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# Error calculus and regularity of Poisson functionals: the lent particle method.

Nicolas BOULEAU\*

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## Abstract

We propose a new method to apply the Lipschitz functional calculus of local Dirichlet forms to Poisson random measures.

## Résumé

**Calcul d'erreur et régularité des fonctionnelles de Poisson : la méthode de la particule prêtée.** Nous proposons une nouvelle méthode pour appliquer le calcul fonctionnel lipschitzien des formes de Dirichlet locales aux mesures aléatoires de Poisson.

## 1 Notation and basic formulae.

Let us consider a local Dirichlet structure with carré du champ  $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$  where  $(X, \mathcal{X}, \nu)$  is a  $\sigma$ -finite measured space called *bottom-space*. Singletons are in  $\mathcal{X}$  and  $\nu$  is diffuse,  $\mathbf{d}$  is the domain of the Dirichlet form  $\epsilon[u] = 1/2 \int \gamma[u] d\nu$ . We denote  $(a, \mathcal{D}(a))$  the generator in  $L^2(\nu)$  (cf. [3]).

A random Poisson measure associated to  $(X, \mathcal{X}, \nu)$  is denoted  $N$ .  $\Omega$  is the configuration space of countable sums of Dirac masses on  $X$  and  $\mathcal{A}$  is the  $\sigma$ -field generated by  $N$ , of law  $\mathbb{P}$  on  $\Omega$ . The space  $(\Omega, \mathcal{A}, \mathbb{P})$  is called *the up-space*. We write  $N(f)$  for  $\int f dN$ . If  $p \in [1, \infty[$  the set  $\{e^{iN(f)} : f \text{ real}, f \in L^1 \cap L^2(\nu)\}$  is total in  $L^p_{\mathbb{C}}(\Omega, \mathcal{A}, \mathbb{P})$ . We put  $\tilde{N} = N - \nu$ . The relation  $\mathbb{E}(\tilde{N}f)^2 = \int f^2 d\nu$  extends and gives sense to  $\tilde{N}(f)$ ,  $f \in L^2(\nu)$ . The Laplace functional and the differential calculus with  $\gamma$  yield

$$(1) \quad \forall f \in \mathbf{d}, \forall h \in \mathcal{D}(a) \quad \mathbb{E}[e^{i\tilde{N}(f)}(\tilde{N}(a[h]) + \frac{i}{2}N(\gamma[f, h]))] = 0.$$

## 2 Product, particle by particle, of a Poisson random measure by a probability measure.

Given a probability space  $(R, \mathcal{R}, \rho)$ , let us consider a Poisson random measure  $N \odot \rho$  on  $(X \times R, \mathcal{X} \times \mathcal{R})$  with intensity  $\nu \times \rho$  such that for  $f \in L^1(\nu)$  and  $g \in L^1(\rho)$  if

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$N(f) = \sum f(x_n)$  then  $(N \odot \rho)(fg) = \sum f(x_n)g(r_n)$  where the  $r_n$ 's are i.i.d. independent of  $N$  with law  $\rho$ . Calling  $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$  the product of all the factors  $(R, \mathcal{R}, \rho)$  involved in the construction of  $N \odot \rho$ , we obtain the following properties : For an  $\mathcal{A} \times \mathcal{X} \times \mathcal{R}$ -measurable and positive function  $F$ ,  $\hat{\mathbb{E}} \int F(\omega, x, r) N \odot \rho(dxdr) = \int F d\rho dN$   $\mathbb{P}$ -a.s.

Let us denote by  $\mathbb{P}_N$  the measure  $\mathbb{P}(d\omega)N_\omega(dx)$  on  $(\Omega \times X, \mathcal{A} \times \mathcal{X})$ . We have

**Lemma 2.1** *Let  $F$  be  $\mathcal{A} \times \mathcal{X} \times \mathcal{R}$ -measurable,  $F \in \mathcal{L}^2(\mathbb{P}_N \times \rho)$  and such that  $\int F(\omega, x, r) \rho(dr) = 0$   $\mathbb{P}_N$ -a.s., then  $\int F d(N \odot \rho)$  is well defined, belongs to  $L^2(\mathbb{P} \times \hat{\mathbb{P}})$  and*

$$(2) \quad \hat{\mathbb{E}} \left( \int F d(N \odot \rho) \right)^2 = \int F^2 dN d\rho \quad \mathbb{P}\text{-a.s.}$$

