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FERMIONS AND DISCRETE SYMMETRIES IN QUANTUM FIELD THEORY.

I. GENERALITIES & THE PROPAGATOR FOR ONE FLAVOR.

Q. Duret ^{1 2} & B. Machet ^{1 3}

Abstract: Starting from Wigner's symmetry representation theorem, we give a general account of discrete symmetries P , C , T and their products, focusing on fermions in Quantum Field Theory. We deal in full generality with unitary and antiunitary operators and put a special emphasis on the linearity and unitarity of charge conjugation. We provide the rules of transformation of Weyl spinors, both at the classical level (grassmanian functions) and quantum level (operators). Making use of Wightman's definition of invariance, we outline ambiguities linked to the notion of classical fermionic Lagrangian. We then present the general constraints cast on the fermionic propagator for one flavor by P , C , T and their products; we show that propagating a Majorana fermion is incompatible with the breaking of both C and CP .

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Keywords: spinors, discrete symmetries, propagator



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1 Introduction

Fermions are usually treated, in most aspects of their phenomenology, as classical, though anticommuting, objects. They are generally endowed with a mass matrix though, for coupled systems¹, this can only be a linear approximation in the vicinity of one among the physical poles of their full (matricial) propagator [1] [2]. In this perspective, the study of neutral kaons [1], and more specially of the role held, there, by discrete symmetries P , C , T and their products, has shown that subtle differences occur between the “classical” treatment obtained from a Lagrangian and a mass matrix, and the full quantum treatment dealing with their propagator. Using a classical approximation for fermions is *a priori* still more subject to caution since, in particular, their anticommutation is of quantum origin. This is why, after the work [1], we decided to perform a not less exhaustive study of coupled fermionic systems in Quantum Field Theory, dealing especially with the propagator approach. Treating fermions on a rigorous ground is all the more important as the very nature, Dirac or Majorana, of neutrinos is still unknown, and that all theoretical results used up to now, concerning specially flavor mixing, rely on a classical Lagrangian (mass matrix).

The second and third parts of this work are dedicated to general statements concerning the discrete symmetries parity P , charge conjugation C , time reversal T , and their products. It does not pretend to be original, but makes a coherent synthesis of results scattered in the literature, and which sometimes contradict each other. Starting from Wigner’s representation theorem [3] and Wightman’s point of view for symmetry transformations [4], we give the general rules of transformations of operators and of their hermitian conjugates by any unitary or antiunitary transformation. We then specialize to transforming Weyl spinors by P , C , T and their products, first when they are considered at the classical level (grassmannian functions), then at the quantum level (anticommuting operators). We put a special emphasis on the properties of unitarity and linearity of the charge conjugation operator, which is sometimes erroneously considered to be antilinear.

The fourth part deals with the concept of invariance of a given theory, still essentially following Wightman [4]. By taking the simple example of fermionic mass terms (Dirac and Majorana), we exhibit ambiguities and inconsistencies that arise in the transformations of a classical Lagrangian by antiunitary transformations. This motivates, like for neutral kaons [7], the propagator approach, which is the only safe way of deducing unambiguously the constraints cast by symmetry transformations on a Quantum Field Theory².

The fifth and last part of this work is dedicated to the propagator of a single fermion (one flavor) and its antiparticle, from which it cannot be separated, in Quantum Field Theory. We derive in full generality all constraints cast on it by P , C , T , PC , PCT . We show in particular that an observed fermion can be Majorana only if C and CP are both unbroken.

This study is largely unfinished since the case of several flavors of fermions is not investigated here. This necessary extension, which will give access to the essential issue of flavor mixing, in connection with discrete symmetries, is currently under investigation (we recall that results concerning mixing at the quantum level have already been obtained, by less general techniques, in [6] and [7]).

2 Generalities

In this paper we shall note equivalently $\xi^\alpha \xrightarrow{C} -i(\eta^{\dot{\alpha}})^* \equiv (\xi^\alpha)^c = -i(\eta^{\dot{\alpha}})^* \equiv C.\xi^\alpha = -i(\eta^{\dot{\alpha}})^*$, where ξ^α is a Weyl spinor (see Appendix A.1).

The corresponding fermionic field operators will be put into square brackets, for example $[\xi^\alpha]$, $[\xi^\alpha]^U$, the last being the transformed by the transformation U . Formally $[\xi^\alpha]^U = (\xi^\alpha)^U$.

The transition amplitude between two fermionic states is noted $\langle \chi | \psi \rangle$; this defines a scalar product and the corresponding norm $\langle \psi | \psi \rangle$ is real positive. The scalar product satisfies

$$\langle \psi | \chi \rangle^* = \langle \chi | \psi \rangle; \tag{1}$$

¹Both quarks and leptons form coupled systems through the Higgs sector.

²and more generally its Green functions, from which the S-matrix can be in principle reconstructed [4].

we consider furthermore ([5]) that representations of the Poincaré group satisfy ³

$$\langle \psi | \chi \rangle^* = \langle \psi^* | \chi^* \rangle . \quad (2)$$

2.0.1 The symmetry representation theorem of Wigner [3]

A symmetry transformation is defined as a transformation on the states (ray representations) $\Psi \rightarrow \Psi'$ that preserve transition probabilities

$$|\langle \Psi'_1 | \Psi'_2 \rangle|^2 = |\langle \Psi_1 | \Psi_2 \rangle|^2 . \quad (3)$$

The so-called ‘‘symmetry representation theorem’’ states ⁴: *any symmetry transformation can be represented on the Hilbert space of physical states by an operator that is either linear and unitary, or antilinear and antiunitary.*

Since we have to deal with unitary as well as antiunitary operators, it is important to state their general properties and how they operate on fermionic field operators. A unitary operator \mathcal{U} and an antiunitary operator \mathcal{A} satisfy respectively

$$\forall \psi, \chi \quad \langle \mathcal{U}\psi | \mathcal{U}\chi \rangle = \langle \psi | \chi \rangle, \quad \langle \mathcal{A}\psi | \mathcal{A}\chi \rangle = \langle \chi | \psi \rangle = \langle \psi | \chi \rangle^* . \quad (4)$$

Both preserve the probability transition $|\langle \psi | \chi \rangle|^2 = |\langle \mathcal{U}\psi | \mathcal{U}\chi \rangle|^2 = |\langle \mathcal{A}\psi | \mathcal{A}\chi \rangle|^2$.

2.0.2 Antiunitarity and antilinearity

An antilinear operator is an operator that complex conjugates any c-number on its right

$$\mathcal{A} \text{ antilinear} \Leftrightarrow \mathcal{A}(c | \psi \rangle) = c^* \mathcal{A} | \psi \rangle . \quad (5)$$

An antiunitary operator is also antilinear. Let us indeed consider the antiunitary operator \mathcal{A} .

$\langle \mathcal{A}\psi | \mathcal{A} | \lambda\chi \rangle = \langle \mathcal{A}\psi | \mathcal{A}\lambda\chi \rangle = \langle \lambda\chi | \psi \rangle = \lambda^* \langle \chi | \psi \rangle = \lambda^* \langle \mathcal{A}\psi | \mathcal{A} | \chi \rangle$ shows that \mathcal{A} is antilinear.

2.0.3 Unitarity and linearity

In the same way, one shows that: a unitary operator is linear.

2.0.4 Symmetry transformations: Wightman’s point of view

Wightman [4] essentially deals with vacuum expectation values of strings of field operators. The transformed $\hat{\mathcal{O}}$ of an operator \mathcal{O} is defined through the transformation that changes the state ϕ into $\hat{\phi}$

$$\langle \hat{\phi} | \mathcal{O} | \hat{\phi} \rangle = \langle \phi | \hat{\mathcal{O}} | \phi \rangle \quad (6)$$

One has accordingly:

* for a unitary transformation \mathcal{U}

$$\hat{\mathcal{O}} = \mathcal{U}^{-1} \mathcal{O} \mathcal{U}, \quad (7)$$

³For fermionic scalar products, we refer the reader to [8].

⁴We refer the reader to [9] for a careful demonstration of this theorem.

* for a antiunitary transformation \mathcal{A} ^{5 6}

$$\begin{aligned}\hat{\mathcal{O}} &= (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger \\ &= \mathcal{A}^\dagger\mathcal{O}^\dagger(\mathcal{A}^{-1})^\dagger = \mathcal{A}^\dagger\mathcal{O}^\dagger\mathcal{A}.\end{aligned}\quad (9)$$

This is the demonstration.

* For \mathcal{U} unitary ($\mathcal{U}\mathcal{U}^\dagger = 1 = \mathcal{U}^\dagger\mathcal{U}$):

$$\langle \mathcal{U}\psi | \mathcal{O} | \mathcal{U}\chi \rangle = \langle \psi | \mathcal{U}^\dagger\mathcal{O}\mathcal{U} | \chi \rangle = \langle \psi | \mathcal{U}^{-1}\mathcal{O}\mathcal{U} | \chi \rangle, \text{ q.e.d.}$$

* For \mathcal{A} antiunitary:

- first, we demonstrate the important relation

$$\forall(\psi, \chi) \quad \langle \mathcal{A}\psi | \mathcal{A}\mathcal{O}\mathcal{A}^{-1} | \mathcal{A}\chi \rangle = \langle \chi | \mathcal{O}^\dagger | \psi \rangle. \quad (10)$$

Indeed:

$$\langle \mathcal{A}\psi | \mathcal{A}\mathcal{O}\mathcal{A}^{-1} | \mathcal{A}\chi \rangle = \langle \mathcal{A}\psi | \mathcal{A}\mathcal{O} | \chi \rangle = \langle \mathcal{A}\psi | \mathcal{A}(\mathcal{O}\chi) \rangle \stackrel{(4)}{=} \langle \mathcal{O}\chi | \psi \rangle = \langle \chi | \mathcal{O}^\dagger | \psi \rangle;$$

- one has then, in particular ⁷

$$\langle \mathcal{A}\psi | \mathcal{O} | \mathcal{A}\chi \rangle = \langle \mathcal{A}\psi | \mathcal{A}(\mathcal{A}^{-1}\mathcal{O}\mathcal{A})\mathcal{A}^{-1} | \mathcal{A}\chi \rangle = \langle \chi | (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger | \psi \rangle, \quad (12)$$

which yields the desired result for $\psi = \chi$ ⁸.

According to (9), an extra hermitian conjugation occurs in the transformation of an operator by an antiunitary transformation⁹.

2.0.5 General constraints

$$\langle \hat{\phi} | \mathcal{O}^\dagger | \hat{\phi} \rangle \stackrel{(6)}{=} \langle \phi | \widehat{\mathcal{O}}^\dagger | \phi \rangle \text{ evaluates also as } \langle \hat{\phi} | \mathcal{O}^\dagger | \hat{\phi} \rangle = \langle \hat{\phi} | \mathcal{O} | \hat{\phi} \rangle \stackrel{(6)}{=} \langle \phi | \hat{\mathcal{O}} | \phi \rangle^*$$

= $\langle \phi | (\hat{\mathcal{O}})^\dagger | \phi \rangle$, such that, comparing the two expressions one gets

$$\widehat{\mathcal{O}}^\dagger = (\hat{\mathcal{O}})^\dagger, \quad (13)$$

which is a constraint that must be satisfied by any operator \mathcal{O} transformed by unitary as well as antiunitary symmetry transformations. (13) can easily be checked explicitly. $[\psi]$ being the field operator associated with the grassmanian function ψ , one has:

* for a unitary transformation \mathcal{U} :

$$\begin{aligned}\widehat{[\psi]}^\dagger &\stackrel{(7)}{=} \mathcal{U}^{-1} [\psi]^\dagger \mathcal{U}, \\ \widehat{[\psi]}^\dagger &\stackrel{(13)}{=} ((\widehat{[\psi]})^\dagger)^\dagger \stackrel{(7)}{=} (\mathcal{U}^{-1} [\psi] \mathcal{U})^\dagger \mathcal{U} \mathcal{U}^\dagger \stackrel{=1=\mathcal{U}^\dagger\mathcal{U}}{=} \mathcal{U}^{-1} [\psi]^\dagger \mathcal{U};\end{aligned}\quad (14)$$

⁵The last equality in (9) comes from the property, demonstrated by Weinberg [9], that an antiunitary operator must also satisfy the relation $\mathcal{A}\mathcal{A}^\dagger = 1 = \mathcal{A}^\dagger\mathcal{A}$ (see Appendix B). So, in particular, one has $(\mathcal{A}^{-1})^\dagger\mathcal{A}^{-1} = 1 \Rightarrow (\mathcal{A}^{-1})^\dagger = \mathcal{A}$.

⁶Because of (9), for $\mathcal{O} = \mathcal{O}_1\mathcal{O}_2 \dots \mathcal{O}_n$

$$\begin{aligned}[\mathcal{O}_1\mathcal{O}_2 \dots \mathcal{O}_n]^\ominus &= (\mathcal{A}^{-1}\mathcal{O}_1\mathcal{O}_2 \dots \mathcal{O}_n\mathcal{A})^\dagger = (\mathcal{A}^{-1}\mathcal{O}_1\mathcal{A}\mathcal{A}^{-1}\mathcal{O}_2\mathcal{A}\mathcal{A}^{-1} \dots \mathcal{A}\mathcal{A}^{-1}\mathcal{O}_n\mathcal{A})^\dagger \\ &= (\mathcal{A}^{-1}\mathcal{O}_n\mathcal{A})^\dagger \dots (\mathcal{A}^{-1}\mathcal{O}_2\mathcal{A})^\dagger (\mathcal{A}^{-1}\mathcal{O}_1\mathcal{A})^\dagger \\ &= [\mathcal{O}_n]^\ominus \dots [\mathcal{O}_2]^\ominus [\mathcal{O}_1]^\ominus;\end{aligned}\quad (8)$$

antiunitarity implies that the order of operators has to be swapped when calculating the transformed of a string of operators.

⁷When the *in* and *out* states are different, one can write accordingly

$$\langle \mathcal{A}\psi | \mathcal{O} | \mathcal{A}\chi \rangle = \langle \chi | \hat{\mathcal{O}} | \psi \rangle = \langle \chi | (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger | \psi \rangle \quad (11)$$

The *in* and *out* states have to be swapped in the expressions on the r.h.s., ensuring that all terms in (11) are linear in ψ and antilinear in χ .

⁸One cannot use (164) to transform $\langle \chi | (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger | \psi \rangle$ into $\langle \psi | \mathcal{A}^{-1}\mathcal{O}\mathcal{A} | \chi \rangle$ because $\mathcal{A}^{-1}\mathcal{O}\mathcal{A}$ acts linearly and should thus this considered as a unitary operator.

⁹See [4], eq.(1-30).

* for a antiunitary transformation \mathcal{A} :

$$\begin{aligned} \widehat{[\psi]^\dagger} &\stackrel{(9)}{=} (\mathcal{A}^{-1} [\psi]^\dagger \mathcal{A})^\dagger = \mathcal{A}^\dagger [\psi] \mathcal{A}, \\ \widehat{[\psi]^\dagger} &\stackrel{(13)}{=} (\mathcal{A}^\dagger [\psi]^\dagger \mathcal{A})^\dagger = \mathcal{A}^\dagger [\psi] \mathcal{A}. \end{aligned} \quad (15)$$

Since $[\psi]$ and $[\psi]^\dagger$ are respectively associated with the grassmanian functions ψ and ψ^* , (13) also casts constraints on the transformation of grassmanian functions:

$$\widehat{\psi^*} = (\widehat{\psi})^*. \quad (16)$$

3 Discrete symmetries

3.1 Parity transformations

We adopt the convention $P^2 = -1$ [10]. Then the transformation of spinors are

$$\begin{aligned} \xi^\alpha(\vec{x}, t) &\xrightarrow{P} i\eta_{\dot{\alpha}}(-\vec{x}, t) \quad , \quad \eta_{\dot{\alpha}}(\vec{x}, t) \xrightarrow{P} i\xi^\alpha(-\vec{x}, t), \\ \xi_\alpha(\vec{x}, t) &\xrightarrow{P} -i\eta^{\dot{\alpha}}(-\vec{x}, t) \quad , \quad \eta^{\dot{\alpha}}(\vec{x}, t) \xrightarrow{P} -i\xi_\alpha(-\vec{x}, t). \end{aligned} \quad (17)$$

The parity transformed of the complex conjugates are defined [10] as the complex conjugates of the parity transformed

$$P.(\xi^\alpha)^* = (P.\xi^\alpha)^*; \quad (18)$$

this ensures in particular that the constraints (13) and (16) are satisfied. It yields

$$\begin{aligned} (\xi^\alpha)^*(\vec{x}, t) &\xrightarrow{P} -i(\eta_{\dot{\alpha}})^*(-\vec{x}, t) \quad , \quad (\eta_{\dot{\alpha}})^*(\vec{x}, t) \xrightarrow{P} -i(\xi^\alpha)^*(-\vec{x}, t), \\ (\xi_\alpha)^*(\vec{x}, t) &\xrightarrow{P} i(\eta^{\dot{\alpha}})^*(-\vec{x}, t) \quad , \quad (\eta^{\dot{\alpha}})^*(\vec{x}, t) \xrightarrow{P} i(\xi_\alpha)^*(-\vec{x}, t). \end{aligned} \quad (19)$$

For Dirac bi-spinors (see Appendix A), one gets

$$P.\psi_D = U_P \psi_D, \quad U_P = i\gamma^0, \quad U_P^\dagger = -U_P = U_P^{-1}, \quad U_P^2 = -1, \quad U_P^\dagger U_P = 1. \quad (20)$$

3.1.1 Parity transformation on fermionic field operators

Going to field operators, one uses (7), for unitary operators

$$[\xi^\alpha]^P = P^{-1}[\xi^\alpha]P \quad (21)$$

to get

$$\begin{aligned} P^{-1}\xi^\alpha(\vec{x}, t)P &= i\eta_{\dot{\alpha}}(-\vec{x}, t) \quad , \quad P^{-1}\eta_{\dot{\alpha}}(\vec{x}, t)P = i\xi^\alpha(-\vec{x}, t), \\ P^{-1}\xi_\alpha(\vec{x}, t)P &= -i\eta^{\dot{\alpha}}(-\vec{x}, t) \quad , \quad P^{-1}\eta^{\dot{\alpha}}(\vec{x}, t)P = -i\xi_\alpha(-\vec{x}, t), \\ P^{-1}(\xi^\alpha)^\dagger(\vec{x}, t)P &= -i(\eta_{\dot{\alpha}})^\dagger(-\vec{x}, t) \quad , \quad P^{-1}(\eta_{\dot{\alpha}})^\dagger(\vec{x}, t)P = -i(\xi^\alpha)^\dagger(-\vec{x}, t), \\ P^{-1}(\xi_\alpha)^\dagger(\vec{x}, t)P &= i(\eta^{\dot{\alpha}})^\dagger(-\vec{x}, t) \quad , \quad P^{-1}(\eta^{\dot{\alpha}})^\dagger(\vec{x}, t)P = i(\xi_\alpha)^\dagger(-\vec{x}, t), \end{aligned} \quad (22)$$

which satisfies the constraint (13). The following constraint then arises

$$(P^{-1})^2 \xi^\alpha P^2 = -\xi^\alpha. \quad (23)$$

Indeed: $(P^{-1})^2 \xi^\alpha P^2 = P^{-1}(P^{-1}\xi^\alpha P)P \stackrel{(22)}{=} P^{-1}i\eta_{\dot{\alpha}}P \stackrel{\text{linear}}{=} iP^{-1}\eta_{\dot{\alpha}}P \stackrel{(22)}{=} -\xi^\alpha$.

Taking the hermitian conjugate of the first equation of the first line in (22) and comparing it with the first equation of the third line, it is also immediate to check that $(PP^\dagger)\mathcal{O}(PP^\dagger)^{-1} = \mathcal{O}$, $\mathcal{O} = \xi^\alpha \dots$, which is correct for P unitary or antiunitary.

3.2 Charge Conjugation as a linear (unitary) operator

C is the operation which transforms a particle into its antiparticle, and *vice versa*, without changing its spin and momentum (see for example [11] p.17); it satisfies $C^2 = 1$ [10]

A Dirac fermion and its charge conjugate transform alike [10] and satisfy the same equation; the charge conjugate satisfies

$$C.\psi_D = V_C \overline{\psi_D}^T, \quad (24)$$

where V_C is a unitary operator

$$V_C = \gamma^2 \gamma^0, \quad (V_C)^\dagger V_C = 1 = (V_C)^2; \quad (25)$$

this action on Dirac fermions is generally taken as the definition of C . Equivalently

$$C.\psi_D = U_C \psi_D^*, \quad U_C = V_C \gamma^0 = \gamma^2, \quad U_C^\dagger U_C = 1 = -(U_C)^2. \quad (26)$$

Naively considering (24) (as often done) entails

$$C.(\lambda \psi_D) = \lambda^* C.(\psi_D), \quad (27)$$

which leads to consider that C acts antilinearly on ψ_D . We show below, after eq. (31), that this is a mistake and that C should act linearly, otherwise PCT becomes linear and unitary, which is wrong.

