''/S, S''/S) \in \mathcal{E}_2^\#(R/S)} \mathbb{Z} \end{array}$$

is commutative, where $\pi_{R,S}$ is the projection map obtained by identifying sections (T'', S'') of P such that $S \leq S'' \leq T'' \leq R$ with sections $(T''/S, S''/S)$ of R/S . This shows that it is enough to suppose that the group P is either elementary abelian of rank 3, or isomorphic to X_{p^3} . These special cases are detailed below.

To prove that conversely, Conditions 1, 2, and 3 characterize the image of \mathcal{D}_P , observe first that by Theorem 1.1 of [9], the map

$$\prod_{(T,S)} \text{Defres}_{T/S}^P : D(P) \longrightarrow \varprojlim_{(T,S)} D(T/S)$$

is an isomorphism, where T/S runs through all sections of P which are either elementary abelian p -groups of rank ≤ 3 or extraspecial groups of order p^3 and exponent p .

Suppose that Theorem 4.5 is true when P is elementary abelian of rank at most 3, or extraspecial of exponent p . Let P be an arbitrary p -group, and consider a sequence $v = (v_{T,S})_{(T,S) \in \mathcal{E}_2^\sharp(P)}$ fulfilling the conditions of Theorem 4.5.

If $(V,U) \in \mathcal{X}_3(P)$, then the correspondence $(T,S) \mapsto (T/U, S/U)$ is a one to one correspondence between the set of elements (T,S) of $\mathcal{E}_2^\sharp(P)$ such that $U \leq S \leq T \leq V$, and $\mathcal{E}_2^\sharp(V/U)$. Through this bijection, the sequence of integers $v_{T,S}$, for $U \leq S \leq T \leq V$, yields a sequence of integers fulfilling the conditions of Theorem 4.5 for the group V/U , hence an element in the image of the map $\mathcal{D}_{V/U}$. In other words, there is a unique element $w_{V,U} \in 2D(V/U)$ such that

$$\text{Defres}_{T/S}^{V/U} w_{V,U} = v_{T,S} \cdot 2\Omega_{T/S} ,$$

for all $(T,S) \in \mathcal{E}_2^\sharp(P)$ with $U \leq S \leq T \leq V$.

Now the uniqueness of $w_{V,U}$ shows that $w_{xV, xU} = {}^x w_{V,U}$ for any $x \in P$, and that $\text{Defres}_{V'/U'}^{V/U} w_{V,U} = w_{V',U'}$ whenever (V,U) and (V',U') are in $\mathcal{X}_3(P)$ and $U \leq U' \leq V' \leq V$. In other words, the sequence $(w_{V,U})_{(V,U) \in \mathcal{X}_3(P)}$ is an element of $\varprojlim_{\mathcal{X}_3(P)} 2D$. By

Lemma 4.2, there exists an element $t \in 2D(P)$ such that

$$w_{V,U} = \text{Defres}_{V/U}^P t ,$$

for any $(V,U) \in \mathcal{X}_3(P)$. Then obviously $\mathcal{D}_P(t) = v$, and v lies in the image of \mathcal{D}_P , as was to be shown.

So the only thing left to check is that Theorem 4.5 holds when P is elementary abelian of rank at most 3, or isomorphic to X_{p^3} . This is a case by case verification, using the following lemma :

4.6. Lemma. *Let P be a finite p -group, and X be a finite P -set.*

1. *If T/S is a section of P , then*

$$\text{Defres}_{T/S}^P \Omega_X = \Omega_{X^S} ,$$

where X^S denotes the set of fixed points of S on X , viewed as a T/S -set.

2. *If moreover $(T,S) \in \mathcal{E}_2^\sharp(P)$, then*

$$\text{Defres}_{T/S}^P (2\Omega_X) = \left(\sum_{\substack{S \leq V \leq T \\ X^V \neq \emptyset}} \mu(S, V) \right) 2\Omega_{T/S} .$$

$$\text{In other words } \mathcal{D}_P(2\Omega_X)_{T,S} = \sum_{\substack{S \leq V \leq T \\ X^V \neq \emptyset}} \mu(S, V) .$$

Proof. Assertion 1 follows from Section 4 of [2]. For Assertion 2, note that by Assertion 1 and Lemma 5.2.3 of [2], since T/S is abelian,

$$\text{Defres}_{T/S}^P 2\Omega_X = \sum_{\substack{S \leq U \leq V \leq T \\ X^V \neq \emptyset}} \mu(U, V) \cdot 2\Omega_{T/U} ,$$

and that $2\Omega_{T/U} = 0$ in $D(T/S)$ unless $U = S$. □

Now there are four cases :

- If $|P| \leq p$, there is nothing to do, since the map \mathcal{D}_P is an isomorphism $\{0\} \rightarrow \{0\}$.
- If $P \cong (\mathbb{Z}/p\mathbb{Z})^2$, then $2D(P) \cong \mathbb{Z}$, and $\mathcal{E}_2^\sharp(P) = \{(P, \mathbf{1})\}$. In this case, there is no condition on the image of \mathcal{D}_P , and \mathcal{D}_P is an isomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$. So Theorem 4.5 holds in this case.
- If $P \cong (\mathbb{Z}/p\mathbb{Z})^3$, then $\mathcal{E}_2^\sharp(P)$ consists of $p^2 + p + 1$ sections (P, R) , for $|R| = p$, and $p^2 + p + 1$ sections $(Q, \mathbf{1})$, for $|Q| = p^2$. The group $2D(P)$ is a free abelian group, with basis

$$\{2\Omega_{P/\mathbf{1}}\} \sqcup \{2\Omega_{P/R} \mid |R| = p\}.$$

The following arrays gives the values of the sequence $v = \mathcal{D}_P(u)$ for the element u in its first column on the left :

