# Entropy of semiclassical measures in dimension 2

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# ENTROPY OF SEMICLASSICAL MEASURES IN DIMENSION 2

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Abstract. We study the asymptotic properties of eigenfunctions of the Laplacian in the case of a compact Riemannian surface of Anosov type or of nonpositive sectional curvature. In both cases, we show that the Kolmogorov-Sinai entropy of a semiclassical measure $\mu$ for the geodesic flow $g^t$ is bounded from below by half of the Ruelle upper bound, i.e.

$$h_{KS}(\mu, g) \geq \frac{1}{2}\int_{S^*M} \chi^+(\rho) d\mu(\rho).$$

## 1. Introduction

In quantum mechanics, the semiclassical principle asserts that in the high energy limit, one should observe classical phenomena. Our main concern will be the study of this property when the classical system is said to be chaotic.

Let $M$ be a compact $\mathcal{C}^\infty$ Riemannian surface. For all $x \in M$, $T^*_xM$ is endowed with a norm $\|.\|_x$ given by the metric over $M$. The geodesic flow $g^t$ over $T^*M$ is defined as the Hamiltonian flow corresponding to the Hamiltonian $H(x,\xi) := \frac{\|\xi\|_x^2}{2}$. This last quantity corresponds to the classical kinetic energy in the case of the absence of potential. As any observable, this quantity can be quantized via pseudodifferential calculus and the quantum operator corresponding to $H$ is $-\frac{\hbar^2\Delta}{2}$ where $\hbar$ is proportional to the Planck constant and $\Delta$ is the Laplace Beltrami operator acting on $L^2(M)$.

Our main result concerns the influence of the classical Hamiltonian behavior on the spectral asymptotic properties of $\Delta$. More precisely, our main interest is the study of the measure $|\psi_\hbar(x)|^2 dx$ where $\psi_\hbar$ is an eigenfunction of $-\frac{\hbar^2\Delta}{2}$ associated to the eigenvalue 1:

$$-\frac{\hbar^2\Delta}{2}\psi_\hbar = \psi_\hbar.$$

This is equivalent to the study of large eigenvalues of $\Delta$. As $M$ is a compact Riemannian manifold, the family $-\hbar^{-2}$ forms a discrete subsequence that tends to infinity. One natural question is to study the (weak) limits of the probability measure $|\psi_\hbar(x)|^2 dx$ as $\hbar$ tends to 0. This means studying the asymptotic behavior of the probability to find a particle in $x$ when the system is in the state $\psi_\hbar$.

In order to study the influence of the Hamiltonian flow, we first need to lift this measure to the cotangent bundle. This can be achieved thanks to pseudodifferential calculus. In fact there exists a procedure of quantization that gives us an operator $\mathrm{Op}_\hbar(a)$ on the phase space $L^2(M)$ for any observable $a(x,\xi)$ in a certain class of symbols. Then a natural way to lift the previous measure is to define the following quantity:

$$\mu_\hbar(a) = \int_{T^*M} a(x,\xi) d\mu_\hbar(x,\xi) := \langle \psi_\hbar, \mathrm{Op}_\hbar(a)\psi_\hbar\rangle_{L^2(M)}.$$

This formula gives a distribution $\mu_\hbar$ on the space $T^*M$ and describes now the distribution in position and velocity.

Let $(\psi_{\hbar_k})$ be a sequence of orthonormal eigenfunctions of the Laplacian corresponding to the eigenvalues $-\hbar_k^{-2}$ such that the corresponding sequence of distributions $\mu_k$ on $T^*M$ converges as $k$ tends to infinity to a limit $\mu$. Such a limit is called a semiclassical measure. Using standard facts of pseudodifferential calculus, it can be shown that $\mu$ is a probability measure that does not

depend on the choice of the quantization $\mathrm{Op}_\hbar$ and that is carried on the unit energy layer

$$S^*M := \left\{ (x,\xi) : H(x,\xi) = \frac{1}{2} \right\}.$$

Moreover, another result from semiclassical analysis, known as the Egorov property, states that for any fixed $t$:

$$\forall a \in \mathcal{C}_c^\infty(T^*M),\ U^{-t}\mathrm{Op}_\hbar(a)U^t = \mathrm{Op}_\hbar(a \circ g^t) + O_t(\hbar), \tag{1}$$

where $U^t$ denotes the quantum propagator $e^{\frac{\imath t\hbar\Delta}{2}}$. Precisely, it says that for fixed times, the quantum evolution is related to the classical evolution under the geodesic flow. From this, it can be deduced that $\mu$ is invariant under the geodesic flow. One natural question to ask is what measures supported on $S^*M$ are in fact semiclassical measures. The corresponding question in quantum chaos is: when the classical behavior is said to be chaotic, what is the set of semiclassical measures? A first result in this direction has been found by Shnirelman [30], Zelditch [33], Colin de Verdière [11]:

**Theorem 1.1.** *Let $(\psi_k)$ be an orthonormal basis of $L^2(M)$ composed of eigenfunctions of the Laplacian. Moreover, suppose the geodesic flow on $S^*M$ is ergodic with respect to Liouville measure. Then, there exists a subsequence $(\mu_{k_p})_p$ of density one that converges to the Liouville measure on $S^*M$ as $p$ tends to infinity.*

By 'density one', we mean that $\frac{1}{n}\sharp\{p : 1 \leq k_p \leq n\}$ tends to one as $n$ tends to infinity. This theorem states that, in the case of an ergodic geodesic flow, almost all eigenfunctions concentrate on the Liouville measure in the high energy limit. This phenomenon is called quantum ergodicity and has many extensions. The Quantum Unique Ergodicity Conjecture states that the set of semiclassical measures should be reduced to the Liouville measure in the case of Anosov geodesic flow [27]. This question still remains widely open. In fact, in the case of negative curvature, there are many measures invariant under the geodesic flow: for example, there exists an infinity of closed geodesics (each of them carrying naturally an invariant measure). In recent papers, Lindenstrauss proved a particular form of the conjecture, the Arithmetic Quantum Unique Ergodicity [25]. Precisely, he proved that for a sequence of Hecke eigenfunctions of the Laplacian on an arithmetic surface, $|\psi|^2 dx$ converges to the Lebesgue measure on the surface. This result is actually the best-known positive result towards the conjecture.
In order to understand the phenomenon of quantum chaos, many people started to study toy models as the cat map (a typical hyperbolic automorphism of $\mathbb{T}^2$). These dynamical systems provide systems with similar dynamical properties to the geodesic flow on a manifold of negative curvature. Moreover, they can be quantized using Weyl formalism and the question of Quantum Ergodicity naturally arises. For example, Bouzouina and de Bièvre proved the Quantum Ergodicity property for the quantized cat map [8]. However, de Bièvre, Faure and Nonnenmacher proved that in this case, the Quantum Unique Ergodicity is too optimistic [18]. In fact, they constructed a sequence of eigenfunctions that converges to $\frac{1}{2}(\delta_0 + \mathrm{Leb})$, where $\delta_0$ is the Dirac measure on 0 and Leb is the Lebesgue measure on $\mathbb{T}^2$. Faure and Nonnenmacher also proved that if we split the semiclassical measure into its pure point, Lebesgue and singular continuous components, $\mu = \mu_{\mathrm{pp}} + \mu_{\mathrm{Leb}} + \mu_{\mathrm{sc}}$, then $\mu_{\mathrm{pp}}(\mathbb{T}^2) \leq \mu_{\mathrm{Leb}}(\mathbb{T}^2)$ and in particular $\mu_{\mathrm{pp}}(\mathbb{T}^2) \leq 1/2$ [19]. As in the case of geodesic flow, there is an arithmetic point of view on this problem. Recently, Kelmer proved that in the case of $\mathbb{T}^{2d}$ ($d \geq 2$, for a generic family of symplectic matrices), either there exists isotropic submanifold invariant under the $2d$ cat map or one has Arithmetic Quantum Unique Ergodicity [23]. Moreover, in the first case, he showed that we can construct semiclassical measure equal to Lebesgue on the isotropic submanifold.

1.1. **Statement of the main result.** In recent papers [2], [5], Anantharaman and Nonnenmacher got concerned with the study of the localization of eigenfunctions on $M$ as in the case of the toy models. They tried to understand it via the Kolmogorov-Sinai entropy. This paper is in the same spirit and our main result gives an information on the set of semiclassical measures in the case of a surface $M$ of nonpositive sectional curvature. More precisely, we give an information on the localization (or complexity) of a semiclassical measure:

**Theorem 1.2.** *Let $M$ be a $\mathcal{C}^\infty$ Riemannian surface of nonpositive sectional curvature and $\mu$ a semiclassical measure. Then,*

$$h_{KS}(\mu, g) \geq \frac{1}{2}\int_{S^*M} U^u(\rho)d\mu(\rho), \tag{2}$$

*where $h_{KS}$ is the Kolmogorv-Sinai entropy and $U^u(\rho)$ is the unstable Riccati solution at $\rho$.*

We recall that the lower bound can be expressed in term of the Lyapunov exponent [20] as

$$h_{KS}(\mu, g) \geq \frac{1}{2}\int_{S^*M} \chi^+(\rho)d\mu(\rho), \tag{3}$$

where $\chi^+(\rho)$ is the upper Lyapunov exponent at the point $\rho$ [7]. In order to comment this result, let us recall a few facts about the Kolmogorov-Sinai (also called metric) entropy. It is a nonnegative number associated to a flow $g$ and a $g$-invariant measure $\mu$, that estimates the complexity of $\mu$ with respect to this flow. For example, a measure carried by a closed geodesic will have entropy zero. In particular, this theorem shows that the support of a semiclassical measure cannot be reduced to a closed unstable geodesic. Moreover, this lower bound seems to be the optimal result we can prove using this method and only the dynamical properties of $M$. In fact, in the case of the toy models some of the counterexamples that have been constructed (see [18], [23]) have entropy equal to $\frac{1}{2}\int_{S^*M} \chi^+(\rho)d\mu(\rho)$. We underline that our inequality is also coherent with the 'scars' constructed by Donnelly [14]. In fact, his 'scars' are supported on closed stable geodesics (included in flat parts of a surface of nonpositive curvature) and have zero entropy. Recall also that a standard theorem of dynamical systems due to Ruelle [28] asserts that, for any invariant measure $\mu$ under the geodesic flow:

$$h_{KS}(\mu, g) \leq \int_{S^*M} \chi^+(\rho)d\mu(\rho) \tag{4}$$

with equality if and only if $\mu$ is the Liouville measure in the case of an Anosov flow [24]. We can make a last observation on the assumptions of the theorem: it is not known whether the geodesic flow is ergodic or not for the Liouville measure on a surface of nonpositive curvature. The best result in this direction is that there exists an open invariant subset $U$ of positive Liouville measure such that the restriction $g_{|U}$ is ergodic with respect to Liouville [7].
The lower bound of theorem 1.2 was conjectured to hold for any semiclassical measure for an Anosov manifold in any dimension by Anantharaman [2]. In fact, Anantharaman proved that in any dimension, the entropy of a semiclassical measure should be bounded from below by a (not really explicit) positive constant [2]. Then, Anantharaman and Nonnenmacher showed that inequality (2) holds in the case of the Walsh Baker's map [4] and in the case of constant negative curvature in all dimension [5]. In the general case of an Anosov flow on a manifold of dimension $d$, Anantharaman, Koch and Nonnenmacher [3] proved a lower bound using the same method:

$$h_{KS}(\mu, g) \geq \int_{S^*M} \sum_{j=1}^{d-1} \chi_j^+(\rho)d\mu(\rho) - \frac{(d-1)\lambda_{\max}}{2}.$$

where $\lambda_{\max} := \lim_{t\to\pm\infty} \frac{1}{t}\log\sup_{\rho\in S^*M} |d_\rho g^t|$ is the maximal expansion rate of the geodesic flow and the $\chi_j^+$'s are the positive Lyapunov exponents [7]. In particular if $\lambda_{\max}$ is very large, the previous inequality can be trivial. However, they conjectured inequality (3) should hold in the general case of manifolds without conjugate points by replacing $\chi^+$ by the sum of nonnegative Lyapunov exponents [5], [3]. Our main result answers this conjecture in the particular case of surfaces of nonpositive curvature (which are particular surfaces without conjugate points) and our proof is really specific to this case. In fact, we use the continuity of $U^u(\rho)$ (which is false in the general case of surfaces without conjugate points [6]) and a strong result by Green [21] and Eberlein [15] about uniform divergence of Jacobi fields which only holds in dimension 2. We note that it could also be an interesting question to look for an extension of this result to ergodic billiards for instance.

In this paper, we also show that the main inequality holds for Anosov surfaces as conjectured by Anantharaman [2]:

**Theorem 1.3.** *Let $M$ be a $\mathcal{C}^\infty$ Riemannian surface and $\mu$ a semiclassical measure. Suppose the geodesic flow $(g^t)_t$ has the Anosov property. Then,*

$$h_{KS}(\mu, g) \geq \frac{1}{2}\left|\int_{S^*M} \log J^u(\rho) d\mu(\rho)\right|, \tag{5}$$

*where $J^u(\rho)$ is the unstable Jacobian at the point $\rho$.*

The lower bound can also be written in terms of the upper Lyapunov exponent [7]. In this theorem, the hypothesis are slightly different. However the procedure of the proof is the same as for theorem 1.2 but simpler in a way. So we will first prove theorem 1.3 and the modifications that need to be made to prove theorem 1.2 will be given in appendix A of the paper. Now let us discuss briefly the main ideas of our proof of theorem 1.3.

1.2. **Heuristic of the proof.** The procedure developed in [3] uses a result known as the entropic uncertainty principle [26]. To use this principle in the semiclassical limit, we need to understand the precise link between the classical evolution and the quantum one for large times. Typically, we have to understand Egorov theorem (1) for large range of times of order $t \sim |\log \hbar|$ (i.e. have a uniform remainder term of (1) for a large range of times). For a general symbol $a$ in $\mathcal{C}_c^\infty(T^*M)$, we can only expect to have a uniform Egorov property for times $t$ in the range of times $[-\frac{1}{2}|\log\hbar|/\lambda_{\max}, \frac{1}{2}|\log\hbar|/\lambda_{\max}]$ [9]. However, if we only consider this range of times, we do not take into account that the unstable jacobian can be very different between two points of $S^*M$. In this paper, we would like to say that the range of times for which the Egorov property holds depends also on the support of the symbol $a(x,\xi)$ we consider. For particular families of symbol of small support (that depends on $\hbar$), we show that we have a 'local' Egorov theorem with an allowed range of times that depends on our symbol (see (66) for example). To make this heuristic idea work, we first try to reparametrize the flow [12] in order to have a uniform expansion rate on the manifold. We define $\overline{g}^\tau(\rho) := g^t(\rho)$ where

$$\tau := -\int_0^t \log J^u(g^s\rho) ds. \tag{6}$$

This new flow $\overline{g}$ has the same trajectories as $g$. However, the 'velocity of motion' along the trajectory at $\rho$ is $|\log J^u(\rho)|$-greater for $\overline{g}$ than for $g$. We underline here that the unstable direction is of dimension 1 (as $M$ is a surface) and it is crucial because it implies that $\log J^u$ exactly measures the expansion rate in the unstable direction at each point[1]. As a consequence, this new flow $\overline{g}$ has a uniform expansion rate. Once this reparametrization is done, we use the following formula to recover $t$ knowing $\tau$:

$$t_\tau(\rho) = \inf\left\{s > 0 : -\int_0^s \log J^u(g^{s'}\rho) ds' \geq \tau\right\}. \tag{7}$$

The number $t_\tau(\rho)$ can be thought of as a stopping time corresponding to $\rho$. We consider now $\tau = \frac{1}{2}|\log\hbar|$. For a given symbol $a(x,\xi)$ localized near a point $\rho$, $t_{\frac{1}{2}|\log\hbar|}(\rho)$ is exactly the range of times for which we can expect Egorov to hold. This new flow seems in a way more adapted to our problem. Moreover, we can define a $\overline{g}$-invariant measure $\overline{\mu}$ corresponding to $\mu$ [12]. The measure $\overline{\mu}$ is absolutely continuous with respect to $\mu$ and verifies $\frac{d\overline{\mu}}{d\mu}(\rho) = \log J^u(\rho)/\int_{S^*M}\log J^u(\rho)d\mu(\rho)$. We can apply the classical result of Abramov:

$$h_{KS}(\mu, g) = \left|\int_{S^*M} \log J^u(\rho) d\mu(\rho)\right| h_{KS}(\overline{\mu}, \overline{g}).$$

[1] In fact, for the Anosov case, the crucial point is that at each point $\rho$ of $S^*M$, the expansion rate is the same in any direction, i.e. $dg^{-1}_{|E^u(g^1\rho)}$ is of the form $J^u(\rho)^{\frac{1}{d-1}} v_\rho$ where $d$ is the dimension of the manifold $M$ and $v_\rho$ is an isometry. The proof of theorem 1.3 can be immediately adapted to Anosov manifolds of higher dimensions satisfying this isotropic expansion property (for example manifolds of constant negative curvature).

To prove theorem 1.3, we would have to show that $h_{KS}(\overline{\mu}, \overline{g}) \geq 1/2$. However, the flow $\overline{g}$ has no reason to be a Hamiltonian flow to which corresponds a quantum propagator $\overline{U}$. As a consequence, there is no particular reason that this inequality should be a consequence of [5]. In the quantum case, there is also no obvious reparametrization we can make as in the classical case. However, we will reparametrize the quantum propagator starting from a discrete reparametrization of the geodesic flow and by introducing a small parameter of time $\eta$. To have an artificial discrete reparametrization of the geodesic flow, we will introduce a suspension set [12]. Then, in this setting, we will define discrete analogues of the previous quantities (6) and (7) that will be precised in the paper. It will allow us to prove a lower bound on the entropy of a certain reparametrized flow and then using Abramov theorem [1] deduce the expected lower bound on the entropy of a semiclassical measure.
Finally, we would like to underline that in a recent paper [22], Gutkin also used a version of the Abramov theorem to prove an analogue of theorem 1.3 in the case of toy models with an unstable direction of dimension 1.