In terms of Weyl fermions (see Appendix A), one has

$$\psi_D \equiv \begin{pmatrix} \xi^\alpha \\ \eta_\beta \end{pmatrix} \xrightarrow{C} -i \begin{pmatrix} \eta^{\dot{\alpha}*} \\ \xi_\beta^* \end{pmatrix} = -i \begin{pmatrix} g^{\dot{\alpha}\beta} \eta_\beta^* \\ g_{\alpha\beta} \xi^{\beta*} \end{pmatrix} = \begin{pmatrix} -\sigma_{\dot{\alpha}\beta}^2 \eta_\beta^* \\ \sigma_{\alpha\beta}^2 \xi^{\beta*} \end{pmatrix} = \gamma^2 \begin{pmatrix} \xi^\alpha \\ \eta_\beta \end{pmatrix}^* = \gamma^2 \psi_D^*, \quad (28)$$

and, so

$$\begin{aligned} \xi^\alpha &\xrightarrow{C} -i \eta^{\dot{\alpha}*} & \eta_{\dot{\alpha}} &\xrightarrow{C} -i \xi_\alpha^*, \\ \xi_\alpha &\xrightarrow{C} -i \eta_{\dot{\alpha}}^* & \eta^{\dot{\alpha}} &\xrightarrow{C} -i \xi^{\alpha*}. \end{aligned} \quad (29)$$

The transformation of complex conjugates fields results from the constraint (16), which imposes

$$\begin{aligned} (\xi^\alpha)^* &\xrightarrow{C} i \eta^{\dot{\alpha}} & (\eta_{\dot{\alpha}})^* &\xrightarrow{C} i \xi_\alpha, \\ (\xi_\alpha)^* &\xrightarrow{C} i \eta_{\dot{\alpha}} & (\eta^{\dot{\alpha}})^* &\xrightarrow{C} i \xi^{\alpha}. \end{aligned} \quad (30)$$

It is now easy to show that (recall that $U_C^2 = -1$ from (26))

$$C^2 = 1, C \text{ unitary and linear.} \quad (31)$$

One gets then: $C.C.\xi^\alpha = C.(-i(\eta^{\dot{\alpha}})^*) \stackrel{\text{linear}}{=} (-i)C.(\eta^{\dot{\alpha}})^* \stackrel{(30)}{=} \xi^\alpha$, which entails, as needed, $C^2 = 1$.

If (16) is satisfied (that is, accordingly, if (30) is true together with (29)), but if we take C antilinear (thus antiunitary), by operating a second time with C on the l.h.s. of (29) or (30), one finds that it can only satisfy $C^2 = -1$ instead of $C^2 = 1$. The commutation and anticommutation relations with other symmetry transformations P and T are also changed¹⁰, which swaps in particular the sign of $(PCT)^2$. Furthermore, since T is antilinear and P is linear, this makes PCT linear, thus unitary, which is wrong. (16) is thus only compatible with unitarity and linearity for C .

If (16) is not satisfied, that is, if the signs of (30) are swapped, one can keep $C^2 = 1$ at the price of taking C antilinear. Then it can only be non-unitary, which is in conflict with all assertions. Also, unless we abandon the natural correspondence $\psi \leftrightarrow [\psi], \psi^* \leftrightarrow [\psi]^\dagger$ between fields and operators, (13) cannot be

¹⁰With our conventions, we have $CP = PC, (PC)^2 = -1$, and $(PCT)^2 = 1$.

satisfied either, which creates a problem with Wightman's definition (6) of the transformed of an operator by a symmetry transformation (in the sense of Wigner).

So, despite C complex conjugates a Dirac spinor, it should act linearly $C.\lambda\psi = \lambda C.\psi$. (24) and its consequence (27) should not be considered as the basic equations defining C transformation; they should be supplemented by the condition of unitarity (hence linearity). This brings no trouble with the property that a fermion and its charge conjugate transform alike and satisfy the same equation [10]. Indeed, if ψ_D is a Dirac fermion, $\lambda\psi_D$ satisfies the same Dirac equation since the latter is linear in ψ ; if ψ^C transforms alike by Lorentz and satisfies the same Dirac equation, too, both $\lambda\psi^C$ and $\lambda^*\psi^C$ also do. Linearity or antilinearity is not fixed by the two conditions "transforming alike by Lorentz" and "satisfying the same equation", such that this property must be determined by other criteria ¹¹.

See also appendix D, where a careful analysis is done of the pitfalls that accompany the use of γ matrices in the expression of the discrete transformations P , C and T .

3.2.1 Charge conjugation on field operators

The transition from (29) and (30) for grassmanian functions to the transformations for field operators is done according to (7) for *unitary* operators, through the correspondence $\mathcal{U}\psi \leftrightarrow \mathcal{U}^{-1}[\psi]\mathcal{U}$. One gets

$$\begin{aligned} C^{-1}\xi^\alpha C &= -i(\eta^{\dot{\alpha}})^\dagger & , & & C^{-1}\eta_{\dot{\alpha}} C &= -i(\xi_\alpha)^\dagger, \\ C^{-1}\xi_\alpha C &= -i(\eta_{\dot{\alpha}})^\dagger & , & & C^{-1}\eta^{\dot{\alpha}} C &= -i(\xi^\alpha)^\dagger, \\ C^{-1}(\xi^\alpha)^\dagger C &= i(\eta^{\dot{\alpha}}) & , & & C^{-1}(\eta_{\dot{\alpha}})^\dagger C &= i(\xi_\alpha), \\ C^{-1}(\xi_\alpha)^\dagger C &= i(\eta_{\dot{\alpha}}) & , & & C^{-1}((\eta^{\dot{\alpha}})^\dagger)^\dagger C &= i(\xi^\alpha). \end{aligned} \quad (32)$$

Hermitian conjugating the first equation of the first line of (32) immediately shows its compatibility with the first equation of the third line: $C^\dagger(\xi^\alpha)^\dagger(C^{-1})^\dagger = i\eta^{\dot{\alpha}} = C^{-1}(\xi^\alpha)^\dagger C \Rightarrow (\xi^\alpha)^\dagger = CC^\dagger(\xi^\alpha)^\dagger(C^{-1})^\dagger C^{-1}$, which entails $CC^\dagger = \pm 1$ which is correct for C unitary (or antiunitary). We would find an inconsistency if the sign of the last four equations was swapped.

Since C is linear, one immediately gets

$$(C^{-1})^2 \mathcal{O} C^2 = C^{-1}(C^{-1} \mathcal{O} C)C = \mathcal{O}, \mathcal{O} = \xi^\alpha \dots \quad (33)$$

3.3 PC transformation

Combining (17), (29) and (30), and using, when needed, the linearity of C , one gets

$$\begin{aligned} \xi^\alpha(\vec{x}, t) &\xrightarrow{PC} \xi_\alpha^*(-\vec{x}, t) & , & & \xi_\alpha(\vec{x}, t) &\xrightarrow{PC} -\xi^{\alpha*}(-\vec{x}, t), \\ \eta_{\dot{\alpha}}(\vec{x}, t) &\xrightarrow{PC} \eta^{\dot{\alpha}*}(-\vec{x}, t) & , & & \eta^{\dot{\alpha}}(-\vec{x}, t) &\xrightarrow{PC} -\eta_{\dot{\alpha}}^*(-\vec{x}, t), \end{aligned} \quad (34)$$

and

$$\begin{aligned} (\xi^\alpha)^*(\vec{x}, t) &\xrightarrow{PC} \xi_\alpha(-\vec{x}, t) & , & & (\xi_\alpha)^*(\vec{x}, t) &\xrightarrow{PC} -\xi^\alpha(-\vec{x}, t), \\ (\eta_{\dot{\alpha}})^*(\vec{x}, t) &\xrightarrow{PC} \eta^{\dot{\alpha}}(-\vec{x}, t) & , & & (\eta^{\dot{\alpha}})^*(\vec{x}, t) &\xrightarrow{PC} -\eta_{\dot{\alpha}}(-\vec{x}, t). \end{aligned} \quad (35)$$

One easily checks that $(PC)^2 = -1$.

Like for charge conjugation, one has

$$PC.(\xi^\alpha)^* = (PC.\xi^\alpha)^*. \quad (36)$$

¹¹In [12] (p. 85), C is considered to be antilinear. In [13] it is instead stated (see p.70) that C is unitary and linear even though it takes ψ into ψ^* . We agree with [13].

For a Dirac fermion, one has

$$\begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\beta}} \end{pmatrix} \xrightarrow{PC} \begin{pmatrix} \xi_\alpha^* \\ \eta^{\dot{\beta}*} \end{pmatrix} = \begin{pmatrix} g_{\alpha\beta} \xi^{\beta*} \\ g^{\dot{\beta}\dot{\gamma}} \eta_{\dot{\gamma}}^* \end{pmatrix} = \begin{pmatrix} (i\sigma^2)_{\alpha\beta} \xi^{\beta*} \\ (-i\sigma^2)_{\dot{\beta}\dot{\gamma}} \eta_{\dot{\gamma}}^* \end{pmatrix} = i \begin{pmatrix} (\eta_{\dot{\alpha}})^c \\ (\xi^\beta)^c \end{pmatrix} = i\gamma^0 \gamma^2 \begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\beta}} \end{pmatrix}^*, \quad (37)$$

equivalently

$$PC.\psi_D = V_{PC} \bar{\psi}^T = U_P V_C \bar{\psi}^T = U_{PC} \psi^* = U_P U_C \psi^*. \quad (38)$$

Majorana fermions (see subsection 3.6) $\psi_M^\pm = \begin{pmatrix} \xi^\alpha \\ \pm(-i)\xi_\beta^* \end{pmatrix}$ and $\chi_M^\pm = \begin{pmatrix} \pm(-i)(\eta^{\dot{\beta}})^* \\ \eta_{\dot{\beta}} \end{pmatrix}$ have *PC parity* $\pm i$ ¹².

3.4 Time reversal

The time reversed $\langle \chi(t') | \psi(t) \rangle T$ of a transition matrix element $\langle \chi(t') | \psi(t) \rangle$, $t < t'$ is defined by $\langle \chi(t) | \psi(t') \rangle^* = \langle \psi(t') | \chi(t) \rangle$, $t > t'$; the complex conjugation is made necessary by $t < t'$ and the fact that *in* states must occur at a time smaller than *out* states; the arrow of time is not modified when one defines the time-reversed of a transition matrix element.

The operator T is accordingly antiunitary, hence antilinear:

$$\langle TA | TB \rangle = \langle B | A \rangle \Rightarrow T \text{ antiunitary}, \quad (39)$$

In Quantum Mechanics, time reversal must change grassmanian functions into their complex conjugate (see for example the argumentation concerning Schrödinger's equation in [11]). According to [10], the *grassmanian functions* transform by time inversion according to

$$\begin{aligned} \psi_D(\vec{x}, t) &\xrightarrow{T} T.\psi_D(\vec{x}, t) = V_T \overline{\psi_D(\vec{x}, -t)}^T; \\ V_T &= i\gamma^3 \gamma^1 \gamma^0, \quad V_T^\dagger V_T = 1 = V_T^2, \quad V_T^\dagger = V_T = V_T^{-1}, \end{aligned} \quad (40)$$

which shows that T is antilinear when it acts on grassmanian functions. So doing, $T.\psi_D$ and ψ_D satisfy time reversed equations. One also defines

$$U_T = V_T \gamma^0 = i\gamma^3 \gamma^1 = -U_T^*, U_T^\dagger = U_T = U_T^{-1}, U_T^\dagger U_T = U_T^2 = 1. \quad (41)$$

$$T.\psi_D = U_T \psi_D^* = i\gamma^3 \gamma^1 \psi_D^*. \quad (42)$$

This yields for Weyl fermions

$$\begin{aligned} \xi^\alpha(\vec{x}, t) &\xrightarrow{T} -i\xi_\alpha^*(\vec{x}, -t), & \xi_\alpha(\vec{x}, t) &\xrightarrow{T} i\xi^{\alpha*}(\vec{x}, -t), \\ \eta_{\dot{\alpha}}(\vec{x}, t) &\xrightarrow{T} i\eta^{\dot{\alpha}*}(\vec{x}, -t), & \eta^{\dot{\alpha}}(\vec{x}, t) &\xrightarrow{T} -i\eta_{\dot{\alpha}}^*(\vec{x}, -t). \end{aligned} \quad (43)$$

The constraint (16) then entails

$$\begin{aligned} (\xi^\alpha)^*(\vec{x}, t) &\xrightarrow{T} i\xi_\alpha(\vec{x}, -t), & (\xi_\alpha)^*(\vec{x}, t) &\xrightarrow{T} -i\xi^\alpha(\vec{x}, -t), \\ (\eta_{\dot{\alpha}})^*(\vec{x}, t) &\xrightarrow{T} -i\eta^{\dot{\alpha}}(\vec{x}, -t), & (\eta^{\dot{\alpha}})^*(\vec{x}, t) &\xrightarrow{T} i\eta_{\dot{\alpha}}(\vec{x}, -t). \end{aligned} \quad (44)$$

One has

$$T^2 = 1, \quad CT = -TC, \quad PT = TP \quad (45)$$

¹²For example, $PC. \begin{pmatrix} \xi^\alpha \\ (\eta_{\dot{\beta}})^c \end{pmatrix} = \begin{pmatrix} \xi_\alpha^* \\ i\xi^\beta \end{pmatrix} = i\gamma^0 \begin{pmatrix} \xi^\alpha \\ (\eta_{\dot{\beta}})^c \end{pmatrix}$.

3.4.1 Time reversal on fermionic field operators

The transition to field operators is done according to (9) for antiunitary transformations, through the correspondence $(\mathcal{A}\psi)^\dagger \leftrightarrow \mathcal{A}^{-1}[\psi]\mathcal{A}$, which involves an extra hermitian conjugation with respect to the transformations of grassmanian functions ([4], eq.(1-30)):

$$\begin{aligned} T^{-1}\xi^\alpha(\vec{x}, t)T &= i\xi_\alpha(\vec{x}, -t) & , & & T^{-1}\eta_{\dot{\alpha}}(\vec{x}, t)T &= -i\eta^{\dot{\alpha}}(\vec{x}, -t), \\ T^{-1}\xi_\alpha(\vec{x}, t)T &= -i\xi^\alpha(\vec{x}, -t) & , & & T^{-1}\eta^{\dot{\alpha}}(\vec{x}, t)T &= i\eta_{\dot{\alpha}}(\vec{x}, -t), \\ T^{-1}(\xi^\alpha)^\dagger(\vec{x}, t)T &= -i(\xi_\alpha)^\dagger(\vec{x}, -t) & , & & T^{-1}(\xi_\alpha)^\dagger(\vec{x}, t)T &= i(\xi^\alpha)^\dagger(\vec{x}, -t), \\ T^{-1}(\eta_{\dot{\alpha}})^\dagger(\vec{x}, t)T &= i(\eta^{\dot{\alpha}})^\dagger(\vec{x}, -t) & , & & T^{-1}(\eta^{\dot{\alpha}})^\dagger(\vec{x}, t)T &= -i(\eta_{\dot{\alpha}})^\dagger(\vec{x}, -t). \end{aligned} \quad (46)$$

Since T is antilinear, one finds immediately that, though $T^2 = 1$, one must have

$$(T^{-1})^2 \mathcal{O} T^2 = T^{-1}(T^{-1} \mathcal{O} T)T = -\mathcal{O}, \quad \mathcal{O} = \xi^\alpha \dots \quad (47)$$

3.5 PCT transformation

Combining the previous results, using the linearity of P and C , one gets for the grassmanian functions ¹³

$$\begin{aligned} \xi^\alpha(x) &\xrightarrow{PCT} i\xi^\alpha(-x) & , & & \eta_{\dot{\alpha}}(x) &\xrightarrow{PCT} -i\eta_{\dot{\alpha}}(-x), \\ \xi_\alpha(x) &\xrightarrow{PCT} i\xi_\alpha(-x) & , & & \eta^{\dot{\alpha}}(x) &\xrightarrow{PCT} -i\eta^{\dot{\alpha}}(-x), \\ \psi_D(x) &\xrightarrow{PCT} i\gamma^5\psi_D(-x), \end{aligned} \quad (48)$$

where the overall sign depends on the order in which the operators act; here they are supposed to act in the order: first T , then C and last P . When acting on bispinors, one has $CT = -TC$ and $PT = TP$ ¹⁴. So, using also $CP = PC$, one gets $(PCT)(PCT) = (PCT)(P(-)TC) = (PCT)(-TPC)$. $T^2 = 1$, $C^2 = 1$, $P^2 = -1$ (our choice) and $PC = CP$ entail

$$(PCT)^2 = 1. \quad (49)$$

Note that, both C and T introducing complex conjugation, the latter finally disappears and PCT introduces no complex conjugation for the grassmanian functions. This is why one has

$$PCT.\psi_D(x) = U_\Theta\psi_D(-x), \quad (50)$$

$$U_\Theta = U_P U_C U_T = -\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^5, \quad U_\Theta U_\Theta^\dagger = 1 = -U_\Theta^2, \quad U_\Theta^\dagger = -U_\Theta. \quad (51)$$

For the complex conjugate fields, the constraint (16) gives

$$\begin{aligned} (\xi^\alpha)^*(x) &\xrightarrow{PCT} -i(\xi^\alpha)^*(-x) & , & & (\eta_{\dot{\alpha}})^*(x) &\xrightarrow{PCT} i(\eta_{\dot{\alpha}})^*(-x), \\ (\xi_\alpha)^*(x) &\xrightarrow{PCT} -i(\xi_\alpha)^*(-x) & , & & (\eta^{\dot{\alpha}})^*(x) &\xrightarrow{PCT} i(\eta^{\dot{\alpha}})^*(-x), \\ \psi_D^* &\xrightarrow{PCT} -i\gamma^5\psi_D^*, \end{aligned} \quad (52)$$

such that (this only occurs for P and PCT)

$$PCT.(\xi^\alpha)^* = (PCT.\xi^\alpha)^* \Leftrightarrow U_\Theta(\xi^\alpha)^* \equiv ((\xi^\alpha)^*)^\Theta = (U_\Theta\xi^\alpha)^* \equiv ((\xi^\alpha)^\Theta)^*. \quad (53)$$

Since P and C are unitary and T antiunitary, PCT is antiunitary, thus *antilinear*. So, despite no complex conjugation is involved $\Theta.\lambda\xi^\alpha = \lambda^*\Theta.\xi^\alpha$ ¹⁵.

¹³Examples:

$PCT.\xi^\alpha = PC.(T.\xi^\alpha) = PC.(-i\xi_\alpha^*) = P.(-i)C.\xi_\alpha^* = (-i)P.C.\xi_\alpha^* = (-i)P.i\eta_{\dot{\alpha}} = P.\eta_{\dot{\alpha}} = i\xi^\alpha$;
 $PCT.(\xi^\alpha)^* = PC.(T.(\xi^\alpha)^*) = PC.(i\xi_\alpha) = P.iC.\xi_\alpha = iP.(-i)(\eta_{\dot{\alpha}})^* = P.(\eta_{\dot{\alpha}})^* = -i(\xi^\alpha)^*$.

¹⁴We disagree with [10] who states that T and P anticommute.

¹⁵This is to be put in correspondence with C , which is *linear* despite complex conjugation is involved.

3.5.1 PCT operation on fermionic field operators

Since Θ is antiunitary, one has, according to (9)

$$\begin{aligned}\Theta^{-1}\xi^\alpha(x)\Theta &= -i(\xi^\alpha)^\dagger(-x) & , & & \Theta^{-1}\xi_\alpha(x)\Theta &= -i(\xi_\alpha)^\dagger(-x), \\ \Theta^{-1}\eta_{\dot{\alpha}}(x)\Theta &= i(\eta_{\dot{\alpha}})^\dagger(-x) & , & & \Theta^{-1}\eta^{\dot{\alpha}}(x)\Theta &= i(\eta^{\dot{\alpha}})^\dagger(-x), \\ \Theta^{-1}(\xi^\alpha)^\dagger(x)\Theta &= i\xi^\alpha(-x) & , & & \Theta^{-1}(\xi_\alpha)^\dagger(x)\Theta &= i\xi_\alpha(-x), \\ \Theta^{-1}(\eta_{\dot{\alpha}})^\dagger(x)\Theta &= -i\eta_{\dot{\alpha}}(-x) & , & & \Theta^{-1}(\eta^{\dot{\alpha}})^\dagger(x)\Theta &= -i\eta^{\dot{\alpha}}(-x).\end{aligned}\quad (54)$$

and, using the antilinearity of Θ , one gets

$$(\Theta^{-1})^2 \mathcal{O} \Theta^2 = \Theta(\Theta^{-1} \mathcal{O} \Theta)\Theta = -\mathcal{O}, \mathcal{O} = \xi^\alpha \dots \quad (55)$$

3.6 Majorana fermions

A Majorana fermion is a bi-spinor which is a C eigenstate (it is a special kind of Dirac fermion with half as many degrees of freedom); since $C^2 = 1$, the only two possible eigenvalues are $C = +1$ and $C = -1$; thus, a Majorana fermions must satisfy (see (28)) one of the two possible Majorana conditions

$$* -i\eta^{\dot{\alpha}*} = \pm\xi^\alpha \Leftrightarrow \eta^{\dot{\alpha}} = \pm(-i)\xi^{\alpha*} \Leftrightarrow \eta_{\dot{\beta}} = \pm(-i)\xi_{\beta}^*;$$

$$* -i\xi_{\beta}^* = \pm\eta_{\dot{\beta}}, \text{ which is the same condition as above;}$$

so,

$$\psi_M^\pm = \begin{pmatrix} \xi^\alpha \\ \pm(-i)\xi_{\beta}^* \end{pmatrix} = \begin{pmatrix} \xi^\alpha \\ \pm(-i)g_{\alpha\beta}\xi^{\beta*} \end{pmatrix} = \begin{pmatrix} \xi^\alpha \\ \pm\sigma_{\alpha\beta}^2\xi^{\beta*} \end{pmatrix}; \quad (56)$$

the + sign in the lower spinor corresponds to $C = +1$ and the - sign to $C = -1$ ¹⁶.