	$v_{P,R}$	$v_{Q,\mathbf{1}}$
$2\Omega_{P/\mathbf{1}}$	0	1
$2\Omega_{P/R'}$	$\begin{cases} 0 & \text{if } R' \neq R \\ 1 & \text{if } R' = R \end{cases}$	$\begin{cases} 0 & \text{if } R' < Q \\ 1 & \text{if } R' \not< Q \end{cases}$

The image of the element $u = m_{\mathbf{1}} \cdot 2\Omega_{P/\mathbf{1}} + \sum_{|R'|=p} m_{R'} \cdot 2\Omega_{P/R'}$ by the map \mathcal{D}_P is equal to the sequence $v = (v_{T,S})$, where

$$(4.7) \quad v_{P,R} = m_R \quad v_{Q,\mathbf{1}} = m_{\mathbf{1}} + \sum_{R \not< Q} m_R$$

If $Q \neq Q'$ are subgroups of order p^2 of P , then $QQ' = P$, and

$$v_{Q,\mathbf{1}} + \sum_{\mathbf{1} < X < Q} v_{P,X} = m_{\mathbf{1}} + \sum_{|X|=p} m_X = v_{Q',\mathbf{1}} + \sum_{\mathbf{1} < X < Q'} v_{P,X},$$

so Condition 2 of 4.5 holds for the sections $(Q, \mathbf{1})$ and $(Q', \mathbf{1})$ of P . Since P is abelian, Conditions 1 and 3 of 4.5 are obviously satisfied.

Conversely, suppose that Condition 2 hold for a sequence $v = (v_{T,S})_{(T,S) \in \mathcal{E}_2^\sharp(P)}$. This sequence is in the image of \mathcal{D}_P if and only if there exist integers $m_{\mathbf{1}}, m_{R'}$, for $|R'| = p$, such that 4.7 hold.

The first equation gives $m_R = v_{P,R}$, and then the second one gives

$$m_{\mathbf{1}} = v_{Q,\mathbf{1}} - \sum_{R \not< Q} v_{P,R}.$$

This is consistent if the right hand side does not depend on Q , i.e. if for any subgroups $Q \neq Q'$ of order p^2 of P

$$v_{Q,\mathbf{1}} - \sum_{R \not< Q} v_{P,R} = v_{Q',\mathbf{1}} - \sum_{R \not< Q'} v_{P,R},$$

or equivalently

$$v_{Q,\mathbf{1}} + \sum_{R < Q} v_{P,R} = v_{Q',\mathbf{1}} + \sum_{R < Q'} v_{P,R}.$$

This is precisely Condition 2 of 4.5 for the section $(Q, \mathbf{1})$ and $(Q', \mathbf{1})$, since $QQ' = P$ in this case. Thus Theorem 4.5 holds for $P \cong (\mathbb{Z}/p\mathbb{Z})^3$.

• If $P \cong X_{p^3}$, then $\mathcal{E}_2^\sharp(P)$ consists of the section (P, Z) , where Z is the centre of P , and of $p+1$ sections $(Q, \mathbf{1})$, where Q is a subgroup of index p in P . The group $D(P)$ is equal to $D^\Omega(P)$, since $p \neq 2$, so it is generated by the elements $\Omega_{P/\mathbf{1}}$, $\Omega_{P/X}$, for $|X| = p$, and $\Omega_{P/Q}$, for $|Q| = p^2$, which have order 2 in $D(P)$. Thus $2D(P)$ is generated by the elements $2\Omega_{P/\mathbf{1}}$ and $2\Omega_{P/X}$, for $|X| = p$. The following array gives the values of the sequence $v = \mathcal{D}_P(u)$ for the element u in its first column on the left, where R denotes a non central subgroup of order p of P :

	$v_{P,Z}$	$v_{Q,\mathbf{1}}$
$2\Omega_{P/\mathbf{1}}$	0	1
$2\Omega_{P/Z}$	1	0
$2\Omega_{P/R}$	0	$\begin{cases} 1 & \text{if } R \not\leq Q \\ 1-p & \text{if } R < Q \end{cases}$

The values in this table can be computed using Lemma 4.6 : for example

$$\text{Res}_Q^P 2\Omega_{P/R} = \left(\sum_{\substack{\mathbf{1} \leq V \leq Q \\ V \leq PR}} \mu(\mathbf{1}, V) \right) 2\Omega_{Q/\mathbf{1}} .$$

If $R \not\leq Q$, then there is only one term in the summation, for $V = \mathbf{1}$, and $\mu(\mathbf{1}, V) = 1$ in this case. And if $R \leq Q$, then there are p additional terms, obtained for the p distinct conjugates V of R in P , and $\mu(\mathbf{1}, V) = -1$ for each of them. This gives the value $1-p$ in this case.

Now if $u = m_{\mathbf{1}} \cdot 2\Omega_{P/\mathbf{1}} + m_Z \cdot 2\Omega_{P/Z} + \sum_{[R]} m_R \cdot 2\Omega_{P/R}$ (where the brackets around R mean that R runs through a set of representatives of conjugacy classes of non central subgroups of order p of P), then the sequence $v = \mathcal{D}_P(u)$ is given by :

$$(4.8) \quad v_{P,Z} = m_Z \quad v_{Q,\mathbf{1}} = m_{\mathbf{1}} + \sum_{[R] \not\leq Q} m_R + (1-p) \sum_{[R] < Q} m_R ,$$

The second equation is equivalent to

$$(4.9) \quad v_{Q,\mathbf{1}} = m_{\mathbf{1}} + \sum_{[R]} m_R - p \sum_{[R] < Q} m_R .$$

It follows that $v_{Q,\mathbf{1}} \equiv v_{Q',\mathbf{1}} \pmod{p}$, for any subgroups Q and Q' of order p^2 in P . This shows that Condition 3 of 4.5 holds for the sections $(Q, \mathbf{1})$ and $(Q', \mathbf{1})$ of P . Condition 2 is obviously satisfied in this case, since P has no subquotient isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$.