1.3. **Organization of the paper.** In section 2, we briefly recall properties we will need about entropy in the classical and quantum settings. In particular, we recall the version of Abramov theorem we will need. In section 3, we describe the assumptions we make on the manifold $M$ and introduce some notations. In section 4, we draw a precise outline of the proof of theorem 1.3 and state some results that we will prove in the following sections. Sections 5 and 6 are devoted to the detailed proofs of the results we admitted in section 4. In appendix A, we explain which points of the proof need to be modified in the case of nonpositive curvature (theorem 1.2). Finally, sections 7 and appendix B are devoted to results of semiclassical analysis that are quite technical and that we will use at different points of the paper (in particular in section 6).

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## 2. Classical and quantum entropy

2.1. **Kolmogorov-Sinai entropy.** Let us recall a few facts about Kolmogorov-Sinai (or metric) entropy that can be found for example in [32]. Let $(X, \mathcal{B}, \mu)$ be a measurable probability space and $P := (P_\alpha)_{\alpha \in I}$ a finite measurable partition of $X$, i.e. a finite collection of measurable subsets that forms a partition. Each $P_\alpha$ is called an atom of the partition. Assuming $0 \log 0 = 0$, one defines the entropy of the partition as:

$$H(\mu, P) := -\sum_{\alpha \in I} \mu(P_\alpha) \log \mu(P_\alpha) \geq 0. \tag{8}$$

Given two measurable partitions $P := (P_\alpha)_{\alpha \in I}$ and $Q := (Q_\beta)_{\beta \in K}$, one says that $P$ is a refinement of $Q$ if every element of $Q$ can be written as the union of elements of $P$ and it can be shown that $H(\mu, Q) \leq H(\mu, P)$. Otherwise, one denotes $P \vee Q := (P_\alpha \cap Q_\beta)_{\alpha \in I, \beta \in K}$ their join (which is still a partition) and one has $H(\mu, P \vee Q) \leq H(\mu, P) + H(\mu, Q)$ (subadditivity property). Let $T$ be a measure preserving transformation of $X$. The $n$-refined partition $\vee_{i=0}^{n-1} T^{-i} P$ of $P$ with respect to $T$ is then the partition made of the atoms $(P_{\alpha_0} \cap \cdots \cap T^{-(n-1)} P_{\alpha_{n-1}})_{\alpha \in \{1, \cdots, K\}^n}$. We define the entropy with respect to this refined partition:

$$H_n(\mu, T, P) = -\sum_{|\alpha|=n} \mu(P_{\alpha_0} \cap \cdots \cap T^{-(n-1)} P_{\alpha_{n-1}}) \log \mu(P_{\alpha_0} \cap \cdots \cap T^{-(n-1)} P_{\alpha_{n-1}}). \tag{9}$$

Using the subadditivity property of entropy, we have for any integers $n$ and $m$:

$$H_{n+m}(\mu, T, P) \leq H_n(\mu, T, P) + H_m(T^{n\sharp}\mu, T, P) = H_n(\mu, T, P) + H_m(\mu, T, P). \tag{10}$$

For the last equality, it is important to underline that we really use the $T$-invariance of the measure $\mu$. A classical argument for subadditive sequences allows us to define the following quantity:

$$h_{KS}(\mu, T, P) := \lim_{n\to\infty} \frac{H_n(\mu, T, P)}{n}. \tag{11}$$

It is called the Kolmogorov Sinai entropy of $(T, \mu)$ with respect to the partition $P$. The Kolmogorov Sinai entropy $h_{KS}(\mu, T)$ of $(\mu, T)$ is then defined as the supremum of $h_{KS}(\mu, T, P)$ over all partitions $P$ of $X$. Finally, it can be denoted that this quantity can be infinite (not in our case thanks to Ruelle inequality (4) for instance). Note also that if, for all index $(\alpha_0, \cdots, \alpha_{n-1})$, $\mu(P_{\alpha_0} \cap \cdots \cap T^{-(n-1)} P_{\alpha_{n-1}}) \leq Ce^{-\beta n}$ with $C$ positive constant, then $h_{KS}(\mu, T) \geq \beta$: the metric entropy measures the exponential decrease of the atoms of the refined partition.

2.2. **Quantum entropy.** One can defined a quantum counterpart to the metric entropy. Let $\mathcal{H}$ be an Hilbert space. We call a partition of identity $(\tau_\alpha)_{\alpha\in I}$ a family of operators that satisfies the following relation:

$$\sum_{\alpha\in I} \tau_\alpha^* \tau_\alpha = \mathrm{Id}_{\mathcal{H}}. \tag{12}$$

Then, one defines the quantum entropy of a normalized vector $\psi$ as:

$$h_\tau(\psi) := -\sum_{\alpha\in I} \|\tau_\alpha \psi\|^2 \log \|\tau_\alpha \psi\|^2. \tag{13}$$

Finally, one has the following generalization of a theorem from [5] (the proof immediately generalizes to this case), known as the entropic uncertainty principle [26]:

**Theorem 2.1.** *Let $O_\beta$ be a family of bounded operators and $U$ a unitary operator of an Hilbert space $(\mathcal{H}, \|.\|)$. Let $\delta'$ be a positive number. Given $\tau_\alpha$ and $\pi_\beta$ two partitions of identity and $\psi$ a normalized vector in $\mathcal{H}$ such that*

$$\|(Id - O_\beta)\pi_\beta \psi\| \leq \delta'.$$

*Suppose both partitions are of cardinal less than $\mathcal{N}$, then:*

$$h_\tau(U\psi) + h_\pi(\psi) \geq -2\log\left(c_O(U) + \mathcal{N}\delta'\right),$$

*where $c_O(U) = \max_{\alpha,\beta}\left(\|\tau_\alpha U \pi_\beta^* O_\beta\|\right)$, with $\|\tau_\alpha U \pi_\beta^* O_\beta\|$ the operator norm in $\mathcal{H}$.*

2.3. **Entropy of a special flow.** In the previous papers of Anantharaman, Koch and Nonnenmacher (see [3] for example), the main difficulty that was faced to prove main inequality (5) was that the value of $\log J^u(\rho)$ could change a lot depending on the point of the energy layer they looked at. As was mentioned (see section 1.2), we will try to adapt their proof and take into account the changes of the value of $\log J^u(\rho)$. To do this, we will, in a certain way, reparametrize the geodesic flow. Before explaining precisely this strategy, let us recall a classical fact of dynamical system for reparametrization of measure preserving transformations known as the Abramov theorem.
First, let us define a special flow (see [1], [12]). Let $(X, \mathcal{B}, \mu)$ be a probability space, $T$ an automorphism of $X$ and $f$ a measurable function such that $f(x) > a > 0$ for all $x$ in $X$. The function $f$ is called a roof function. We are interested in the set:

$$\overline{X} := \{(x, s) : x \in X, 0 \leq s < f(x)\}. \tag{14}$$

$\overline{X}$ is equipped with the $\sigma$-algebra by restriction of the $\sigma$-algebra on the cartesian product $X \times \mathbb{R}$. For $A$ measurable, one defines $\overline{\mu}(A) := \frac{1}{\int_X f d\mu} \int\int_A d\mu(x) ds$ and $\overline{\mu}(\overline{X}) = 1$.

**Definition 2.2.** The special flow under the automorphism $T$, constructed by the function $f$ is the flow $(\overline{T}^t)$ that acts on $\overline{X}$ in the following way, for $t \geq 0$:

$$\overline{T}^t(x, s) := \left(T^n x, s + t - \sum_{k=0}^{n-1} f\left(T^k x\right)\right), \tag{15}$$

where $n$ is the only integer such that $\sum_{k=0}^{n-1} f\left(T^k x\right) \leq s+t < \sum_{k=0}^{n} f\left(T^k x\right)$.

For $t<0$, one puts, if $s+t>0$:

$$\overline{T}^t(x,s) := (x, s+t),$$

and otherwise,

$$\overline{T}^t(x,s) := \left(T^{-n}x, s+t+\sum_{k=-n}^{-1} f\left(T^k x\right)\right),$$

where $n$ is the only integer such that $-\sum_{k=-n}^{-1} f\left(T^k x\right) \leq s+t < -\sum_{k=-n+1}^{-1} f\left(T^k x\right)$.

*Remark.* A suspension semi-flow can also be defined from an endomorphism.

It can be shown that this special flow preserves the measure $\overline{\mu}$ if $T$ preserves $\mu$ [12]. Finally, we can state Abramov theorem for special flows [1]:

**Theorem 2.3.** *With the previous notations, one has, for all $t \in \mathbb{R}$:*

$$h_{KS}\left(\overline{T}^t, \overline{\mu}\right) = \frac{|t|}{\int_X f d\mu} h_{KS}(T,\mu). \tag{16}$$

## 3. Classical setting of the paper

Before starting the main lines of the proof, we want to describe the classical setting for our surface $M$ and introduce notations that will be useful in the paper. We suppose the geodesic flow over $T^*M$ to have the Anosov property for the first part of the paper (we will treat the adaptation to the nonpositive curvature case in appendix A). This means that for any $\lambda > 0$, the geodesic flow $g^t$ is Anosov on the energy layer $\mathcal{E}(\lambda) := H^{-1}(\lambda) \subset T^*M$ and in particular, the following decomposition holds for all $\rho \in \mathcal{E}(\lambda)$:

$$T_\rho \mathcal{E}(\lambda) = E^u(\rho) \oplus E^s(\rho) \oplus \mathbb{R} X_H(\rho),$$

where $X_H$ is the Hamiltonian vector field associated to $H$, $E^u$ the unstable space and $E^s$ the stable space [10]. It can be denoted that in the setting of this article, they are all one dimensional spaces. The unstable Jacobian $J^u(\rho)$ at the point $\rho$ is defined as the Jacobian of the restriction of $g^{-1}$ to the unstable subspace $E^u(g^1\rho)$:

$$J^u(\rho) := \det\left(dg^{-1}_{|E^u(g^1\rho)}\right).$$

For $\theta$ small positive number ($\theta$ will be fixed all along the paper), one defines $\mathcal{E}^\theta := H^{-1}(]1/2-\theta, 1/2+\theta[)$. As the geodesic flow is Anosov, we can suppose there exist $0 < a_0 < b_0$ such that for all $\rho \in \mathcal{E}^\theta$, $a_0 \leq -\log J^u(\rho) \leq b_0$.

*Remark.* In fact, in the general setting of an Anosov flow, we can only suppose that there exists $k_0 \in \mathbb{N}$ such that $\det\left(dg^{-k_0}_{|E^u(g^{k_0}\rho)}\right) < 1$ for all $\rho \in \mathcal{E}^\theta$. So, to be in the correct setting, we should take $g^{k_0}$ instead of $g$ in the paper. In fact, as $h_{KS}(\mu, g^{k_0}) = k_0 h_{KS}(\mu, g)$ and

$$-\int_{S^*M} \log\det\left(dg^{-k_0}_{|E^u(g^{k_0}\rho)}\right) d\mu(\rho) = -k_0 \int_{S^*M} \log\det\left(dg^{-1}_{|E^u(g^1\rho)}\right) d\mu(\rho),$$

theorem 1.3 follows for $k_0 = 1$ from the case $k_0$ large. However, in order to avoid too many notations, we will suppose $k_0 = 1$.

Let $\epsilon$ and $\eta$ be small positive constants lower than the injectivity radius of the manifold. We choose $\eta$ small enough to have $(2+\frac{b_0}{a_0})b_0\eta \leq \frac{\epsilon}{2}$ (this property will only be used in the proof of proposition 5.3). We denote $f$ the function $-\log J^u$ and remark that there exists $\varepsilon > 0$ such that if $d(\rho,\rho') \leq \varepsilon$ with $\rho, \rho' \in \mathcal{E}^\theta$, then $|f(\rho) - f(\rho')| \leq a_0\epsilon$.

Consider a partition $M = \bigsqcup_{i=1}^K O_i$ of diameter smaller than $\delta$. Let $(\Omega_i)_{i=1}^K$ be a finite open cover of $M$ such that for all $1 \leq i \leq K$, $O_i \subsetneq \Omega_i$. For $\gamma \in \{1, \cdots, K\}^2$, define an open subset of $T^*M$:

$$U_\gamma := (T^*\Omega_{\gamma_0} \cap g^{-\eta} T^*\Omega_{\gamma_1}) \cap \mathcal{E}^\theta.$$

We choose the partition $(O_i)_i$ and the open cover $(\Omega_i)_{i=1}^K$ of $M$ such that $(U_\gamma)_{\gamma \in \{1,\cdots,K\}^2}$ is a finite open cover of diameter smaller than $\varepsilon$ of $\mathcal{E}^\theta$. Then, we define the following quantity, called the discrete Jacobian in time $\eta$:

$$J_\eta^u(\gamma) := \sup\{J^u(\rho) : \rho \in U_\gamma\}, \tag{17}$$

if the previous set is non empty, $e^{-\Lambda}$ otherwise (where $\Lambda$ is a very large constant). Outline that $J_\eta^u(\gamma)$ depends on $\eta$ as $U_\gamma$ depends on $\eta$. The definition can seem quite asymmetric as we consider the supremum of $J^u(\rho)$ and not of $J_\eta^u(\rho)$. However, this choice makes things easier for our analysis. For simplicity of notations, we also define:

$$f(\gamma) := -\eta \log J_\eta^u(\gamma) \leq \eta b_0 \leq \frac{\epsilon}{2}, \tag{18}$$

where the upper bound follows from the previous hypothesis. Moreover, we have for all $\rho \in U_\gamma$,

$$|f(\gamma) + \eta \log J^u(\rho)| \leq a_0 \eta \epsilon. \tag{19}$$

*Remark.* This last inequality shows that even if our choice for $J_\eta^u(\gamma)$ seems quite asymmetric, it allows to have an explicit bound in $\eta$ for quantity (19) and it will be quite useful. With a more symmetric choice, we would not have been able to get an explicit bound in $\eta$ for (19).

## 4. Outline of the proof

Let $(\psi_{\hbar_k})$ be a sequence of orthonormal eigenfunctions of the Laplacian corresponding to the eigenvalues $-\hbar_k^{-2}$ such that the corresponding sequence of distributions $\mu_k$ on $T^*M$ converges as $k$ tends to infinity to the semiclassical measure $\mu$. For simplicity of notations and to fit semiclassical analysis notations, we will denote $\hbar$ tends to 0 the fact that $k$ tends to infinity and $\psi_\hbar$ and $\hbar^{-2}$ the corresponding eigenvector and eigenvalue. As was already mentioned, we will for simplicity concentrate first on the proof of the main inequality (3) in the case of an Anosov flow and we will give in appendix A the adaptations that need to be made in the case of nonpositive curvature. To prove this inequality, we will in particular give a symbolic interpretation of a semiclassical measure and apply the previous results on special flows to this measure.
Let $\epsilon' > 4\epsilon$ be a positive number, where $\epsilon$ was defined in 3. The link between the two quantities $\epsilon$ and $\epsilon'$ will only be used in section 7 to define $\nu$. In the following of the paper, the Ehrenfest time $n_E(\hbar)$ will be the quantity:

$$n_E(\hbar) := [(1 - \epsilon')|\log \hbar|]. \tag{20}$$

We also consider a smaller non integer time:

$$T_E(\hbar) := (1 - \epsilon) n_E(\hbar). \tag{21}$$

### 4.1. Quantum partitions of identity.

In order to find a lower bound on the metric entropy of the semiclassical measure $\mu$, we would like to apply the entropic uncertainty principle (theorem 2.1) and see what informations it will give (when $\hbar$ tends to 0) on the metric entropy of the semiclassical measure $\mu$. To do this, we define quantum partitions of identity corresponding to a given partition of the manifold.

4.1.1. *Partitions of identity.* In section 3, we considered a partition of small diameter $(O_i)_i$ of $M$. We also defined $(\Omega_i)_i$ a corresponding finite open cover of small diameter of $M$. By convolution of the characteristic functions $\mathbf{1}_{O_i}$, we obtain $\mathcal{P} = (P_i)_{i=1,..K}$ a smooth partition of unity on $M$ i.e. for all $x \in M$:

$$\sum_{i=1}^K P_i^2(x) = 1.$$

We assume that for all $1 \leq i \leq K$, $P_i$ is an element of $\mathcal{C}_c^\infty(\Omega_i)$. To this classical partition corresponds a quantum partition of identity of $L^2(M)$. In fact, if $P_i$ denotes the multiplication operator by $P_i(x)$ on $L^2(M)$, then one has:

$$\sum_{i=1}^{K} P_i^* P_i = \mathrm{Id}_{L^2(M)}. \tag{22}$$

4.1.2. *Refinement of the quantum partition under the Schrödinger flow.* Like in the classical setting of entropy (9), we would like to make a refinement of the quantum partition. To do this refinement, we use the Schrödinger propagation operator $U^t = e^{\frac{it\hbar\Delta}{2}}$. We define $A(t) := U^{-t}AU^t$, where $A$ is an operator on $L^2(M)$. To fit as much as possible with the metric entropy (see definition (9) and Egorov property (1)), we define the following operators:

$$\tau_\alpha = P_{\alpha_k}(k\eta)\cdots P_{\alpha_1}(\eta)P_{\alpha_0} \tag{23}$$

and

$$\pi_\beta = P_{\beta_{-k}}(-k\eta)\cdots P_{\beta_{-2}}(-2\eta)P_{\beta_0}P_{\beta_{-1}}(-\eta), \tag{24}$$

where $\alpha = (\alpha_0, \cdots, \alpha_k)$ and $\beta = (\beta_{-k}, \cdots, \beta_0)$ are finite sequences of symbols such that $\alpha_j \in [1, K]$ and $\beta_{-j} \in [1, K]$. We can remark that the definition of $\pi_\beta$ is the analogue for negative times of the definition of $\tau_\alpha$. The only difference is that we switch the two first terms $\beta_0$ and $\beta_{-1}$. The reason of this choice will appear later in the application of the quantum uncertainty principle (see equality (40) in section 5.3). One can see that for fixed $k$, using the Egorov property (1):

$$\|P_{\alpha_k}(k\eta)\cdots P_{\alpha_1}(\eta)P_{\alpha_0}\psi_\hbar\|^2 \to \mu(P_{\alpha_k}^2 \circ g^{k\eta} \times \cdots P_{\alpha_1}^2 \circ g^\eta \times P_{\alpha_0}^2) \text{ as } \hbar \text{ tends to } 0. \tag{25}$$

This last quantity is the one used to compute $h_{KS}(\mu, g^\eta)$ (with the notable difference that the $P_j$ are here smooth functions instead of characteristic functions: see (9)). As was discussed in the heuristic of the proof 1.2, we will have to understand for which range of times $k\eta$, the Egorov property can be be applied. In particular, we will study for which range of times, the operator $\tau_\alpha$ is a pseudodifferential operator of symbol $P_{\alpha_k} \circ g^{k\eta} \times \cdots P_{\alpha_1} \circ g^\eta \times P_{\alpha_0}$ (see (25)). In [5] and [3], they only considered $k\eta \leq |\log\hbar|/\lambda_{\max}$ where $\lambda_{\max} := \lim_{t\to\pm\infty} \frac{1}{t}\log\sup_{\rho\in S^*M} |d_\rho g^t|$. This choice was not optimal and in the following, we try to define sequences $\alpha$ for which we can say that $\tau_\alpha$ is a pseudodifferential operator.