The Majorana conditions linking ξ and η are

$$\xi^\alpha \stackrel{C=\pm 1}{=} \pm(-i)(\eta^{\dot{\alpha}})^* \Leftrightarrow \eta_{\dot{\beta}} \stackrel{C=\pm 1}{=} \pm(-i)(\xi_{\beta})^*; \quad (57)$$

using formulæ(29,30) for the charge conjugates of Weyl fermions, they also write

$$\xi^\alpha \stackrel{C=\pm 1}{=} \pm(\xi^\alpha)^c, \quad \eta_{\dot{\beta}} \stackrel{C=\pm 1}{=} \pm(\eta_{\dot{\beta}})^c. \quad (58)$$

A Majorana bi-spinor can accordingly also be written¹⁷

$$\chi_M^\pm = \begin{pmatrix} \pm(-i)(\eta^{\dot{\beta}})^* \\ \eta_{\dot{\beta}} \end{pmatrix}, \quad (60)$$

¹⁶Remark: Arguing that $(-i)(\xi_{\beta})^*$ transforms like a right fermion, we can call $\omega_{\dot{\beta}} = (-i)(\xi_{\beta})^*$, and the Majorana fermion ψ_M^+ rewrites $\psi_M^+ = \begin{pmatrix} \xi^\alpha \\ \omega_{\dot{\beta}} \end{pmatrix}$. If we then calculate its charge conjugate according to the standard rules (29), one gets $\psi_M^+ \xrightarrow{C} \begin{pmatrix} -i(\omega^{\dot{\alpha}})^\dagger \\ -i(\xi_\alpha)^* \end{pmatrix} \equiv \begin{pmatrix} \xi^\alpha \\ -i(\xi_\alpha)^* \end{pmatrix}$, which shows that it is indeed a $C = +1$ eigenstate. The argumentation becomes trivial if one uses for Majorana fermions the same formula for charge conjugation as the one at the extreme right of (28) for Dirac fermions $(\psi_M)^c = \gamma^2(\psi_M)^*$, $(\chi_M)^c = \gamma^2(\chi_M)^*$.

¹⁷The Majorana spinors ψ_M^\pm and χ_M^\pm can also be written

$$\psi_M^\pm = \begin{pmatrix} \xi^\alpha \\ \pm(-i)(\xi^\alpha)^{CP} \end{pmatrix}, \quad \chi_M^\pm = \begin{pmatrix} \pm(-i)(\eta_{\dot{\beta}})^{CP} \\ \eta_{\dot{\beta}} \end{pmatrix}; \quad (59)$$

they involve one Weyl spinor and its CP conjugate (see subsection 3.3).

which is identical to ψ_M^\pm by the relations (57). By charge conjugation, using (29), $\psi_M^+ \xleftrightarrow{C} \chi_M^+, \psi_M^- \xleftrightarrow{C} -\chi_M^-$.

A so-called Majorana mass term writes

$$\begin{aligned} \overline{\psi}_M \psi_M &\equiv \psi_M^\dagger \gamma^0 \psi_M \equiv \pm i [-(\xi^\alpha)^* (\xi_\alpha)^* + \xi_\alpha \xi^\alpha] = \pm i [(\xi_\alpha)^* (\xi^\alpha)^* + \xi_\alpha \xi^\alpha] \\ \text{or } \overline{\psi}_M \gamma^5 \psi_M &\equiv \psi_M^\dagger \gamma^0 \gamma^5 \psi_M \equiv \mp i [(\xi^\alpha)^* (\xi_\alpha)^* + \xi_\alpha \xi^\alpha] = \mp i [(-\xi_\alpha)^* (\xi^\alpha)^* + \xi_\alpha \xi^\alpha]. \end{aligned} \quad (61)$$

Along the same lines, Majorana kinetic terms write $\overline{\psi}_M \gamma^\mu \overleftrightarrow{p}_\mu \psi_M$ or $\overline{\psi}_M \gamma^\mu \gamma^5 \overleftrightarrow{p}_\mu \psi_M$; they rewrite in terms of Weyl spinors (using (162))

$$\begin{aligned} \overline{\psi}_M \gamma^\mu \overleftrightarrow{p}_\mu \psi_M &= \psi_M^\dagger \begin{pmatrix} \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} & 0 \\ 0 & \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} \end{pmatrix} \psi_M \\ &= (\xi^\alpha)^* \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} \xi^\beta + (\pm (-i) (\xi_\alpha)^*)^* \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} (\pm (-i) \xi_\beta^*) \\ &= (\xi^\alpha)^* \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} \xi^\beta + \xi_\alpha \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} \xi_\beta^*, \end{aligned} \quad (62)$$

and

$$\begin{aligned} \overline{\psi}_M \gamma^\mu \gamma^5 \overleftrightarrow{p}_\mu \psi_M &= \psi_M^\dagger \begin{pmatrix} \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} & 0 \\ 0 & \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} \end{pmatrix} \gamma^5 \psi_M \\ &= (\xi^\alpha)^* \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} \xi^\beta - (\pm (-i) (\xi_\alpha)^*)^* \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} (\pm (-i) \xi_\beta^*) \\ &= (\xi^\alpha)^* \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} \xi^\beta - \xi_\alpha \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} \xi_\beta^*. \end{aligned} \quad (63)$$

A Dirac fermion can always be written as the sum of two Majorana's (the first has $C = +1$ and the second $C = -1$):
$$\begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\beta}} \end{pmatrix} = \frac{1}{2} \left[\begin{pmatrix} \xi^\alpha - i(\eta^{\dot{\alpha}})^* \\ -i\xi_\beta^* + \eta_{\dot{\beta}} \end{pmatrix} + \begin{pmatrix} \xi^\alpha + i(\eta^{\dot{\alpha}})^* \\ i\xi_\beta^* + \eta_{\dot{\beta}} \end{pmatrix} \right].$$

While a Dirac fermion \pm its charge conjugate is always a Majorana fermion ($C = \pm 1$), any Majorana fermion (*i.e.* a general bi-spinor which is a C eigenstate) cannot be uniquely written as the sum of a given Dirac fermion \pm its charge conjugate; suppose indeed that a $C = +1$ Majorana fermion is written

like the sum of a Dirac fermion + its charge conjugate
$$\begin{pmatrix} \theta^\alpha \\ -i\theta_\beta^* \end{pmatrix} = \begin{pmatrix} \xi^\alpha - i(\eta^{\dot{\alpha}})^* \\ \eta_{\dot{\beta}} - i\xi_\beta^* \end{pmatrix};$$
 since the

two corresponding equations are not independent, ξ and η cannot be fixed, but only the combination $\xi^\alpha - i(\eta^{\dot{\alpha}})^* \sim \xi^\alpha - i\eta^{\dot{\alpha}}$; so, while a Majorana fermion can indeed always be written as the sum of a Dirac fermion + its charge conjugate, this decomposition is not unique; infinitely many different Dirac fermions can be used for this purpose.

A Majorana fermion can always be written as the sum of a left fermion \pm its charge conjugate, or the sum of a right fermion \pm its charge conjugate. Let us demonstrate the first case only, since the second goes exactly along the same lines.

$$\begin{aligned} \psi_M^\pm &= \begin{pmatrix} \xi^\alpha \\ \pm(-i)\xi_\beta^* \end{pmatrix} = \begin{pmatrix} \xi^\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \pm(-i)\xi_\beta^* \end{pmatrix} = \psi_L \pm \gamma^2 \psi_L^* = \psi_L \pm (\psi_L)^c, \\ \psi_L &= \begin{pmatrix} \xi^\alpha \\ 0 \end{pmatrix} = \frac{1 + \gamma^5}{2} \psi_D. \end{aligned} \quad (64)$$

We recall that Majorana fermions have CP parity $= \pm i$ (see subsection 3.3); they are *not* CP eigenstates (a γ^0 matrix comes into play in the definition of CP parity).

4 Invariance

4.1 Wightman's point of view [4]

The invariance of a “theory” is expressed by the invariance of the vacuum and the invariance of all n -point functions; \mathcal{O} is then a product of fields at different space-time points and ($\hat{\mathcal{O}}$ being the transformed of \mathcal{O})

$$|0\rangle = |\hat{0}\rangle, \langle 0|\mathcal{O}|0\rangle = \langle 0|\hat{\mathcal{O}}|0\rangle. \quad (65)$$

* in the case of a unitary transformation \mathcal{U} ,

$$\langle 0|\mathcal{O}|0\rangle \stackrel{sym}{=} \langle 0|\mathcal{O}^U|0\rangle \stackrel{vacuum\ inv}{=} \langle 0^U|\mathcal{O}^U|0^U\rangle, \mathcal{O}^U = \mathcal{U}^{-1}\mathcal{O}\mathcal{U}; \quad (66)$$

taking the example of parity and if $\mathcal{O} = \phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)$, one has

$\mathcal{O}^P = P^{-1}\mathcal{O}P = \phi_1(t_1, -\vec{x}_1)\phi_2(t_2, -\vec{x}_2)\dots\phi_n(t_n, -\vec{x}_n)$, such that parity invariance writes

$$\langle 0|\phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)|0\rangle = \langle 0|\phi_1(t_1, -\vec{x}_1)\phi_2(t_2, -\vec{x}_2)\dots\phi_n(t_n, -\vec{x}_n)|0\rangle. \quad (67)$$

* in the case of a antiunitary transformation \mathcal{A} ,

$$\begin{aligned} \langle 0|\mathcal{O}|0\rangle &\stackrel{sym}{=} \langle 0|\mathcal{O}^A|0\rangle = \langle 0^A|\mathcal{O}^A|0^A\rangle; \\ \mathcal{O}^A &= (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger \Rightarrow \\ \langle 0|\mathcal{O}|0\rangle &\stackrel{sym}{=} = \langle 0|(\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger|0\rangle = \langle 0|\mathcal{A}^{-1}\mathcal{O}\mathcal{A}|0\rangle^*; \end{aligned} \quad (68)$$

taking the example of $\Theta = PCT$, with $\mathcal{O} = \phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)$, one has

$\mathcal{O}^\Theta = (\Theta^{-1}\mathcal{O}\Theta)^\dagger = (\Theta^{-1}\phi_n\Theta)^\dagger \dots (\Theta^{-1}\phi_2\Theta)^\dagger (\Theta^{-1}\phi_1\Theta)^\dagger = \phi_n^\Theta \dots \phi_2^\Theta \phi_1^\Theta$.

For fermions [4]

$$\phi(x)^\Theta \equiv \pm\phi(-x) = (\Theta^{-1}\phi(x)\Theta)^\dagger, \quad (69)$$

such that PCT invariance expresses as (of course the sign is unique and must be precisely determined)

$$\begin{aligned} \langle 0|\phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)|0\rangle &\stackrel{sym}{=} \pm \langle 0|\phi_n(-x_n)\dots\phi_2(-x_2)\phi_1(-x_1)|0\rangle \\ &= \pm \langle 0|\phi_1^*(-x_1)\phi_2^*(-x_2)\dots\phi_n^*(-x_n)|0\rangle^* \\ &= \pm \langle 0|(\Theta^{-1}\phi_1(x_1)\Theta)(\Theta^{-1}\phi_2(x_2)\Theta)\dots(\Theta^{-1}\phi_n(x_n)\Theta)|0\rangle^*. \end{aligned} \quad (70)$$

It is enough to change $x_i \rightarrow -x_i$ and to read all Green functions from right to left instead of reading them from left to right (like Pauli).

For a general antiunitary transformation \mathcal{A} , the last line of (68) expressing the invariance also reads, since the vacuum is supposed to be invariant by \mathcal{A}^{-1} as well as by \mathcal{A} :

$$\begin{aligned} \langle 0|\mathcal{O}|0\rangle &\equiv \langle 0|\mathcal{O}0\rangle \\ &= \langle \mathcal{A}^{-1}0|(\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger|\mathcal{A}^{-1}0\rangle = \langle \mathcal{A}^{-1}0|(\mathcal{A}^{-1}\mathcal{O}\mathcal{A})|\mathcal{A}^{-1}0\rangle^* = \langle \mathcal{A}^{-1}0|\mathcal{A}^{-1}(\mathcal{O}0)\rangle^*; \end{aligned} \quad (71)$$

requesting that, for any ϕ , $\langle \phi|\mathcal{O}|\phi\rangle = \langle \phi|(\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger|\phi\rangle$ would be much stronger a condition.

Wightman's expression of the invariance is weaker than requesting $\mathcal{O} = \hat{\mathcal{O}}$, since it occurs only for VEV's and not when sandwiched between any state ϕ .

4.2 The condition $\mathcal{O} = \hat{\mathcal{O}}$

It is often used to express the invariance of a theory with (Lagrangian or) Hamiltonian \mathcal{O} by the transformation under consideration.

* For unitary transformations, this condition is equivalent to

$$\mathcal{O} = \mathcal{U}^{-1}\mathcal{O}\mathcal{U} \Leftrightarrow [\mathcal{U}, \mathcal{O}] = 0; \quad (72)$$

* For antiunitary transformations it yields (we use the property that, for unitary as well as for antiunitary operators $\mathcal{U}^{-1} = \mathcal{U}^\dagger$ and $\mathcal{A}^{-1} = \mathcal{A}^\dagger$, see footnote 5 and Appendix B)

$$\mathcal{O} = (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger = \mathcal{A}^{-1}\mathcal{O}^\dagger\mathcal{A} \Leftrightarrow \mathcal{A}\mathcal{O} = \mathcal{O}^\dagger\mathcal{A}. \quad (73)$$

Note that this is similar (apart from the exchange $\Theta \leftrightarrow \Theta^{-1}$) to the condition proposed in [14] (p.322) as the “*CPT*” theorem for any Lagrangian density $\mathcal{L}(x)$ considered as a hermitian *operator*

$$\Theta\mathcal{L}(x)\Theta^{-1} = \mathcal{L}^\dagger(-x). \quad (74)$$

So, that the Hamiltonian commutes with the symmetry transformation can eventually be accepted when this transformation is unitary (and we have already mentioned that this statement is stronger than Wightman’s expression for invariance); however, when the transformation is antiunitary, one must be more careful.

Requesting that the transformed states should satisfy the same equations as the original ones is only true for unitary transformations. It is not in the case of antiunitary operations like *T* (or *PCT*) since a time reversed fermion does not satisfy the same equation as the original fermion but the time-reversed equation.

4.3 Hamiltonian. Lagrangian.

4.3.1 The case of a unitary transformation

• Invariance of the Hamiltonian

In Quantum Mechanics, a system is said to be invariant by a unitary transformation \mathcal{U} if the transformed of the eigenstates of the Hamiltonian H have the same energies as the original states

$$H\psi = E\psi \text{ and } H\mathcal{U}\psi = E\mathcal{U}\psi; \quad (75)$$

since \mathcal{U} is unitary, it is in particular linear, such that $E\mathcal{U}\psi = \mathcal{U}E\psi = \mathcal{U}H\psi$; this is why the invariance of the theory is commonly expressed by

$$H = \mathcal{U}^{-1}H\mathcal{U} \Leftrightarrow [\mathcal{U}, H] = 0. \quad (76)$$

Defining, according to Wightman, the transformed \hat{H} of the Hamiltonian H by $\hat{H} = \mathcal{U}^{-1}H\mathcal{U}$, we see the the invariance condition (76) also rewrites $\hat{H} = H$. No special condition of reality is required for E .

• Invariance of the Lagrangian

The Lagrangian approach is often more convenient in Quantum Field Theory; it determines the (classical) equations of motion, and also the perturbative expansion.

The Lagrangian density $\mathcal{L}(x)$ is written $\langle \Psi(x) | L(x) | \Psi(x) \rangle$, where L is an operator and $\Psi(x)$ is a “vector” of different fields.

A reasonable definition for the invariance of the theory if that the transformed $\mathcal{U}\Psi$ of Ψ satisfies the same equation as Ψ ; since $\mathcal{L}(x)$ and $e^{i\alpha}\mathcal{L}(x)$ will provide the same (classical) dynamics, one expresses this invariance by

$$\langle \mathcal{U}\Psi(x) | L(x) | \mathcal{U}\Psi(x) \rangle = e^{i\alpha} \langle \Psi(x) | L(x) | \Psi(x) \rangle = e^{i\alpha}\mathcal{L}(x). \quad (77)$$

Due to the unitarity of \mathcal{U} , this is equivalent to $\langle \Psi(x) | \mathcal{U}^{-1}L(x)\mathcal{U} | \Psi(x) \rangle = e^{i\alpha} \langle \Psi(x) | L(x) | \Psi(x) \rangle$ or, owing to the fact that Ψ can be anything,

$$L\mathcal{U} = e^{i\alpha}\mathcal{U}L. \quad (78)$$

If one applies this rule to a mass term, and consider the mass (scalar) as an operator, the unitarity of \mathcal{U} entails that a scalar as well as the associated operator should stay unchanged. This leaves only the possibility $\alpha = 0$. The condition (78) reduces accordingly to the vanishing of the commutator $[L, \mathcal{U}]$. Wightman's definition (6) of the transformed $\hat{L} = \mathcal{U}^{-1}L\mathcal{U}$ of the operator L makes this condition equivalent to $\hat{L} = L$. No condition of reality (hermiticity) is required on L .

4.3.2 The case of antiunitary transformations

The situation is more tricky, since, in particular, the states transformed by a antiunitary transformation (for example T) do not satisfy the same classical equations as the original states (in the case of T , they satisfy the time-reversed equations).

This why it is more convenient to work with each bilinear present in the Lagrangian or Hamiltonian, which we write for example $\langle \phi | \mathcal{O} | \chi \rangle$. ϕ, ξ can be fermions or bosons, \mathcal{O} a scalar, a derivative operator Taking the example of PCT , this bilinear transforms into $\langle \Theta\phi | \mathcal{O} | \Theta\chi \rangle \stackrel{(12)}{=} \langle \chi | \hat{\mathcal{O}} | \phi \rangle = \langle \chi | (\Theta^{-1}\mathcal{O}\Theta)^\dagger | \phi \rangle$.

Application: Dirac and Majorana mass terms

• Problems with a classical fermionic Lagrangian

In view of all possible terms compatible with Lorentz invariance, we work in a basis which can accommodate, for example, both a Dirac fermion and its antiparticle. Accordingly, For a single Dirac fermion (and its antiparticle), we introduce the 4-vector of Weyl fermions

$$\psi = \begin{pmatrix} n_L \\ n_R \end{pmatrix} = \begin{pmatrix} \xi^\alpha \\ (\xi^\beta)^c \\ (\eta_\gamma)^c \\ \eta_\delta \end{pmatrix} \equiv \begin{pmatrix} \xi^\alpha \\ -i(\eta^\beta)^* \\ -i(\xi_\gamma)^* \\ \eta_\delta \end{pmatrix} \underset{\sim}{\text{Lorentz}} \begin{pmatrix} \xi^\alpha \\ \eta^\beta \\ \xi_\gamma \\ \eta_\delta \end{pmatrix}, \quad (79)$$

where $\underset{\sim}{\text{Lorentz}}$ means “transforms like (by Lorentz)”

Let us study the transform by PCT of a Dirac-type mass term $m_D \xi^{\alpha*}(x) \eta_{\dot{\alpha}}(x) = \langle \xi^\alpha(x) | m_D | \eta_{\dot{\alpha}}(x) \rangle$ and of a Majorana-type mass term $m_M \xi^{\alpha*}(x) (\eta_{\dot{\alpha}})^c(x) = \langle \xi^\alpha(x) | m_M | (\eta_{\dot{\alpha}})^c(x) \rangle$.