Suppose now conversely that a sequence $v = (v_{T,S})_{(T,S) \in \mathcal{E}_2^\sharp(P)}$ is given, and that Conditions 1 and 3 of 4.5 hold. Then v lies in the image of \mathcal{D}_P if and only if there exist integers $m_{\mathbf{1}}$, m_Z , m_R (invariant by P -conjugation), such that Equations 4.8 hold.

The first equation in 4.8 gives $m_Z = v_{P,Z}$, and the second one gives

$$m_{\mathbf{1}} - v_{Q,\mathbf{1}} + \sum_{[R]} m_R = p \sum_{[R] < Q} m_R .$$

All subgroups of order p of Q different from Z are conjugate in P . Denoting by R_Q one of them, this equation becomes

$$(4.10) \quad m_{\mathbf{1}} - v_{Q,\mathbf{1}} + \sum_{[R]} m_R = pm_{R_Q} .$$

Summing this relation over Q yields

$$(p+1)m_{\mathbf{1}} - \sum_Q v_{Q,\mathbf{1}} + (p+1) \sum_{[R]} m_R = p \sum_{[R]} m_R ,$$

thus

$$\sum_{[R]} m_R = \sum_Q v_{Q,\mathbf{1}} - (p+1)m_{\mathbf{1}} .$$

Now 4.10 yields

$$pm_{R_Q} = \sum_{Q' \neq Q} v_{Q',\mathbf{1}} - pm_{\mathbf{1}} .$$

By Condition 3, the sum $\sum_{Q' \neq Q} v_{Q',\mathbf{1}}$ is congruent to $pv_{Q,\mathbf{1}}$ modulo p , i.e. to 0. Since $Q = R_Q Z$, this gives finally

$$m_R = \frac{1}{p} \left(\sum_{Q' \neq RZ} v_{Q',\mathbf{1}} \right) - m_{\mathbf{1}} .$$

Conversely, if this holds for any R , then equation 4.9 holds : indeed, in this case

$$\sum_{[R]} m_R = \sum_Q v_{Q,\mathbf{1}} - (p+1)m_{\mathbf{1}} ,$$

thus

$$\begin{aligned} m_{\mathbf{1}} + \sum_{[R]} m_R - p \sum_{[R] < Q} m_R &= m_{\mathbf{1}} + \sum_Q v_{Q,\mathbf{1}} - (p+1)m_{\mathbf{1}} - pm_{R_Q} \\ &= \sum_Q v_{Q,\mathbf{1}} - pm_{\mathbf{1}} - \left(\sum_{Q' \neq Q} v_{Q',\mathbf{1}} \right) + pm_{\mathbf{1}} \\ &= v_{Q,\mathbf{1}} . \end{aligned}$$

Thus 4.9 holds, and Theorem 4.5 also, when $P = X_p^3$. □

5. Proof of Theorem 2.15

Let P be a finite p -group. Clearly $T(P)$ is the kernel of r_P , and $\text{Im } r_P \leq \text{Ker } h_P$, by Proposition 2.14. So the only thing to show is that this inclusion is an equality.

Let $u \in \text{Ker } h_P$. It means that there exists a P -invariant function $E \mapsto m_E$ from $\mathcal{A}_{\geq 2}(P)$ to \mathbb{Z} such that for any $E < F$ in $\mathcal{A}_{\geq 2}(P)$

$$w_{E,F} = m_E - m_F ,$$

where the integer $w_{E,F}$ is defined by the equality

$$w_{E,F} \cdot \Omega_{E/\mathbf{1}} = \text{Res}_E^F \sigma_F \text{Res}_F^P u - \sigma_E \text{Res}_E^P u .$$

In other words

$$\text{Res}_E^F (m_F \cdot \Omega_{F/\mathbf{1}} + \sigma_F \text{Res}_F^P u) = m_E \cdot \Omega_{E/\mathbf{1}} + \sigma_E \text{Res}_E^P u .$$

Set $w_E = m_E \cdot \Omega_{E/\mathbf{1}} + \sigma_E \text{Res}_E^P u$, for $E \in \mathcal{A}_{\geq 2}(P)$. Then $\text{Res}_E^F w_F = w_E$, for any $E < F$ in $\mathcal{A}_{\geq 2}(P)$, and ${}^x(w_E) = w_{x_E}$ for any $x \in P$ and $E \in \mathcal{A}_{\geq 2}(P)$. Moreover, for any $E \in \mathcal{A}_{\geq 2}(P)$

$$r_E(w_E) = r_E \sigma_E \text{Res}_E^P u = \text{Res}_E^P u,$$

since $r_E(\Omega_{E/\mathbf{1}}) = 0$, and since σ_E is a section of r_E . It means that for any subgroup $Y \neq \mathbf{1}$ of E

$$\text{Def}_{E/Y}^E w_E = \text{Res}_{E/Y}^{N_P(Y)/Y} u_Y.$$

If $(T, S) \in \mathcal{E}_2^\sharp(P)$, define an integer $v_{T,S}$ by

$$(5.1) \quad \begin{cases} \text{Res}_{T/S}^{N_P(S)/S}(2u_S) = v_{T,S} \cdot 2\Omega_{T/S} & \text{if } S \neq \mathbf{1} \\ 2w_T = v_{T,\mathbf{1}} \cdot 2\Omega_{T/\mathbf{1}} & \text{if } S = \mathbf{1} \end{cases}$$

This sequence of integers $(v_{T,S})_{(T,S) \in \mathcal{E}_2^\sharp(P)}$ satisfies some of the conditions of Theorem 4.5. Indeed :

- If $x \in P$ and $(T, S) \in \mathcal{E}_2^\sharp(P)$, then $v_{xT, xS} = v_{T,S}$: this is because ${}^x(w_E) = w_{x_E}$ for any $E \in \mathcal{A}_{\geq 2}(P)$, and because ${}^x(u_Q) = u_{x_Q}$ for any subgroup $Q \neq \mathbf{1}$ of P . Thus Condition 1 of Theorem 4.5 holds.