4.1.3. *Index family adapted to the variation of the unstable Jacobian.* Let $\alpha = (\alpha_0, \alpha_1, \cdots)$ be a sequence (finite or infinite) of elements of $\{1, \cdots, K\}$ whose length is larger than 1. We define the following quantity (see (18)):

$$f(\alpha) := f(\alpha_0, \alpha_1).$$

We also define a natural shift on these sequences (again finite or infinite and larger than 1 in length):

$$\sigma((\alpha_0, \alpha_1, \cdots)) := (\alpha_1, \cdots).$$

In the paper, we will try to use the symbol $x$ for infinite sequences and $\alpha$ for finite ones. For negative times, we define the analogous functions, for $\beta := (\cdots, \beta_{-1}, \beta_0)$:

$$\overline{f}(\beta) := \overline{f}(\beta_{-1}, \beta_0)$$

and the backward shift

$$T(\beta) := (\cdots, \beta_{-1}).$$

Let $\alpha$ and $\beta$ be as previously (finite or infinite). We define:

$$\beta.\alpha := (\cdots, \beta_{-1}, \beta_0, \alpha_0, \alpha_1, \cdots).$$

The same obviously works for any sequences of the form $(\cdots, \beta_{p-1}, \beta_p)$ and $(\alpha_q, \alpha_{q+1}, \cdots)$. Then, as described in section 5, index families depending on the value of the unstable Jacobian can be defined as follows:

$$I^\eta(\hbar) := I^\eta(T_E(\hbar)) = \left\{ (\alpha_0, \cdots, \alpha_k) : k \geq 3, \sum_{i=1}^{k-2} f\left(\sigma^i\alpha\right) \leq T_E(\hbar) < \sum_{i=1}^{k-1} f\left(\sigma^i\alpha\right) \right\}, \tag{26}$$

$$K^\eta(\hbar) := K^\eta(T_E(\hbar)) = \left\{ (\beta_{-k}, \cdots, \beta_0) : k \geq 3, \sum_{i=1}^{k-2} \overline{f}\left(T^i\beta\right) \leq T_E(\hbar) < \sum_{i=1}^{k-1} \overline{f}\left(T^i\beta\right) \right\}. \tag{27}$$

These sets define the maximal sequences for which we can expect to have Egorov property for the corresponding $\tau_\alpha$. The sums used to define these sets are in a way a discrete analogue of the integral in the inversion formula (7) defined in the introduction[2]. The sums used to define the allowed sequences are in fact Riemann sums (with small parameter $\eta$) corresponding to the integral (6). We can think of the time $|\alpha|\eta$ as a stopping time for which property (25) will hold (for a symbol $a$ corresponding to $\alpha$).

A good way of thinking of these families of words is by keeping in mind picture 1. On this figure, we draw the case $K = 4$. The biggest square has sides of length 1. Each square represents an element of $I^\eta(\hbar)$ and each square with sides of length $1/2^k$ represents a sequence of length $k+1$ (for $k \geq 0$). If we denote $C(\alpha)$ the square that represents $\alpha$, then we can represent the sequences $\alpha.\gamma$ for each $\gamma$ in $\{1, \cdots, 4\}$ by subdividing the square $C(\alpha)$ in 4 squares of same size. Finally, by definition of $I^\eta(\hbar)$, we can remark that if $\alpha.\gamma$ is represented in the subdivision (for $\gamma$ in $\{1, \cdots, 4\}$), then $\alpha.\gamma'$ is represented in the subdivision for each $\gamma'$ in $\{1, \cdots, 4\}$. Families of operators can

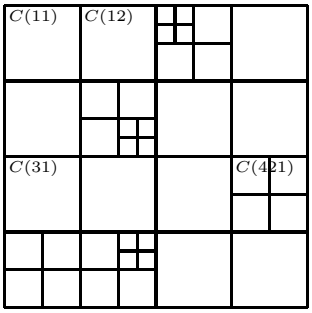


FIGURE 1. *Refinement of variable size*

be associated to these families of index: $(\tau_\alpha)_{\alpha \in I^\eta(\hbar)}$ and $(\pi_\beta)_{\beta \in K^\eta(\hbar)}$. One can show that these partitions form quantum partitions of identity (see section 5):

**Proposition 4.1.**

$$\sum_{\alpha \in I^\eta(\hbar)} \tau_\alpha^* \tau_\alpha = Id_{L^2(M)} \text{ and } \sum_{\beta \in K^\eta(\hbar)} \pi_\beta^* \pi_\beta = Id_{L^2(M)}.$$

4.2. **Symbolic interpretation of semiclassical measures.** Now that we have defined these partitions of variable size, we want to show that they are adapted to compute the entropy of a certain measure with respect to some reparametrized flow associated to the geodesic flow. To do this, we start by giving a symbolic interpretation of the quantum partitions. Denote $\Sigma := \{1, \cdots, K\}^{\mathbb{N}}$ and $\mathcal{C}_i$ the subset of sequences $(x_n)_{n \in \mathbb{N}}$ such that $x_0 = i$. Define also:

$$[\alpha_0, \cdots, \alpha_k] := \mathcal{C}_{\alpha_0} \cap \cdots \cap \sigma^{-k} \mathcal{C}_{\alpha_k},$$

where $\sigma$ is the shift $\sigma((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$ (it fits the notations of the previous section). The set $\Sigma$ is then endowed with the probability measure (not necessarily $\sigma$-invariant):

$$\mu_\hbar^\Sigma\left([\alpha_0, \cdots, \alpha_k]\right) = \mu_\hbar^\Sigma\left(\mathcal{C}_{\alpha_0} \cap \cdots \cap \sigma^{-k}\mathcal{C}_{\alpha_k}\right) = \|P_{\alpha_k}(k\eta) \cdots P_{\alpha_0}\psi_\hbar\|^2.$$

Using the property (12), it is clear that this definition assures the compatibility conditions to define a probability measure:

$$\sum_{\alpha_{k+1}} \mu_\hbar^\Sigma\left([\alpha_0, \cdots, \alpha_{k+1}]\right) = \mu_\hbar^\Sigma\left([\alpha_0, \cdots, \alpha_k]\right).$$

[2]In the higher dimension case mentioned in the footnote of section 1.2, we should take $(d-1)T_E(\hbar)$ (where $d$ is the dimension of $M$) instead of $T_E(\hbar)$ in the definition of $I^\eta(\hbar)$ and $K^\eta(\hbar)$.

Then, we can define the special flow, in the sense of Abramov (section 2.3), associated to this probability measure. To do this, the suspension set (14) is defined as:

$$\overline{\Sigma} := \{(x,s) : \alpha \in \Sigma, 0 \leq s < f(x)\}. \tag{28}$$

Recall that the roof function $f$ is defined as $f(x) := f(x_0, x_1)$. We define a probability measure $\overline{\mu}_\hbar^{\overline{\Sigma}}$ on $\overline{\Sigma}$:

$$\overline{\mu}_\hbar^{\overline{\Sigma}} = \mu_\hbar^\Sigma \times \frac{dt}{\sum_{\alpha\in\{1,\cdots,K\}^2} f(\alpha)\|P_\alpha \psi_\hbar\|^2} = \mu_\hbar^\Sigma \times \frac{dt}{\sum_{\alpha\in\{1,\cdots,K\}^2} f(\alpha)\mu_\hbar^\Sigma([\alpha])}. \tag{29}$$

The semi-flow (15) associated to $\sigma$ is for time $s$:

$$\overline{\sigma}^s(x,t) := \left(\sigma^{n-1}(x), s+t-\sum_{j=0}^{n-2} f\left(\sigma^j x\right)\right), \tag{30}$$

where $n$ is the only integer such that $\sum_{j=0}^{n-2} f\left(\sigma^j x\right) \leq s+t < \sum_{j=0}^{n-1} f\left(\sigma^j x\right)$. In the following, we will only consider time 1 of the flow and its iterates and we will denote $\overline{\sigma} := \overline{\sigma}^1$.

*Remark.* It can be underlined that the same procedure holds for the partition $(\pi_\beta)$. The only differences are that we have to consider $\Sigma_- := \{1,\cdots,K\}^{-\mathbb{N}}$, $T((x_n)_{n\leq 0}) = (x_{n-1})_{n\leq 0}$ and that the corresponding measure is, for $k \geq 1$:

$$\mu_\hbar^{\Sigma_-}([\beta_{-k},\cdots,\beta_0]) = \mu_\hbar^{\Sigma_-}\left(T^{-k}\mathcal{C}_{\beta_{-k}} \cap \cdots \cap \mathcal{C}_{\beta_0}\right) = \|P_{\beta_{-k}}(-k\eta)\cdots P_{\beta_0}P_{\beta_{-1}}(-\eta)\psi_\hbar\|^2.$$

For $k = 0$, one should take the only possibility to assure the compatibility condition:

$$\mu_\hbar^{\Sigma_-}([\beta_0]) = \sum_{j=1}^K \mu_\hbar^{\Sigma_-}([\beta_{-1},\beta_0]).$$

The definition is quite different from the positive case but in the semiclassical limit, it will not change anything as $P_{\beta_0}$ and $P_{\beta_{-1}}(-\eta)$ commute.

Now let $\alpha$ be an element of $I^\eta(\hbar)$. Define:

$$\tilde{\mathcal{C}}_\alpha := \mathcal{C}_{\alpha_0} \cap \cdots \cap \sigma^{-k}\mathcal{C}_{\alpha_k}. \tag{31}$$

This new family of subsets forms a partition of $\Sigma$. Then, a partition $\overline{\mathcal{C}}_\hbar$ of $\overline{\Sigma}$ can be defined starting from the partition $\tilde{\mathcal{C}}$ and $[0, f(\alpha)[$. An atom of this suspension partition is an element of the form $\overline{\mathcal{C}}_\alpha = \tilde{\mathcal{C}}_\alpha \times [0, f(\alpha)[$. For $\overline{\Sigma}^-$ (the suspension set corresponding to $\Sigma_-$), we define an analogous partition $\overline{\mathcal{C}}_\hbar^-$. Finally, with this interpretation, equality (48) from section 5.3 (which is just a careful adaptation of the uncertainty principle) can be read as follows:

$$H\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\mathcal{C}}_\hbar\right) + H\left(\overline{\mu}_\hbar^{\overline{\Sigma}^-}, \overline{\mathcal{C}}_\hbar^-\right) \geq (1-\epsilon')(1-\epsilon)|\log\hbar| + C, \tag{32}$$

where $H$ is defined by (8). To fit as much as possible with the setting of the classical metric entropy, we expect $\overline{\mathcal{C}}_\hbar$ to be a refined partition under the special flow. It is not exactly the case but we can prove the following lemma (see proposition 5.3):

**Lemma 4.2.** *There exists an explicit partition $\overline{\mathcal{C}}$ of $\overline{\Sigma}$, independent of $\hbar$ such that $\vee_{i=0}^{n_E(\hbar)-1}\overline{\sigma}^{-i}\overline{\mathcal{C}}$ is a refinement of the partition $\overline{\mathcal{C}}_\hbar$. Moreover, let $n$ be a fixed positive integer. Then, an atom of the refined partition $\vee_{i=0}^{n-1}\overline{\sigma}^{-i}\overline{\mathcal{C}}$ is of the form $\tilde{\mathcal{C}}_\alpha \times B(\alpha)$, where $\alpha = (\alpha_0,\cdots,\alpha_k)$ is a $k+1$-uple such that $(\alpha_0,\cdots,\alpha_k)$ verifies $n(1-\epsilon) \leq \sum_{j=0}^{k-1} f\left(\sigma^j\alpha\right) \leq n(1+\epsilon)$ and $B(\alpha)$ is a subinterval of $[0, f(\alpha)[$.*

Then applying basic properties of $H$ (see section 2.1), one finds that:

$$H\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\mathcal{C}}_\hbar\right) \leq H\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \vee_{i=0}^{n_E(\hbar)-1}\overline{\sigma}^{-i}\overline{\mathcal{C}}\right) = H_{n_E(\hbar)}\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\sigma}, \overline{\mathcal{C}}\right). \tag{33}$$

To conclude this symbolic interpretation of quantum entropy, with natural notations, inequality (32) together with (33) give the following proposition:

**Proposition 4.3.** *With the previous notations, one has the following inequality:*

$$\frac{1}{n_E(\hbar)}\left(H_{n_E(\hbar)}\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\sigma}, \overline{\mathcal{C}}\right) + H_{n_E(\hbar)}\left(\overline{\mu}_\hbar^{\overline{\Sigma}^-}, \overline{T}, \overline{\mathcal{C}}_-\right)\right) \geq (1-\epsilon) + \frac{C}{n_E(\hbar)}. \tag{34}$$

The quantum entropic uncertainty principle gives an information on the entropy of a special flow. Now, we would like to let $\hbar$ tends to 0 to find a lower on the metric entropy of a limit measure (that we will precise in section 4.3) with respect to $\overline{\sigma}$. However, both $n_E(\hbar)$ and $\mu_\hbar$ depend on $\hbar$ and we have to be careful before passing to the semiclassical limit.

4.3. **Subadditivity of the entropy.** The Egorov property (1) implies that $\mu_\hbar^\Sigma$ tends to a measure $\mu^\Sigma$ on $\Sigma$ (as $\hbar$ tends to 0) defined as follows:

$$\mu^\Sigma\left([\alpha_0, \cdots, \alpha_k]\right) = \mu\left(P_{\alpha_k}^2 \circ g^{k\eta} \times \cdots \times P_{\alpha_0}^2\right). \tag{35}$$

Using the property of partition, this defines a probability measure on $\Sigma$. To this probability measure corresponds a probability measure $\overline{\mu}^{\overline{\Sigma}}$ on the suspension set $\overline{\Sigma}$. It is an immediate corollary that $\overline{\mu}^{\overline{\Sigma}}$ is the limit of the probability measure $\overline{\mu}_\hbar^{\overline{\Sigma}}$. Moreover, using Egorov one more time, one can check that the measure $\mu^\Sigma$ is $\sigma$-invariant and using results about special flows [12], $\overline{\mu}^{\overline{\Sigma}}$ is $\overline{\sigma}$-invariant. The same works for $\mu_\hbar^{\Sigma^-}$ and $\overline{\mu}_\hbar^{\overline{\Sigma}^-}$.

*Remark.* In the following, we will often prove properties in the case of $\Sigma$. The proofs are the same in the case of $\Sigma_-$.

As $n_E(\hbar)$ and $\mu_\hbar$ depend both on $\hbar$, we cannot let $\hbar$ tend to 0 if we want to keep an information about the metric entropy. In fact, the left quantity in (34) does not tend a priori to the Kolmogorov-Sinai entropy. We want to proceed as in the classical case (see (10)) and prove a subadditivity property. This will allow to replace $n_E(\hbar)$ by a fixed $n_0$ (see below) in the left hand side of (34). This is done with the following theorem that will be proved in section 6:

**Theorem 4.4.** *Let $\overline{C}$ be the partition of lemma (4.2). There exists a function $R(n_0, \hbar)$ on $\mathbb{N}\times(0,1]$ such that*

$$\forall n_0 \in \mathbb{N}, \quad \lim_{\hbar\to 0} |R(n_0, \hbar)| = 0.$$

*Moreover, for any $\hbar \in (0,1]$ and any $n_0, m \in \mathbb{N}$ such that $n_0 + m \leq n_E(\hbar)$, one has:*

$$H_{n_0+m}\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\sigma}, \overline{\mathcal{C}}\right) \leq H_{n_0}\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\sigma}, \overline{\mathcal{C}}\right) + H_m\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\sigma}, \overline{\mathcal{C}}\right) + R(n_0, \hbar).$$

*The same holds for $\Sigma_-$.*

This theorem says that the entropy satisfies almost the subadditivity property (see (10)) for time lower than the Ehrenfest time. It is an analogue of a theorem from [5] (proposition 2.8) except that we have taken into account the fact that the unstable jacobian varies on the surface and that we can make our semiclassical analysis for larger time than in [5]. The proof of this theorem is the object of section 6 and 7 (where semiclassical analysis for 'local Ehrenfest time' is performed). Then, one can apply the standard argument for subadditive sequences. Let $n_0$ be a fixed integer in $\mathbb{N}$ and write the euclidian division $n_E(\hbar) = qn_0 + r$ with $r < n_0$. The previous theorem then implies:

$$\frac{H_{n_E(\hbar)}\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\sigma}, \overline{\mathcal{C}}\right)}{n_E(\hbar)} \leq \frac{H_{n_0}\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\sigma}, \overline{\mathcal{C}}\right)}{n_0} + \frac{H_r\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\sigma}, \overline{\mathcal{C}}\right)}{n_E(\hbar)} + \frac{R(n_0, \hbar)}{n_0}.$$

As $r$ stays uniformly bounded in $n_0$, the inequality (34) becomes:

$$\frac{1}{n_0}\left(H_{n_0}\left(\overline{\mu}_\hbar^{\overline{\Sigma}}, \overline{\sigma}, \overline{\mathcal{C}}\right) + H_{n_0}\left(\overline{\mu}_\hbar^{\overline{\Sigma}^-}, \overline{\sigma}_-, \overline{\mathcal{C}}_-\right)\right) \geq (1-\epsilon) + \frac{C(n_0)}{n_E(\hbar)} - 2\frac{R(n_0, \hbar)}{n_0}. \tag{36}$$