* m_D and m_M we first consider as operators sandwiched between fermionic grassmanian functions. The two mass terms transform respectively into $\langle \Theta \xi^\alpha(x) | m_D | \Theta \eta_{\dot{\alpha}}(x) \rangle$ and $\langle \Theta \xi^\alpha(x) | m_M | \Theta (\eta_{\dot{\alpha}})^c(x) \rangle$. We now use (12), which transforms these two expressions into $\langle \eta_{\dot{\alpha}} | m_D^\Theta | \xi^\alpha \rangle$ and $\langle (\eta_{\dot{\alpha}})^c | m_M^\Theta | \xi^\alpha \rangle$. Since Θ is antilinear, $\Theta^{-1}m\Theta = m^* \Rightarrow m^\Theta \equiv (\Theta^{-1}m\Theta)^\dagger = m$. So the two mass terms transform respectively into $m_D \langle \eta_{\dot{\alpha}} | \xi^\alpha \rangle \equiv m_D \eta_{\dot{\alpha}}^* \xi^\alpha$ and $m_M \langle \eta_{\dot{\alpha}}^c | \xi^\alpha \rangle \equiv m_M (\eta_{\dot{\alpha}}^c)^* \xi^\alpha$. Notice that $\eta_{\dot{\alpha}}^* \xi^\alpha$ is (using anticommutation) $(-)$ the complex conjugate of $\xi^{\alpha*} \eta_{\dot{\alpha}}$ and like wise, that $(\eta_{\dot{\alpha}}^c)^* \xi^\alpha$ is $(-)$ the complex conjugate of $\xi^{\alpha*} \eta_{\dot{\alpha}}^c$.

The Lagrangian density also a priori involves Dirac and Majorana mass terms $\mu_D \eta_{\dot{\alpha}}^* \xi^\alpha$ and $\mu_M (\eta_{\dot{\alpha}}^c)^* \xi^\alpha$, such that PCT invariance requires $m_D = \mu_D$ and $m_M = \mu_M$ ¹⁸.

¹⁸If the Lagrangian (Hamiltonian) is furthermore real, it should match its complex conjugate (see Appendix C). The c.c. of the Dirac mass terms are $m_D^* \xi^\alpha \eta_{\dot{\alpha}}^* + \mu_D^* \eta_{\dot{\alpha}} \xi^{\alpha*} \stackrel{\text{anticom}}{=} -m_D^* \eta_{\dot{\alpha}}^* \xi^\alpha - \mu_D^* \xi^{\alpha*} \eta_{\dot{\alpha}}$ and the c.c. of the Majorana mass term are $m_M^* \xi^\alpha (\eta_{\dot{\alpha}}^c)^* + \mu_M^* (\eta_{\dot{\alpha}}^c) \xi^{\alpha*} \stackrel{\text{anticom}}{=} -m_M^* (\eta_{\dot{\alpha}}^c)^* \xi^\alpha - \mu_M^* \xi^{\alpha*} (\eta_{\dot{\alpha}}^c)$. Using (29) to replace $\eta_{\dot{\alpha}}^c$ by $(-i)\xi_\alpha^*$, the reality of the Lagrangian is seen to require $m_D = -\mu_D^*$ and $m_M = -\mu_M^*$.

So, combining the two, we see that a real and PCT invariant (classical) Lagrangian should satisfy $m_D = \mu_D$ imaginary and $m_M = \mu_M$ imaginary.

* If we instead consider that $m\phi^*\chi \xrightarrow{PCT} m(\Theta\phi^*)\Theta\chi$ we obtain, using (48)(52), that the Dirac mass term transforms into $m_d(-i\xi^{\alpha*})(-i\eta_{\dot{\alpha}})$, that is, it changes sign by PCT . The Majorana mass term transforms into $m_M(-i\xi^{\alpha*})\Theta(-i\xi_{\dot{\alpha}}^*) \stackrel{antilin}{=} m_M(-i\xi^{\alpha*})(+i)\Theta\xi_{\dot{\alpha}}^* = (-i\xi_{\dot{\alpha}}^*)(+i)(-i\xi_{\dot{\alpha}}^*) = -i\xi^{\alpha*}\xi_{\dot{\alpha}}^*$, that is, unlike the Dirac mass term, the Majorana mass term does not change sign. This alternative would in particular exclude the simultaneous presence of Dirac and Majorana mass terms (necessary for the see-saw mechanism).

* Conclusion: antiunitary transformations of a classical fermionic Lagrangian are ambiguous and can lead to contradictory statements. Defining a classical fermionic Lagrangian is most probably itself problematic¹⁹.

• Quantum (operator) Lagrangian

Dirac and Majorana mass terms write respectively $[\xi^\alpha]^\dagger[m_D][\eta_{\dot{\alpha}}]$ and $[\xi^\alpha]^\dagger[m_M][\eta_{\dot{\alpha}}^c] \stackrel{(32)}{=} [\xi^\alpha]^\dagger[m_M](-i)[\xi_\alpha]^\dagger$. Using (8), one gets $([\xi^\alpha]^\dagger[m_D][\eta_{\dot{\alpha}}])^\Theta = [\eta_{\dot{\alpha}}]^\Theta[m_D]^\Theta([\xi^\alpha]^\dagger)^\Theta = [\eta_{\dot{\alpha}}]^\Theta[m_D]^\Theta([\xi^\alpha]^\Theta)^\dagger = -i[\eta_{\dot{\alpha}}][m_D](-i)[\xi^\alpha]^\dagger$, such that, using the anticommutation of fermionic operators, the Dirac mass term transforms by Θ into itself.

As far as the Majorana mass term is concerned, it transforms into $([\xi^\alpha]^\dagger[m_M][\eta_{\dot{\alpha}}^c])^\Theta = ([\eta_{\dot{\alpha}}^c])^\Theta[m_M]^\Theta([\xi^\alpha]^\dagger)^\Theta = (-i[\xi_\alpha]^\dagger)^\Theta[m_M]^\Theta([\xi^\alpha]^\dagger)^\Theta$. One uses again (8) to evaluate $(-i[\xi_\alpha]^\dagger)^\Theta = ([\xi_\alpha]^\dagger)^\Theta(-i)^\Theta = (-i)[\xi_\alpha]^\dagger(-i) = -[\xi_\alpha]^\dagger$. So, finally, the Majorana mass term transforms into $-[\xi_\alpha]^\dagger m_M(-i)[\xi^\alpha]^\dagger \stackrel{anticom}{=} -i[\xi^\alpha]^\dagger m_M[\xi_\alpha]^\dagger$, that is, like the Dirac mass term, into itself.

The same conclusions are obtained in the propagator formalism.

5 The fermionic propagator and discrete symmetries (1 flavor)

The fermionic propagator $\Delta(x)$ is a matrix with a Lorentz tensorial structure, the matrix elements of which are the vacuum expectation values of \mathcal{T} -products of two fermionic operators:

$$\mathcal{T}\psi(x)\chi(y) = \theta(x^0 - y^0)\psi(x)\chi(y) - \theta(y^0 - x^0)\chi(y)\psi(x); \quad (80)$$

the Lorentz indices of the two operators yield the tensorial structure of the matrix elements.

If, for example, one works in the fermionic basis $(\psi_1, \psi_2, \psi_3, \psi_4)$, and if $\alpha, \beta \dots$ denote their Lorentz indices, the propagator is a 4×4 matrix $\Delta(x)$ such that

$$\Delta_{ij}^{\alpha\beta}(x) = \langle \psi_i^\alpha | \Delta(x) | \psi_j^\beta \rangle = \langle 0 | \mathcal{T}(\psi_i)^\alpha(\frac{x}{2})(\psi_j^\dagger)^\beta(-\frac{x}{2}) | 0 \rangle. \quad (81)$$

Supposing

$$\langle \psi_i^\alpha | \psi_j^\beta \rangle = \delta_{ij}\delta^{\alpha\beta}, \quad (82)$$

we shall also use the notation,

$$\begin{aligned} \Delta(x) &= \sum_{i,j} | \psi_i^\alpha \rangle \Delta_{ij}^{\alpha\beta}(x) \langle \psi_j^\beta | \\ &= \left(| \psi_1^\alpha \rangle \quad | \psi_2^\alpha \rangle \quad | \psi_3^\alpha \rangle \quad | \psi_4^\alpha \rangle \right) \Delta_{ij}^{\alpha\beta}(x) \begin{pmatrix} \langle \psi_1^\beta | \\ \langle \psi_2^\beta | \\ \langle \psi_3^\beta | \\ \langle \psi_4^\beta | \end{pmatrix}; \end{aligned} \quad (83)$$

since one indeed finds $\langle \psi_i^\alpha | \Delta(x) | \psi_j^\beta \rangle = \Delta_{ij}^{\alpha\beta}(x)$.

¹⁹Let us also mention the arbitrariness that results from adding to a mass matrix any vanishing anticommutator.

In the basis (79) in which we are working, the fermionic propagator is a 4×4 matrix which involves the following types of \mathcal{T} products:

* mass-like propagators:

$$\begin{aligned} &< 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle \text{ and } < 0 | \mathcal{T} (\xi^\alpha)^c(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle \text{ (Dirac-like),} \\ &< 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle \text{ and } < 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) (\xi^\beta)^\dagger(-x) | 0 \rangle \text{ (Dirac-like),} \\ &< 0 | \mathcal{T} \xi^\alpha(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle, < 0 | \mathcal{T} (\xi^\alpha)^c(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle \text{ (Majorana-like),} \\ &< 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) (\xi^\beta)^\dagger(-x) | 0 \rangle, < 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle \text{ (Majorana-like);} \end{aligned}$$

* kinetic-like propagators:

$$\begin{aligned} &< 0 | \mathcal{T} \xi^\alpha(x) (\xi^\beta)^\dagger(-x) | 0 \rangle \text{ and } < 0 | \mathcal{T} (\xi^\alpha)^c(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle \text{ (diagonal),} \\ &< 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle \text{ and } < 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle \text{ (diagonal),} \\ &< 0 | \mathcal{T} \xi^\alpha(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle \text{ and } < 0 | \mathcal{T} (\xi^\alpha)^c(x) (\xi^\beta)^\dagger(-x) | 0 \rangle \text{ (non-diagonal),} \\ &< 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle \text{ and } < 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle \text{ (non-diagonal).} \end{aligned}$$

Any propagator is a non-local functional of two fields, which are evaluated at two different space-time points; a consequence is that, unlike for the Lagrangian, which is a local functional of the fields, one cannot implement constraints coming from the anticommutation of fermions. Likewise, a propagator has no hermiticity (or reality) property, and no corresponding constraint exist²⁰. So, the only constraints that can be cast on the propagator come from discrete symmetries and their combinations: C , CP , PCT . The mass eigenstates, which are determined from the propagator are accordingly expected to be less constrained than the eigenstates of any quadratic Lagrangian²¹.

5.1 PCT constraints

All demonstrations proceed along the following steps.

Suppose that we want to deduce PCT constraints for $< 0 | \mathcal{T} \psi(x) \chi^\dagger(-x) | 0 \rangle$. The information that we have from (54) is: there exist ϕ and ω such that $\psi(x) = \Theta \phi^\dagger(-x) \Theta^{-1}$, $\chi^\dagger(-x) = \Theta \omega(x) \Theta^{-1}$ ²², the vacuum is supposed to be invariant $| 0 \rangle = | \Theta 0 \rangle$, and Θ is antiunitary, which entails (10)²³. We have accordingly

$$\begin{aligned} &< 0 | \mathcal{T} \psi(x) \chi^\dagger(-x) | 0 \rangle = < 0 | \mathcal{T} \Theta \phi^\dagger(-x) \Theta^{-1} \Theta \omega(x) \Theta^{-1} | 0 \rangle \\ &\text{invariance of the vacuum} \quad < \Theta 0 | \mathcal{T} \Theta \phi^\dagger(-x) \Theta^{-1} \Theta \omega(x) \Theta^{-1} | \Theta 0 \rangle = < \Theta 0 | \mathcal{T} \Theta \phi^\dagger(-x) \omega(x) \Theta^{-1} | \Theta 0 \rangle \\ &\stackrel{(10)}{=} < 0 | \theta(t) \omega^\dagger(x) \phi(-x) - \theta(-t) \phi(-x) \omega^\dagger(x) | 0 \rangle = - < 0 | \mathcal{T} \phi(-x) \omega^\dagger(x) | 0 \rangle. \end{aligned}$$

5.1.1 Constraints on mass-like terms

$$\begin{aligned} * \text{ Majorana - like } &< 0 | \mathcal{T} \xi^\alpha(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle &= < 0 | \mathcal{T} \xi^\alpha(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 \rangle \\ & &= - < 0 | \mathcal{T} ((\eta_{\dot{\beta}})^c)^\dagger(x) \xi^\alpha(-x) | 0 \rangle; \\ * \text{ Majorana - like } &< 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) (\xi^\beta)^\dagger(-x) | 0 \rangle &= < 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(-x) (\xi^\beta)^\dagger(x) | 0 \rangle \\ & &= - < 0 | \mathcal{T} (\xi^\beta)^\dagger(x) (\eta_{\dot{\alpha}})^c(-x) | 0 \rangle; \\ * \text{ Majorana - like } &< 0 | \mathcal{T} (\xi^\alpha)^c(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle &= < 0 | \mathcal{T} (\xi^\alpha)^c(-x) (\eta_{\dot{\beta}})^\dagger(x) | 0 \rangle \\ & &= - < 0 | \mathcal{T} (\eta_{\dot{\beta}})^\dagger(x) (\xi^\alpha)^c(-x) | 0 \rangle; \\ * \text{ Majorana - like } &< 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle &= < 0 | \mathcal{T} \eta_{\dot{\alpha}}(-x) ((\xi^\beta)^c)^\dagger(x) | 0 \rangle \\ & &= - < 0 | \mathcal{T} ((\xi^\beta)^c)^\dagger(x) \eta_{\dot{\alpha}}(-x) | 0 \rangle; \end{aligned}$$

²⁰Only the spectral function has positivity properties.

²¹and any mass matrix, which can only be eventually introduced in a linear approximation to the inverse propagator in the vicinity of one of its poles [2].

²²For example, from (54), one gets $\xi^\alpha = \Theta(-i(\xi^\alpha)^\dagger) \Theta^{-1}$.

²³ Θ , though antiunitary, does not act on the θ functions of the \mathcal{T} -product because they are real.

$$\begin{aligned}
* \text{ Dirac-like } & \langle 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle = \langle 0 | \mathcal{T} \xi^\alpha(-x) (\eta_{\dot{\beta}})^\dagger(x) | 0 \rangle \\
& = - \langle 0 | \mathcal{T} (\eta_{\dot{\beta}})^\dagger(x) \xi^\alpha(-x) | 0 \rangle; \\
* \text{ Dirac-like } & \langle 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) (\xi^\beta)^\dagger(-x) | 0 \rangle = \langle 0 | \mathcal{T} \eta_{\dot{\alpha}}(-x) (\xi^\beta)^\dagger(x) | 0 \rangle \\
& = - \langle 0 | \mathcal{T} (\xi^\beta)^\dagger(x) \eta_{\dot{\alpha}}(-x) | 0 \rangle; \\
* \text{ Dirac-like } & \langle 0 | \mathcal{T} (\xi^\alpha)^c(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle = \langle 0 | \mathcal{T} (\xi^\alpha)^c(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 \rangle \\
& = - \langle 0 | \mathcal{T} ((\eta_{\dot{\beta}})^c)^\dagger(x) (\xi^\alpha)^c(-x) | 0 \rangle; \\
* \text{ Dirac-like } & \langle 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle = \langle 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(-x) ((\xi^\beta)^c)^\dagger(x) | 0 \rangle \\
& = - \langle 0 | \mathcal{T} ((\xi^\beta)^c)^\dagger(x) (\eta_{\dot{\alpha}})^c(-x) | 0 \rangle.
\end{aligned} \tag{84}$$

We give the demonstration of the first (Majorana-like) line of (84).

$$\begin{aligned}
& \langle 0 | \mathcal{T} \xi^\alpha(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle = \langle 0 | \mathcal{T} \xi^\alpha(x) i \xi_\beta(-x) | 0 \rangle = i \langle 0 | \mathcal{T} \xi^\alpha(x) \xi_\beta(-x) | 0 \rangle \\
& = i \langle 0 | \mathcal{T} \Theta(-i(\xi^\alpha)^\dagger(-x)) \Theta^{-1} \Theta(-i(\xi_\beta)^\dagger(x)) \Theta^{-1} | 0 \rangle \\
& \stackrel{\text{invariance of the vacuum}}{=} i \langle \Theta 0 | \mathcal{T} \Theta(-i(\xi^\alpha)^\dagger(-x)) \Theta^{-1} \Theta(-i(\xi_\beta)^\dagger(x)) \Theta^{-1} | \Theta 0 \rangle \\
& = i \langle \Theta 0 | \mathcal{T} \Theta(-i(\xi^\alpha)^\dagger(-x)) (-i(\xi_\beta)^\dagger(x)) \Theta^{-1} | \Theta 0 \rangle \\
& = -i \langle \Theta 0 | \mathcal{T} \Theta(\xi^\alpha)^\dagger(-x) (\xi_\beta)^\dagger(x) \Theta^{-1} | \Theta 0 \rangle \\
& \stackrel{\text{antiunitarity}^{(10)}}{=} -i \langle 0 | \theta(t) \xi_\beta(x) (\xi^\alpha)(-x) | 0 \rangle + i \langle 0 | \theta(-t) \xi^\alpha(-x) \xi_\beta(x) | 0 \rangle \\
& = +i \langle 0 | \mathcal{T} \xi^\alpha(-x) \xi_\beta(x) | 0 \rangle = \langle 0 | \mathcal{T} \xi^\alpha(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 \rangle.
\end{aligned}$$

All these propagators are accordingly left invariant²⁴ by the 4-inversion $x \rightarrow -x$, or, in Fourier space, they are invariant when $p_\mu \rightarrow -p_\mu$.

5.1.2 Constraints on kinetic-like terms

$$\begin{aligned}
* \text{ Diagonal } & \langle 0 | \mathcal{T} \xi^\alpha(x) (\xi^\beta)^\dagger(-x) | 0 \rangle = - \langle 0 | \mathcal{T} \xi^\alpha(-x) (\xi^\beta)^\dagger(x) | 0 \rangle \\
& = \langle 0 | \mathcal{T} (\xi^\beta)^\dagger(x) \xi^\alpha(-x) | 0 \rangle; \\
* \text{ Diagonal } & \langle 0 | \mathcal{T} (\xi^\alpha)^c(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle = - \langle 0 | \mathcal{T} (\xi^\alpha)^c(-x) ((\xi^\beta)^c)^\dagger(x) | 0 \rangle \\
& = \langle 0 | \mathcal{T} (\xi^\beta)^c)^\dagger(x) (\xi^\alpha)^c(-x) | 0 \rangle; \\
* \text{ Diagonal } & \langle 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle = - \langle 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 \rangle \\
& = \langle 0 | \mathcal{T} ((\eta_{\dot{\beta}})^c)^\dagger(x) (\eta_{\dot{\alpha}})^c(-x) | 0 \rangle; \\
* \text{ Diagonal } & \langle 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle = - \langle 0 | \mathcal{T} \eta_{\dot{\alpha}}(-x) (\eta_{\dot{\beta}})^\dagger(x) | 0 \rangle \\
& = \langle 0 | \mathcal{T} (\eta_{\dot{\beta}})^\dagger(x) \eta_{\dot{\alpha}}(-x) | 0 \rangle; \\
* \text{ Non-diagonal } & \langle 0 | \mathcal{T} \xi^\alpha(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle = - \langle 0 | \mathcal{T} \xi^\alpha(-x) ((\xi^\beta)^c)^\dagger(x) | 0 \rangle \\
& = \langle 0 | \mathcal{T} ((\xi^\beta)^c)^\dagger(x) \xi^\alpha(-x) | 0 \rangle; \\
* \text{ Non-diagonal } & \langle 0 | \mathcal{T} (\xi^\alpha)^c(x) (\xi^\beta)^\dagger(-x) | 0 \rangle = - \langle 0 | \mathcal{T} (\xi^\alpha)^c(-x) (\xi^\beta)^\dagger(x) | 0 \rangle \\
& = \langle 0 | \mathcal{T} (\xi^\beta)^\dagger(x) (\xi^\alpha)^c(-x) | 0 \rangle; \\
* \text{ Non-diagonal } & \langle 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle = - \langle 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(-x) (\eta_{\dot{\beta}})^\dagger(x) | 0 \rangle
\end{aligned}$$

²⁴This is not much information, but it is correct. Consider indeed the usual Feynman propagator in Fourier space for a Dirac fermion with mass m

$$\int d^4 x e^{ipx} \langle 0 | \mathcal{T} \begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\alpha}} \end{pmatrix} (x) \begin{pmatrix} (\xi^\beta)^\dagger & (\eta_{\dot{\beta}})^\dagger \end{pmatrix} (-x) \gamma^0 | 0 \rangle = \frac{p_\mu \gamma^\mu + m}{p^2 - m^2} = \frac{1}{p^2 - m^2} \begin{pmatrix} m & p_\mu \overline{\sigma}^\mu \\ p_\mu \sigma^\mu & m \end{pmatrix}; \tag{85}$$

it yields in particular (the γ^0 in (85) makes $\gamma_{\alpha,\beta}^\mu$ appear)

$$\int d^4 x e^{ipx} \langle 0 | \mathcal{T} \xi^\alpha(x) \eta_{\dot{\beta}}(-x) | 0 \rangle = \frac{p_\mu \gamma_{\alpha\beta}^\mu + m \delta_{\alpha\beta}}{p^2 - m^2}, \alpha, \beta = 1, 2. \tag{86}$$

PCT invariance tells us that, in a Dirac mass-like propagator, the p^μ term is not present, and the remaining term is diagonal in α, β ; and, indeed, $\gamma_{\alpha\beta}^\mu$ vanishes $\forall \alpha, \beta = 1, 2$, while the term proportional to m is diagonal in α, β .

$$\begin{aligned}
& = \langle 0 | \mathcal{T}(\eta_{\dot{\beta}})^\dagger(x)(\eta_{\dot{\alpha}})^c(-x) | 0 \rangle; \\
* \text{ Non - diagonal } \langle 0 | \mathcal{T}\eta_{\dot{\alpha}}(x)((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle & = - \langle 0 | \mathcal{T}\eta_{\dot{\alpha}}(-x)((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 \rangle > \\
& = \langle 0 | \mathcal{T}((\eta_{\dot{\beta}})^c)^\dagger(x)\eta_{\dot{\alpha}}(-x) | 0 \rangle.
\end{aligned} \tag{87}$$

In Fourier space, all these propagators must accordingly be odd in p_μ . We check like above on the Dirac propagator that it is indeed the case. One gets for example (the γ^0 in (85) now makes $\gamma_{\alpha,\beta+2}^\mu$ appear)

$$\int d^4x e^{ipx} \langle 0 | \mathcal{T}\xi^\alpha(x)(\xi^\beta)^\dagger(-x) | 0 \rangle = \frac{p_\mu \gamma_{\alpha\beta+2}^\mu + m\delta_{\alpha\beta+2}}{p^2 - m^2}, \alpha, \beta = 1, 2, \tag{88}$$

in which only the terms linear in p_μ are present, which are indeed odd in p_μ as predicted by *PCT* invariance.