The argument consists in considering  $F_n$  satisfying

$$\mathbb{E} \int F_n^2 d\nu d\rho < +\infty \text{ and } \mathbb{E} \int (\int |F_n| d\nu)^2 d\rho < +\infty \text{ and then using the relation}$$

$$\hat{\mathbb{E}} \left( \int F_n d(N \odot \rho) \right)^2 = \left( \int F_n d\rho dN \right)^2 - \int (\int F_n d\rho)^2 dN + \int F_n^2 d\rho dN \quad \mathbb{P}\text{-a.s.}$$

### 3 Construction by Friedrichs' method and expression of the gradient.

a) We suppose the space by  $\mathbf{d}$  of the bottom structure is separable, then a gradient exists (cf. [3] Chap. V, p.225 *et seq.*). We denote it  $\flat$  and choose it with values in the space  $L^2(R, \mathcal{R}, \rho)$ . Thus, for  $u \in \mathbf{d}$  we have  $u^\flat \in L^2(\nu \times \rho)$ ,  $\gamma[u] = \int (u^\flat)^2 d\rho$  and  $\flat$  satisfies the chain rule. We suppose in addition, what is always possible, that  $\flat$  takes its values in the subspace orthogonal to the constant 1, i.e.

$$(3) \quad \forall u \in \mathbf{d} \quad \int u^\flat d\rho = 0 \quad \nu\text{-a.s.}$$

This hypothesis is important here as in many applications (cf. [2] Chap V §4.6). We suppose also, but this is not essential (cf. [3] p44)  $1 \in \mathbf{d}_{loc}$   $\gamma[1] = 0$  so that  $1^\flat = 0$ .

b) We define a pre-domain  $\mathcal{D}_0$  dense in  $L^2_{\mathbb{C}}(\mathbb{P})$  by

$$\mathcal{D}_0 = \left\{ \sum_{p=1}^m \lambda_p e^{i\hat{N}(f_p)}; m \in \mathbb{N}^*, \lambda_p \in \mathbb{C}, f_p \in \mathcal{D}(a) \cap L^1(\nu) \right\}.$$

c) We introduce the creation operator inspired from quantum mechanics (see [7], [8], [9], [1], [5],[6] and [10] among others) defined as follows

$$(4) \quad \varepsilon_x^+(\omega) \text{ equals } \omega \text{ if } x \in \text{supp}(\omega), \text{ and equals } \omega + \varepsilon_x \text{ if } x \notin \text{supp}(\omega)$$

so that

$$(5) \quad \varepsilon_x^+(\omega) = \omega \quad N_\omega\text{-a.e. } x \quad \text{and} \quad \varepsilon_x^+(\omega) = \omega + \varepsilon_x \quad \nu\text{-a.e. } x$$

This map is measurable and the Laplace functional shows that for an  $\mathcal{A} \times \mathcal{X}$ -measurable  $H \geq 0$ ,

$$(6) \quad \mathbb{E} \int \varepsilon^+ H d\nu = \mathbb{E} \int H dN.$$

Let us remark also that by (5), for  $F \in \mathcal{L}^2(\mathbb{P}_N \times \rho)$

$$(7) \quad \int \varepsilon^+ F d(N \odot \rho) = \int F d(N \odot \rho) \quad \mathbb{P} \times \hat{\mathbb{P}}\text{-a.s.}$$

d) We defined a gradient  $\sharp$  for the up-structure on  $\mathcal{D}_0$  by putting for  $F \in \mathcal{D}_0$

$$(8) \quad F^\sharp = \int (\varepsilon^+ F)^\flat d(N \odot \rho)$$

this definition being justified by the fact that for  $\mathbb{P}$ -a.e.  $\omega$  the map  $y \mapsto F(\varepsilon_y^+(\omega)) - F(\omega)$  is in  $\mathbf{d}$ ,  $\varepsilon^+ F$  belongs to  $L^\infty(\mathbb{P}) \otimes \mathbf{d}$  algebraic tensor product, and  $(\varepsilon^+ F - F)^\flat = (\varepsilon^+ F)^\flat \in L^2(\mathbb{P}_N \times \rho)$ .