4.4. **Application of the Abramov theorem.** Using inequality (36), we can conclude using Abramov theorem (16). Making $\hbar$ tend to 0, one finds that (as was mentioned at the beginning of 4.3):

$$\frac{1}{n_0}\left(H_{n_0}\left(\overline{\mu}^{\overline{\Sigma}},\overline{\sigma},\overline{\mathcal{C}}\right)+H_{n_0}\left(\overline{\mu}^{\overline{\Sigma}_-},\overline{T},\overline{\mathcal{C}}_-\right)\right)\geq(1-\epsilon).$$

The Abramov theorem holds for automorphisms so one can look at the natural extension $\Sigma'=\{1,\cdots,K\}^{\mathbb{Z}}$ of $\Sigma$ and the same inequality holds:

$$\frac{1}{n_0}\left(H_{n_0}\left(\overline{\mu}^{\overline{\Sigma'}},\overline{\sigma'},\overline{\mathcal{C}'}\right)+H_{n_0}\left(\overline{\mu}^{\overline{\Sigma'}},\overline{\sigma'}^{-1},\overline{\mathcal{C}'}_-\right)\right)\geq(1-\epsilon), \tag{37}$$

where $\sigma'$ is the two-sided shift on $\Sigma'$ and $\overline{\mathcal{C}'}$ is the lift of the partition $\overline{\mathcal{C}}$ to the natural extension. In view of section 5, we have an exact expression for $\mathcal{C}$ in terms of the functions $(P_i)_i$ (see proposition 5.3). The measure $\overline{\mu}^{\overline{\Sigma'}}$ is $\overline{\sigma'}$-invariant as $\mu^{\Sigma'}$ is $\sigma'$-invariant [12]. In the previous inequality, there is still one notable difference with the metric entropy: we consider smooth partitions of identity $(P_i)_i$ (as it was necessary to make the semiclassical analysis). To return to the classical case, the procedure of [5] can be adapted using the exact form of the partition $\overline{\mathcal{C}}$ (see proposition 5.3). Recall that each $P_i$ is an element of $\mathcal{C}_c^\infty(\Omega_i)$ and that we considered a partition $M=\bigsqcup_i O_i$ of small diameter $\delta$, where each $O_i\subsetneq\Omega_i$ (see section 3). We suppose it is small enough so that $\mu$ does not charge the boundary of the $O_i$. By convolution of the $\mathbf{1}_{O_i}$, we obtained the smooth partition $(P_i)_i$ of identity of diameter smaller than $2\delta$. The previous inequality does not depend on the derivatives of the $P_i$. Regarding also the form of the partition $\overline{\mathcal{C}}$ (see lemma 4.2), we can replace the smooth functions $P_i$ by the characteristic functions $\mathbf{1}_{O_i}$ in inequality (37). One can let $n_0$ tend to infinity and find:

$$2h_{KS}\left(\overline{\mu}^{\Sigma'},\overline{\sigma'}\right)\geq h_{KS}\left(\overline{\mu}^{\Sigma'},\overline{\sigma'},\overline{\mathcal{C}'}\right)+h_{KS}\left(\overline{\mu}^{\Sigma'},\overline{\sigma'}^{-1},\overline{\mathcal{C}'_-}\right)\geq(1-\epsilon).$$

We used the fact that $h_{KS}(\mu,T^{-1})=h_{KS}(\mu,T)$ for the metric entropy of a dynamical system $(X,\mu,T)$. Then, using Abramov theorem (16), the previous inequality implies:

$$\eta h_{KS}(\mu,g)=h(\mu,g^\eta)=h_{KS}\left(\mu^{\Sigma'},\sigma'\right)\geq-\frac{1}{2}(1-\epsilon)\sum_{\gamma\in\{1,\cdots,K\}^2}\eta\log J^u(\gamma)\,\mu^{\Sigma'}([\gamma]).$$

After division by $\eta$ and letting the diameter of the partition and $\epsilon$ tend to 0, one gets:

$$h_{KS}(\mu,g)\geq\frac{1}{2}\left|\int_{S^*M}\log J^u(\rho)d\mu(\rho)\right|.\square$$

## 5. Partitions of variable size

In this section, we define precisely the index families $I^\eta$ and $K^\eta$ depending on the unstable jacobian used in section 4. These families are used to construct quantum partitions of identity and partitions adapted to the special flow (see section 5.2). In the last section, we apply the uncertainty principle to eigenfunctions of the Laplacian for these quantum partitions of variable size.

5.1. **Stopping time.** Let $t$ be a real positive number that will be greater than $2b_0\eta$. Define index families as follows (see section 4.1.3 for definitions of $f$, $\sigma$, $\overline{f}$ and $T$):

$$I^\eta(t):=\left\{\alpha=(\alpha_0,\cdots,\alpha_k):k\geq3,\sum_{i=1}^{k-2}f\left(\sigma^i\alpha\right)\leq t<\sum_{i=1}^{k-1}f\left(\sigma^i\alpha\right)\right\},$$

$$K^\eta(t):=\left\{\beta=(\beta_{-k},\cdots,\beta_0):k\geq3,\sum_{i=1}^{k-2}\overline{f}\left(T^i\beta\right)\leq t<\sum_{i=1}^{k-1}\overline{f}\left(T^i\beta\right)\right\}.$$

Let $x$ be an element of $\{1,\cdots,K\}^{\mathbb{N}}$. We denote $k_t(x)$ the unique integer $k$ such that

$$\sum_{i=1}^{k-2}f\left(\sigma^ix\right)\leq t<\sum_{i=1}^{k-1}f\left(\sigma^ix\right).$$

In the probability language, $k_t$ is a stopping time in the sense that the property $\{k_t(x) \leq k\}$ depends only on the $k+1$ first letters of $x$. For a finite word $\alpha = (\alpha_0, \cdots, \alpha_k)$, we say that $k = k_t(\alpha)$ if $\alpha$ satisfies the previous inequality. With these notations, $I^\eta(t) := \{\alpha : |\alpha| = k_t(\alpha)+1\}$. The same holds for $K^\eta(t)$.

*Remark.* This stopping time $k_t(\alpha)$ for $t \sim \frac{n_E(\hbar)}{2}$ will be the time for which we will later try to make the Egorov property work. Precisely, we will prove an Egorov property for some symbols corresponding to the sequence $\alpha$ (see (66) for example).

5.2. **Partitions associated.**

5.2.1. *Partitions of identity.* Let $\alpha = (\alpha_0, \cdots, \alpha_k)$ be a finite sequence. Recall that we denoted $\tau_\alpha := P_{\alpha_k}(k\eta)\cdots P_{\alpha_0}$, where $A(s) := U^{-s}AU^s$. In [5] and [3], they used quantum partitions of identity by considering $(\tau_\alpha)_{|\alpha|=k}$. In our paper, we consider a slightly different partition that is more adapted to the variations of the unstable jacobian:

**Lemma 5.1.** *Let $t$ be in $[2b_0\eta, +\infty[$. The family $(\tau_\alpha)_{\alpha\in I^\eta(t)}$ is a partition of identity:*

$$\sum_{\alpha\in I^\eta(t)} \tau_\alpha^*\tau_\alpha = Id_{L^2(M)}.$$

*Proof.* We define for each $1 \leq l \leq N$ (where $N+1$ is the size of the longest word of $I^\eta(t)$):

$$I_l^\eta(t) := \{\alpha = (\alpha_0, \cdots, \alpha_l) : \exists \gamma = (\gamma_{l+1}, \cdots, \gamma_k),\ N \geq k > l \text{ s.t. } \alpha.\gamma \in I^\eta(t)\}.$$

For $l = N$, this set is empty. We want to to show that for each $2 \leq l \leq N$, we have:

$$\sum_{\alpha\in I^\eta(t), |\alpha|=l+1} \tau_\alpha^*\tau_\alpha + \sum_{\alpha\in I_l^\eta(t)} \tau_\alpha^*\tau_\alpha = \sum_{\alpha\in I_{l-1}^\eta(t)} \tau_\alpha^*\tau_\alpha. \tag{38}$$

To prove this equality we use the fact that $\sum_{\gamma=1}^K P_\gamma(l)^* P_\gamma(l) = \mathrm{Id}_{L^2(M)}$ to write:

$$\sum_{\alpha\in I_{l-1}^\eta(t)} \tau_\alpha^*\tau_\alpha = \sum_{\gamma=1}^K \sum_{\alpha\in I_{l-1}^\eta(t)} \tau_{\alpha.\gamma}^*\tau_{\alpha.\gamma}.$$

We split then this sum in two parts to find equality (38). To conclude the proof, we write:

$$\sum_{\alpha\in I^\eta(t)} \tau_\alpha^*\tau_\alpha = \sum_{k=2}^N \sum_{\alpha\in I^\eta(t), |\alpha|=k+1} \tau_\alpha^*\tau_\alpha$$

As $t > 2b_0\eta \geq \max_\gamma f(\gamma)$, the set $I_1^\eta(t)$ is equal to $\{1, \cdots, K\}^2$. By induction from $N$ to 1 using equality (38) at each step, we find then:

$$\sum_{\alpha\in I^\eta(t)} \tau_\alpha^*\tau_\alpha = \mathrm{Id}_{L^2(M)}.$$

□

*Remark.* A step of the induction can be easily understood by looking at figure 2 where each square represents an index over which the sum is made (as it was explained for figure 1).

Following the same procedure, we denote $\pi_\beta = P_{\beta_{-k}}(-k\eta)\cdots P_{\beta_0}P_{\beta_{-1}}(-\eta)$ for $\beta$ in $K^\eta(t)$. These operators follow the relation: $\sum_{\beta\in K} \pi_\beta^*\pi_\beta = \mathrm{Id}_{L^2(M)}$. As was mentioned in section 4.1.2, because of a technical reason that will appear in the application of the entropic uncertainty principle (see (40)), the two definitions are slightly different.

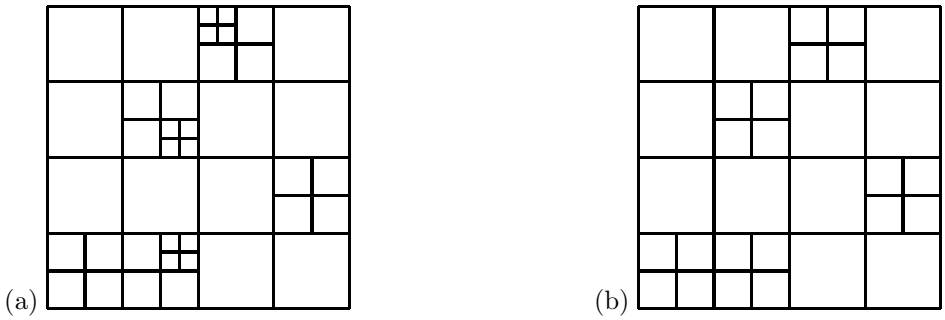


FIGURE 2. *A step of the induction*

5.2.2. *Partitions of* $\{1,\cdots,K\}^{\mathbb{N}}$ *associated.* In this section, we would like to consider some partitions of $\Sigma := \{1,\cdots,K\}^{\mathbb{N}}$ and of $\overline{\Sigma}$ (see (28)) associated to the family $I^\eta(1)$. Recall that:

$$I^\eta(1) := \left\{ \alpha = (\alpha_0,\cdots,\alpha_k) : k \geq 3, \sum_{i=1}^{k-2} f\left(\sigma^i\alpha\right) \leq 1 < \sum_{i=1}^{k-1} f\left(\sigma^i\alpha\right) \right\}.$$

For $\alpha \in I^\eta(1)$, it can be easily remarked that $\sum_{j=0}^{k-1} f\left(\sigma^j\alpha\right) > 1$. It means that there exists a unique integer $k' \leq k$ such that:

$$\sum_{j=0}^{k'-2} f\left(\sigma^j\alpha\right) \leq 1 < \sum_{j=0}^{k'-1} f\left(\sigma^j\alpha\right).$$

In the following, $k$ and $k'$ will be often denoted $k(\alpha) = k_1(\alpha)$ and $k'(\alpha)$ to remember the dependence in $\alpha$. The following lemma can be easily shown:

**Lemma 5.2.** *Let* $\alpha \in I^\eta(1)$. *One has* $|k(\alpha) - k'(\alpha)| \leq \frac{b_0}{a_0} + 1$.

*Proof.* Suppose $k' + 1 < k$ (otherwise it is trivial). Write:

$$\sum_{j=1}^{k-2} f\left(\sigma^j\alpha\right) - \sum_{j=0}^{k'-1} f\left(\sigma^j\alpha\right) \leq 1 - 1 \text{ implies } \sum_{j=k'}^{k-2} f\left(\sigma^j\alpha\right) \leq f\left(\alpha\right).$$

And finally, one finds $(k - 2 - k' + 1)a_0\eta \leq b_0\eta$. □

Let $\alpha$ be an element of $I^\eta(1)$. We make a partition of the interval $[0, f(\alpha)[$ under a form that will be useful (as it is adapted to the dynamics of the special flow). Motivated by the definition of a special flow, let us divide it as follows for $k = k(\alpha)$ and $k' = k'(\alpha)$:

$$I_{k'-2}(\alpha) = [0, \sum_{j=0}^{k'-1} f\left(\sigma^j\alpha\right) - 1[, \cdots I_{p-2}(\alpha) = [\sum_{j=0}^{p-2} f\left(\sigma^j\alpha\right) - 1, \sum_{j=0}^{p-1} f\left(\sigma^j\alpha\right) - 1[, \cdots$$

$$I_{k-2}(\alpha) = [\sum_{j=0}^{k-2} f\left(\sigma^j\alpha\right) - 1, f\left(\alpha\right)[,$$

where $k'(\alpha) \leq p \leq k(\alpha)$. If $k(\alpha) = k'(\alpha)$, one puts $I_{k'-2}(\alpha) = I_{k-2}(\alpha) = [0, f(\alpha)[$.
A partition $\tilde{\mathcal{C}}$ of $\Sigma$ can be defined. It is composed of the following atoms:

$$\tilde{\mathcal{C}}_\gamma := \mathcal{C}_{\gamma_0} \cap \cdots \cap \sigma^{-k}\mathcal{C}_{\gamma_k},$$

where $\gamma$ be an element of $I^\eta(1)$. A partition $\overline{\mathcal{C}}$ of $\overline{\Sigma}$ can be constructed starting from the partition $\tilde{\mathcal{C}}$ and the partition of $[0, f(\gamma)[$. In fact, let $\gamma$ be in $I^\eta(1)$ and $k'(\gamma) \leq p \leq k(\gamma)$. An atom of the partition $\overline{\mathcal{C}}$ is defined as $\overline{\mathcal{C}}_{\gamma,p} = \tilde{\mathcal{C}}_\gamma \times I_{p-2}(\gamma)$. The choice of these specific intervals allows to know

the exact action of $\overline{\sigma}$ on each atom of the partition: for $(x,t) \in \overline{\mathcal{C}}_{\gamma,p}$, $\overline{\sigma}(x,t) = (\sigma^{p-1}(x), 1+t-\sum_{j=0}^{p-2} f(\sigma^j x))$.

5.2.3. *Partitions adapted to the special flow.* In this section, lemma 4.2 is shown and proves in particular that the previous partitions are well adapted to the special flow on $\overline{\Sigma}$. Lemma 4.2 can be written precisely:

**Proposition 5.3.** *Let $n$ be a positive integer. Let $(\gamma_i, p_i)_{0\leq i\leq n-1}$ be a family of couples such that $\gamma_i \in I^\eta(1)$ and $k'(\gamma_i) \leq p_i \leq k(\gamma_i)$. There exists $\alpha'$ in $I^\eta(n(1-\epsilon))$ such that:*

$$\overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \cap \overline{\sigma}^{-(n-1)}\overline{\mathcal{C}}_{\gamma_{n-1},p_{n-1}} \subset \tilde{\mathcal{C}}_{\alpha'} \times [0, f(\gamma_0)[.$$

*Moreover, an atom of the refined partition $\vee_{i=0}^{n-1}\overline{\sigma}^{-i}\overline{\mathcal{C}}$ can be written as follows:*

$$\overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \cap \overline{\sigma}^{-(n-1)}\overline{\mathcal{C}}_{\gamma_{n-1},p_{n-1}} = \tilde{\mathcal{C}}_\alpha \times B(\gamma), \tag{39}$$

*where $\alpha = (\alpha_0, \cdots, \alpha_k)$ is a $k+1$-uple and $B(\gamma)$ a subinterval of $[0, f(\gamma_0)[$ (possibly empty). Finally, $(\alpha_0, \cdots, \alpha_k)$ satisfies $\sum_{j=0}^{k-1} f\left(\sigma^j \alpha\right) \leq n(1+\epsilon)$.*

Before entering the proof, a simple reformulation of the first part of the proposition is that $\vee_{i=0}^{n-1}\overline{\sigma}^{-i}\overline{\mathcal{C}}$ is a refinement of the partition $\left(\tilde{\mathcal{C}}_{\alpha'} \times [0, f(\alpha')[\right)_{\alpha' \in I^\eta(n(1-\epsilon))}$.