Note that *PCT* invariance does not forbid non-diagonal kinetic-like propagators.

5.1.3 Simple assumptions and consequences

PCT symmetry constrains, in Fourier space, all mass-like propagators to be p -even and all kinetic-like propagators to be p -odd; the former can only write $f(p^2)\delta_{\alpha\beta}$ and the latter $g(p^2)p_\mu\sigma_{\alpha\beta}^\mu$ or $h(p^2)p_\mu\bar{\sigma}^{\mu}_{\alpha\beta}$. This is what we will suppose hereafter, and consider, in Fourier space, a propagator

$$\Delta(p) = \left(\begin{array}{c} | \xi^\alpha \rangle \quad | (\xi^\alpha)^c \rangle \quad | (\eta_{\dot{\alpha}})^c \rangle \quad | \eta_{\dot{\alpha}} \rangle \\ \left(\begin{array}{cc} \alpha_1(p^2) & a_1(p^2) \\ b_1(p^2) & \beta_1(p^2) \end{array} \right) p_\mu \bar{\sigma}^{\mu}_{\alpha\beta} \quad \left(\begin{array}{cc} m_{L1}(p^2) & \mu_1(p^2) \\ m_1(p^2) & m_{R1}(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \left(\begin{array}{cc} m_{L2}(p^2) & m_2(p^2) \\ \mu_2(p^2) & m_{R2}(p^2) \end{array} \right) \delta_{\alpha\beta} \quad \left(\begin{array}{cc} \beta_2(p^2) & b_2(p^2) \\ a_2(p^2) & \alpha_2(p^2) \end{array} \right) p_\mu \sigma^{\mu}_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} \langle \xi^\beta | \\ \langle (\xi^\beta)^c | \\ \langle (\eta_{\dot{\beta}})^c | \\ \langle \eta_{\dot{\beta}} | \end{array} \right). \tag{89}$$

This ansatz enables to get explicit constraints on the propagator. It is motivated by the fact that, classically, the (quadratic) Lagrangian, which is the inverse propagator, has this same Lorentz structure

$$L = \left(\begin{array}{c|c} K_1(p_-)_{\alpha\beta} & M_1 \delta_{\alpha\beta} \\ \hline M_2 \delta_{\alpha\beta} & K_2(p_+)_{\alpha\beta} \end{array} \right). \tag{90}$$

An important property is that it automatically satisfies the *PCT* constraints (84) (87). For mass-like propagators, which are invariant by the 4-inversion $x \rightarrow -x$ it is a triviality; for kinetic like propagators, the “-” signs which occur in the r.h.s.’s of (87) are canceled by the one which comes from the differential operator p_μ acting on $(-x)$ instead of x . We consider accordingly that (89) expresses the invariance of the propagator by *PCT*.

From now onwards we shall always use the form (89) for the propagator, considering therefore that it is *PCT* invariant. It includes sixteen complex parameters. We will see how individual discrete symmetries and their products reduce this number.

5.2 Charge conjugate fields

By using the definitions of charge conjugate fields

$$\begin{aligned}
\xi^\alpha & = g^{\alpha\gamma} \xi_\gamma = -i\sigma_{\alpha\gamma}^2 \xi_\gamma = -i\sigma_{\alpha\gamma}^2 (-i)((\eta_{\dot{\gamma}})^c)^\dagger = -\sigma_{\alpha\gamma}^2 ((\eta_{\dot{\gamma}})^c)^\dagger, \\
\eta_{\dot{\beta}} & = g_{\beta\delta} \eta^{\dot{\beta}} = i\sigma_{\beta\delta}^2 \eta^{\dot{\beta}} = i\sigma_{\beta\delta}^2 (-i)((\xi^\delta)^c)^\dagger = \sigma_{\beta\delta}^2 ((\xi^\delta)^c)^\dagger.
\end{aligned} \tag{91}$$

one can bring additional constraints to the ones obtained from expressing the invariance by a discrete symmetry like PCT . We first give the example of a Dirac-like propagator:

$$\begin{aligned} & \langle 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle = \langle 0 | \mathcal{T}(-) \sigma_{\alpha\gamma}^2 ((\eta_{\dot{\gamma}})^c)^\dagger(x) \left(\sigma_{\beta\delta}^2 ((\xi^\delta)^c)^\dagger(-x) \right)^\dagger | 0 \rangle \\ & = \sigma_{\alpha\gamma}^2 \sigma_{\beta\delta}^2 \langle 0 | \mathcal{T} ((\eta_{\dot{\gamma}})^c)^\dagger(x) (\xi^\delta)^c(-x) | 0 \rangle = (\delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\beta} \delta_{\delta\gamma}) \langle 0 | \mathcal{T} ((\eta_{\dot{\gamma}})^c)^\dagger(x) (\xi^\delta)^c(-x) | 0 \rangle = \\ & - \langle 0 | \mathcal{T} (\xi^\alpha)^c(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 \rangle + \delta_{\alpha\beta} \langle 0 | \mathcal{T} (\xi^\gamma)^c(-x) ((\eta_{\dot{\gamma}})^c)^\dagger(x) | 0 \rangle. \end{aligned}$$

The r.h.s. of the corresponding PCT constraint in the first line of (84) writes the same but for the exchange $x \rightarrow (-x)$. If we now use the ansatz (89) which implements PCT invariance, one gets

$$\mu_1(p^2) \delta_{\alpha\beta} = -(\delta_{\beta\gamma} \delta_{\alpha\delta} - \delta_{\alpha\beta} \delta_{\delta\gamma}) m_1(p^2) \delta_{\delta\gamma} = \delta_{\alpha\beta} m_1(p^2), \quad (92)$$

equivalently

$$m_1(p^2) = \mu_1(p^2). \quad (93)$$

Likewise, one gets $m_2(p^2) = \mu_2(p^2)$.

For Majorana-like propagator, using the definitions (91) of charge conjugate fields, one gets

$$\begin{aligned} \langle 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}}^c)^\dagger(-x) | 0 \rangle & = \langle 0 | \mathcal{T} (\eta_{\dot{\beta}}^c)^\dagger(x) \xi^\alpha(-x) | 0 \rangle - \delta_{\alpha\beta} \langle 0 | \mathcal{T} (\eta_{\dot{\gamma}}^c)^\dagger(x) \xi^\gamma(-x) | 0 \rangle \\ & = - \langle 0 | \mathcal{T} \xi^\alpha(-x) (\eta_{\dot{\beta}}^c)^\dagger(x) | 0 \rangle + \delta_{\alpha\beta} \langle 0 | \mathcal{T} \xi^\gamma(-x) (\eta_{\dot{\gamma}}^c)^\dagger(x) | 0 \rangle, \end{aligned} \quad (94)$$

while, with the same procedure, its transformed by PCT in the r.h.s. of (84) becomes

$$- \langle 0 | \mathcal{T} (\eta_{\dot{\beta}}^c)^\dagger(x) \xi^\alpha(-x) | 0 \rangle = - \langle 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}}^c)^\dagger(-x) + \delta_{\alpha\beta} \langle 0 | \mathcal{T} \xi^\gamma(x) (\eta_{\dot{\gamma}}^c)^\dagger(-x) | 0 \rangle. \quad (95)$$

One only gets tautologies such that no additional constraint arises.

We implement the same procedure for kinetic-like terms, for example $\langle 0 | \mathcal{T} \xi^\alpha(x) (\xi^\beta)^\dagger(-x) | 0 \rangle = \langle 0 | \mathcal{T} (\xi^\beta)^\dagger(x) \xi^\alpha(-x) | 0 \rangle$. Using $\xi^\alpha = -\sigma_{\alpha\gamma}^2 ((\eta_{\dot{\gamma}})^c)^\dagger$ and $(\xi^\beta)^\dagger = \sigma_{\beta\delta}^2 (\eta_{\dot{\delta}})^c$ and (89), one gets

$$\begin{aligned} \alpha_1(p^2) p_\mu \overline{\sigma^\mu}_{\alpha\beta} & = -(\delta_{\beta\gamma} \delta_{\alpha\delta} - \delta_{\alpha\beta} \delta_{\delta\gamma}) \beta_2(p^2) p_\mu \sigma^\mu_{\delta\gamma} \\ & = -\beta_2(p^2) (p_\mu \sigma^\mu_{\alpha\beta} - \delta_{\alpha\beta} p_\mu \text{Tr} \sigma_\mu) \\ & = -\beta_2(p^2) (p_\mu \sigma^\mu_{\alpha\beta} - \delta_{\alpha\beta} (2p_0 + 0 \times p^i)) \\ & = -\beta_2(p^2) (-p_0 \sigma_{\alpha\beta}^0 + \vec{p} \cdot \vec{\sigma}_{\alpha\beta}) \\ & = \beta_2(p^2) p_\mu \overline{\sigma^\mu}_{\alpha\beta}, \end{aligned} \quad (96)$$

which entails

$$\alpha_1(p^2) = \beta_2(p^2). \quad (97)$$

Likewise, one gets $\alpha_2(p^2) = \beta_1(p^2)$, and, for the non-diagonal kinetic-like propagators, $a_1(p^2) = a_2(p^2)$, $b_1(p^2) = b_2(p^2)$.

So, after making use of the definition of charge conjugate fields, (89) expressing the PCT invariance of the propagator rewrites

$$\Delta_{PCT}(p) = \left(\begin{array}{cccc} | \xi^\alpha \rangle & | (\xi^\alpha)^c \rangle & | (\eta_{\dot{\alpha}})^c \rangle & | \eta_{\dot{\alpha}} \rangle \end{array} \right) \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & u(p^2) \\ v(p^2) & \beta(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} & \left(\begin{array}{cc} m_{L1}(p^2) & \mu_1(p^2) \\ \mu_1(p^2) & m_{R1}(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} m_{L2}(p^2) & \mu_2(p^2) \\ \mu_2(p^2) & m_{R2}(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \alpha(p^2) & v(p^2) \\ u(p^2) & \beta(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} \langle \xi^\beta | \\ \langle (\xi^\beta)^c | \\ \langle (\eta_{\dot{\beta}})^c | \\ \langle \eta_{\dot{\beta}} | \end{array} \right). \quad (98)$$

PCT symmetry has finally reduced the total number of arbitrary functions necessary to describe one flavor of fermions from sixteen to ten.

5.3 C constraints

C is a unitary operator and we may use directly (32) in the expression of the propagator. This is an example of demonstration, in which we suppose that the vacuum is invariant by C .

$$\begin{aligned} & \langle 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle = \langle C 0 | \mathcal{T} C(-i(\eta^{\dot{\alpha}})^\dagger)(x) C^{-1} C(i\xi_\beta)(-x) C^{-1} | C 0 \rangle \\ & = \langle C 0 | \mathcal{T} C(\eta^{\dot{\alpha}})^\dagger(x) \xi_\beta(-x) C^{-1} | C 0 \rangle = \langle 0 | \mathcal{T} C^\dagger C(\eta^{\dot{\alpha}})^\dagger(x) \xi_\beta(-x) C^{-1} C | 0 \rangle \\ & = \langle 0 | \mathcal{T} (\eta^{\dot{\alpha}})^\dagger(x) \xi_\beta(-x) | 0 \rangle = \langle 0 | \mathcal{T} ((\xi^\alpha)^c)^\dagger(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle. \end{aligned}$$

By using (89) expressing PCT invariance, one gets accordingly

$$\Delta_{C+PCT}(p) = \left(\begin{array}{c} | \xi^\alpha \rangle \quad | (\xi^\alpha)^c \rangle \quad | (\eta_{\dot{\alpha}})^c \rangle \quad | \eta_{\dot{\alpha}} \rangle \\ \hline \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} \quad \left| \quad \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \right. \\ \hline \left. \left(\begin{array}{cc} \sigma(p^2) & m(p^2) \\ m(p^2) & \sigma(p^2) \end{array} \right) \delta_{\alpha\beta} \quad \left| \quad \left(\begin{array}{cc} \beta(p^2) & b(p^2) \\ b(p^2) & \beta(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \right. \end{array} \right) \left(\begin{array}{c} \langle \xi^\beta | \\ \langle (\xi^\beta)^c | \\ \langle (\eta_{\dot{\beta}})^c | \\ \langle \eta_{\dot{\beta}} | \end{array} \right). \quad (99)$$

All 2×2 submatrices are in particular symmetric.

Combining now (98) and (99), a $C + PCT$ invariant propagator, after using the definition of charge conjugate fields, can finally be reduced to

$$\Delta_{C+PCT}(p) = \left(\begin{array}{c} | \xi^\alpha \rangle \quad | (\xi^\alpha)^c \rangle \quad | (\eta_{\dot{\alpha}})^c \rangle \quad | \eta_{\dot{\alpha}} \rangle \\ \hline \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} \quad \left| \quad \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \right. \\ \hline \left. \left(\begin{array}{cc} \sigma(p^2) & m(p^2) \\ m(p^2) & \sigma(p^2) \end{array} \right) \delta_{\alpha\beta} \quad \left| \quad \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \right. \end{array} \right) \left(\begin{array}{c} \langle \xi^\beta | \\ \langle (\xi^\beta)^c | \\ \langle (\eta_{\dot{\beta}})^c | \\ \langle \eta_{\dot{\beta}} | \end{array} \right), \quad (100)$$

in which the number of arbitrary functions has now been reduced to six.

5.4 P constraints

In momentum space, the parity transformed of $p_\mu \sigma^\mu \equiv (p_0 \sigma^0 + \vec{p} \cdot \vec{\sigma})$ is $(p_0 \sigma^0 - \vec{p} \cdot \vec{\sigma}) \equiv p_\mu \overline{\sigma^\mu}$.

Using (22) and the assumption (89) expressing PCT invariance, and supposing the vacuum invariant by parity, one gets

$$\Delta_{P+PCT}(p) = \left(\begin{array}{c} | \xi^\alpha \rangle \quad | (\xi^\alpha)^c \rangle \quad | (\eta_{\dot{\alpha}})^c \rangle \quad | \eta_{\dot{\alpha}} \rangle \\ \hline \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ b(p^2) & \beta(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} \quad \left| \quad \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ m(p^2) & \sigma(p^2) \end{array} \right) \delta_{\alpha\beta} \right. \\ \hline \left. \left(\begin{array}{cc} \sigma(p^2) & m(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \quad \left| \quad \left(\begin{array}{cc} \beta(p^2) & b(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \right. \end{array} \right) \left(\begin{array}{c} \langle \xi^\beta | \\ \langle (\xi^\beta)^c | \\ \langle (\eta_{\dot{\beta}})^c | \\ \langle \eta_{\dot{\beta}} | \end{array} \right). \quad (101)$$

A $P + C + PCT$ invariant propagator writes

$$\Delta_{P+C+PCT}(p) = \left(\begin{array}{c} |\xi^\alpha \rangle \quad |(\xi^\alpha)^c \rangle \quad |(\eta_{\dot{\alpha}})^c \rangle \quad |\eta_{\dot{\alpha}} \rangle \\ \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} \quad \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \quad \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \begin{pmatrix} \langle \xi^\beta | \\ \langle (\xi^\beta)^c | \\ \langle (\eta_{\dot{\beta}})^c | \\ \langle \eta_{\dot{\beta}} | \end{pmatrix}. \quad (102)$$

The expressions above can be further reduced by using the definition of charge conjugate fields, which leads to (98) as the expression of PCT invariance. So doing, a $P + PCT$ invariant propagator writes

$$\Delta_{P+PCT}(p) = \left(\begin{array}{c} |\xi^\alpha \rangle \quad |(\xi^\alpha)^c \rangle \quad |(\eta_{\dot{\alpha}})^c \rangle \quad |\eta_{\dot{\alpha}} \rangle \\ \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ b(p^2) & \alpha(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \quad \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \sigma(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \left(\begin{array}{cc} \sigma(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \quad \left(\begin{array}{cc} \alpha(p^2) & b(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} \end{array} \right) \begin{pmatrix} \langle \xi^\beta | \\ \langle (\xi^\beta)^c | \\ \langle (\eta_{\dot{\beta}})^c | \\ \langle \eta_{\dot{\beta}} | \end{pmatrix}; \quad (103)$$

and one finds again the expression (102) for a $P + C + PCT$ invariant propagator.

5.5 CP constraints

Using (34), (89), and supposing the vacuum invariant by CP , one gets

$$\Delta_{CP+PCT}(p) = \left(\begin{array}{c} |\xi^\alpha \rangle \quad |(\xi^\alpha)^c \rangle \quad |(\eta_{\dot{\alpha}})^c \rangle \quad |\eta_{\dot{\alpha}} \rangle \\ \left(\begin{array}{cc} \alpha(p^2) & u(p^2) \\ v(p^2) & \beta(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} \quad \left(\begin{array}{cc} m_L(p^2) & \mu(p^2) \\ m(p^2) & m_R(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \left(\begin{array}{cc} m_L(p^2) & \mu(p^2) \\ m(p^2) & m_R(p^2) \end{array} \right) \delta_{\alpha\beta} \quad \left(\begin{array}{cc} \alpha(p^2) & u(p^2) \\ v(p^2) & \beta(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \begin{pmatrix} \langle \xi^\beta | \\ \langle (\xi^\beta)^c | \\ \langle (\eta_{\dot{\beta}})^c | \\ \langle \eta_{\dot{\beta}} | \end{pmatrix}. \quad (104)$$

It can be further constrained by using the definition of charge conjugate fields which makes the PCT constraint be (98), to

$$\Delta_{CP+PCT}(p) = \left(\begin{array}{c} |\xi^\alpha \rangle \quad |(\xi^\alpha)^c \rangle \quad |(\eta_{\dot{\alpha}})^c \rangle \quad |\eta_{\dot{\alpha}} \rangle \\ \left(\begin{array}{cc} \alpha(p^2) & u(p^2) \\ u(p^2) & \beta(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} \quad \left(\begin{array}{cc} m_L(p^2) & \mu(p^2) \\ \mu(p^2) & m_R(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \left(\begin{array}{cc} m_L(p^2) & \mu(p^2) \\ \mu(p^2) & m_R(p^2) \end{array} \right) \delta_{\alpha\beta} \quad \left(\begin{array}{cc} \alpha(p^2) & u(p^2) \\ u(p^2) & \beta(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \begin{pmatrix} \langle \xi^\beta | \\ \langle (\xi^\beta)^c | \\ \langle (\eta_{\dot{\beta}})^c | \\ \langle \eta_{\dot{\beta}} | \end{pmatrix}. \quad (105)$$

One then gets 4 symmetric 2×2 sub-blocks.

5.6 Eigenstates of a $C + PCT$ invariant propagator

We do not consider any PCT violation, because, if this occurred, the very foundations of local Quantum Field Theory would be undermined, and the meaning of our conclusions itself could thus strongly be cast in doubt.