- Suppose that (T, S) and (T', S) are elements of $\mathcal{E}_2^\sharp(P)$ such that $T \leq N_P(T')$. There are two cases to consider :

(a) If $S \neq \mathbf{1}$, then for any section $(V, U) \in \mathcal{E}_2^\sharp(N_P(S)/S)$

$$\begin{aligned} v_{V,U} \cdot \Omega_{V/U} &= \text{Res}_{V/U}^{N_P(U)/U}(2u_U) \\ &= \text{Res}_{V/U}^{N_P(S,U)/U} \text{Res}_{N_P(S,U)/U}^{N_P(U)/U}(2u_U) \\ &= \text{Res}_{V/U}^{N_P(S,U)/U} \text{Defres}_{N_P(S,U)/U}^{N_P(S)/S}(2u_S) \\ &= \text{Defres}_{V/U}^{N_P(S)/S}(2u_S). \end{aligned}$$

It follows that the sequence $(v_{V,U})_{(V,U) \in \mathcal{E}_2^\sharp(N_P(S)/S)}$ is equal to $\mathcal{D}_{N_P(S)/S}(2u_S)$, hence it is in the image of the map $\mathcal{D}_{N_P(S)/S}$. Thus if $TT'/S \cong (\mathbb{Z}/p\mathbb{Z})^3$, then Condition 2 of Theorem 4.5 holds for the sections (T, S) and (T', S) of $N_P(S)/S$. And if $TT'/S \cong X_{p^3}$, then Condition 3 of Theorem 4.5 holds, for a similar reason.

(b) If $S = \mathbf{1}$, then set $F = TT'$. If $F \cong (\mathbb{Z}/p\mathbb{Z})^3$, then consider a section $(V, U) \in \mathcal{E}_2^\sharp(F)$. If $U = \mathbf{1}$, then

$$\text{Defres}_{V/U}^F 2w_F = \text{Res}_V^F 2w_F = 2w_V = v_{V,\mathbf{1}} \cdot 2\Omega_{V/\mathbf{1}}.$$

And if $U \neq \mathbf{1}$, then

$$\begin{aligned} \text{Defres}_{V/U}^F 2w_F &= \text{Res}_{V/U}^{F/U} \text{Def}_{F/U}^F 2w_F \\ &= \text{Res}_{V/U}^{F/U} \text{Res}_{F/U}^{N_P(U)/U} 2u_U \\ &= \text{Res}_{V/U}^{N_P(U)/U} 2u_U \\ &= v_{V,U} \cdot 2\Omega_{V/U}. \end{aligned}$$

It follows that the sequence $(v_{V,U})_{(V,U) \in \mathcal{E}_2^\sharp(F)}$ is equal to $\mathcal{D}_F(2w_F)$. In particular, Condition 2 of Theorem 4.5 is fulfilled for the sections $(T, \mathbf{1})$ and $(T', \mathbf{1})$ of F .

Hence the sequence $(v_{T,S})_{(T,S) \in \mathcal{E}_2^\sharp(P)}$ fulfills all the conditions of Theorem 4.5, except possibly Condition 3 for sections $(T, \mathbf{1})$ and $(T', \mathbf{1})$ such that $T \leq N_P(T')$ and $TT' \cong X_{p^3}$. This situation is handled by the following lemma :

5.2. Lemma. *Let P be a finite p -group, and $(v_{T,S})_{(T,S) \in \mathcal{E}_2^\sharp(P)}$ be a sequence of integers such that :*

1. *If $x \in P$ and $(T, S) \in \mathcal{E}_2^\sharp(P)$, then $v_{xT, xS} = v_{T,S}$.*
2. *If (T, S) and (T', S) are in $\mathcal{E}_2^\sharp(P)$, if $T \leq N_P(T')$ and if $TT'/S \cong (\mathbb{Z}/p\mathbb{Z})^3$, then*

$$v_{T,S} + \sum_{S < X < T} v_{TT',X} = v_{T',S} + \sum_{S < X < T'} v_{TT',X} .$$

3. *If $S \neq \mathbf{1}$, if (T, S) and (T', S) are in $\mathcal{E}_2^\sharp(P)$, if $T \leq N_P(T')$ and if $TT'/S \cong X_{p^3}$, then*

$$v_{T,S} \equiv v_{T',S} \pmod{p} .$$

Then :

- (i) *If $T, T' \in \mathcal{A}_{=2}(P)$, if T and T' are in the same connected component of $\mathcal{A}_{\geq 2}(P)$, if $T \leq N_P(T')$ and $TT' \cong X_{p^3}$, then $v_{T,\mathbf{1}} \equiv v_{T',\mathbf{1}} \pmod{p}$.*
- (ii) *There exists a sequence of integers $(y_T)_{T \in \mathcal{A}_{=2}(P)}$ such that*
 - (a) *If $x \in P$ and $T \in \mathcal{A}_{=2}(P)$, then $y_{xT} = y_T$.*
 - (b) *If $T, T' \in \mathcal{A}_{=2}(P)$, if $T \leq N_P(T')$ and $TT' \cong (\mathbb{Z}/p\mathbb{Z})^3$, then $y_T = y_{T'}$.*
 - (c) *If $T, T' \in \mathcal{A}_{=2}(P)$, if $T \leq N_P(T')$ and $TT' \cong X_{p^3}$, then*

$$y_T + v_{T,\mathbf{1}} \equiv y_{T'} + v_{T',\mathbf{1}} \pmod{p}$$

Proof. The proof of Assertion (i) goes by induction on $|P|$, starting with the case where P is cyclic, where there is nothing to prove. Assume then that Hypotheses 1), 2), and 3) imply Assertion 1, for any p -group of order strictly smaller than $|P|$. Let T and T' be elementary abelian subgroups of rank 2 of P , such that $T \leq N_P(T')$ and $TT' \cong X_{p^3}$. Set $X = TT'$, and denote by Z the centre of X .