*Proof.* We begin by proving the second part of the proposition. First suppose the considered atom is a non empty atom of $\vee_{i=0}^{n-1}\overline{\sigma}^{-i}\overline{\mathcal{C}}$ (otherwise the result is trivial by taking $B(\gamma)$ empty).
Let $(x,t)$ be an element of $\overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \cap \overline{\sigma}^{-(n-1)}\overline{\mathcal{C}}_{\gamma_{n-1},p_{n-1}}$. We denote $k_j = k(\gamma_j)$. The sequence $x$ is of the form $(\gamma_0^0, \cdots, \gamma_0^{k_0}, x')$ and $t$ belongs to $I_{p_0-2}(\gamma_0)$. We recall that for $(x,t) \in \overline{\mathcal{C}}_{\gamma_0,p_0}$:

$$\overline{\sigma}(x,t) = \left(\sigma^{p_0-1}(x), 1+t-\sum_{j=0}^{p_0-2} f\left(\sigma^j x\right)\right).$$

Necessarily, one has $\gamma_1 = (\gamma_0^{p_0-1}, \cdots, \gamma_0^{k_0}, \gamma_1^{k_0-p_0+2}, \cdots, \gamma_1^{k_1})$. Proceeding by induction, one finds that $x = (\gamma_0^0, \cdots, \gamma_0^{k_0}, \gamma_1^{k_0-p_0+2}, \cdots, \gamma_{n-1}^{k_{n-1}}, x")$. Define then $\alpha = (\gamma_0^0, \cdots, \gamma_0^{k_0}, \gamma_1^{k_0-p_0+2}, \cdots, \gamma_{n-1}^{k_{n-1}})$ and:

$$B(\gamma) := \left\{t \in [0, f(\gamma_0)[ : \exists x \text{ st } (x,t) \in \overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \cap \overline{\sigma}^{-(n-1)}\overline{\mathcal{C}}_{\gamma_{n-1},p_{n-1}}\right\}.$$

The first inclusion $\overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \cap \overline{\sigma}^{-(n-1)}\overline{\mathcal{C}}_{\gamma_{n-1},p_{n-1}} \subset \tilde{\mathcal{C}}_\alpha \times B(\gamma)$ is clear.
Now we will prove the converse inclusion. Consider $(x,t)$ an element of $\overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \overline{\sigma}^{-(n-1)}\overline{\mathcal{C}}_{\gamma_{n-1},p_{n-1}}$. The only thing to prove is that $(X,t) = ((\gamma_0^0, \cdots, \gamma_0^{k_0}, \gamma_1^{k_0-p_0+2}, \cdots, \gamma_{n-1}^{k_{n-1}}, x'), t)$ is still an element of $\overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \overline{\sigma}^{-(n-1)}\overline{\mathcal{C}}_{\gamma_{n-1},p_{n-1}}$, for every $x'$ in $\{1, \cdots, K\}^{\mathbb{N}}$. We proceed by induction and suppose $(X,t)$ belongs to $\overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \overline{\sigma}^{-(j-1)}\overline{\mathcal{C}}_{\gamma_{j-1},p_{j-1}}$ for some $j < n$. We have to verify that $\overline{\sigma}^j(X,t)$ belongs to $\overline{\mathcal{C}}_{\gamma_j,p_j}$. As $(X,t)$ belongs to $\overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \overline{\sigma}^{-(j-1)}\overline{\mathcal{C}}_{\gamma_{j-1},p_{j-1}}$, we have:

$$\overline{\sigma}^j(X,t) = \left(\sigma^{p_0+\cdots+p_{j-1}-j}(X), j+t-\sum_{i=0}^{p_0+\cdots+p_{j-1}-j-1} f(\sigma^i X)\right).$$

It has already been mentioned that for all $i$, $(\gamma_i^0, \cdots, \gamma_i^{k_i-p_i+1}) = (\gamma_{i-1}^{p_{i-1}-1}, \cdots, \gamma_{i-1}^{k_i})$ (as the considered atom is not empty). It follows that $\sigma^{p_0+\cdots+p_{j-1}-j}(X)$ belongs to $\tilde{\mathcal{C}}_{\gamma_j}$. We know that $\overline{\sigma}^j(x,t)$ is an element of $\overline{\mathcal{C}}_{\gamma_j,p_j}$ and as a consequence:

$$j+t-\sum_{i=0}^{p_0+\cdots+p_{j-1}-j-1} f(\sigma^i X) = j+t-\sum_{i=0}^{p_0+\cdots+p_{j-1}-j-1} f(\sigma^i x) \in I_{p_j-2}(\gamma_j).$$

By induction, equality (39) is true. For each $0 \leq j \leq n-1$, $t$ belongs to $B(\gamma)$ implies that:

$$t \in I_{p_j-2}(\gamma_j) - j + \sum_{i=0}^{p_0+\cdots+p_{j-1}-j-1} f(\sigma^i \alpha).$$

The set $B(\gamma)$ is then defined as the intersection of $n$ subintervals of $[0, f(\gamma_0)[$ and is in fact a subinterval of $[0, f(\gamma_0)[$.

It remains now to prove upper and lower bounds on $\sum_{j=0}^{k-1} f\left(\sigma^j \alpha\right)$. Recall that:

$$\alpha = (\gamma_0^0, \cdots, \gamma_0^{k_0}, \gamma_1^{k_0-p_0+2}, \cdots, \gamma_1^{k_1}, \cdots, \gamma_{n-1}^{k_{n-1}}).$$

As $0 \leq f(\gamma) \leq \frac{\epsilon}{2}$ for all $\gamma$ (finite or infinite subsequence: see inequality (18)), we have then:

$$\sum_{j=0}^{k-1} f\left(\sigma^j \alpha\right) \leq \sum_{l=0}^{n-2} \sum_{j=0}^{k_l-2} f\left(\sigma^j \gamma_l\right) + \sum_{j=0}^{k_{n-1}-1} f\left(\sigma^j \gamma_{n-1}\right) \leq n(1+\epsilon).$$

For the lower bound, the same kind of procedure works with a little more care. For $\gamma_0$:

$$\sum_{j=1}^{k_0-1} f(\sigma^j \alpha) = \sum_{j=1}^{k_0-1} f(\sigma^j \gamma_0) > 1 > 1-\epsilon.$$

and for $1 \leq l \leq n-1$, one has using lemma 5.2:

$$\sum_{j=k_{l-1}-p_{l-1}+1}^{k_l-1} f(\sigma^j \gamma_l) > 1 - (k_{l-1} - p_{l-1} + 1) b_0 \eta > 1 - (2 + \frac{b_0}{a_0}) b_0 \eta > 1 - \epsilon,$$

where the relations between $\epsilon$, $\eta$, $a_0$ and $b_0$ are defined in section 3. A lower bound on $\sum_{j=1}^{k-1} f(\sigma^j \alpha)$ is $n(1-\epsilon)$. Considering a word $\alpha'$ starting like $\alpha$ but of smaller size, the first part of the proposition is proved as:

$$I^\eta(n(1-\epsilon)) := \left\{ (\alpha'_0, \cdots, \alpha'_k) : k \geq 2, \sum_{j=1}^{k-2} f\left(\sigma^j \alpha'\right) \leq n(1-\epsilon) < \sum_{j=1}^{k-1} f\left(\sigma^j \alpha'\right) \right\}.$$

□

*Remark.* As a final comment on this section, we underline again that all the proofs have been written in the case of $\{1, \cdots, K\}^{\mathbb{N}}$.

5.3. **Uncertainty principle for eigenfunctions of the Laplacian.** In the previous section 5.2, we have seen that the partitions of variable size are well adapted to the reparametrized flow (used in the Abramov theorem). In section 4, we used also the fact that we have a lower bound (32) on the entropy of $\overline{\mu_\hbar^\Sigma}$ with respect to the suspension application $\overline{\sigma}$. The goal of this last section is to prove this lower bound (precisely proposition 5.7). To do this, we use the entropic uncertainty principle (theorem 2.1).

5.3.1. *Energy cutoff.* Before applying the uncertainty principle, we proceed to sharp energy cutoffs so as to get precise lower bounds on the quantum entropy (as it was done in [2], [5] and [3]). These cutoffs are made in our microlocal analysis in order to get as good exponential decrease as possible of the norm of the refined quantum partition. This cutoff in energy is possible because even if the distributions $\mu_\hbar$ are defined on $T^*M$, they concentrate on the energy layer $S^*M$. The following energy localization is made in a way to compactify the phase space and in order to preserve the semiclassical measure.

Let $\delta$ be a positive number less than 1 and $\chi_\delta(t)$ in $\mathcal{C}^\infty(\mathbb{R}, [0,1])$. Moreover, $\chi_\delta(t) = 1$ for $|t| \leq e^{-\delta/2}$

and $\chi_\delta(t) = 0$ for $|t| \geq 1$. As in [5], the sharp $\hbar$-dependent cutoffs are then defined in the following way:

$$\forall \hbar \in (0,1),\ \forall n \in \mathbb{N},\ \forall \rho \in T^*M, \quad \chi^{(n)}(\rho, \hbar) := \chi_\delta(e^{-n\delta}\hbar^{-1+\delta}(H(\rho) - 1/2)).$$

For $n$ fixed, the cutoff $\chi^{(n)}$ is localized in an energy interval of length $2e^{n\delta}\hbar^{1-\delta}$ centered around the energy layer $\mathcal{E}$. In this paper, indices $n$ will satisfy $2e^{n\delta}\hbar^{1-\delta} << 1$. It implies that the widest cutoff is supported in an energy interval of microscopic length and that $n \leq K_\delta |\log \hbar|$, where $K_\delta \leq \delta^{-1}$. Using then a non standard pseudodifferential calculus (see [5] for a brief reminder of the procedure from [31]), one can quantize these cutoffs into pseudodifferential operators. We will denote $\mathrm{Op}(\chi^{(n)})$ the quantization of $\chi^{(n)}$. The main properties of this quantization are recalled in section B.2. In particular, the quantization of these cutoffs preserves the eigenfunctions of the Laplacian:

$$\|\psi_\hbar - \mathrm{Op}(\chi^{(n)})\psi_\hbar\| = O(\hbar^\infty)\|\psi_\hbar\|.$$

5.3.2. *Applying the entropic uncertainty principle.* To get bound on the entropy of the suspension measure, the entropic uncertainty principle should not be applied to the family of operators $(\tau_\alpha)_{\alpha \in I^\eta(\hbar)}$ directly but it will be applied several times to get terms of the form $c_\alpha \|\tau_\alpha \psi_\hbar\|^2$ (see (32)) instead of one of the form $\|\tau_\alpha \psi_\hbar\|^2$ in the formula of the quantum entropy (13). We remind that the $c_\alpha$ correspond to the Lebesgues part of the suspension measure $\overline{\mu}_\hbar^\Sigma$. They are defined by

$$c_\alpha = \frac{f(\alpha)}{\sum_{\gamma \in \{1,\cdots,K\}^2} f(\gamma)\mu_\hbar^\Sigma([\gamma])}.$$

The previous goal can be achieved after defining new families of quantum partitions.
Let $\gamma = (\gamma_0, \gamma_1)$ be an element of $\{1, \cdots, K\}^2$. Denote then:

$$I_\hbar(\gamma) := \{(\alpha') : \gamma.\alpha' \in I^\eta(\hbar)\},$$

$$K_\hbar(\gamma) := \{(\beta') : \beta'.\gamma \in K^\eta(\hbar)\}.$$

The following partitions of identity can be associated to them, for $\alpha' \in I_\hbar(\gamma)$ and $\beta' \in K_\hbar(\gamma)$:

$$\tilde{\tau}_{\alpha'} = P_{\alpha'_n}(n\eta) \cdots P_{\alpha'_2}(2\eta),$$

$$\tilde{\pi}_{\beta'} = P_{\beta'_{-n}}(-n\eta) \cdots P_{\beta'_{-2}}(-2\eta).$$

For analogous reasons as the case of $I^\eta(\hbar)$, the families $(\tilde{\tau}_{\alpha'})_{\alpha' \in I_\hbar(\gamma)}$ and $(\tilde{\pi}_{\beta'})_{\beta' \in I_\hbar(\gamma)}$ form quantum partitions of identity for every $\gamma = (\gamma_0, \gamma_1) \in \{1, \cdots, K\}^2$.
Given these new quantum partitions of identity, the entropic principle should be applied for given initial conditions in times 0 and 1. Let $\|\psi_\hbar\| = 1$ be a fixed element of the sequence of eigenfunctions of the Laplacian defined earlier, associated to the eigenvalue $\frac{1}{\hbar^2}$.
Let $\gamma = (\gamma_0, \gamma_1)$ be an element of $\{1, \cdots, K\}^2$. Define:

$$P_\gamma := P_{\gamma_1} P_{\gamma_0}(-\eta).$$

We underline that for $\alpha' \in I_\hbar(\gamma)$ and $\beta' \in K_\hbar(\gamma)$:

$$\tilde{\tau}_{\alpha'} U^{-\eta} P_\gamma = \tau_{\gamma.\alpha'} U^{-\eta} \text{ and } \tilde{\pi}_{\beta'} P_\gamma = \pi_{\beta'.\gamma}, \tag{40}$$

where $\gamma.\alpha' \in I^\eta(\hbar)$ and $\beta'.\gamma \in K^\eta(\hbar)$ by definition. In equality (40) appears the fact that the definitions of $\tau$ and $\pi$ are slightly different (see (23) and (24)). It is due to the fact that we want to compose $\tilde{\tau}$ and $\tilde{\pi}$ with the same operator $P_\gamma$. Suppose that $\|P_\gamma \psi_\hbar\|$ is not equal to 0 (otherwise the obtained result is trivial). We apply the quantum uncertainty principle (2.1) to the partitions of identity: $(\tilde{\tau}_{\alpha'})_{\alpha' \in I_\hbar(\gamma)}$ and $(\tilde{\pi}_{\beta'})_{\beta' \in K_\hbar(\gamma)}$. These two partitions are of cardinality $\mathcal{N} \simeq \hbar^{-K_0}$ where $K_0$ is some fixed positive number (depending on the cardinality of the partition $K$, on $a_0$, on $b_0$ and $\eta$). We choose $\mathrm{Op}(\chi^{(k')})$ for the family of bounded bounded operators $O_{\beta'}$ (where $k'$ is the length of $\beta'$) and $\delta' = \|P_\gamma \psi_\hbar\|^{-1}\hbar^L$ (see corollary B.2) such that $\hbar^{L-K_0} << h^{1/2(1-\epsilon')(1-\epsilon)}$. The isometry chosen is $U^{-\eta}$ and the normalized vector is $\tilde{\psi}_\hbar := \frac{P_\gamma \psi_\hbar}{\|P_\gamma \psi_\hbar\|}$. Applying the entropic uncertainty principle (2.1), one gets:

**Corollary 5.4.**

$$h_{\tilde{\tau}}(U^{-\eta}\tilde{\psi}_\hbar) + h_{\tilde{\pi}}(\tilde{\psi}_\hbar) \geq -2\log\left(c_\chi^\gamma(U^{-\eta}) + \hbar^{L-K_0}\|P_\gamma\psi_\hbar\|^{-1}\right),$$

*where* $c_\chi^\gamma(U^{-\eta}) = \max_{\alpha'\in I_\hbar(\gamma),\beta'\in K_\hbar(\gamma)} \left(\|\tilde{\tau}_{\alpha'}U^{-\eta}\tilde{\pi}^*_{\beta'}Op(\chi^{(k')})\|\right).$

First, remark that the quantity $c_\chi^\gamma(U^{-\eta})$ can be easily replaced by

$$c_\chi(U^{-\eta}) := \max_{\gamma\in\{1,\cdots,K\}^2}\ \max_{\alpha\in I_\hbar(\gamma),\beta\in K_\hbar(\gamma)} \left(\|\tilde{\tau}_\alpha U^{-\eta}\tilde{\pi}^*_\beta \mathrm{Op}(\chi^{(k')})\|\right), \tag{41}$$

which is independent of $\gamma$. Then, it can be noted that the quantity $h_{\tilde{\tau}}(U^{-\eta}\tilde{\psi}_\hbar)$ can be written:

$$h_{\tilde{\tau}}(U^{-\eta}\tilde{\psi}_\hbar) = -\sum_{\alpha'\in I_\hbar(\gamma)} \|\tilde{\tau}_{\alpha'}U^{-\eta}\tilde{\psi}_\hbar\|^2 \log\|\tilde{\tau}_{\alpha'}U^{-\eta}P_\gamma\psi_\hbar\|^2 + \sum_{\alpha'\in I_\hbar(\gamma)} \|\tilde{\tau}_{\alpha'}U^{-\eta}\tilde{\psi}_\hbar\|^2 \log\|P_\gamma\psi_\hbar\|^2.$$

Using the fact that $\psi_\hbar$ is an eigenvector of $U^\eta$ and that $(\tilde{\tau}_{\alpha'})_{\alpha'\in I_\hbar(\gamma)}$ is a partition of identity, one has:

$$h_{\tilde{\tau}}(U^{-\eta}\tilde{\psi}_\hbar) = -\frac{1}{\|P_\gamma\psi_\hbar\|^2}\sum_{\alpha'\in I_\hbar(\gamma)} \|\tau_{\gamma.\alpha'}\psi_\hbar\|^2 \log\|\tau_{\gamma.\alpha'}\psi_\hbar\|^2 + \log\|P_\gamma\psi_\hbar\|^2.$$

The same holds for $h_{\tilde{\pi}}(\tilde{\psi}_\hbar)$ (using here equality (40)):

$$h_{\tilde{\pi}}(\tilde{\psi}_\hbar) = -\frac{1}{\|P_\gamma\psi_\hbar\|^2}\sum_{\beta'\in K_\hbar(\gamma)} \|\pi_{\beta'.\gamma}\psi_\hbar\|^2 \log\|\pi_{\beta'.\gamma}\psi_\hbar\|^2 + \log\|P_\gamma\psi_\hbar\|^2.$$

Then, the quantity

$$-\sum_{\alpha'\in I_\hbar(\gamma)} \|\tau_{\gamma.\alpha'}\psi_\hbar\|^2 \log\|\tau_{\gamma.\alpha'}\psi_\hbar\|^2 - \sum_{\beta'\in K_\hbar(\gamma)} \|\pi_{\beta'.\gamma}\psi_\hbar\|^2 \log\|\pi_{\beta'.\gamma}\psi_\hbar\|^2 \tag{42}$$

is bounded from below by:

$$-2\|P_\gamma\psi_\hbar\|^2\log\left(c_\chi(U^{-\eta})\|P_\gamma\psi_\hbar\| + \hbar^{L-K_0}\right) \geq -2\|P_\gamma\psi_\hbar\|^2\log\left(c_\chi(U^{-\eta}) + \hbar^{L-K_0}\right) \tag{43}$$

as $\|P_\gamma\psi_\hbar\| \leq 1$. As was already mentioned, this lower bound is trivial in the case where $\|P_\gamma\psi_\hbar\|$ is equal to 0. Using the fact that:

$$c_\gamma = \frac{f(\gamma)}{\sum_{\gamma'\in\{1,\cdots,K\}^2} f(\gamma')\|P_{\gamma'}\psi_\hbar\|^2}, \tag{44}$$

one easily checks that $\sum_{\gamma\in\{1,\cdots,K\}^2} c_\gamma\|P_\gamma\psi_\hbar\|^2 = 1$. If we multiply (42) and (43) by $c_\gamma$ and make the sum over all $\gamma$ in $\{1,\cdots,K\}^2$, we find:

$$-\sum_{\alpha\in I^\eta(\hbar)} c_\alpha\|\tau_\alpha\psi_\hbar\|^2\log\|\tau_\alpha\psi_\hbar\|^2 - \sum_{\beta\in K^\eta(\hbar)} c_\beta\|\pi_\beta\psi_\hbar\|^2\log\|\pi_\beta\psi_\hbar\|^2 \geq -2\log\left(c_\chi(U^{-\eta}) + \hbar^{L-K_0}\right).$$

Finally, we have the following property:

**Corollary 5.5.** *Define:*

$$h^c_\tau(\psi_\hbar) := -\sum_{\alpha\in I^\eta(\hbar)} c_\alpha\|\tau_\alpha\psi_\hbar\|^2\log\left(c_\alpha\|\tau_\alpha\psi_\hbar\|^2\right),\ \ h^c_\pi(\psi_\hbar) := -\sum_{\beta\in K^\eta(\hbar)} c_\beta\|\pi_\beta\psi_\hbar\|^2\log\left(c_\beta\|\pi_\beta\psi_\hbar\|^2\right) \tag{45}$$

*where* $c_\alpha = c_{\alpha_0,\alpha_1}$ *and* $c_\beta = c_{\beta_{-1},\beta_0}$. *One has:*

$$h^c_\tau(\psi_\hbar) + h^c_\pi(\psi_\hbar) \geq -2\log\left(c_\chi(U^{-\eta}) + \hbar^{L-K_0}\right) - \log\left(\max_\gamma c_\gamma\right). \tag{46}$$

To prove this corollary, it has been used that $\sum_{\alpha\in I^\eta(\hbar)} c_\alpha\|\tau_\alpha\psi_\hbar\|^2 = 1$ and $\sum_{\beta\in K^\eta(\hbar)} c_\beta\|\pi_\beta\psi_\hbar\|^2 = 1$. Precisely it was used to take into account the introduction of the weight $c_\alpha$ in the expression (45) of $h^c$. This new quantity is slightly different from the one used in the definition of the quantum entropy (see (13)). In fact, we have introduced some weights $c_\gamma$ in each term of the sum. However, if we think of the definition of $\overline{\mu_\hbar^\Sigma}$ (which is of the form $C\mu_\hbar^\Sigma \times$ Leb: see (29)), this quantity is more adapted to compute the classical entropy of $\overline{\mu_\hbar^\Sigma}$ than the usual quantum entropy.