We look here for the eigenstates of the 4×4 matrix in (100)

$$\Delta_{C+PCT}(p^2) = \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \sigma_{\alpha\beta}^\mu & \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} \sigma(p^2) & m(p^2) \\ m(p^2) & \sigma(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \overline{\sigma}_{\alpha\beta}^\mu \end{array} \right). \quad (106)$$

The three symmetric matrices $\begin{pmatrix} \rho & \mu \\ \mu & \rho \end{pmatrix}$, $\begin{pmatrix} \sigma & m \\ m & \sigma \end{pmatrix}$ and $\begin{pmatrix} \alpha & a \\ a & \alpha \end{pmatrix}$ can be simultaneously diagonalized by a unitary matrix U according to

$$\begin{aligned} U^T \begin{pmatrix} \rho & \mu \\ \mu & \rho \end{pmatrix} U &= \begin{pmatrix} (\rho + \mu)e^{2i\varphi} & \\ & (\rho - \mu)e^{-2i\varphi} \end{pmatrix}, \\ U^T \begin{pmatrix} \alpha & a \\ a & \alpha \end{pmatrix} U &= \begin{pmatrix} (\alpha + a)e^{2i\varphi} & \\ & (\alpha - a)e^{-2i\varphi} \end{pmatrix}, \\ U &= \frac{1}{\sqrt{2}} e^{i\omega} \begin{pmatrix} e^{i\varphi} & -e^{-i\varphi} \\ e^{i\varphi} & e^{-i\varphi} \end{pmatrix}. \end{aligned} \quad (107)$$

We can choose the particular case

$$U = U_0 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \quad (108)$$

Call the initial basis

$$\langle n_L | = \begin{pmatrix} \langle \xi^\alpha | \\ \langle (\xi^\beta)^c | \end{pmatrix} \equiv \begin{pmatrix} \langle \xi^\alpha | \\ \langle -i(\eta^{\dot{\beta}})^\dagger | \end{pmatrix}, \quad \langle n_R | = \begin{pmatrix} \langle (\eta^{\dot{\alpha}})^c | \\ \langle \eta_{\dot{\beta}} | \end{pmatrix} \equiv \begin{pmatrix} \langle -i(\xi_\alpha)^\dagger | \\ \langle \eta_{\dot{\beta}} | \end{pmatrix}, \quad (109)$$

one has

$$\left(| \xi^\alpha \rangle \quad | (\xi^\beta)^c \rangle \quad | (\eta_{\dot{\gamma}})^c \rangle \quad | \eta_{\dot{\delta}} \rangle \right) = \left(| n_L \rangle \quad | n_R \rangle \right). \quad (110)$$

Define the new basis by

$$\begin{aligned} \langle N_L | &= U_0^\dagger \langle n_L |, & \langle N_R | &= U_0^\dagger \langle n_R |, \\ | N_L \rangle &= U_0 | n_L \rangle, & | N_R \rangle &= U_0 | n_R \rangle. \end{aligned} \quad (111)$$

One has explicitly

$$\langle N_L | = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle \xi^\alpha - i(\eta^{\dot{\alpha}})^\dagger | \\ \langle -\xi^\alpha - i(\eta^{\dot{\alpha}})^\dagger | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle \xi^\alpha + (\xi^\alpha)^c | \\ \langle -\xi^\alpha + (\xi^\alpha)^c | \end{pmatrix},$$

$$\langle N_R | = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle -i(\xi_\alpha)^\dagger + \eta_{\dot{\alpha}} | \\ \langle +i(\xi_\alpha)^\dagger + \eta_{\dot{\alpha}} | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle \eta_{\dot{\alpha}} + (\eta_{\dot{\alpha}})^c | \\ \langle \eta_{\dot{\alpha}} - (\eta_{\dot{\alpha}})^c | \end{pmatrix}, \quad (112)$$

and one can write

$$\langle N_L | = \begin{pmatrix} \langle \chi^\alpha | \\ \langle (-i)(\omega^{\dot{\beta}})^\dagger | \end{pmatrix}, \langle N_R | = \begin{pmatrix} \langle (-i)(\chi_\alpha)^\dagger | \\ \langle \omega_{\dot{\beta}} | \end{pmatrix}. \quad (113)$$

In this new basis, the propagator writes (using (from (108)) $U_0^T U_0 = 1$)

$$\Delta_{C+PCT}(p^2) = \begin{pmatrix} | N_L \rangle & | N_R \rangle \end{pmatrix} \left(\begin{array}{c|c} \begin{pmatrix} \alpha(p^2) + a(p^2) & \\ & \alpha(p^2) - a(p^2) \end{pmatrix} p_\mu \sigma_{\alpha\beta}^\mu & \begin{pmatrix} \rho(p^2) + \mu(p^2) & \\ & \rho(p^2) - \mu(p^2) \end{pmatrix} \delta_{\alpha\beta} \\ \hline \begin{pmatrix} \sigma(p^2) + m(p^2) & \\ & \sigma(p^2) - m(p^2) \end{pmatrix} \delta_{\alpha\beta} & \begin{pmatrix} \alpha(p^2) + a(p^2) & \\ & \alpha(p^2) - a(p^2) \end{pmatrix} p_\mu \overline{\sigma}^\mu_{\alpha\beta} \end{array} \right) \begin{pmatrix} \langle N_L | \\ \langle N_R | \end{pmatrix}. \quad (114)$$

Remember that $|u\rangle\langle v|$ corresponds, in our notation, to a propagator $\langle 0 | \mathcal{T} u(x) v^\dagger(-x) | 0 \rangle$.

One introduces the Majorana fermions (see subsection 3.6)

$$\begin{aligned} X_M^\pm &= \begin{pmatrix} \chi^\alpha \\ \pm(-i)(\chi_\alpha)^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi^\alpha + (\xi^\alpha)^c \\ \pm(\eta_{\dot{\alpha}} + (\eta_{\dot{\alpha}})^c) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi^\alpha - i(\eta_{\dot{\alpha}})^\dagger \\ \pm(\eta_{\dot{\alpha}} - i(\xi_\alpha)^\dagger) \end{pmatrix}, \\ \Omega_M^\pm &= \begin{pmatrix} \pm(-i)(\omega^{\dot{\beta}})^\dagger \\ \omega_{\dot{\beta}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm(-\xi^\beta + (\xi^\beta)^c) \\ \eta_{\dot{\beta}} - (\eta_{\dot{\beta}})^c \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm(-\xi^\beta - i(\eta^{\dot{\beta}})^\dagger) \\ \eta_{\dot{\beta}} + i(\xi_\beta)^\dagger \end{pmatrix}. \end{aligned} \quad (115)$$

5.6.1 Kinetic-like propagators

They can be rewritten

$$\begin{aligned} \int d^4 x e^{ipx} \langle 0 | \mathcal{T} \chi^\alpha(x) (\chi^\beta)^\dagger(-x) | 0 \rangle &= (\alpha(p^2) + a(p^2)) p_\mu \sigma_{\alpha\beta}^\mu, \\ \int d^4 x e^{ipx} \langle 0 | \mathcal{T} (\chi_\alpha)^\dagger(x) \chi_\beta(-x) | 0 \rangle &= (\alpha(p^2) + a(p^2)) p_\mu \overline{\sigma}^\mu_{\alpha\beta}, \\ \int d^4 x e^{ipx} \langle 0 | \mathcal{T} (\omega^{\dot{\alpha}})^\dagger(x) \omega^{\dot{\beta}}(-x) | 0 \rangle &= (\alpha(p^2) - a(p^2)) p_\mu \sigma_{\alpha\beta}^\mu, \\ \int d^4 x e^{ipx} \langle 0 | \mathcal{T} \omega_{\dot{\alpha}}(x) (\omega_{\dot{\beta}})^\dagger(-x) | 0 \rangle &= (\alpha(p^2) - a(p^2)) p_\mu \overline{\sigma}^\mu_{\alpha\beta}, \end{aligned} \quad (116)$$

5.6.2 Mass-like propagators

They write

$$\begin{aligned} \int d^4 x e^{ipx} \langle 0 | \mathcal{T} \chi^\alpha(x) i \chi_\beta(-x) | 0 \rangle &= \delta_{\alpha\beta} (\rho(p^2) + \mu(p^2)), \\ \int d^4 x e^{ipx} \langle 0 | \mathcal{T} (-i)(\chi_\alpha)^\dagger(x) (\chi^\beta)^\dagger(-x) | 0 \rangle &= \delta_{\alpha\beta} (\sigma(p^2) + m(p^2)), \\ \int d^4 x e^{ipx} \langle 0 | \mathcal{T} (-i)(\omega^{\dot{\alpha}})^\dagger(x) (\omega_{\dot{\beta}})^\dagger(-x) | 0 \rangle &= \delta_{\alpha\beta} (\rho(p^2) - \mu(p^2)), \\ \int d^4 x e^{ipx} \langle 0 | \mathcal{T} \omega_{\dot{\alpha}}(x) i \omega^{\dot{\beta}}(-x) | 0 \rangle &= \delta_{\alpha\beta} (\sigma(p^2) - m(p^2)). \end{aligned} \quad (117)$$

5.6.3 Conclusion

When C and PCT invariance holds, the fermion propagator decomposes into the propagators for the Majorana fermions X and Ω (115) (note that we have introduced below the “bar” fields instead of the \dagger fields, thus a γ^0 matrix)

$$\begin{aligned} \int d^4x e^{ipx} \langle 0 | \mathcal{T} X_{M\alpha}^\pm(x) \overline{X_{M\beta}^\pm(-x)} | 0 \rangle &= \begin{pmatrix} (\rho(p^2) + \mu(p^2))\delta_{\alpha\beta} & (\alpha(p^2) + a(p^2))p_\mu \sigma_{\alpha\beta}^\mu \\ (\alpha(p^2) + a(p^2))p_\mu \overline{\sigma}_{\alpha\beta}^\mu & (\sigma(p^2) + m(p^2))\delta_{\alpha\beta} \end{pmatrix}, \\ \int d^4x e^{ipx} \langle 0 | \mathcal{T} \Omega_{M\alpha}^\pm(x) \overline{\Omega_{M\beta}^\pm(-x)} | 0 \rangle &= \begin{pmatrix} (\rho(p^2) - \mu(p^2))\delta_{\alpha\beta} & (\alpha(p^2) - a(p^2))p_\mu \sigma_{\alpha\beta}^\mu \\ (\alpha(p^2) - a(p^2))p_\mu \overline{\sigma}_{\alpha\beta}^\mu & (\sigma(p^2) - m(p^2))\delta_{\alpha\beta} \end{pmatrix}. \end{aligned} \quad (118)$$

(118) also writes

$$\begin{aligned} \frac{1}{2} \int d^4x e^{ipx} \left(\langle 0 | \mathcal{T} X_{M\alpha}^\pm(x) \overline{X_{M\beta}^\pm(-x)} | 0 \rangle + \langle 0 | \mathcal{T} \Omega_{M\alpha}^\pm(x) \overline{\Omega_{M\beta}^\pm(-x)} | 0 \rangle \right) \\ = \begin{pmatrix} \rho(p^2)\delta_{\alpha\beta} & \alpha(p^2)p_\mu \sigma_{\alpha\beta}^\mu \\ \alpha(p^2)p_\mu \overline{\sigma}_{\alpha\beta}^\mu & \sigma(p^2)\delta_{\alpha\beta} \end{pmatrix}, \\ \frac{1}{2} \int d^4x e^{ipx} \left(\langle 0 | \mathcal{T} X_{M\alpha}^\pm(x) \overline{X_{M\beta}^\pm(-x)} | 0 \rangle - \langle 0 | \mathcal{T} \Omega_{M\alpha}^\pm(x) \overline{\Omega_{M\beta}^\pm(-x)} | 0 \rangle \right) \\ = \begin{pmatrix} \mu(p^2)\delta_{\alpha\beta} & a(p^2)p_\mu \sigma_{\alpha\beta}^\mu \\ a(p^2)p_\mu \overline{\sigma}_{\alpha\beta}^\mu & m(p^2)\delta_{\alpha\beta} \end{pmatrix}. \end{aligned} \quad (119)$$

So, when $C + PCT$ invariance is realized, the most general fermion propagator is equivalent to two Majorana propagators.

The determinant of $\Delta(p^2)$ (114) is the products of the determinants of the matrices in the r.h.s. of (118); so, the poles of the two Majorana propagators in (118) are also poles of $\Delta(p^2)$, and the physical states (eigenstates of the propagator at its poles) are the Majorana fermions X and Ω .

5.7 Conditions for propagating Majorana eigenstates

5.7.1 General conditions for diagonalizing a CPT invariant propagator

We consider the most general PCT invariant propagator (98).

We are only concerned here with neutral fermions, for which diagonalizing each 2×2 sub-matrix of the propagator is meaningful: for charged fermions, this would mix in the same state fermions of different charges, which is impossible as soon as we assume that electric charge is conserved.

The two diagonal 2×2 sub-blocks involve differential operators, with one dotted and one undotted spinor index, factorized by simple functions of space-time. We will suppose that, inside each of these sub-blocks, the four differential operators are identical, such that their elements only differ by the functions of space-time. When we speak about diagonalizing these matrices, this concerns accordingly the space-time functions; then the differential operators follow naturally.

The mass-like sub-blocks are diagonal in spinor indices and involve only functions of space-time.

The propagator \mathcal{P} writes

$$\mathcal{P} = \left(\begin{array}{c|c} |n_L\rangle & |n_R\rangle \end{array} \right) \left(\begin{array}{c|c} K_1 & M_1 \\ \hline M_2 & K_2 \end{array} \right) \left(\begin{array}{c} \langle n_L| \\ \langle n_R| \end{array} \right). \quad (120)$$

K_1, K_2, M_1 and M_2 have *a priori* no special properties, are not hermitian nor symmetric. There always exist U_1 and U_2 , which have no reason to be unitary, such that

$$U_1^{-1}K_1U_1 = \Delta_1 \text{ diagonal}, \quad U_2^{-1}K_2U_2 = \Delta_2 \text{ diagonal}, \quad (121)$$

such that the propagator rewrites

$$\begin{aligned} \mathcal{P} &= \left(\begin{array}{c|c} |n_L\rangle_{U_1} & |n_R\rangle_{U_2} \end{array} \right) \left(\begin{array}{c|c} \Delta_1 & U_1^{-1}M_1U_2 \\ \hline U_2^{-1}M_2U_1 & \Delta_2 \end{array} \right) \left(\begin{array}{c} U_1^{-1}\langle n_L| \\ U_2^{-1}\langle n_R| \end{array} \right) \\ &= \left(\begin{array}{c|c} |\mathfrak{N}_L\rangle & |\mathfrak{N}_R\rangle \end{array} \right) \left(\begin{array}{c|c} \Delta_1 & U_1^{-1}M_1U_2 \\ \hline U_2^{-1}M_2U_1 & \Delta_2 \end{array} \right) \left(\begin{array}{c} \langle N_L| \\ \langle N_R| \end{array} \right), \\ \text{with} \quad &\langle N_L| = U_1^{-1}\langle n_L|, \langle N_R| = U_2^{-1}\langle n_R|, |\mathfrak{N}_L\rangle = |n_L\rangle_{U_1}, |\mathfrak{N}_R\rangle = |n_R\rangle_{U_2}. \end{aligned} \quad (122)$$

We look for M_1 and M_2 .

The propagator can be diagonalized \Leftrightarrow

$$U_1^{-1}M_1U_2 = D_1 \text{ diagonal}, \quad U_2^{-1}M_2U_1 = D_2 \text{ diagonal}. \quad (123)$$

That $[D_1, D_2] = 0$ entails in particular

$$U_1^{-1}M_1M_2U_1 = D_1D_2 \text{ diagonal} = D_2D_1 = U_2^{-1}M_2M_1U_2, \quad (124)$$

which coincides with the commutation of M_1 and M_2 only when $U_1 = U_2$.

Since $[\Delta_1, D_1D_2] = 0 = [\Delta_2, D_1D_2]$, one also gets $U_1^{-1}[K_1, M_1M_2]U_1 = 0 = U_2^{-1}[K_2, M_2M_1]U_2$, which entails

$$[K_1, M_1M_2] = 0 = [K_2, M_2M_1]. \quad (125)$$

(121), (123), (124) and (125) are the conditions that K_1, K_2, M_1 and M_2 must satisfy for the propagator to be diagonalizable; they are must less stringent than the commutation of the four of them.

In practice: One supposes that M_1 and M_2 fulfill condition (125). To determine U_1 and U_2 , one can accordingly use indifferently (121) or (124): U_1 diagonalizes K_1 or M_1M_2 , U_2 diagonalizes K_2 or M_2M_1 . Supposing that (124) is satisfied, M_1M_2 and of M_2M_1 are constrained to have the same eigenvalues, which may give additional restrictions on M_1 and M_2 .

Once U_1 and U_2 are determined, call

$$\mathcal{M}_1 = U_1^{-1}M_1U_2, \quad \mathcal{M}_2 = U_2^{-1}M_2U_1. \quad (126)$$

(124) entails that, in particular, \mathcal{M}_1 and \mathcal{M}_2 must commute. Since U_1 diagonalizes M_1M_2 and U_2 diagonalizes M_2M_1 , $\mathcal{M}_1\mathcal{M}_2$ and $\mathcal{M}_2\mathcal{M}_1$ are diagonal.

Write $\mathcal{M}_1 = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix}$ and $\mathcal{M}_2 = \begin{pmatrix} \mathfrak{p} & \mathfrak{q} \\ \mathfrak{r} & \mathfrak{s} \end{pmatrix}$; by direct inspection, one finds that the two products

$\mathcal{M}_1\mathcal{M}_2$ and $\mathcal{M}_2\mathcal{M}_1$ are diagonal either if \mathcal{M}_1 and \mathcal{M}_2 are diagonal, or if $\mathcal{M}_2 = t \begin{pmatrix} \mathfrak{d} & -\mathfrak{b} \\ -\mathfrak{c} & \mathfrak{a} \end{pmatrix}$, that

is, is proportional to \mathcal{M}_1^{-1} ; in this last case, $\mathcal{M}_1\mathcal{M}_2 = \mathcal{M}_2\mathcal{M}_1$ is proportional to the unit matrix, which means that the eigenvalues of M_1M_2 are all identical (and so are the eigenvalues of M_2M_1).

We are looking for more: the conditions that must satisfy M_1 and M_2 for \mathcal{M}_1 and \mathcal{M}_2 to be separately diagonal. We attempt to find them by putting the additional restriction that the eigenstates are Majorana fermions.