If there is a proper subgroup Q of P containing X , and such that T and T' are in the same connected component of $\mathcal{A}_{\geq 2}(Q)$, then $v_{T,\mathbf{1}} \equiv v_{T',\mathbf{1}} \pmod{p}$, by induction, since Hypotheses 1), 2), and 3) obviously hold for Q if they hold for P . It is the case in particular if $\mathcal{A}_{\geq 2}(Q)$ is connected.

Suppose that there exists a subgroup C of order p in $C_P(X)$, not contained in X (i.e. different from Z). Then the center $T'' = C \times Z$ of the subgroup $Q = C \times X$ of P is not cyclic. Hence $\mathcal{A}_{\geq 2}(Q)$ is connected, and Q contains T and T' . Thus I can suppose that $Q = P$, and then T'' is equal to the centre of P . It is elementary abelian of rank 2. Moreover $TT'' \cong (\mathbb{Z}/p\mathbb{Z})^3$, since T and T'' are elementary abelian of rank 2 and centralize each other, and since $T \cap T'' = T \cap X \cap T'' = Z$. Hypothesis 2, applied to the sections $(T, \mathbf{1})$ and $(T', \mathbf{1})$ of P yields

$$(5.3) \quad v_{T,\mathbf{1}} - v_{T',\mathbf{1}} = \sum_{\mathbf{1} < F < T''} v_{TT'',F} - \sum_{\mathbf{1} < F < T} v_{TT'',F} .$$

Now $TT'' \trianglelefteq P$ since $|P : TT''| = p$, and $T \trianglelefteq P$, since $T \trianglelefteq X$ and $C \leq C_P(X)$. Hence P acts by conjugation on the set of subgroups F such that $\mathbf{1} < F < T$, and $F = Z$ is the unique fixed point under this action. Now Hypothesis 1 implies that

$$\sum_{\mathbf{1} < F < T} v_{TT'', F} \equiv v_{TT'', Z} \pmod{p},$$

and Equation 5.3 yields

$$(5.4) \quad v_{T, \mathbf{1}} - v_{T'', \mathbf{1}} \equiv \sum_{\substack{\mathbf{1} < F < T'' \\ F \neq Z}} v_{TT'', F} \pmod{p}.$$

The same argument applies with T' instead of T , so

$$(5.5) \quad v_{T', \mathbf{1}} - v_{T'', \mathbf{1}} \equiv \sum_{\substack{\mathbf{1} < F < T'' \\ F \neq Z}} v_{T'T'', F} \pmod{p}.$$

Now if $\mathbf{1} < F < T''$ and $F \neq Z$, the group P/F has order p^3 and exponent p (since P has exponent p), and it is non abelian (since $F \not\leq [P, P] = Z$). Hence $P/F \cong X_{p^3}$. Since $P = (TT'')(T'T'')$, Hypothesis 3, applied to the sections (TT'', F) and $(T'T'', F)$ yields $v_{TT'', F} \equiv v_{T'T'', F} \pmod{p}$. This shows that the right hand sides of 5.4 and 5.5 are congruent modulo p . So are the left hand sides, thus $v_{T, \mathbf{1}} - v_{T'', \mathbf{1}} \equiv v_{T', \mathbf{1}} - v_{T'', \mathbf{1}} \pmod{p}$, and $v_{T, \mathbf{1}} \equiv v_{T', \mathbf{1}} \pmod{p}$.

Hence I can suppose that Z is the only subgroup of order p of $C_P(X)$. In particular, the centre of P is cyclic, and Z is the only subgroup of order p in this centre. Moreover, since $T \neq T'$ and T, T' are in the same connected component of $\mathcal{A}_{\geq 2}(P)$, the groups T and T' are not maximal elementary abelian subgroups, thus P has p -rank at least equal to 3, and T and T' are in the big component \mathcal{C} of $\mathcal{A}_{\geq 2}(P)$.

In this case, there is a normal subgroup T_0 of P which is elementary abelian of rank 2, and $T_0 \in \mathcal{C}$. Moreover $T_0 > Z$.

If $T_0 \not\leq X$, then $T_0 \cap X = Z$. Then $|T_0 X| = p^4$, and $|T_0 X : X| = p$. Thus T_0 normalizes X . Moreover, if Y is a subgroup of index p of X , then $Y > Z$, and $|T_0 Y| = p^3$, thus $|T_0 Y : Y| = p$, and T_0 normalizes Y . It follows that the image of T_0 in the group $\text{Out}(X)$ of outer automorphisms of X , which is isomorphic to $GL_2(\mathbb{F}_p)$, is a p -subgroup stabilizing every line. So this image is trivial, and T_0 acts on X by inner automorphisms. Let $t \in T_0 - X$. Then there exists $y \in X$ such that $y^{-1}t \in C_P(X)$. In particular $y^{-1}t$ centralizes y , so t centralizes y , and then $(y^{-1}t)^p = (y^{-1})^p t^p = 1$. Moreover $y \neq t$, since $t \notin X$. Hence $y^{-1}t$ has order p . Since Z is the only subgroup of order p of $C_P(X)$, it follows that $y^{-1}t \in Z$, so $t \in X$. This contradiction shows that $T_0 \leq X$.