5.3.3. *Exponential decrease of the atoms of the quantum partition.* Now that we have lower bound (46), we give an estimate on the exponential decrease of the atoms of the quantum partition. As in [2], [5], [3], one has[3]:

**Theorem 5.6.** [2] [5] [3] *For every $\mathcal{K} > 0$ ($\mathcal{K} \leq C_\delta$), there exists $\hbar_\mathcal{K}$ and $C_\mathcal{K}$ such that uniformly for all $\hbar \leq \hbar_\mathcal{K}$, for all $k + k' \leq \mathcal{K}|\log\hbar|$,*

$$\|P_{\alpha_k}U^\eta P_{\alpha_{k-1}}\cdots U^\eta P_{\alpha_0}U^{3\eta}P_{\alpha'_k}U^\eta\cdots P_{\alpha'_0}Op(\chi^{(k')})\|_{L^2(M)}$$

$$\leq C_\mathcal{K}\hbar^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\left(\sum_{j=0}^{k-1}f(\sigma^j\alpha)+\sum_{j=0}^{k'-1}f(\sigma^j\alpha')\right)\right). \tag{47}$$

Outline that the crucial role of the sharp energy cutoff appears in particular to prove this theorem. In fact, without the cutoff, the previous norm operator could have only be bounded by 1 and the entropic uncertainty principle would have been empty. The previous inequality (47) allows to give an estimate on the quantity (41) (as it allows us to bound $c_\chi(U^{-\eta})$). In fact, one has, for each $\gamma \in \{1,\cdots,K\}^2$:

$$\|\tilde\tau_\alpha U^{-\eta}\tilde\pi_\beta^*\mathrm{Op}(\chi^{(k')})\| = \|P_{\alpha_k}U^\eta P_{\alpha_{k-1}}\cdots U^\eta P_{\alpha_2}U^{3\eta}P_{\beta_{-2}}U^\eta\cdots P_{\beta_{-k'}}\mathrm{Op}(\chi^{(k')})\|,$$

where $(\alpha_2,\cdots,\alpha_k) \in I_\hbar(\gamma)$ and $(\beta_{-k'},\cdots,\beta_{-2}) \in K_\hbar(\gamma)$. Using the definition of the sets $I^\eta(\hbar)$ (26) and $K^\eta(\hbar)$ (27), one has $k + k' \leq \frac{2}{a_0\eta}|\log\hbar|$. Using theorem (5.6) with $\mathcal{K} = \frac{2}{a_0\eta}$, one has:

$$\|\tilde\tau_\alpha U^{-\eta}\tilde\pi_\beta^*\mathrm{Op}(\chi^{(k')})\| \leq C_\mathcal{K}\hbar^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\left(\sum_{j=2}^{k-1}f(\sigma^j\alpha)+\sum_{j=2}^{k'-1}\overline{f}(T^j\beta)\right)\right),$$

where $C_\mathcal{K}$ does not depend on $\hbar$. Using again the definition of the sets $I^\eta(\hbar)$ (26) and $K^\eta(\hbar)$ (27), one has:

$$c_\chi(U^{-\eta}) = \max_{\gamma\in\{1,\cdots,K\}^2}\max_{\alpha\in I_\hbar(\gamma),\beta\in K_\hbar(\gamma)}\left(\|\tilde\tau_\alpha U^{-\eta}\tilde\pi_\beta^*\mathrm{Op}(\chi^{(k')})\|\right) \leq \tilde C_\mathcal{K}\hbar^{\frac{1}{2}(1-\epsilon')(1-\epsilon)},$$

where $\tilde C_\mathcal{K}$ does not depend on $\hbar$. The main inequality (46) for the quantum entropy can be rewritten and the discussion of this section can be summarized finally by the following proposition:

**Proposition 5.7.** *Define:*

$$h_\tau^c(\psi_\hbar) := -\sum_{\alpha\in I^\eta(\hbar)} c_\alpha\|\tau_\alpha\psi_\hbar\|^2\log\left(c_\alpha\|\tau_\alpha\psi_\hbar\|^2\right),\ h_\pi^c(\psi_\hbar) := -\sum_{\beta\in K^\eta(\hbar)} c_\beta\|\pi_\beta\psi_\hbar\|^2\log\left(c_\beta\|\pi_\beta\psi_\hbar\|^2\right)$$

*where $c_\alpha = c_{\alpha_0,\alpha_1}$, $c_\beta = c_{\beta_{-1},\beta_0}$ and $c_\gamma = \frac{f(\gamma)}{\sum_{\gamma\in\{1,\cdots,K\}^2} f(\gamma)\|P_\gamma\psi_\hbar\|^2}$. One has:*

$$h_\tau^c(\psi_\hbar) + h_\pi^c(\psi_\hbar) \geq (1-\epsilon')(1-\epsilon)|\log\hbar| + C, \tag{48}$$

*where $C \in \mathbb{R}$ does not depend on $\hbar$.*

[3]In the higher dimension case mentioned in the footnote of section 1.2, we should replace $\hbar^{-\frac{1}{2}}$ (where $d$ is the dimension of $M$) by $\hbar^{-\frac{d-1}{2}}$ in inequality (47).

This lower bound is the one we used in section 4 to get lower bound on the classical entropy. In fact, this last inequality gives a lower bound on $h_\tau^c$ and $h_\pi^c$ and we already explained in section 4.2 that these quantities are in fact the classical entropy of the measure $\overline{\mu}_\hbar^\Sigma$ for a specific partition (see (31) and (32)).

## 6. Subadditivity of the quantum entropy

As was mentioned in section 4 and proved in section 5, the uncertainty principle gives (if we forget the backward side $\Sigma_-$):

$$\frac{1}{n_E(\hbar)} H_{n_E(\hbar)}\left(\overline{\mu}_\hbar^\Sigma, \overline{\sigma}, \overline{\mathcal{C}}\right) \geq \frac{1}{2}(1-\epsilon).$$

To prove our main theorem 1.3, we need to show that this lower bound holds for a fixed $n_0$ on the quantity $\frac{1}{n_0} H_{n_0}\left(\overline{\mu}_\hbar^\Sigma, \overline{\sigma}, \overline{\mathcal{C}}\right)$ (as we need to let $\hbar$ tend to 0 independently of $n$ to recover the semiclassical measure $\overline{\mu}^\Sigma$: see section 4.3). To do this we want to reproduce the classical argument for the existence of the metric entropy (see (10)), i.e. we need to prove a subadditivity property for logarithmic time:

**Theorem 6.1.** *Let $\overline{\mathcal{C}}$ be the partition of proposition 5.3. There exists a function $R(n_0, \hbar)$ on $\mathbb{N} \times (0,1]$ such that*

$$\forall n_0 \in \mathbb{N}, \qquad \lim_{\hbar \to 0} |R(n_0, \hbar)| = 0.$$

*Moreover, for any $\hbar \in (0,1]$, any $n_0, m \in \mathbb{N}$ such that $n_0 + m \leq T_E(\hbar)$, one has:*

$$H_{n_0+m}\left(\overline{\mu}_\hbar^\Sigma, \overline{\sigma}, \overline{\mathcal{C}}\right) \leq H_{n_0}\left(\overline{\mu}_\hbar^\Sigma, \overline{\sigma}, \overline{\mathcal{C}}\right) + H_m\left(\overline{\mu}_\hbar^\Sigma, \overline{\sigma}, \overline{\mathcal{C}}\right) + R(n_0, \hbar).$$

A key point to prove the subadditivity property in the case of the metric entropy is that the measure is invariant under the dynamics (see (10)). In our case, invariance of the semiclassical measure under the geodesic flow is a consequence of the Egorov property (1): to prove that subadditivity almost holds (in the sense of the previous theorem), we will have to prove an Egorov property for logarithmic times. We will see that with our choice of 'local' Ehrenfest time, this will be possible and the previous theorem will then hold. The proof of this theorem is the subject of this section (and it also uses results from section 7).
Let $n_0$ and $m$ be two positive integers such that que $m + n_0 \leq T_E(\hbar)$. One has:

$$H\left(\vee_{i=0}^{n+n_0-1} \overline{\sigma}^{-i}\overline{\mathcal{C}}, \overline{\mu}_\hbar^\Sigma\right) = H\left(\vee_{i=0}^{n-1} \overline{\sigma}^{-i}\overline{\mathcal{C}} \vee \vee_{i=n}^{n_0+n-1} \overline{\sigma}^{-i}\overline{\mathcal{C}}, \overline{\mu}_\hbar^\Sigma\right).$$

Using classical properties of the metric entropy, one has (see section 2.1):

$$H_{n+n_0}\left(\overline{\sigma}, \overline{\mu}_\hbar^\Sigma, \overline{\mathcal{C}}\right) \leq H_n\left(\overline{\sigma}, \overline{\mu}_\hbar^\Sigma, \overline{\mathcal{C}}\right) + H_{n_0}\left(\overline{\sigma}, \overline{\sigma}^n \sharp \overline{\mu}_\hbar^\Sigma, \overline{\mathcal{C}}\right).$$

Using proposition 6.2 and the continuity of the function $x \log x$ on $[0,1]$, there exists a function $R(n_0, \hbar)$ with the properties of theorem 6.1 such that $H_{n_0}\left(\overline{\sigma}, \overline{\sigma}^n \sharp \overline{\mu}_\hbar^\Sigma, \overline{\mathcal{C}}\right) = H_{n_0}\left(\overline{\sigma}, \overline{\mu}_\hbar^\Sigma, \overline{\mathcal{C}}\right) + R(n_0, \hbar)$ and thus:

$$H_{n+n_0}\left(\overline{\sigma}, \overline{\mu}_\hbar^\Sigma, \overline{\mathcal{C}}\right) \leq H_n\left(\overline{\sigma}, \overline{\mu}_\hbar^\Sigma, \overline{\mathcal{C}}\right) + H_{n_0}\left(\overline{\sigma}, \overline{\mu}_\hbar^\Sigma, \overline{\mathcal{C}}\right) + R(n_0, \hbar). \square \tag{49}$$

So the crucial point to prove this theorem is to show that the measure of the atoms of the refined partition is almost invariant under $\overline{\sigma}$ (proposition 6.2). In the following of this section, $A$ is defined as:

$$A = \overline{\mathcal{C}}_{\gamma_0, p_0} \cap \cdots \cap \overline{\sigma}^{-(n_0-1)} \overline{\mathcal{C}}_{\gamma_{n_0-1}, p_{n_0-1}}.$$

*Remark.* In this section, only the case of $\{1, \cdots, K\}^{\mathbb{N}}$ is treated. The proof of the backward case $\{1, \cdots, K\}^{-\mathbb{N}}$ works in the same way.

6.1. **Pseudo-invariance of the measure of the atoms of the partitions.** From this point, our main goal is to show the pseudo invariance of the atoms of the refined partition. More precisely:

**Proposition 6.2.** *Let $m, n_0$ be two positive integers such that $m+n_0 \leq T_E(\hbar)$. Consider an atom of the refined partition $A = \overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \cap \overline{\sigma}^{-(n_0-1)}\overline{\mathcal{C}}_{\gamma_{n_0-1},p_{n_0-1}}$. One has:*

$$\overline{\mu}_\hbar^{\overline{\Sigma}}\left(\overline{\sigma}^{-m}A\right) = \overline{\mu}_\hbar^{\overline{\Sigma}}(A) + O(\hbar^{(1-2\nu)/6}),$$

*with a uniform constant in $n_0$ and $m$ in the allowed interval. The constant $\nu < 1/2$ is the one defined by theorem 7.1.*

This result says that the measure $\overline{\mu}_\hbar^{\overline{\Sigma}}$ is almost $\overline{\sigma}$ invariant for logarithmic times. As a consequence, the classical argument (see (10)) for subadditivity of the entropy can be applied as long as we consider times where the pseudo invariance holds (see (49)).
Let $A$ be as in the proposition. From proposition 5.3, there exists $(\alpha_0, \cdots, \alpha_k)$ and $B(\gamma)$ such that:

$$A = \left(\mathcal{C}_{\alpha_0} \cap \cdots \sigma^{-k}\mathcal{C}_{\alpha_k}\right) \times B(\gamma).$$

Still from proposition 5.3, one knows that $B(\gamma)$ is a subinterval of $[0, f(\gamma_0)[$. Moreover, the following property on $\alpha$ holds:

$$n_0(1-\epsilon) \leq \sum_{j=0}^{k-1} f(\sigma^j\alpha) \leq n_0(1+\epsilon). \tag{50}$$

The plan of the proof of proposition 6.2 is the following. First, we will give an exact expression in terms of $\alpha$ and $B(\gamma)$ of $\overline{\mu}_\hbar^{\overline{\Sigma}}\left(\overline{\sigma}^{-m}A\right)$. Then, we will see how to prove the proposition making the simplifying assumption that all operators $(P_i(k\eta))_{i,k}$ commute. Finally, we will estimate the error term due to the fact that operators do not exactly commute.

6.1.1. *Computation of $\overline{\mu}_\hbar^{\overline{\Sigma}}(\overline{\sigma}^{-m}A)$.* We choose a positive integer $m$. As a first step of the proof, we want to give a precise formula for the measure of $\overline{\sigma}^{-m}A$. To do this, we have to determine the shape of the set $\overline{\sigma}^{-m}A$. Let us then define:

$$\overline{\Sigma}_p^m := \left\{ (x,t) \in \overline{\Sigma} : \sum_{j=0}^{p-2} f(\sigma^j x) \leq m + t < \sum_{j=0}^{p-1} f(\sigma^j x) \right\}.$$

We underline that because $m \geq 1$, we have that $\overline{\Sigma}_p^m$ is empty for $p \leq 3$. One has then $\overline{\Sigma} = \bigsqcup_{p\geq 3} \overline{\Sigma}_p^m$ and as a consequence:

$$\begin{aligned}
\overline{\sigma}^{-m}A &= \bigsqcup_{p\geq 3} \left(\overline{\Sigma}_p^m \cap \overline{\sigma}^{-m}A\right) \\
&= \bigsqcup_{p\geq 3} \left\{ (x,t) \in \overline{\Sigma}_p^m : m + t - \sum_{j=0}^{p-2} f(\sigma^j x) \in B(\gamma), (x_{p-1}, \cdots, x_{p+k-1}) = \alpha \right\}.
\end{aligned}$$

Note that $t \in B(\gamma) - m + \sum_{j=0}^{p-2} f(\sigma^j x)$ together with $(x_{p-1}, \cdots, x_{p+k-1}) = \alpha$ imply that $\sum_{j=0}^{p-2} f(\sigma^j x) \leq m + t < \sum_{j=0}^{p-1} f(\sigma^j x)$. It allows to rewrite:

$$\overline{\sigma}^{-m}A = \bigsqcup_{p\geq 3} \left\{ (x,t) \in \Sigma \times \mathbb{R}_+ : 0 \leq t < f(x), t \in B(\gamma) - m + \sum_{j=0}^{p-2} f(\sigma^j x), (x_{p-1}, \cdots, x_{p+k-1}) = \alpha \right\}.$$

Finally, one can write the measure of this suspension set:

$$\overline{\mu}_\hbar^{\overline{\Sigma}}\left(\overline{\sigma}^{-m}A\right) = \sum_{p\geq 1} \sum_{\substack{|\beta| = p+k \\ (\beta_{p-1}, \cdots, \beta_{p+k-1}) = \alpha}} c_{\beta,\alpha}(m) \| P_{\beta_{k+p-1}}((k+p-1)\eta) P_{\beta_{k+p-2}}((k+p-2)\eta) \cdots P_{\beta_0}\psi_\hbar \|^2,$$

where

$$c_{\beta,\alpha}(m) = \lambda \text{ Leb}\left( B(\gamma) \cap [m - \sum_{j=0}^{p-2} f(\sigma^j \beta), m - \sum_{j=1}^{p-2} f(\sigma^j \beta)[ \right)$$

with $\lambda^{-1} := \sum_{\gamma' \in \{1,\cdots,K\}^2} f(\gamma') \mu_\hbar^\Sigma([\gamma'])$ the normalization constant of the measure. Outline that the previous sum is finite with at most $2b_0/a_0$ non zeros terms in each string $\beta$ (as $c_{\bullet,\alpha}(m)$ is zero except a finite number of times). For simplicity of the following of the proof, we reindex the previous expressions:

(51)
$$\overline{\mu}_\hbar^\Sigma\left(\overline{\sigma}^{-m} A\right) = \sum_{p \geq 3} \sum_{\substack{|\beta| = p+k \\ (\beta_0, \cdots, \beta_k) = \alpha}} c_{\beta,\alpha}(m) \| P_{\beta_k}((k+p-1)\eta) P_{\beta_{k-1}}((k+p-2)\eta) \cdots P_{\beta_{-p+1}} \psi_\hbar \|^2,$$

where $c_{\beta,\alpha}(m) = \lambda \text{ Leb}\left( B(\gamma) \cap [m - \sum_{j=0}^{p-2} f(\sigma^j \beta), m - \sum_{j=1}^{p-2} f(\sigma^j \beta)[ \right)$ with $\lambda$ defined as previously. Then, to prove proposition 6.2, we have to show that the previous quantity (51) is equal to:

$$\lambda \text{ Leb}\left(B(\gamma)\right) \| P_{\alpha_k}(k\eta) \cdots P_{\alpha_0} \psi_\hbar \|_{L^2}^2 + O_{L^2}(\hbar^{(1-2\nu)/6}).$$

6.1.2. *If everything commutes...* We will now use our explicit expression for $\overline{\mu}_\hbar^\Sigma\left(\overline{\sigma}^{-m} A\right)$ (see (51)) and verify it is equal to $\overline{\mu}_\hbar^\Sigma(A)$ under the simplifying assumption that all the involved pseudo-differential operators commute. In the next section, we will then give an estimate of the error term due to the fact that the operators do not exactly commute. In order to prove the pseudo invariance, denote:

$$K_m(\alpha) := \{\beta = (\beta_{-p+1}, \cdots, \beta_k) : (\beta_0, \cdots, \beta_k) = \alpha, c_{\beta,\alpha}(m) \neq 0\}$$

and

$$K_m^{(q)}(\alpha) := \{(\beta_{-q+1}, \cdots, \beta_k) : \exists \gamma = (\gamma_{-p+1}, \cdots, \gamma_{-q}) \text{ s.t. } q < p,\ \gamma.\beta \in K_m(\alpha)\}.$$

With these notations, we can write (51) as follows:

$$\overline{\mu}_\hbar^\Sigma\left(\overline{\sigma}^{-m} A\right) = \sum_{\beta \in K_m(\alpha)} c_{\beta,\alpha}(m) \|\tau_\beta \psi_\hbar\|^2 = \sum_{p=3}^{N} \sum_{\beta \in K_m(\alpha) : |\beta| = k+p} c_{\beta,\alpha}(m) \|\tau_\beta \psi_\hbar\|^2. \tag{52}$$

Recall that by definition (see (23)) $\tau_\beta := P_{\beta_k}((k+p-1)\eta) P_{\beta_{k-1}}((k+p-2)\eta) \cdots P_{\beta_{-p+1}}$. For simplicity of notations, let us denote $B(\gamma) = [a, b[$ (where $a$ and $b$ obviously depend on $\gamma$). A last notation we define is for $\beta$ such that $|\beta| = k+q$ and $\sigma^{q-1}\beta = \alpha$:

$$\overline{c}_{\beta,\alpha}(m) := \lambda \text{ Leb}\left( [a, b[ \cap [a, m - \sum_{j=1}^{q-2} f(\sigma^j \beta)[ \right), \tag{53}$$

where $\lambda$ is the normalization constant of the measure previously defined. Suppose now that all the operators $(P_i(k\eta))_{i,k}$ commute. We have the following lemma:

**Lemma 6.3.** *If all the operators $(P_i(k\eta))_{i,k}$ commute, then one has, for $2 \leq q \leq N$:*

$$\sum_{\beta \in K_m(\alpha) : |\beta| = k+q} c_{\beta,\alpha}(m) \|\tau_\beta \psi_\hbar\|^2 + \sum_{\beta \in K_m^{(q)}(\alpha)} \overline{c}_{\beta,\alpha}(m) \|\tau_\beta \psi_\hbar\|^2 = \sum_{\beta \in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m) \|\tau_\beta \psi_\hbar\|^2.$$

*Proof.* Let $2 \leq q \leq N$. Consider $\beta$ an element of $K_m^{(q-1)}(\alpha)$. Using the property of partition of identity, we have:

$$\sum_{\beta \in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m) \|\tau_\beta \psi_\hbar\|^2 = \sum_{\gamma=1}^{K} \sum_{\beta \in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m) \|P_\gamma(-\eta) \tau_\beta \psi_\hbar\|^2.$$

For each $1 \leq \gamma \leq K$, we have the following property for $\overline{c}_{\gamma.\beta,\alpha}(m)$ (as $f \geq 0$):

$$\overline{c}_{\beta,\alpha}(m) = \overline{c}_{\gamma.\beta,\alpha}(m) + c_{\gamma.\beta,\alpha}(m).$$

We can write then:

$$\sum_{\beta\in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m)\|\tau_\beta\psi_\hbar\|^2 = \sum_{\gamma=1}^{K}\sum_{\beta\in K_m^{(q-1)}(\alpha)} (\overline{c}_{\gamma.\beta,\alpha}(m)+c_{\gamma.\beta,\alpha}(m))\|P_\gamma(-\eta)\tau_\beta\psi_\hbar\|^2.$$

Notice that we have assumed the operators commute, we have:

$$P_\gamma(-\eta)P_{\beta_k}((k+q-2)\eta)\cdots P_{\beta_{-q+2}}\psi_\hbar = P_{\beta_k}((k+q-1)\eta)\cdots P_{\beta_{-q+2}}P_\gamma(-\eta)\psi_\hbar. \tag{54}$$

As a consequence, we have:

$$\sum_{\beta\in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m)\|\tau_\beta\psi_\hbar\|^2 = \sum_{\gamma=1}^{K}\sum_{\beta\in K_m^{(q-1)}(\alpha)} (\overline{c}_{\gamma.\beta,\alpha}(m)+c_{\gamma.\beta,\alpha}(m))\|\tau_\beta P_\gamma(-\eta)\psi_\hbar\|^2.$$

By definition of the different sets $K_m$ and as $\psi_\hbar$ is an eigenfunction of the Laplacian, this last equality allows to conclude the proof of the lemma. □

Proceeding then by induction from $N$ to 1 (see equality (52)) and using the previous lemma at each step, we can conclude that if all the operators commute:

$$\overline{\mu}_\hbar^{\overline{\Sigma}}\left(\overline{\sigma}^{-m}A\right) = \overline{\mu}_\hbar^{\overline{\Sigma}}(A).$$

6.1.3. *Estimates of the error terms.* Regarding to the previous section, we have to see what is exactly the error term we forgot at each step of the recurrence and we have to verify that it is bounded by some positive power of $\hbar$. Precisely, we have to understand what is the error term in equation (54) if we do not suppose anymore that all the operators commute. Precisely, the error term we have to take into account in (54) is:

$$R_{\beta,\gamma,\hbar} = \sum_{j=-q+2}^{k} P_{\beta_k}((k+q-2)\eta)\cdots P_{\beta_{j+1}}((j+q-1)\eta)R^j(\beta,\gamma)P_{\beta_{j-1}}((j+q-3)\eta)\cdots P_{\beta_{-q+2}}\psi_\hbar,$$

where $R^j(\beta,\gamma) = [P_\gamma(-\eta), P_{\beta_j}((j+q-2)\eta)]$ is the bracket of the two operators. We denote $R^j_{\beta,\gamma,\hbar}$ each term of the previous sum. The error term we forgot at each step $q$ of the induction in the previous section is then:

$$E(\hbar,q) := \sum_{\gamma=1}^{K}\sum_{\beta\in K_m^{(q-1)}(\alpha)} \left(\langle R_{\beta,\gamma,\hbar}, P_\gamma(-\eta)\tau_\beta\psi_\hbar\rangle + \langle \tau_\beta P_\gamma(-\eta)\psi_\hbar, R_{\beta,\gamma,\hbar}\rangle\right). \tag{55}$$

So for each step $q$ of the induction to prove the pseudo invariance of the measure, a first error term we have to estimate is of the form:

$$\sum_{j=-q+2}^{k}\sum_{\gamma=1}^{K}\sum_{\beta\in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m)\left\langle R^j_{\beta,\gamma,\hbar}, P_\gamma(-\eta)\tau_\beta\psi_\hbar\right\rangle. \tag{56}$$

Using Cauchy Schwarz inequality twice and the fact that $0 \le \overline{c}_{\beta,\alpha}(m) \le \mathrm{Leb}(B(\gamma)) \le b_0\eta$, this last quantity is bounded by

$$\lambda b_0\eta\left(\sum_{j=-q+2}^{k}\sum_{\gamma=1}^{K}\sum_{\beta\in K_m^{(q-1)}(\alpha)} \|R^j_{\beta,\gamma,\hbar}\|^2\right)^{\frac{1}{2}}\left(\sum_{j=-q+2}^{k}\sum_{\gamma=1}^{K}\sum_{\beta\in K_m^{(q-1)}(\alpha)} \|P_\gamma(-\eta)\tau_\beta\psi_\hbar\|^2\right)^{\frac{1}{2}}. \tag{57}$$

The last term of the product is bounded as:

$$\sum_{j=-q+2}^{k}\sum_{\gamma=1}^{K}\sum_{\beta\in K_m^{(q-1)}(\alpha)} \|P_\gamma(-\eta)\tau_\beta\psi_\hbar\|^2 \le (k+q)K\sum_{|\beta|=k+q-1}\|\tau_\beta\psi_\hbar\|^2 = (k+q)K = O(|\log\hbar|).$$

As a consequence, the error term (57) is bounded by:

$$C|\log\hbar|\left(\sum_{j=-q+2}^{k}\sum_{\gamma=1}^{K}\sum_{\beta\in K_m^{(q-1)}(\alpha)}\|R^j_{\beta,\gamma,\hbar}\|^2\right)^{\frac{1}{2}},$$

where $C$ is some positive uniform constant (depending only on the partition and on $\eta$). We extend now the definition of $R^j(\beta,\gamma)$ (previously defined as $[P_\gamma(-\eta),P_{\beta_j}((j+q-2)\eta)]$ for $\beta$ in $K_m^{(q-1)}(\alpha)$) to any word $\beta$ of length $k+q-1$. If $j+q-1$ letters of $\beta$ are also the $j+q-1$ first letters of a word $\beta'$ in $K_m^{(q-1)}(\alpha)$, we take $R^j(\beta,\gamma):=[P_\gamma(-\eta),P_{\beta_j}((j+q-2)\eta)]$. Otherwise, we take $R^j(\beta,\gamma):=\hbar\,\mathrm{Id}_{L^2(M)}$. We define then for any sequence of length $k+q-1$:

$$R^j_{\beta,\gamma,\hbar}=P_{\beta_k}((k+q-2)\eta)\cdots P_{\beta_{j+1}}((j+q-1)\eta)R^j(\beta,\gamma)P_{\beta_{j-1}}((j+q-3)\eta)\cdots P_{\beta_{-q+2}}\psi_\hbar.$$

In theorem 6.4 from the section 6.2, we will prove in particular that for every $\beta$ of size $q+k-1$ and for each $-q+2\le j\le k$:

(58)
$$\|R^j(\beta,\gamma)P_{\beta_{j-1}}((j+q-3)\eta)\cdots P_{\beta_{-q+2}}\psi_\hbar\|_{L^2(M)}\le C\hbar^{1-2\nu}\|P_{\beta_{j-1}}((j+q-3)\eta)\cdots P_{\beta_{-q+2}}\psi_\hbar\|_{L^2(M)},$$

where $C$ is a uniform constant for $n_0$ and $m$ positive integers such that $n_0+m\le T_E(\hbar)$ and $\nu<1/2$ (defined in section 7). The theorem 6.4 can be applied as $\sum_{j=0}^{k+q-2}f(\sigma^j\beta)\le(n_0+m)(1+\epsilon)\le n_E(\hbar)$ (see (50) and (53)). Using bound (58) and the property of partition of identity, we have:

$$\sum_{|\beta|=k+q-1}\|R^j_{\beta,\gamma,\hbar}\|^2=O(\hbar^{2(1-2\nu)}).$$

The error term (57) (and as a consequence (56)) is then bounded by:

$$\tilde{C}|\log\hbar|\left(\sum_{j=-q+2}^{k}\sum_{\gamma=1}^{K}\sum_{|\beta|=k+q-1}\|R^j_{\beta,\gamma,\hbar}\|^2\right)^{\frac{1}{2}}=O(\hbar^{\frac{1-2\nu}{4}}).$$

Looking at equation (55), we see that the other error term for the step $q$ of the induction can be estimated with the same method and is also a $O(\hbar^{\frac{1-2\nu}{4}})$. As the number $N$ of steps in the induction is a $O(|\log\hbar|)$, the error term we forgot in the previous section (due to the fact that the operators do not commute) is a $O(\hbar^{\frac{1-2\nu}{6}})$. This concludes the proof of proposition 6.2.□

6.2. **Commutation of pseudodifferential operators.** In order to complete the proof of the pseudo invariance of the measure (proposition 6.2), we need to prove inequality (58). It will be a consequence of (60) below. Once we have proved this inequality, the subadditivity property will be completely proved. The exact property we need is stated by the following theorem:

**Theorem 6.4.** *Let $(\gamma_0,\cdots,\gamma_k)$ be such that*

$$\sum_{j=0}^{k-1}f(\sigma^j\gamma)\le n_E(\hbar). \tag{59}$$

*One has:*

(60)
$$\left\|[P_{\gamma_k}(k\eta),P_{\gamma_0}]\,P_{\gamma_{k-1}}((k-1)\eta)\cdots P_{\gamma_1}(\eta)\psi_\hbar\right\|_{L^2}\le C\hbar^{1-2\nu}\left\|P_{\gamma_{k-1}}((k-1)\eta)\cdots P_{\gamma_1}(\eta)\psi_\hbar\right\|_{L^2},$$

*where $\nu<1/2$ is defined in section 7, $C$ is a constant depending on the partition and uniform in all $\gamma$ satisfying (59).*

Let $\gamma$ be a finite sequence as in the previous theorem. Denote $t(\gamma)=\sum_{j=0}^{k(\gamma)-1}f(\sigma^j\gamma)$. This quantity is less than $n_E(\hbar)$ in the setting of theorem 6.4. There exists a unique integer $l(\gamma)<k(\gamma)$

such that:

$$\sum_{j=0}^{l(\gamma)-2} f(\sigma^j\gamma) \leq \frac{t(\gamma)}{2} < \sum_{j=0}^{l(\gamma)-1} f(\sigma^j\gamma).$$

In the following, the dependence of $l$ and $k$ in $\gamma$ will be often omitted for simplicity of notations and will be recalled only when it is necessary. This definition allows to write the quantity we want to bound

$$\left\|[P_{\gamma_k}(k\eta), P_{\gamma_0}]\, P_{\gamma_{k-1}}((k-1)\eta)\cdots P_{\gamma_1}(\eta)\psi_\hbar\right\|_{L^2}$$

in the following way:

(61)
$$\left\|[P_{\gamma_k}((k-l+1)\eta), P_{\gamma_0}((-l+1)\eta)]\, P_{\gamma_{k-1}}((k-l)\eta)\cdots P_{\gamma_l}(\eta)P_{\gamma_{l-1}}\cdots P_{\gamma_1}((-l+2)\eta)\psi_\hbar\right\|_{L^2}.$$

The reason why we choose to write the quantity we want to bound in (60) in the previous form instead of its original form is to have a more symmetric situation for our semiclassical analysis. To prove the bound in theorem 6.4, a class of symbols taken from [13] will be used (see (79) for a definition) and results about them are recalled in appendix B. Before starting the proof, using proposition B.3, we can restrict ourselves to observables carried on a thin energy strip around the energy layer $\mathcal{E}^\theta$. It means that the quantity we want to bound is the following norm:

(62)
$$\left\|\left[\hat{P}_{\gamma_k}((k-l+1)\eta), \hat{P}_{\gamma_0}((-l+1)\eta)\right]\hat{P}_{\gamma_{k-1}}((k-l)\eta)\cdots\hat{P}_{\gamma_l}(\eta)\hat{P}_{\gamma_{l-1}}\cdots\hat{P}_{\gamma_{k-1}}((-l+2)\eta)\psi_\hbar\right\|_{L^2},$$

where $\hat{P}_i$ is now equal to $\mathrm{Op}_\hbar(P_i^f)$, where $P_i^f$ is compactly supported in $T^*\Omega_i \cap \mathcal{E}^\theta$ (see proposition B.3).