For  $F, G \in \mathcal{D}_0$  of the form

$$F = \sum_p \lambda_p e^{i\tilde{N}(f_p)} = \Phi(\tilde{N}(f_1), \dots, \tilde{N}(f_m)) \quad G = \sum_q \mu_q e^{i\tilde{N}(g_q)} = \Psi(\tilde{N}(g_1), \dots, \tilde{N}(g_n))$$

we compute using (2), (3) and (7) (in the spirit of prop. 1 of [9] or lemma 1.2 of [6])

$$(9) \quad \hat{\mathbb{E}}[F^\sharp \overline{G^\sharp}] = \sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p) - i\tilde{N}(g_q)} \gamma[f_p, g_q]$$

and we have

**Proposition 3.1** *If we put  $A_0[F] = \sum_p \lambda_p e^{i\tilde{N}(f_p)} (i\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p]))$  it comes*

$$(10) \quad \mathbb{E}[A_0[F] \overline{G}] = -\frac{1}{2} \mathbb{E} \sum_{p,q} \Phi'_p \overline{\Psi'_q} N(\gamma[f_p, g_q]).$$

In order to show that  $A_0[F]$  does not depend on the form of  $F$ , by (10) it is enough to show that the expression  $\sum_{p,q} \Phi'_p \overline{\Psi'_q} N(\gamma[f_p, g_q])$  depends only on  $F$  and  $G$ . But this comes from (9) since  $F^\sharp$  and  $G^\sharp$  depend only on  $F$  and  $G$ .

By this proposition,  $A_0$  is symmetric on  $\mathcal{D}_0$ , negative, and the argument of Friedrichs applies (cf [3] p4),  $A_0$  extends uniquely to a selfadjoint operator  $(A, \mathcal{D}(A))$  which defines a closed positive (hermitian) quadratic form  $\mathcal{E}[F] = -\mathbb{E}[A[F] \overline{F}]$ . By (10) contractions operate and (cf. [3])  $\mathcal{E}$  is a Dirichlet form which is local with carré du champ denoted  $\Gamma$  and the up-structure obtained  $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$  satisfies

$$(11) \quad \forall f \in \mathbf{d}, \quad \tilde{N}(f) \in \mathbb{D} \quad \text{and} \quad \Gamma[\tilde{N}(f)] = N(\gamma[f])$$

The operator  $\sharp$  extends to a gradient for  $\Gamma$  as a closed operator from  $L^2(\mathbb{P})$  into  $L^2(\mathbb{P} \times \hat{\mathbb{P}})$  with domain  $\mathbb{D}$  which satisfies the chain rule and may be computed on functionals  $\Phi(\tilde{N}(f_1), \dots, \tilde{N}(f_m))$ ,  $\Phi$  Lipschitz and  $\mathcal{C}^1$  and their limits in  $\mathbb{D}$  (as done in [4]).

Formula (8) for  $\sharp$  can be extended from  $\mathcal{D}_0$  to  $\mathbb{D}$ . Let us introduce the space  $\underline{\mathbb{D}}$  closure of  $\mathcal{D}_0 \otimes \mathbf{d}$  for the norm

$$\|H\|_{\underline{\mathbb{D}}} = (\mathbb{E} \int \gamma[H(\omega, \cdot)](x) N(dx))^{1/2} + \mathbb{E} \int |H(\omega, x)| \xi(x) N(dx)$$

where  $\xi > 0$  is a fixed function such that  $N(\xi) \in L^2(\mathbb{P})$ .

**Theorem 3.1** *The formula  $F^\sharp = \int (\varepsilon^+ F)^\flat d(N \odot \rho)$  decomposes as follows*

$$F \in \mathbb{D} \xrightarrow{\varepsilon^+} \varepsilon^+ F \in \underline{\mathbb{D}} \xrightarrow{\flat} (\varepsilon^+ F)^\flat \in L_0^2(\mathbb{P}_N \times \rho) \xrightarrow{d(N \odot \rho)} F^\sharp \in L^2(\mathbb{P} \times \hat{\mathbb{P}})$$

where each operator is continuous on the range of the preceding one,  $L_0^2(\mathbb{P}_N \times \rho)$  denoting the closed subspace of  $L^2(\mathbb{P}_N \times \rho)$  of  $\rho$ -centered elements, and we have

$$(12) \quad \Gamma[F] = \hat{\mathbb{E}}|F^\sharp|^2 = \int \gamma[\varepsilon^+ F] dN.$$