Since the congruences $v_{T, \mathbf{1}} \equiv v_{T_0, \mathbf{1}} \pmod{p}$ and $v_{T', \mathbf{1}} \equiv v_{T_0, \mathbf{1}} \pmod{p}$ imply the congruence $v_{T, \mathbf{1}} \equiv v_{T', \mathbf{1}} \pmod{p}$, it is enough to suppose that $T_0 = T$, thus $T \trianglelefteq P$. Let F be an elementary abelian subgroup of rank 3 of P containing T' : such a subgroup exists, since T' is not a maximal element of $\mathcal{A}_{\geq 2}(P)$. Set $T'' = C_F(T)$. Then $|F : T''|$ divide p , since F/T'' is a p -subgroup of $\text{Aut}(T) \cong GL_2(\mathbb{F}_p)$. Moreover $F \not\leq C_P(T)$, since $F > T'$. Thus $|F : T''| = p$, and $T'' \cong (\mathbb{Z}/p\mathbb{Z})^2$. Moreover $T' \neq T''$, since $T' \not\leq C_P(T)$, thus $F = T'T''$.

Now F centralizes T' , and normalizes T . Thus F normalizes $TT' = X$. Moreover $F \cap X = T'$, since $T' \leq F \cap X$, and since F and X are distinct subgroups of order p^3 of P , for F is abelian and X is not. Hence $|FX : F| = p$, so FX normalizes F . Thus X

normalizes F , and X also normalizes $C_P(T)$ since $T \trianglelefteq P$. It follows that X normalizes $F \cap C_P(T) = C_F(T) = T''$. Obviously X also normalizes its subgroup T' .

Hypothesis 2, applied to the sections $(T', \mathbf{1})$ and $(T'', \mathbf{1})$ of P , yields

$$(5.6) \quad v_{T', \mathbf{1}} - v_{T'', \mathbf{1}} = \sum_{1 < Y < T''} v_{T'T'', Y} - \sum_{1 < Y < T'} v_{T'T'', Y}.$$

Since X normalizes T' and T'' , and since any subgroup of order p normalized by X is centralized by X , Hypothesis 1 yields

$$\sum_{1 < Y < T''} v_{T'T'', Y} \equiv \sum_{\substack{1 < Y < T'' \\ Y \leq C_P(X)}} v_{T'T'', Y} \pmod{p}.$$

But $T'' \cap C_P(X) = Z$, since $Z \leq T'' \cap C_P(X)$, and since Z is the only subgroup of order p of $C_P(X)$. Thus

$$(5.7) \quad \sum_{1 < Y < T''} v_{T'T'', Y} \equiv v_{T'T'', Z} \pmod{p}.$$

The same argument, applied with T' instead of T'' , since $T' \cap C_P(X) = Z$, yields

$$(5.8) \quad \sum_{1 < Y < T'} v_{T'T'', Y} \equiv v_{T'T'', Z} \pmod{p}.$$

Now it follows from 5.6, 5.7 and 5.8 that $v_{T', \mathbf{1}} - v_{T'', \mathbf{1}} \equiv 0 \pmod{p}$, i.e.

$$(5.9) \quad v_{T', \mathbf{1}} \equiv v_{T'', \mathbf{1}} \pmod{p}.$$

Now the group TT'' is also elementary abelian of rank 3 : indeed, the group T'' centralizes T , and $T'' \cap T = Z$ since $T'' \cap T \geq Z$ and $T'' \neq T$ for $T \not\leq F \leq C_P(T')$. Then Hypothesis 2, for the sections $(T, \mathbf{1})$ and $(T'', \mathbf{1})$, yields

$$(5.10) \quad v_{T, \mathbf{1}} - v_{T'', \mathbf{1}} = \sum_{1 < Y < T''} v_{TT'', Y} - \sum_{1 < Y < T} v_{TT'', Y}.$$

The group X normalizes T and T'' , and $T \cap C_P(X) = Z = T'' \cap C_P(X)$. The same argument as above yields

$$(5.11) \quad v_{T, \mathbf{1}} \equiv v_{T'', \mathbf{1}} \pmod{p}.$$

Thus $v_{T, \mathbf{1}} \equiv v_{T', \mathbf{1}} \pmod{p}$, by 5.9 and 5.11, and this completes the proof of Assertion (i).

For Assertion 2, there is nothing to do if P has no normal subgroup $T_0 \cong (\mathbb{Z}/p\mathbb{Z})^2$, since then P is cyclic, and $\mathcal{A}_{\geq 2}(P) = \emptyset$. If P is not cyclic, fix such a normal subgroup T_0 of P , and denote by \mathcal{C} the connected component of T_0 in $\mathcal{A}_{\geq 2}(P)$. Thus \mathcal{C} is the big component if P has p -rank at least 3, and $\mathcal{C} = \{T_0\}$ otherwise. Define the sequence $(y_T)_{T \in \mathcal{A}_{=2}(P)}$ by

$$y_T = \begin{cases} 0 & \text{if } T \in \mathcal{C} \\ v_{T_0, \mathbf{1}} - v_{T, \mathbf{1}} & \text{otherwise} \end{cases}$$

This sequence obviously fulfills Condition (a) of Lemma 5.2, by Hypothesis 1, and since \mathcal{C} is invariant by P -conjugation. Now if $T, T' \in \mathcal{A}_{=2}(P)$, if $T \leq N_P(T')$ and $TT' \cong (\mathbb{Z}/p\mathbb{Z})^3$, it follows that P has p -rank at least 3, that \mathcal{C} is the big component, and that $T, T' \in \mathcal{C}$. Thus $y_T = y_{T'} = 0$, so Condition (b) of Lemma 5.2 holds. Finally, if $T, T' \in \mathcal{A}_{=2}(P)$, if $T \leq N_P(T')$ and $TT' \cong X_{p^3}$, then there are three cases :