5.7.2 Condition for propagating Majorana fermions

A necessary (but not sufficient) condition for the propagating states to be Majorana is that, by some change of basis, the propagator can be cast in the form

$$\Delta_{Maj}(p^2) = \left(\begin{array}{c|c} \left(\begin{array}{cc} a_1(p^2) & \\ & b_1(p^2) \end{array} \right) p_\mu \sigma_{\alpha\beta}^\mu & \left(\begin{array}{cc} m_1(p^2) & \\ & \mu_1(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} m_2(p^2) & \\ & \mu_2(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} a_2(p^2) & \\ & b_2(p^2) \end{array} \right) p_\mu \overline{\sigma}_{\alpha\beta}^\mu \end{array} \right), \quad (127)$$

with four diagonal 2×2 sub-blocks. Indeed, one can then decompose the propagator into two 4×4 propagators (in a shortened notation) $\begin{pmatrix} a_1 & m_1 \\ m_2 & a_2 \end{pmatrix}$ and $\begin{pmatrix} b_1 & \mu_1 \\ \mu_2 & b_2 \end{pmatrix}$, and the Majorana fermions (see subsection 3.6) are eventually respectively composed with the first components of n_L and n_R , and with the second components of the same set. So, in particular, both kinetic-like and mass-like terms, should be diagonalizable simultaneously²⁵. We note

$$U_1^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad U_2^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad D_1 = \begin{pmatrix} d_1 & 0 \\ 0 & \delta_1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} d_2 & 0 \\ 0 & \delta_2 \end{pmatrix}. \quad (128)$$

One has

$$\begin{aligned} \langle N_L | &= \left(\begin{array}{l} a \langle \xi^\alpha | + b \langle (-i)(\eta^{\dot{\alpha}})^* | \\ c \langle \xi^\alpha | + d \langle (-i)(\eta^{\dot{\alpha}})^* | \end{array} \right), \\ \langle N_R | &= \left(\begin{array}{l} p \langle (-i)\xi_\alpha^* | + q \langle \eta_{\dot{\alpha}} | \\ r \langle (-i)\xi_\alpha^* | + s \langle \eta_{\dot{\alpha}} | \end{array} \right), \\ | \mathfrak{N}_L \rangle &= \frac{1}{ad - bc} \left(\begin{array}{l} d | \xi^\alpha \rangle - c | (-i)(\eta^{\dot{\alpha}})^* \rangle \\ -b | \xi^\alpha \rangle + a | (-i)(\eta^{\dot{\alpha}})^* \rangle \end{array} \right), \\ | \mathfrak{N}_R \rangle &= \frac{1}{ps - qr} \left(\begin{array}{l} s | (-i)\xi_\alpha^* \rangle - r | \eta_{\dot{\alpha}} \rangle \\ -q | (-i)\xi_\alpha^* \rangle + p | \eta_{\dot{\alpha}} \rangle \end{array} \right), \end{aligned} \quad (129)$$

and the question is whether the propagator $\langle 0 | \mathcal{T} \begin{pmatrix} N_L(x) \\ N_R(x) \end{pmatrix} \begin{pmatrix} \mathfrak{N}_L(-x) & \mathfrak{N}_R(-x) \end{pmatrix}^\dagger | 0 \rangle$ can be identified with that of a Majorana fermion and its antifermion (that is, itself). (129) yields in particular the four mass-like propagators

$$\begin{aligned} \langle 0 | \mathcal{T} \left(d\xi^\alpha + ic(\eta^{\dot{\alpha}})^\dagger \right) (x) \left(ip^* \xi_\beta + q^*(\eta_{\dot{\beta}})^\dagger \right) (-x) | 0 \rangle &= (ad - bc) d_1(x) \delta_{\alpha\beta}, \quad (a) \\ \langle 0 | \mathcal{T} \left(-b\xi^\alpha - ia(\eta^{\dot{\alpha}})^\dagger \right) (x) \left(ir^* \xi_\beta + s^*(\eta_{\dot{\beta}})^\dagger \right) (-x) | 0 \rangle &= (ad - bc) \delta_1(x) \delta_{\alpha\beta}, \quad (b) \\ \langle 0 | \mathcal{T} \left(-is(\xi_\alpha)^\dagger - r\eta_{\dot{\alpha}} \right) (x) \left(a^*(\xi^\beta)^\dagger + ib^*\eta^{\dot{\beta}} \right) (-x) | 0 \rangle &= (ps - qr) d_2(x) \delta_{\alpha\beta}, \quad (c) \\ \langle 0 | \mathcal{T} \left(iq(\xi_\alpha)^\dagger + p\eta_{\dot{\alpha}} \right) (x) \left(c^*(\xi^\beta)^\dagger + id^*\eta^{\dot{\beta}} \right) (-x) | 0 \rangle &= (ps - qr) \delta_2(x) \delta_{\alpha\beta}, \quad (d) \end{aligned} \quad (130)$$

which must be the only four non vanishing such propagators since $U_1^{-1} M_1 U_2$ and $U_2^{-1} M_2 U_1$ must be diagonal. We have to identify them with typical mass-like Majorana propagators. For that purpose, we

²⁵Imposing commutation relations between all 2×2 sub-blocks of the propagator is excessive.

have a priori to introduce two Majorana fermions; $X_M^\pm = \begin{pmatrix} \zeta^\alpha \\ \pm(-i)(\zeta_\alpha)^* \end{pmatrix}$, associated, together with its antifermion, to (N_L, N_R) , and $Y_M^\pm = \begin{pmatrix} \chi^\beta \\ \pm(-i)(\chi_\beta)^* \end{pmatrix}$, associated, together with its antifermion, to $(\mathfrak{N}_L, \mathfrak{N}_R)$. An $X - Y$ propagator²⁶ reads (we go to the “bar” fields, which introduces an extra γ^0 ; this has in particular for consequence that “mass-like” propagators now appear on the diagonal)

$$\langle 0 | \mathcal{T} X_M(x) \overline{Y_M}(-x) | 0 \rangle = \begin{pmatrix} \langle 0 | \mathcal{T} \zeta^\alpha(x) (\pm i) \chi_\beta(-x) | 0 \rangle & \langle 0 | \mathcal{T} \zeta^\alpha(x) (\chi^\beta)^\dagger(-x) | 0 \rangle \\ \langle 0 | \mathcal{T} (\zeta_\alpha)^\dagger(x) \chi_\beta(-x) | 0 \rangle & \langle 0 | \mathcal{T} (\mp i) (\zeta_\alpha)^\dagger(x) (\chi^\beta)^\dagger(-x) | 0 \rangle \end{pmatrix}. \quad (131)$$

The four lines of (130) correspond to two mass-like $X - Y$ propagators only if one can associate them into two pairs, such that each pair has the same structure as the diagonal terms of (131). There are accordingly two possibilities: pairing (a) with (c) and (b) with (d), or (a) with (d) and (b) with (c).

* The first possibility requires (κ and λ are proportionality constants) $p = i\lambda a^*$, $q = i\lambda b^*$, $r = -i\kappa c^*$, $s = -i\kappa d^*$, such that

$$U_2^{-1} = i \begin{pmatrix} \lambda a^* & \lambda b^* \\ -\kappa c^* & -\kappa d^* \end{pmatrix}. \quad (132)$$

* The second possibility requires $p = i\rho c^*$, $q = i\rho d^*$, $r = i\theta a^*$, $s = i\theta b^*$ such that

$$U_2^{-1} = i \begin{pmatrix} \rho c^* & \rho d^* \\ \theta a^* & \theta b^* \end{pmatrix}. \quad (133)$$

From now onwards, we furthermore request that a single Majorana fermion propagates in the sense that only \mathcal{T} -products of the type $\langle 0 | \mathcal{T} X^\alpha(x) X_\alpha(-x) | 0 \rangle$ occur, which associates $|\mathcal{N}_L \rangle = |X^\alpha \rangle$ and $\langle N_R | = \langle X_\alpha^* |$. The only possibility is that the coefficients of $|\mathcal{N}_L \rangle$ and $|\mathcal{N}_R \rangle$ in (129) be proportional, and so be the ones of $|\mathcal{N}_R \rangle$ and $|N_L \rangle$ (the two sets of conditions are the same); this gives the supplementary conditions (σ and β are two other proportionality constants) $p = i\sigma d^*$, $q = -i\sigma c^*$, $r = -i\gamma b^*$, $s = i\gamma a^*$, such that

$$U_2^{-1} = i \begin{pmatrix} \sigma d^* & -\sigma c^* \\ -\gamma b^* & \gamma a^* \end{pmatrix}. \quad (134)$$

* First possibility (U_2^{-1} is given by (132) above).

Compatibility between (132) and (134) requires $\frac{q}{p} = \frac{b^*}{a^*} = -\frac{c^*}{d^*} = -\frac{r}{s} = \omega^*$ such that we end up with

$$U_1^{-1} = \begin{pmatrix} a & \omega a \\ -\omega d & d \end{pmatrix}, \quad U_2^{-1} = \begin{pmatrix} p & \omega^* p \\ -\omega^* s & s \end{pmatrix} = \begin{pmatrix} \lambda a^* & \lambda \omega^* a^* \\ \kappa \omega^* d^* & -\kappa d^* \end{pmatrix}. \quad (135)$$

We look for PCT invariant $M_1 = \begin{pmatrix} m_{L1}(x) & \mu_1(x) \\ \mu_1(x) & m_{R1}(x) \end{pmatrix}$ and $M_2 = \begin{pmatrix} m_{L2}(x) & \mu_2(x) \\ \mu_2(x) & m_{R2}(x) \end{pmatrix}$ (see (98)) and their diagonalization according to (123) and (128) by U_1 and U_2 given by (135) and satisfying (125).

²⁶We allow $X \neq Y$; later we shall be more restrictive and request $X = Y$, which better corresponds to the intuitive picture of propagating a definite Majorana fermion.

The equations (121) of diagonalization for the kinetic-like terms $K_1 = \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix}$ and $K_2 = \begin{pmatrix} \alpha & v \\ u & \beta \end{pmatrix}$ (see (98)) yield, for the vanishing of the non-diagonal terms, the conditions

$$\begin{aligned} u - \omega^2 v &= \omega(\alpha - \beta), \\ v - \omega^2 u &= \omega(\alpha - \beta), \\ v - \omega^{*2} u &= \omega^*(\alpha - \beta), \\ u - \omega^{*2} v &= \omega^*(\alpha - \beta). \end{aligned} \quad (136)$$

Likewise, the diagonalization equations (123) for the mass-like terms yield

$$\begin{aligned} \omega^* m_{L1} - \omega m_{R1} &= \mu_1(1 - |\omega|^2), \\ \omega m_{L1} - \omega^* m_{R1} &= \mu_1(1 - |\omega|^2), \\ \omega^* m_{L2} - \omega m_{R2} &= \mu_2(1 - |\omega|^2), \\ \omega m_{L2} - \omega^* m_{R2} &= \mu_2(1 - |\omega|^2). \end{aligned} \quad (137)$$

First, we eliminate the trivial case $\omega = 1$ which brings back to a C invariant propagator.

Subtracting the first or the last two equations of (136) yields $u = v$. One then gets $\alpha - \beta = u \frac{1-\omega^2}{\omega} = u \frac{1-\omega^{*2}}{\omega^*}$, such that ω must be real.

Subtracting the first two equations of (137) also shows that ω must be real as soon as one supposes $m_{L1} + m_{R1} \neq 0$, which we do. Then, one gets $\frac{\mu_1}{m_{L1} - m_{R1}} = \frac{\omega}{1 - \omega^2} = \frac{\mu_2}{m_{L2} - m_{R2}}$. Gathering the results from (136) and (137) leads accordingly to

$$\begin{aligned} K_1 &= u \begin{pmatrix} \alpha & (\alpha - \beta) \frac{\omega}{1 - \omega^2} \\ (\alpha - \beta) \frac{\omega}{1 - \omega^2} & \beta \end{pmatrix} = K_2, \\ M_1 &= \begin{pmatrix} m_{L1} & (m_{L1} - m_{R1}) \frac{\omega}{1 - \omega^2} \\ (m_{L1} - m_{R1}) \frac{\omega}{1 - \omega^2} & m_{R1} \end{pmatrix}, \\ M_2 &= \begin{pmatrix} m_{L2} & (m_{L2} - m_{R2}) \frac{\omega}{1 - \omega^2} \\ (m_{L2} - m_{R2}) \frac{\omega}{1 - \omega^2} & m_{R1} \end{pmatrix}, \end{aligned} \quad (138)$$

and we shall hereafter write $\omega = \tan \vartheta$. The four real symmetric matrices $K_1 = K_2, M_1, M_2$ can be simultaneously diagonalized by the same rotation matrix $U(\vartheta)$ of angle ϑ . After diagonalization, the propagator writes

$$\begin{aligned} \Delta &= \left(|n_L\rangle U \quad |n_R\rangle U \right) \left(\begin{array}{c|c} \delta_+ & \mu_{1+} \\ \hline \delta_- & \mu_{1-} \\ \mu_{2+} & \delta_+ \\ \hline \mu_{2-} & \delta_- \end{array} \right) \begin{pmatrix} U^T \langle n_L | \\ U^T \langle n_R | \end{pmatrix}, \\ \text{with} \quad \delta_{\pm} &= \frac{1}{2} \left(\alpha + \beta \pm \frac{\alpha - \beta}{\cos 2\vartheta} \right), \quad \mu_{1,2,\pm} = \frac{1}{2} \left(m_{L1,2} + m_{R1,2} \pm \frac{m_{L1,2} - m_{R1,2}}{\cos 2\vartheta} \right) \end{aligned} \quad (139)$$

To propagate a Majorana fermion, the condition $\mu_{1+} = \mu_{2+}$ should furthermore be fulfilled. This requires, for arbitrary ϑ , $m_{R1} = m_{R2}, m_{L1} = m_{L2}$ (and thus $\mu_1 = \mu_2$). This corresponds to a propagator (before diagonalization)

$$\Delta = \left(|n_L\rangle \quad |n_R\rangle \right) \left(\begin{array}{cc|cc} \alpha & u & m_L & \mu \\ u & \beta & \mu & m_R \\ \hline m_L & \mu & \alpha & u \\ \mu & m_R & u & \beta \end{array} \right) \begin{pmatrix} \langle n_L | \\ \langle n_R | \end{pmatrix}, \quad \frac{u}{\alpha - \beta} = \frac{\mu}{m_L - m_R}, \quad (140)$$

that is, a CP invariant propagator (see (105)) (the C invariant case corresponds to $\omega = 1$ (see (100)), which has been treated previously). The propagating Majorana fermion are

$$\psi_M = \begin{pmatrix} \cos \vartheta \xi^\alpha - \sin \vartheta (-i(\eta^{\dot{\alpha}})^*) \\ \cos \vartheta (-i(\xi_\gamma)^*) - \sin \vartheta \eta_\gamma \end{pmatrix} \text{ and } \chi_M = \begin{pmatrix} \sin \vartheta \xi^\alpha + \cos \vartheta (-i(\eta^{\dot{\beta}})^*) \\ \sin \vartheta (-i(\xi_\gamma)^*) + \cos \vartheta \eta_\gamma \end{pmatrix}.$$

* Second possibility (U_2^{-1} is given by (133) above). Equating (134), (133) and the expression for U_2^{-1} in (128), one gets $q/p = d^*/c^* = -c^*/d^*$, $s/r = b^*/a^* = -a^*/b^*$, which gives $d = \pm ic$, $b = \pm ia$ and thus

$$U_1^{-1} = \begin{pmatrix} a & \pm ia \\ c & \pm ic \end{pmatrix}, \quad U_2^{-1} = i \begin{pmatrix} \rho c^* & \mp i \rho c^* \\ \pm i \gamma a^* & \gamma a^* \end{pmatrix}. \quad (141)$$

The diagonalization equations (123) for the mass-like terms yield, for the vanishing of the non-diagonal terms, the conditions

$$\begin{aligned} m_{L1} &= -m_{R1}, \\ m_{L2} &= -m_{R2}. \end{aligned} \quad (142)$$

The equations (121) of diagonalization for the kinetic-like terms yield the conditions

$$\begin{aligned} u + v &= \pm i(\alpha - \beta), \\ u + v &= \pm i(\beta - \alpha), \end{aligned} \quad (143)$$

which require $v = -u$, $\beta = \alpha$.

So, the kinetic and mass-like propagators write

$$\begin{aligned} K_1 &= \begin{pmatrix} \alpha & u \\ -u & \alpha \end{pmatrix}, & K_2 &= \begin{pmatrix} \alpha & -u \\ u & \alpha \end{pmatrix}, \\ M_1 &= \begin{pmatrix} m_1 & \mu_1 \\ \mu_1 & -m_1 \end{pmatrix}, & M_2 &= \begin{pmatrix} m_2 & \mu_2 \\ \mu_2 & -m_2 \end{pmatrix}. \end{aligned} \quad (144)$$

K_1 and K_2 , which commute, can be diagonalized simultaneously by a single matrix U . The conditions (125) $[K_1, M_1 M_2] = 0 = [K_2, M_2 M_1]$ require $m_1/m_2 = \mu_1/\mu_2$, such that $M_2 = \chi M_1$. Since $U_1 = U = U_2$, the diagonalization equations (123) for the mass-like propagators rewrite $U^{-1} M_1 U = D_1$, $U^{-1} M_2 U = \chi D_1$, such that the set of four matrices K_1, K_2, M_1, M_2 must commute, which requires $u = 0$. The kinetic-like propagators are thus ‘‘standard’’, *i.e.* proportional to the unit matrix. Before diagonalization, the propagator writes

$$\Delta = \left(\begin{array}{c|c} |n_L\rangle & |n_R\rangle \end{array} \right) \left(\begin{array}{cc|cc} \alpha & & m_1 & \mu_1 \\ & \alpha & \mu_1 & -m_1 \\ \hline \chi m_1 & \chi \mu_1 & \alpha & \\ \chi \mu_1 & -\chi m_1 & & \alpha \end{array} \right) \left(\begin{array}{c} \langle n_L| \\ \langle n_R| \end{array} \right), \quad (145)$$

and, after diagonalization,

$$\Delta = \left(\begin{array}{c|c} |n_L\rangle U & |n_R\rangle U \end{array} \right) \left(\begin{array}{cc|cc} \alpha & & \mu & \\ & \alpha & & -\mu \\ \hline \chi \mu & & \alpha & \\ & -\chi \mu & & \alpha \end{array} \right) \left(\begin{array}{c} U^T \langle n_L| \\ U^T \langle n_R| \end{array} \right),$$

$$\text{with } \mu = \sqrt{m_1^2 + \mu_1^2}. \quad (146)$$

It can propagate Majorana fermions only if $\chi = 1$, such that $M_1 = M_2$. Then, (145) is a special kind of PC invariant propagator (see (105)), which becomes C invariant only when $m_1 = 0$. The two

Majorana fermions have masses $\pm\mu/\alpha$. They are $\psi_M = \begin{pmatrix} \cos \vartheta \xi^\alpha - \sin \vartheta (-i(\eta^\alpha)^*) \\ \cos \vartheta (-i(\xi_\gamma)^*) - \sin \vartheta \eta_\gamma \end{pmatrix}$ and $\chi_M = \begin{pmatrix} \sin \vartheta \xi^\alpha + \cos \vartheta (-i(\eta^\beta)^*) \\ \sin \vartheta (-i(\xi_\gamma)^*) + \cos \vartheta \eta_\gamma \end{pmatrix}$, with $\tan 2\vartheta = \mu_1/m_1$.

5.7.3 Conclusion

For one flavor, a necessary condition for the propagating fermion to be Majorana is either C invariance (which corresponds to $\omega = 1$) or CP invariance²⁷. Reciprocally, the fermion cannot be Majorana (it can only be Dirac) if C and CP are broken^{28 29}.

6 General conclusion

We have gone in this work along the first steps towards the propagator approach to coupled fermions in Quantum Field Theory. We first recalled basic principles, concerning in particular discrete transformations, unitary and antiunitary. After showing on a simple example how ambiguities appear in the classical treatment of a fermionic Lagrangian, we investigated the most general fermionic propagator for one fermion flavor. It is itself a coupled particle-antiparticle system, since the most general couplings between Weyl fermions authorized by Lorentz invariance allows such a situation. We have been in particular able to show that, while the most general $C + CPT$ invariant propagator propagates, as expected, Majorana fermions, a necessary condition for such fermions to propagate is not $C + CPT$ invariance, but (CPT invariance being always assumed) that C and PC are not both broken (C or CP must be unbroken).

We hope to report soon on the case of several flavors and their mixing.

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²⁷Majorana fermions have CP parity = $\pm i$ (see subsection 3.6). The two ($\pm i\gamma^0$) factors cancel in the T -product of the propagator and finally make it CP invariant.

²⁸In the real world with three generations, $K_{\ell 3}$ decays unambiguously show that both C and CP are broken (see for example [11]).

²⁹Majorana fermions being special types of Dirac fermions, the common statement that, in the presence of the most general Dirac + Majorana mass terms, the mass eigenstates are Majorana (see for example [15]) always seemed to us slightly illogical. The conclusion that we draw in the propagator approach is more satisfying: one propagates special types of fermions when some symmetries (C or CP) are unbroken; in the opposite case, that is in the most general case, one instead propagates the most general fermions which are Dirac fermions.

A Notations. Spinors

A.1 Weyl spinors

We adopt the notations of [10], with undotted and dotted indices.

Undotted spinors, contravariant ξ^α or covariant ξ_α can be also called left spinors. Dotted spinors, covariant $\eta_{\dot{\alpha}}$ or contravariant $\eta^{\dot{\alpha}}$ can then be identified as right spinors. They are 2-components complex spinors. The 2-valued spinor indices are not explicitly written.

By an arbitrary transformation of the proper Lorentz group

$$\alpha\delta - \beta\gamma = 1, \quad (147)$$

they transform by

$$\begin{aligned} \xi^{1'} &= \alpha\xi^1 + \beta\xi^2, \\ \xi^{2'} &= \gamma\xi^1 + \delta\xi^2, \\ \eta^{1'} &= \alpha^*\eta^{\dot{1}} + \beta^*\eta^{\dot{2}}, \\ \eta^{2'} &= \gamma^*\eta^{\dot{1}} + \delta^*\eta^{\dot{2}}. \end{aligned} \quad (148)$$

To raise or lower spinor indices, one has to use the metric of $SL(2, C)$

$$g_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_{\alpha\beta}^2; \quad g^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i(\sigma^2)_{\alpha\beta}, \quad (149)$$

and the same for dotted indices. The σ^2 matrix will always be represented with indices down.

$$\xi_\alpha = g_{\alpha\beta}\xi^\beta = i\sigma_{\alpha\beta}^2\xi^\beta, \quad \eta^{\dot{\alpha}} = g^{\dot{\alpha}\dot{\beta}}\eta_{\dot{\beta}} = -i\sigma_{\dot{\alpha}\dot{\beta}}^2\eta_{\dot{\beta}}. \quad (150)$$

One has

$$\xi \cdot \zeta = \xi^\alpha \zeta_\alpha = \xi^1 \zeta^2 - \xi^2 \zeta^1 = -\xi_\alpha \zeta^\alpha \text{ invariant}. \quad (151)$$

By definition, $\eta_{\dot{\alpha}} \sim \xi_\alpha^*$ (transforms as);

$$\eta_{\dot{\alpha}} \sim (g_{\alpha\beta}\xi^\beta)^* = g_{\alpha\beta}(\xi^\beta)^* = i\sigma_{\alpha\beta}^2\xi^{\beta*} : \quad (152)$$

a right-handed Weyl spinor and the complex conjugate of a left-handed Weyl spinor transform alike by Lorentz; likewise, a left-handed spinor transforms like the complex conjugate of a right-handed spinor.