## 4 The lent particle method.

Let us consider, for instance, a real process  $Y_t$  with independent increments and Lévy measure  $\sigma$  integrating  $x^2$ ,  $Y_t$  being supposed centered without Gaussian part. We assume that  $\sigma$  has an l.s.c. density so that a local Dirichlet structure may be constructed on  $\mathbb{R} \setminus \{0\}$  with carré du champ  $\gamma[f] = x^2 f'^2(x)$ . If  $N$  is the random Poisson measure with intensity  $dt \times \sigma$  we have  $\int_0^t h(s) dY_s = \int 1_{[0,t]}(s) h(s) x \tilde{N}(ds dx)$  and the choice done for  $\gamma$  gives  $\Gamma[\int_0^t h(s) dY_s] = \int_0^t h^2(s) d[Y, Y]_s$  for  $h \in L_{loc}^2(dt)$ . In order to study the regularity of the random variable  $V = \int_0^t \varphi(Y_{s-}) dY_s$  where  $\varphi$  is Lipschitz and  $\mathcal{C}^1$ , we have two ways:

a) We may represent the gradient  $\sharp$  as  $Y_t^\sharp = B_{[Y, Y]_t}$  where  $B$  is a standard auxiliary independent Brownian motion. Then by the chain rule  $V^\sharp = \int_0^t \varphi'(Y_{s-})(Y_{s-})^\sharp dY_s + \int_0^t \varphi(Y_{s-}) dB_{[Y]_s}$  now, using  $(Y_{s-})^\sharp = (Y_s^\sharp)_-$ , a classical but rather tedious stochastic computation yields

$$(13) \quad \Gamma[V] = \hat{\mathbb{E}}[V^{\sharp 2}] = \sum_{\alpha \leq t} \Delta Y_\alpha^2 (\int_{] \alpha}^t \varphi'(Y_{s-}) dY_s + \varphi(Y_{\alpha-}))^2.$$

Since  $V$  has real values the *energy image density property* holds, and  $V$  has a density as soon as  $\Gamma[V]$  is strictly positive a.s. what may be discussed using the relation (13).

b) Another more direct way consists in applying the theorem. For this we define  $\flat$  by choosing  $\eta$  such that  $\int_0^1 \eta(r) dr = 0$  and  $\int_0^1 \eta^2(r) dr = 1$  and putting  $f^\flat = x f'(x) \eta(r)$ .

1°. First step. We add a particle  $(\alpha, x)$  i.e. a jump to  $Y$  at time  $\alpha$  with size  $x$  what gives

$$\varepsilon^+ V - V = \varphi(Y_{\alpha-}) x + \int_{] \alpha}^t (\varphi(Y_{s-} + x) - \varphi(Y_{s-})) dY_s$$

2°.  $V^\flat = 0$  since  $V$  does not depend on  $x$ , and

$$(\varepsilon^+ V)^\flat = (\varphi(Y_{\alpha-}) x + \int_{] \alpha}^t \varphi'(Y_{s-} + x) x dY_s) \eta(r) \quad \text{because } x^\flat = x \eta(r).$$

3°. We compute  $\gamma[\varepsilon^+ V] = \int (\varepsilon^+ V)^\flat 2 dr = (\varphi(Y_{\alpha-}) x + \int_{] \alpha}^t \varphi'(Y_{s-} + x) x dY_s)^2$

4°. We take back the particle we gave, because in order to compute  $\int \gamma[\varepsilon^+ V] dN$  the integral in  $N$  confuses  $\varepsilon^+ \omega$  and  $\omega$ .

That gives  $\int \gamma[\varepsilon^+ V] dN = \int (\varphi(Y_{\alpha-}) + \int_{] \alpha}^t \varphi'(Y_{s-}) dY_s)^2 x^2 N(d\alpha dx)$  and (13).

We remark that both operators  $F \mapsto \varepsilon^+ F$ ,  $F \mapsto (\varepsilon^+ F)^\flat$  are non-local, but instead  $F \mapsto \int (\varepsilon^+ F)^\flat d(N \odot \rho)$  and  $F \mapsto \int \gamma[\varepsilon^+ F] dN$  are local : taking back the lent particle gives the locality.

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