- if T and T' are in \mathcal{C} , then $y_T = y_{T'} = 0$, thus $y_T + v_{T,1} = v_{T,1}$, and $y_{T'} + v_{T',1} = v_{T',1}$. But $v_{T,1} \equiv v_{T',1} \pmod{p}$ in this case, by Assertion 1. Thus Condition (c) holds in this case.
- if $T \in \mathcal{C}$ and $T' \notin \mathcal{C}$, then $y_T + v_{T,1} = v_{T,1}$, and $y_{T'} + v_{T',1} = v_{T_0,1}$. But now T and T_0 are both in \mathcal{C} , so $v_{T,1} \equiv v_{T_0,1} \pmod{p}$ in this case, by Assertion 1. Thus Condition (c) holds in this case also. The case $T \notin \mathcal{C}$ and $T' \in \mathcal{C}$ is similar.
- if $T \notin \mathcal{C}$ and $T' \notin \mathcal{C}$, then $y_T + v_{T,1} = v_{T_0,1} = y_{T'} + v_{T',1}$, so Condition (c) holds in this case also.

This completes the proof of Lemma 5.2. \square

End of the proof of Theorem 2.15 : In the beginning of the proof of Theorem 2.15, I started with an element $u \in \varprojlim_{1 < Q \leq P} D(N_P(Q)/Q)$ such that $h_P(u) = 0$. From this

data, in 5.1, I built a sequence of integers $(v_{T,S})_{(T,S) \in \mathcal{E}_2^\sharp(P)}$ fulfilling Hypothesis 1, 2 and 3 of Lemma 5.2. Let $(y_T)_{T \in \mathcal{A}_{=2}(P)}$ denote the sequence of integers provided by this lemma, and define a new sequence of integers $(v'_{T,S})_{(T,S) \in \mathcal{E}_2^\sharp(P)}$ by

$$v'_{T,S} = \begin{cases} v_{T,S} & \text{if } S \neq \mathbf{1} \\ y_T + v_{T,S} & \text{if } S = \mathbf{1} \end{cases} .$$

Then this sequence fulfills Conditions 1, 2, and 3 of Theorem 4.5 : indeed, the new sequence is clearly invariant by conjugation, so Condition 1 is fulfilled. Conditions 2 and 3 for sections (T, S) and (T', S) with $S \neq \mathbf{1}$ are obviously fulfilled, since they are for the sequence $(v_{T,S})$, and since $v_{T,S} = v'_{T,S}$ when $S \neq \mathbf{1}$.

Now if $T, T' \in \mathcal{A}_{=2}(P)$, if $T \leq N_P(T')$ and $TT' \cong (\mathbb{Z}/p\mathbb{Z})^3$, then T and T' are in the same connected component of $\mathcal{A}_{\geq 2}(P)$, and $y_T = y_{T'}$. Thus

$$\begin{aligned} v'_{T,1} + \sum_{1 < Y < T} v'_{TT',Y} &= y_T + v_{T,1} + \sum_{1 < Y < T} v_{TT',Y} \\ &= y_{T'} + v_{T',1} + \sum_{1 < Y < T'} v_{TT',Y} \\ &= v'_{T',1} + \sum_{1 < Y < T'} v'_{TT',Y} , \end{aligned}$$

so Condition 2 is fulfilled.

Finally if $T, T' \in \mathcal{A}_{=2}(P)$, if $T \leq N_P(T')$ and $TT' \cong X_{p^3}$, then

$$v'_{T,1} = y_T + v_{T,1} \equiv y_{T'} + v_{T',1} \pmod{p} ,$$

hence Condition 3 is fulfilled, since $y_{T'} + v_{T',1} = v'_{T',1}$.

By Theorem 4.5, there exists $n \in D(P)$ such that $\mathcal{D}_P(2n) = (v'_{T,S})$. In other words, for any $(T, S) \in \mathcal{E}_2^\sharp(P)$

$$\text{Defres}_{T/S}^P(2n) = v'_{T,S} \cdot 2\Omega_{T/S} .$$

Thus if $S \neq \mathbf{1}$

$$\text{Defres}_{T/S}^P(2n) = v_{T,S} \cdot 2\Omega_{T/S} = \text{Res}_{T/S}^{N_P(S)/S}(2u_S) .$$

Set $t_S = \text{Defres}_{N_P(S)/S}^P(2n) - 2u_S$. Now for any $(V, U) \in \mathcal{E}_2^\#(N_P(S)/S)$

$$\begin{aligned} \text{Defres}_{V/U}^{N_P(S)/S}(t_S) &= \text{Defres}_{V/U}^P(2n) - \text{Defres}_{V/U}^{N_P(S)/S}(2u_S) \\ &= \text{Defres}_{V/U}^P(2n) - \text{Res}_{V/U}^{N_P(S,U)/U} \text{Defres}_{N_P(S,U)/U}^{N_P(S)/S}(2u_S) \\ &= \text{Defres}_{V/U}^P(2n) - \text{Res}_{V/U}^{N_P(S,U)/U} \text{Res}_{N_P(S,U)/U}^{N_P(U)/U}(2u_U) \\ &= \text{Defres}_{V/U}^P(2n) - \text{Res}_{V/U}^{N_P(U)/U}(2u_U) = 0. \end{aligned}$$

It follows that t_S is a torsion element of $D(N_P(S)/S)$, which is also in $2D(N_P(S)/S)$. Since the latter is torsion free, it follows that $t_S = 0$, i.e. that $2u_S = \text{Defres}_{N_P(S)/S}^P(2n)$, for any $S \neq \mathbf{1}$. Equivalently $2u = r_P(2n)$, or $2(u - r_P(n)) = 0$.