A Dirac (bi-)spinor is

$$\xi_D = \begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\alpha}} \end{pmatrix}. \quad (153)$$

A.2 Pauli and Dirac matrices

Since we work with Weyl fermions, we naturally choose the Weyl representation.

Pauli matrices:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad (154)$$

γ matrices

$$\gamma^0 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right), \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (155)$$

and one notes

$$\gamma^\mu = (\gamma^0, \vec{\gamma}) = \gamma^0 \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \overline{\sigma^\mu} \end{pmatrix}, \quad (156)$$

with

$$\sigma^\mu = (\sigma^0, \vec{\sigma}), \quad \overline{\sigma^\mu} = (\sigma^0, -\vec{\sigma}), \quad \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3). \quad (157)$$

$$\begin{aligned} (\gamma^0)^\dagger &= \gamma^0, (\gamma^5)^\dagger = \gamma^5, (\gamma^{1,2,3})^\dagger = -\gamma^{1,2,3}, \\ (\gamma^0)^* &= \gamma^0, (\gamma^5)^* = \gamma^5, (\gamma^{1,3})^* = \gamma^{1,3}, (\gamma^2)^* = -\gamma^2, \\ (\gamma^0)^2 &= 1, (\gamma^5)^2 = 1, (\gamma^{1,2,3})^2 = -1, \\ \gamma^0(\gamma^0)^\dagger &= 1, \gamma^5(\gamma^5)^\dagger = 1, \gamma^{1,2,3}(\gamma^{1,2,3})^\dagger = 1. \end{aligned} \quad (158)$$

One has

$$(\sigma^0)^2 = 1 = (\sigma^i)^2, \{\sigma^i, \sigma^j\} = 2\delta^{ij}. \quad (159)$$

One has the relation

$$\sigma_{\beta\delta}^2 \sigma_{\alpha\gamma}^2 = \delta_{\beta\gamma} \delta_{\alpha\delta} - \delta_{\alpha\beta} \delta_{\delta\gamma}, \quad (160)$$

and the following one is very useful

$$\sigma^2 \sigma^i \sigma^2 = -(\sigma^i)^*, \quad \sigma^2 \sigma^0 \sigma^2 = \sigma^0 \Rightarrow \sigma^2 \sigma^\mu \sigma^2 = (\sigma^0, -\vec{\sigma}^*) = \overline{\sigma^\mu}^*. \quad (161)$$

As far as kinetic terms are concerned,

$$\gamma^0 \gamma^\mu p_\mu = (\gamma^0)^2 p_\mu \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \overline{\sigma^\mu} \end{pmatrix} = \begin{pmatrix} p^0 - \vec{p} \cdot \vec{\sigma} & 0 \\ 0 & p^0 + \vec{p} \cdot \vec{\sigma} \end{pmatrix}. \quad (162)$$

B The adjoint of an antilinear operator

Following Weinberg [9], let us show that the adjoint of an antilinear operator (see (5) for the definition) \mathcal{A} cannot be defined by $\langle \mathcal{A}\psi | \chi \rangle = \langle \psi | \mathcal{A}^\dagger | \chi \rangle$ ³⁰. Indeed, suppose that we can take the usual definition above, and let c be a c-number; using the antilinearity of \mathcal{A} one gets $\langle \mathcal{A}(c\psi) | \chi \rangle = \langle c^*(\mathcal{A}\psi) | \chi \rangle = c \langle \mathcal{A}\psi | \chi \rangle = c \langle \psi | \mathcal{A}^\dagger | \chi \rangle$ is linear in ψ .

But one has also $\langle \mathcal{A}(c\psi) | \chi \rangle = \langle (c\psi) | \mathcal{A}^\dagger | \chi \rangle = \langle \psi | c^* \mathcal{A}^\dagger | \chi \rangle = c^* \langle \psi | \mathcal{A}^\dagger | \chi \rangle$ is antilinear in ψ , which is incompatible with the result above. So, the two expressions cannot be identical and $\langle \mathcal{A}\psi | \chi \rangle \neq \langle \psi | \mathcal{A}^\dagger | \chi \rangle$.

Weinberg ([9] p.51) defines the adjoint by³¹

$$\langle \psi | \mathcal{A}^\dagger | \chi \rangle \equiv \langle \psi | \mathcal{A}^\dagger | \chi \rangle = \langle \mathcal{A}\psi | \chi \rangle^* = \langle \chi | \mathcal{A} | \psi \rangle \equiv \langle \chi | \mathcal{A} | \psi \rangle \quad (164)$$

³⁰This changes nothing to our demonstrations.

³¹So defined, taking $\psi = \chi$, the adjoint satisfies $\langle \psi | \mathcal{A} | \psi \rangle = \langle \psi | \mathcal{A}^\dagger | \psi \rangle$. This entails in particular that, for a antiunitary operator

$$\langle \psi | \mathcal{A}^\dagger | \psi \rangle \neq \langle \psi | \mathcal{A} | \psi \rangle, \quad (163)$$

unless what happens for antiunitary operators (otherwise the matrix element $\langle \psi | \mathcal{A} | \psi \rangle$ of any antiunitary operator could only be real, which is nonsense).

Then, even for an antilinear and antiunitary operator one has ³²

$$\mathcal{A}^\dagger \mathcal{A} = 1. \quad (165)$$

Indeed, $\langle \psi | \mathcal{A}^\dagger \mathcal{A} | \chi \rangle = \langle \psi | \mathcal{A}^\dagger | \mathcal{A} \chi \rangle \stackrel{(164)}{=} \langle \mathcal{A} \chi | \mathcal{A} | \psi \rangle = \langle \mathcal{A} \chi | \mathcal{A} \psi \rangle \stackrel{\text{antiunitarity}}{=} \langle \psi | \chi \rangle$.

By a similar argument, and because \mathcal{A}^\dagger is also antiunitary, one shows that one can also take $\mathcal{A} \mathcal{A}^\dagger = 1$.

So, both linear unitary \mathcal{U} and antilinear antiunitary \mathcal{A} operators satisfy

$$\mathcal{U} \mathcal{U}^\dagger = 1 = \mathcal{U}^\dagger \mathcal{U}, \quad \mathcal{A} \mathcal{A}^\dagger = 1 = \mathcal{A}^\dagger \mathcal{A}. \quad (166)$$

C Classical versus quantum Lagrangian; complex versus hermitian conjugation

In most literature, a fermionic Lagrangian (specially for neutrinos), is completed by its complex conjugate. This is because, at the classical level, a Lagrangian is a scalar and the fields in there are classical fields, not operators.

However, when fields are quantized, they become operators, so does the Lagrangian which is a sum of (local) products of fields, such that, in this case, the complex conjugate should be replaced by the hermitian conjugate.

Consider for example two Dirac fermions $\chi = \begin{pmatrix} \xi^\alpha \\ \eta_\beta \end{pmatrix}$ and $\psi = \begin{pmatrix} \varphi^\alpha \\ \omega_\beta \end{pmatrix}$; a typical mass term in a

classical Lagrangian reads $\overline{\chi L} \psi_R = (\xi^\alpha)^* \omega_{\dot{\alpha}} = \xi^{\dot{\alpha}} \omega_{\dot{\alpha}} = -\omega_{\dot{\alpha}} \xi^{\dot{\alpha}} = \omega^{\dot{\alpha}} \xi_{\dot{\alpha}}$, where we have supposed that ξ and ω anticommute; its complex conjugate reads then $(\overline{\chi L} \psi_R)^* = \omega^\alpha \xi_\alpha = (\omega^{\dot{\alpha}})^* \xi_{\dot{\alpha}}$.

If we now consider operators $(\overline{\chi L} \psi_R) = [\xi^\alpha]^\dagger [\omega_{\dot{\alpha}}] = [\chi_L]^\dagger [\psi_R]$, and its hermitian conjugate is $[\omega_{\dot{\alpha}}]^\dagger [\xi^\alpha] = [\omega_{\dot{\alpha}}^*] [\xi^\alpha]$. Since $([\chi_L]^\dagger [\psi_R])^\dagger = [\psi_R]^\dagger [\chi_L]$, it only ‘coincides’ with the classical complex conjugate if we adopt the convention

$$\psi_R^\dagger \chi_L = (\omega^{\dot{\beta}})^* \xi_\beta, \quad (167)$$

where one has raised the index of ω and lowered the one of ξ . We will hereafter adopt (167).

D On the use of effective expressions for the P , C and T operators when acting on a Dirac fermion

In the body of this paper we have chosen to work with fundamental Weyl fermions ξ^α and $\eta_{\dot{\alpha}}$. In order to determine the action on these of the discrete symmetries P , C and T , we began by expressing their action on Dirac fermions in terms of γ matrices, and, then, deduced from the obtained rules of transformation the ones for each component.

However, one must be very cautious with respect to the expression of the action of P , C and T in terms of Dirac gamma matrices; this notation indeed easily induces into confusion and error, as we show below. It can be specially misleading when calculating the action of various products of these three symmetries. Only a very careful use of this γ notation can prevent one going astray. This is why, in manipulating the symmetry operators, we take as a general principle to strictly use their action on Weyl fermions, associated with the knowledge of their linearity/antilinearity.

Since, nevertheless, using the Dirac formalism is very common among physicists, we also give in the following the correct rules for manipulating, in this language, discrete transformations and their various products.

³²This is in contradiction with [11].

Let K be a transformation having the following expression on a Dirac fermion ψ : $K \cdot \psi = U_K \psi^{(*)}$, where U_K is a matrix which is in general unitary. In the case of the usual transformations P , C and T , U_K may be expressed in terms of γ matrices. One must keep in mind that this does not provide a complete characterization of the corresponding transformation, but only an effective one that must be handled with extreme care. It can indeed be misleading, specially if one relies on “intuition” to infer from this expression the linearity or antilinearity of the transformation under consideration.

The linear/antilinear character of a transformation cannot be deduced from the form it takes when acting on a Dirac fermion, and one must refrain from doing such an inference which is in particular wrong for C and T .

Indeed, $P \cdot \psi = i\gamma^0 \psi$ and P is linear; $C \cdot \psi = \gamma^2 \psi^*$ and C is linear; $T \cdot \psi = i\gamma^3 \gamma^1 \psi^*$ and T is antilinear; $PCT \cdot \psi = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \psi$ and PCT is antilinear.

To illustrate this, let us investigate three *a priori* possible ways of computing the action of PCT , and compare them with the correct result, obtained by applying directly to Weyl fermions the three transformations successively (taking into account the linearity/antilinearity properties of operators):

* the crudest way consists in basically multiplying the U_K 's, without taking into account any action on a spinor (hence neglecting any consideration concerning complex conjugation);

* the second one [10], that we call “Landau” uses as a rule the composition of the symmetry actions on a Dirac spinor;

* eventually, the third one consists of acting with each operator *only on the fermion field itself*, and making careful use of linearity/antilinearity to pass through the possible other terms that occur on the left of ψ . This last method, as we will see by going back to the transformation resulting for each component of ψ , is the only correct one.

- crude : $PCT = U_P U_C U_T = (i\gamma^0)\gamma^2(i\gamma^3\gamma^1) = -\gamma^0\gamma^1\gamma^2\gamma^3$.
- Landau : $PCT\psi = P(C(T\psi)) = i\gamma^0(\gamma^2(i\gamma^3\gamma^1\psi^*))^* = \gamma^0\gamma^1\gamma^2\gamma^3\psi$, hence $PCT = \gamma^0\gamma^1\gamma^2\gamma^3$.
- cautious :

$$\begin{aligned} \psi &\xrightarrow{T} T \cdot \psi = i\gamma^3\gamma^1\psi^* \\ &\xrightarrow{C} C \cdot (i\gamma^3\gamma^1\psi^*) = i\gamma^3\gamma^1 C \cdot \psi^* = i\gamma^3\gamma^1 (C \cdot \psi)^* = i\gamma^3\gamma^1 (\gamma^2)^* \psi = -i\gamma^3\gamma^1\gamma^2\psi \\ &\xrightarrow{P} P \cdot (-i\gamma^3\gamma^1\gamma^2\psi) = -i\gamma^3\gamma^1\gamma^2 P \cdot \psi = -i\gamma^3\gamma^1\gamma^2 (i\gamma^0\psi) = \gamma^3\gamma^1\gamma^2\gamma^0\psi = -\gamma^0\gamma^1\gamma^2\gamma^3\psi. \end{aligned}$$

Similarly,

- crude : $(PCT)^2 = (-\gamma^0\gamma^1\gamma^2\gamma^3)(-\gamma^0\gamma^1\gamma^2\gamma^3) = -\psi$.
- Landau : $(PCT)^2\psi = PCT(PCT\psi) = (\gamma^0\gamma^1\gamma^2\gamma^3)(\gamma^0\gamma^1\gamma^2\gamma^3)\psi = -\psi$.
- cautious :

$$\begin{aligned} (PCT)^2 \cdot \psi &= (PCT) \cdot ((PCT) \cdot \psi) \\ &= (PCT)(-\gamma^0\gamma^1\gamma^2\gamma^3\psi) \\ &= (-\gamma^0\gamma^1\gamma^2\gamma^3)^*(PCT) \cdot \psi \\ &= -\gamma^0\gamma^1(\gamma^2)^*\gamma^3(PCT) \cdot \psi \\ &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\psi \\ &= \psi. \end{aligned}$$

The “cautious” method is the only one which agrees with that directly inferred from the work on Weyl fermions. Nevertheless, it is to be noted that we obtain the correct sign for PCT (though not for $(PCT)^2$) by the crude calculation. So in order to discriminate without any ambiguity between the three ways of manipulating the symmetry operators when acting on a Dirac fermion, *i.e.* to avoid (or minimize) any risk of accidental agreement due to cancellation of two mistakes, we computed systematically the other

products (of two operators) that we can form, and compare the results with the reliable ones obtained directly on the Weyl fermions. The results are summarized below :

	TP	TC	CP
Crude (trivial product of U 's)	$\xi^\alpha \rightarrow -(\eta^{\dot{\alpha}})^*$ $\eta_{\dot{\alpha}} \rightarrow (\xi_\alpha)^*$ $PT = TP$	$\xi^\alpha \rightarrow -\eta_{\dot{\alpha}}$ $\eta_{\dot{\alpha}} \rightarrow \xi^\alpha$ $CT = TC$	$\xi^\alpha \rightarrow -(\xi_\alpha)^*$ $\eta_{\dot{\alpha}} \rightarrow -(\eta^{\dot{\alpha}})^*$ $PC = CP$
Landau (composition)	$\xi^\alpha \rightarrow (\eta^{\dot{\alpha}})^*$ $\eta_{\dot{\alpha}} \rightarrow -(\xi_\alpha)^*$ $PT = -TP$	$\xi^\alpha \rightarrow \eta_{\dot{\alpha}}$ $\eta_{\dot{\alpha}} \rightarrow -\xi^\alpha$ $CT = TC$	$\xi^\alpha \rightarrow (\xi_\alpha)^*$ $\eta_{\dot{\alpha}} \rightarrow (\eta^{\dot{\alpha}})^*$ $PC = CP$
Cautious (our way of computing)	$\xi^\alpha \rightarrow (\eta^{\dot{\alpha}})^*$ $\eta_{\dot{\alpha}} \rightarrow -(\xi_\alpha)^*$ $PT = TP$	$\xi^\alpha \rightarrow -\eta^{\dot{\alpha}}$ $\eta_{\dot{\alpha}} \rightarrow \xi^\alpha$ $CT = -TC$	$\xi^\alpha \rightarrow (\xi_\alpha)^*$ $\eta_{\dot{\alpha}} \rightarrow (\eta^{\dot{\alpha}})^*$ $PC = CP$
Right result (directly from Weyl fermions)	$\xi^\alpha \rightarrow (\eta^{\dot{\alpha}})^*$ $\eta_{\dot{\alpha}} \rightarrow -(\xi_\alpha)^*$ $PT = TP$	$\xi^\alpha \rightarrow -\eta^{\dot{\alpha}}$ $\eta_{\dot{\alpha}} \rightarrow \xi^\alpha$ $CT = -TC$	$\xi^\alpha \rightarrow (\xi_\alpha)^*$ $\eta_{\dot{\alpha}} \rightarrow (\eta^{\dot{\alpha}})^*$ $PC = CP$

Moreover, our way of computing ensures that $T^2 = 1$, in agreement with the result from Weyl spinors, while one encounters problems with the Landau method which leads to $T^2 = -1$. Indeed, $T^2 \cdot \psi = T \cdot (i\gamma^3\gamma^1\psi^*) = -i\gamma^3\gamma^1 T \cdot \psi^* = -i\gamma^3\gamma^1 (T \cdot \psi)^* = -i\gamma^3\gamma^1 (-i)\gamma^3\gamma^1 \psi = \psi$, while Landau's prescription leads to $T^2 \cdot \psi = i\gamma^3\gamma^1 (i\gamma^3\gamma^1\psi^*)^* = i\gamma^3\gamma^1 (-i)\gamma^3\gamma^1 \psi = \gamma^3\gamma^1\gamma^3\gamma^1 \psi = -\psi$.

References

- [1] B. MACHET, V.A. NOVIKOV & M.I. VYSOTSKY: “Binary Systems of Neutral Mesons in Quantum Field Theory”, *hep-ph/0407268*, *Int. J. Mod. Phys. A* 20 (2005) 5399-5452.
- [2] V.A. NOVIKOV: “Binary systems in QM and in QFT: CPT”, *hep-ph/0509126*, published in “La Thuile 2005, Results and perspectives in particle physics” p.321-332.
- [3] E.P. WIGNER: “Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren” (Vieweg, Braunschweig 1931, p.251-3. Reprinted by Edwards Brothers, Ann Arbor 1944); Translated into english by James J. Griffiths: “Group theory and its Application to the Quantum Mechanics of Atomic Spectra” (Academic Press, New York, 1959).
- [4] R.F. STREATER & A.S. WIGHTMAN: “PCT, Spin and Statistics, and All That”, *The Mathematical Physics Monograph Series*, The Benjamin/Cummings Publishing Company, Inc., Advanced Book Program (Reading, Massachusetts, 1972).
- [5] E.P. WIGNER: “Über die Operation des Zeitumkehr in der Quantenmechanik”, *Nachrichten des Gessellschaft der Wissenschaften zu Göttingen Mathematisch-Physikalische Klasse* (1932), 546-559; see formula III p.216; Translated into english by James J. Griffiths: “Group theory and its Application to the Quantum Mechanics of Atomic Spectra” (Academic Press, New York, 1959); see formula (26.6.a) p.328.
- [6] M. BEUTHE: “Oscillations of neutrinos and mesons in Quantum Field Theory”, *hep-ph/0109119 Phys. Rept.* 375 (2003) 105-218.
- [7] Q. DURET, B. MACHET & M.I. VYSOTSKY: “Mixing angles of quarks and leptons in Quantum Field Theory”, *arXiv:0805.4121 [hep-ph]*.
- [8] N.N. BOGOLUBOV, A.A. LOGUNOV, A.I. OKSAK & I.T. TODOROV: “General principles of Quantum Field Theory”, *Mathematical Physics and Applied Mathematics, Volume 10*, Kluwer Academic Publishers (Dordrecht, Boston, London 1990); translated from the Russian (Nauka Publishers, Moscow 1987) by G.G. Gould.
- [9] S. WEINBERG: “The Quantum Theory of Fields”, vol.1, *Foundations*. Cambridge University Press (Cambridge, UK, 1995).
- [10] V. BERESTETSKI, E. LIFSHITZ & L. PITAYEVSKI: “Théorie Quantique Relativiste” tome 1 (Landau & Lifshitz, *Physique Théorique IV*), Editions MIR (Moscou, 1972).
V.B. BERESTETSKY, E.M. LIFSHITZ & L.P. PITAYEVSKY: “Quantum Electrodynamics”, *Course of Theoretical Physics*, 4 (Pergamon Press, Oxford, UK, 1982).
- [11] G.C. BRANCO, L. LAVOURA & J.P. SILVA: “CP violation”, *International Series of Monographs on Physics* 103 (Clarendon Press, Oxford, 1999).
- [12] C. ITZYKSON & J.B. ZUBER: “Quantum Field Theory”, *McGraw-Hill International Editions, Physics Series* (McGraw-Hill, Singapore 1980).
- [13] M.E. PESKIN & D.V. SCHROEDER: “Quantum Field Theory”, *The Advanced Book Program, Perseus Books* (Reading, Massachusetts, 1995).
- [14] T.D. LEE: “Particle Physics and Introduction to Field Theory”, *Contemporary Concepts in Physics, volume 1* (Harwood Academic Publishers, London, UK, 1982).
- [15] S.M. BILENKY, C. GIUNTI & W. GRIMUS: “Phenomenology of Neutrino Oscillation”, *hep-ph/9812360, Prog. Part. Nucl. Phys.* 43 (1999) 1-86.