Now $u - r_P(n)$ is an element of $\varprojlim_{1 < Q \leq P} D_{\text{tors}}(N_P(Q)/Q)$. By Proposition 5.5 of [6], there exists an element $m \in D(P)$ such that $r_P(m) = u - r_P(n)$. It follows that $u = r_P(m + n)$, as was to be shown. This completes the proof of Theorem 2.15. \square

6. Example : the group X_{p^5}

Let P be an extraspecial group of order p^5 and exponent p . The centre Z of P is cyclic of order p , and it is equal to the Frattini subgroup of P . The commutator $P \times P \rightarrow Z$ induces a non degenerate symplectic \mathbb{F}_p -valued scalar product on the factor group $E = P/Z \cong (\mathbb{F}_p)^4$, and the map $Q \mapsto Q/Z$ is a poset isomorphism from the poset of elementary abelian subgroups of P strictly containing Z to the poset \mathcal{E} of non zero totally isotropic subspaces of E . There are $e = \frac{p^4 - 1}{p - 1}$ isotropic lines in E , and the same number of totally isotropic 2-dimensional subspaces. It follows that $|\mathcal{E}|$ is equal to $2e$.

There is a commutative diagram

$$\begin{array}{ccccccc} & & D(P/Z) & \xrightarrow[\cong]{d} & \varprojlim_{Q \geq Z} D(P/Q) & & \\ & & \uparrow \text{Def}_{P/Z}^P & & \uparrow \pi & & \\ 0 & \longrightarrow & T(P) & \longrightarrow & D(P) & \xrightarrow{r_P} & \varprojlim_{1 < Q \leq P} D(N_P(Q)/Q) \xrightarrow{h_P} H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)} \end{array}$$

In this diagram, the group $\varprojlim_{Q \geq Z} D(P/Q)$ is the group of sequences $(u_Q)_{Z \leq Q \leq P}$, where $u_Q \in D(P/Q)$ (note that $Q \trianglelefteq P$ if $Q \geq Z$), such that

$$\forall R \geq Q \geq Z, \text{Def}_{P/R}^{P/Q} u_Q = u_R,$$

and the map π is the projection map on the components $Q \geq Z$. The map d is the product of the deflation maps $\text{Def}_{P/Q}^{P/Z}$, for $Q \geq Z$. It is an isomorphism, since the sequence $(u_Q)_{Q \geq Z}$, where $u_Q \in D(P/Q)$, is in the group $\varprojlim_{Q \geq Z} D(P/Q)$ if and only if $u_Q = \text{Def}_{P/Q}^{P/Z} u_Z$ for any $Q \geq Z$.

The kernel of the map $\text{Def}_{P/Z}^P$ is the set of faithful elements of $D(P)$, and it was denoted by $\partial D(P)$ in [5]. It was shown in this paper (Theorem 9.1) that

$$\partial D(P) \cong \mathbb{Z}^{2e} \oplus \mathbb{Z}/2\mathbb{Z}.$$

The kernel of π consists of the sequences $(u_Q)_{\mathbf{1} < Q \leq P}$ in $\varprojlim_{\substack{\mathbf{1} < Q \leq P \\ Q \cap Z = \mathbf{1}}} D(N_P(Q)/Q)$ for which $u_Q = 0$ if $Q \geq Z$. It was shown in [5] that this is also the group $\varprojlim_{\substack{\mathbf{1} < Q \leq P \\ Q \cap Z = \mathbf{1}}} \partial D(N_P(Q)/Q)$.

All these facts show that there is an exact sequence

$$(6.1) \quad 0 \longrightarrow T(P) \longrightarrow \partial D(P) \xrightarrow{r_P} \varprojlim_{\substack{\mathbf{1} < Q \leq P \\ Q \cap Z = \mathbf{1}}} \partial D(N_P(Q)/Q) \xrightarrow{h_P} H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)},$$

where r_P and h_P are the restrictions of the previously defined maps with the same names to the corresponding subgroups.

If Q is a subgroup of P such that $Q \cap Z = \mathbf{1}$, then Q is elementary abelian of rank at most 2 (see [5] for details). If Q has order p , then $N_P(Q)/Q \cong X_{p^3}$, thus $\partial D(N_P(Q)/Q) \cong \mathbb{Z}^{p+1} \oplus \mathbb{Z}/2\mathbb{Z}$ ([5] Theorem 9.1 or Section 11). If Q has order p^2 , then $N_P(Q)/Q \cong \mathbb{Z}/p\mathbb{Z}$, thus $\partial D(N_P(Q)/Q) \cong \mathbb{Z}/2\mathbb{Z}$.

It follows easily that the group $\varprojlim_{\substack{\mathbf{1} < Q \leq P \\ Q \cap Z = \mathbf{1}}} \partial D(N_P(Q)/Q)$ has free rank at least equal to $e(p+1)$, since it contains the group $\bigoplus_{\substack{|Q|=p \\ Q \neq Z}} 2\partial D(N_P(Q)/Q)$.

Now the group $T(P)$ is free of rank one, generated by $\Omega_{P/\mathbf{1}}$, by Corollary 1.3 of [11], and the group $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$ is isomorphic to $H^1(\mathcal{E}, \mathbb{Z})$. An easy computation, using e.g. Section 6 of [5], shows that this group is free of rank p^4 .

Now the free rank of the image of h_P in the exact sequence 6.1 is at least equal to

$$1 - 2e + e(p+1) = 1 + e(p-1) = p^4,$$

and since this is equal to the free rank of $H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}$, it follows that the free rank of the image of h_P is actually equal to p^4 , and that the free rank of $\varprojlim_{\substack{\mathbf{1} < Q \leq P \\ Q \cap Z = \mathbf{1}}} \partial D(N_P(Q)/Q)$

is equal to $e(p+1)$.

Moreover the map h_P has finite cokernel, and this shows that in this case, the gluing problem does not always have a solution.

6.2. Remark: In this case, a precise description of the map h_P shows that its cokernel is a non trivial finite p -group.

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