6.2.1. *Defining cutoffs.* If we consider quantity (62), we can see that because we consider large times $k\eta$, we can not estimate directly the norm of the bracket $\left[\hat{P}_{\gamma_k}((k-l+1)\eta), \hat{P}_{\gamma_0}((-l+1)\eta)\right]$. However, the quantity we are really interested in is the norm of this bracket on the image of $\hat{P}_{\gamma_{k-1}}((k-l)\eta)\cdots\hat{P}_{\gamma_l}(\eta)\hat{P}_{\gamma_{l-1}}\cdots\hat{P}_{\gamma_{k-1}}((-l+2)\eta)$. So we will introduce some cutoff operators to localize the bracket we want to estimate on the image of $\hat{P}_{\gamma_{k-1}}((k-l)\eta)\cdots\hat{P}_{\gamma_l}(\eta)\hat{P}_{\gamma_{l-1}}\cdots\hat{P}_{\gamma_{k-1}}((-l+2)\eta)$. Then, as was discussed in section 1.2, we will have to verify that it defines a particular family of operators for which the Egorov theorem can be applied for large times.
First, we introduce a new family of functions $(Q_i)_{i=1}^K$ such that such that for each $1 \leq i \leq K$, $Q_i$ belongs to $\mathcal{C}^\infty(T^*\Omega_i \cap \mathcal{E}^\theta)$, $0 \leq Q_i \leq 1$ and $Q_i \equiv 1$ on $\mathrm{supp}P_i^f$. We then define two cutoffs associated to the strings $(\gamma_1, \cdots, \gamma_{l-1})$ and $(\gamma_l, \cdots, \gamma_{k-1})$:

$$Q_{\gamma_{k-1},\cdots,\gamma_l} := Q_{\gamma_l} \circ g^{-(k-l)\eta} \cdots Q_{\gamma_{k-1}} \circ g^{-\eta} \tag{63}$$

and

$$\tilde{Q}_{\gamma_{l-1},\cdots,\gamma_1} := Q_{\gamma_1} \circ g^{\eta} \cdots Q_{\gamma_{l-1}} \circ g^{(l-1)\eta}. \tag{64}$$

The first point of our discussion will be to prove that Egorov theorem can be applied for large times to the pseudodifferential operators corresponding to these two symbols.
We prove the Egorov property for $Q_{\gamma_{k-1},\cdots,\gamma_l}$ for example (the proof works in the same way for the other one). Recall that one has the exact equality, for a symbol $a$:

$$U^{-t}\mathrm{Op}_\hbar(a)U^t - \mathrm{Op}_\hbar(a(t)) = \int_0^t U^{-s}(\mathrm{Diff}a^{t-s})U^s ds, \tag{65}$$

where $a(t) := a \circ g^t$ and $\mathrm{Diff}a^t := \frac{i}{\hbar}[-\hbar^2\Delta, \mathrm{Op}_\hbar(a(t))] - \mathrm{Op}_\hbar(\{H, a(t)\})$. Here, we will consider $a := Q_{\gamma_{k-1},\cdots,\gamma_l}$. One has, for $0 \leq t \leq (k-l+1)\eta$:

$$Q_{\gamma_{k-1},\cdots,\gamma_l}(t) := Q_{\gamma_{k-1},\cdots,\gamma_l} \circ g^t = Q_{\gamma_l} \circ g^{-(k-l)\eta+t} \cdots Q_{\gamma_{k-1}} \circ g^{-\eta+t}.$$

There exists a unique integer $1 \leq j \leq (k-l)$ such that $t - j\eta$ is negative and $t - (j-1)\eta$ is nonnegative. This allows us to rewrite:

$$Q_{\gamma_{k-1},\cdots,\gamma_l}(t) = \left(Q_{\gamma_l} \circ g^{-(k-l-j)\eta} \cdots Q_{\gamma_{k-j}}\right) \circ g^{-j\eta+t} \left(Q_{\gamma_{k-j+1}} \cdots Q_{\gamma_{k-1}} \circ g^{(j-2)\eta}\right) \circ g^{-(j-1)\eta+t}.$$

Using the last part of theorem 7.1 and its subsequent remark, we know that $Q_{\gamma_l}\circ g^{-(k-l-j)\eta}\cdots Q_{\gamma_j}$ and $Q_{\gamma_{j-1}}\cdots Q_{\gamma_{k-1}}\circ g^{(j-2)\eta}$ are symbols of the class $S^{-\infty,0}_\nu$ (see the appendix for a definition of this class of symbols), where $\nu := \frac{1-\epsilon'+4\epsilon}{2}$. Moreover the constants in the bounds of the derivatives are uniform for the words $\gamma$ in the allowed set (see theorem 7.1 and proposition 7.3). As $-\eta \leq t-j\eta < 0 \leq t-(j-1)\eta \leq \eta$ and as the class $S^{-\infty,0}_\nu$ is stable by product, we have then that $Q_{\gamma_1,\cdots,\gamma_{k-l}}(t)$ is in the class $S^{-\infty,0}_\nu$, for $0\leq t\leq (k-l+1)\eta$, with uniform bounds in $t$ and $\gamma$ in the allowed set. As, in [5], we can verify that $\mathrm{Diff}Q^t_{\gamma_{k-1},\cdots,\gamma_l}$ is in $S^{-\infty,\nu-1}_\nu$ and then apply the Calderon-Vaillancourt theorem for $\Psi^{-\infty,\nu-1}_\nu$. As a consequence, there exists a constant $C$ depending only on the family $Q_i$ and on the derivatives of $g^s$ (for $-\eta\leq s\leq\eta$) such that:

$$\forall 0\leq t\leq (k-l+1)\eta,\quad \|\mathrm{Op}_\hbar(Q_{\gamma_{k-1},\cdots,\gamma_l})(t)-\mathrm{Op}_\hbar(Q_{\gamma_{k-1},\cdots,\gamma_l}(t))\|_{L^2}\leq C\hbar^{\frac{1-\nu}{2}}. \tag{66}$$

As we mentioned it in the heuristic of the proof (section 1.2), taking into account the support of the symbol, we have proved a 'local' Egorov property for a range of time that depends on the support of our symbol. Precisely, we have shown that the Egorov property holds until the stopping time defined in section 5.1.

6.2.2. *Proof of theorem 6.4.* Before proving theorem 6.4, we define (in order to have simpler expressions):

$$\psi^\gamma_\hbar := \hat{P}_{\gamma_{k-1}}((k-l)\eta)\cdots\hat{P}_{\gamma_1}((-l+2)\eta)\psi_\hbar.$$

To prove theorem 6.4, we need to bound quantity (62) and precisely to estimate (62), we have to estimate:

$$(62) = \left\|\left[\hat{P}_{\gamma_0}((-l+1)\eta),\hat{P}_{\gamma_k}((k-l+1)\eta)\right]\psi^\gamma_\hbar\right\|_{L^2}. \tag{67}$$

Now we want to introduce our cutoff operators $\mathrm{Op}_\hbar(Q_\bullet)$ in the previous expression:

$$\hat{P}_{\gamma_0}((-l+1)\eta)\hat{P}_{\gamma_k}(k-l+1)\eta) = \hat{P}_{\gamma_0}((-l+1)\eta)\left(Id-\left(\hat{P}_{\gamma_k}\mathrm{Op}_\hbar(Q_{\gamma_{k-1},\cdots,\gamma_l})\right)((k-l+1)\eta)\right)$$
$$+\left(\hat{P}_{\gamma_k}\mathrm{Op}_\hbar(Q_{\gamma_{k-1},\cdots,\gamma_l})\right)((k-l+1)\eta).$$

We will first estimate the norm

$$\left\|\hat{P}_{\gamma_0}((-l+1)\eta)\left(Id-\left(\hat{P}_{\gamma_k}\mathrm{Op}_\hbar(Q_{\gamma_{k-1},\cdots,\gamma_l})\right)((k-l+1)\eta)\right)\psi^\gamma_\hbar\right\|_{L^2}.$$

To do this, we first outline that $\hat{P}_{\gamma_k}$ is in $\Psi^{-\infty,0}(M)$ and $\mathrm{Op}_\hbar(Q_{\gamma_{k-1},\cdots,\gamma_l})$ is in $\Psi^{-\infty,0}_\nu(M)$. Using the standard rules for a product, we know that the previous expression can be transformed as follows:

$$\left\|\hat{P}_{\gamma_0}((-l+1)\eta)\left(Id-\mathrm{Op}_\hbar(P^f_{\gamma_k}Q_{\gamma_{k-1},\cdots,\gamma_l})((k-l+1)\eta)\right)\psi^\gamma_\hbar\right\|_{L^2}+R^1_\gamma(\hbar),$$

where $\|R^1_\gamma(\hbar)\|_{L^2}\leq C\hbar^{1-2\nu}\|\psi^\gamma_\hbar\|_{L^2}$ (where $C$ is independent of $k-l$ as the bounds implied in the derivatives in theorem 7.1 are uniform for words $\gamma$ in the allowed set: see proposition 7.3). We can apply the strategy of the previous section to prove an Egorov property for the operator $\mathrm{Op}_\hbar(P^f_{\gamma_k}Q_{\gamma_{k-1},\cdots,\gamma_l})$. So, up to a $O_{L^2}(\hbar^{\frac{1-\nu}{2}})$, $\mathrm{Op}_\hbar(P^f_{\gamma_k}Q_{\gamma_{k-1},\cdots,\gamma_l})((k-l+1)\eta)$ is equal to the pseudodifferential operator in $\Psi^{-\infty,0}_\nu$:

$$\mathrm{Op}_\hbar\left((P^f_{\gamma_k}Q_{\gamma_{k-1},\cdots,\gamma_l})\circ g^{(k-l+1)\eta}\right)$$

supported in $g^{-\eta}T^*\Omega_{\gamma_l}\cap\cdots\cap g^{-(k-l+1)\eta}T^*\Omega_{\gamma_k}\cap\mathcal{E}^\theta$. Using then theorem 7.1, the following holds:

$$\left(Id-\mathrm{Op}_\hbar\left((P^f_{\gamma_k}Q_{\gamma_{k-1},\cdots,\gamma_l})\circ g^{(k-l+1)\eta}\right)\right)\hat{P}_{\gamma_{k-1}}((k-l)\eta)\cdots\hat{P}_{\gamma_1}((-l+2)\eta)\psi_\hbar = O_{L^2}(\hbar^\infty).$$

Even if the proof of this fact is rather technical, it is intuitively quite clear. In fact, if we suppose that the standard pseudodifferential rules (Egorov, composition) apply, $\hat{P}_{\gamma_{k-1}}((k-l)\eta)\cdots\hat{P}_{\gamma_1}((-l+2)\eta)$ is a pseudodifferential operator compactly supported in $g^{(l-2)\eta}T^*\Omega_{\gamma_1}\cap\cdots\cap g^{(l-k)\eta}T^*\Omega_{\gamma_{k-1}}\cap\mathcal{E}^\theta$. On this set, by definition of the cutoff operators ($Q_i\equiv 1$ on $\mathrm{supp}(P_i)$), $(1-(P^f_{\gamma_k}Q_{\gamma_{k-1},\cdots,\gamma_l})\circ g^{(k-l+1)\eta})$ is equal to 0. As a consequence, we consider the product of two pseudodifferential

operators of disjoint supports: it is $O_{L^2}(\hbar^\infty)$. The statement of theorem 7.1 makes this argument work. To conclude the previous lines of the proof, we have:

$$\left\|\hat{P}_{\gamma_0}((-l+1)\eta)\left(Id-\left(\hat{P}_{\gamma_k}\mathrm{Op}_\hbar(Q_{\gamma_{k-1},\cdots,\gamma_l})\right)((k-l+1)\eta)\right)\psi_\hbar^\gamma\right\|_{L^2}\leq \tilde{C}\hbar^{1-2\nu}\|\psi_\hbar^\gamma\|_{L^2}. \tag{68}$$

Performing this procedure for the other operators, we finally obtain that the only quantity we need to bound to prove theorem 6.4 is the following quantity:

$$\left\|\left[(\hat{P}_{\gamma_k}\mathrm{Op}_\hbar(Q_{\gamma_{k-1},\cdots,\gamma_l}))((k-l+1)\eta),(\hat{P}_{\gamma_0}\mathrm{Op}_\hbar(\tilde{Q}_{\gamma_{l-1},\cdots,\gamma_1}))((-l+1)\eta)\right]\right\|_{L^2}. \tag{69}$$

Using the property of the product on $\Psi_\nu^{-\infty,0}$, we know that, up to a $O_{L^2}(\hbar^{1-2\nu})$, the previous quantity is equal to

$$\left\|\left[\mathrm{Op}_\hbar(P^f_{\gamma_k}Q_{\gamma_{k-1},\cdots,\gamma_l})((k-l+1)\eta),\mathrm{Op}_\hbar(P^f_{\gamma_0}\tilde{Q}_{\gamma_{l-1},\cdots,\gamma_1})((-l+1)\eta)\right]\right\|_{L^2}.$$

Using the same method that in the previous section (which uses theorem 7.1), we can prove an Egorov property for the two pseudodifferential operators that are in the previous bracket and show that, up to a $O_{L^2}(\hbar^{1-2\nu})$, the quantity (69) is equal to:

$$\left\|\left[\mathrm{Op}_\hbar((P^f_{\gamma_k}Q_{\gamma_{k-1},\cdots,\gamma_l})\circ g^{(k-l+1)\eta}),\mathrm{Op}_\hbar((P^f_{\gamma_0}\tilde{Q}_{\gamma_{l-1},\cdots,\gamma_1})\circ g^{(-l+1)\eta})\right]\right\|_{L^2}.$$

Using the pseudodifferential rules in $\Psi_\nu^{-\infty,0}(M)$ (proceeding as in the previous section, the two symbols stay in the good class of symbol using theorem 7.1), we know that the previous bracket is in $\Psi_\nu^{-\infty,2\nu-1}$. Using the Calderon-Vaillancourt theorem, we know that quantity (69) is a $O_{L^2}(\hbar^{1-2\nu})$, where the constant depends only on the partition. This concludes the proof of theorem 6.4.□

## 7. Products of many evolved pseudodifferential operators

The goal of this section is to prove a property used in the proof of theorem 6.4. Precisely, the following theorem states that the product of a large number of evolved pseudodifferential operators remains in a good class of pseudodifferential operators provided the range of times is smaller than the 'local' Ehrenfest time. First, recall that using proposition B.3, we can restrict ourselves to observables carried on a thin energy strip around the energy layer $\mathcal{E}^\theta$. We underline that we do not suppose anymore that this thin energy strip is of size $\hbar^{1-\delta}$: we only need to have a small macroscopic neighborhood of the unit energy layer. Moreover, the class of symbols we will consider will be the class $S_\nu^{-\infty,0}$ (see (79) for a precise definition) with $\nu:=\frac{1-\epsilon'+4\epsilon}{2}$ ($<1/2$, see section 4).

**Theorem 7.1.** *Let $(Q_i)_{i=1}^K$ be a family of smooth functions on $T^*M$ such that for each $1\leq i\leq K$, $Q_i$ belongs to $\mathcal{C}^\infty(T^*\Omega_i\cap\mathcal{E}^\theta)$ and $0\leq Q_i\leq 1$. Consider a family of indices $(\gamma_1,\cdots,\gamma_l)$ such that:*

$$\sum_{j=1}^{l-1}f(\gamma_{j+1},\gamma_j)\leq\frac{n_E(\hbar)}{2}.$$

*Then, for any $1\leq j\leq l$, one has:*

$$Op_\hbar(Q_{\gamma_1})Op_\hbar(Q_{\gamma_2})(-\eta)\cdots Op_\hbar(Q_{\gamma_j})(-(j-1)\eta)=Op_\hbar\left(A^{\gamma_1,\cdots,\gamma_j}\right)(-j\eta)+O_{L^2}(\hbar^\infty),$$

*where $A^{\gamma_1,\cdots,\gamma_j}$ is in the class $S_\nu^{-\infty,0}$. Precisely, one has the following asymptotic expansion:*

$$A^{\gamma_1,\cdots,\gamma_j}\sim\sum_{p\geq 0}\hbar^pA_p^{\gamma_1,\cdots,\gamma_j},$$

*where $A_p^{\gamma_1,\cdots,\gamma_j}$ is in the class $S_\nu^{-\infty,p\epsilon}$ (with the symbols semi norm uniform for $\gamma$ in the allowed set of sequences and $1\leq j\leq l$: see proposition 7.3) and compactly supported in $g^{-\eta}T^*\Omega_{\gamma_j}\cap\cdots g^{-j\eta}T^*\Omega_{\gamma_1}\cap\mathcal{E}^\theta$. Finally the principal symbol $A_0^{\gamma_1,\cdots,\gamma_j}$ is given by the following formula:*

$$A_0^{\gamma_1,\cdots,\gamma_j}=Q_{\gamma_j}\circ g^\eta\cdots Q_{\gamma_2}\circ g^{(j-1)\eta}Q_{\gamma_1}\circ g^{j\eta}.$$

*Remark.* We underline that the asymptotic expansion (except for the order 0 term) is not intrinsically defined as it depends on the choice of coordinates on $M$. We also remark that this theorem holds in particular for the smooth partition of identity we considered previously on the paper. Note also that the the result can be rephrased by saying that $\mathrm{Op}_\hbar(Q_{\gamma_1})(j\eta)\mathrm{Op}_\hbar(Q_{\gamma_2})((j-1)\eta)\cdots\mathrm{Op}_\hbar(Q_{\gamma_j})(\eta)$ is, up to a $O_{L^2}(\hbar^\infty)$, a pseudodifferential operator of the class $\Psi_\nu^{-\infty,0}$ and of well determined support. Under the assumption $\sum_{j=1}^{l-1} f(\gamma_j,\gamma_{j+1}) \leq \frac{n_E(\hbar)}{2}$, we would have proved that $\mathrm{Op}_\hbar(Q_{\gamma_1})(-j\eta)\mathrm{Op}_\hbar(Q_{\gamma_2})(-(j-1)\eta)\cdots\mathrm{Op}_\hbar(Q_{\gamma_j})(-\eta)$ is, up to a $O_{L^2}(\hbar^\infty)$, a pseudodifferential operator of the class $\Psi_\nu^{-\infty,0}$ and of well determined support. These are exactly the properties we used in section 6.2.

The plan of the proof is the following. First, we will construct formally $A^{\gamma_1,\cdots,\gamma_j}$ and its asymptotic expansion in powers of $\hbar$. Then, we will check that these different symbols are in a good class. Finally, we will check that these operators approximate the product we considered. For simplicity of notations, we will forget (for a time) the dependence on $\gamma$ and denote the previous symbol $A^j$ for $l \geq j \geq 1$.

7.1. **Definition of $A_{\gamma_1,\cdots,\gamma_l}$.** In this section, we construct formally the symbol $A^j$. The way to do it is by induction on $j$. First, we will see how to define formally $A^j$ from $A^{j-1}$. Then, using the formulas of the previous section, we will construct the formal order $N$ expansion associated to this $A^j$. We only construct what the order $N$ expansion should be regarding to the formal formulas.

7.1.1. *Definition at each step.* To construct $A^j$, we proceed by induction and at the first step, we consider $\mathrm{Op}_\hbar(Q_{\gamma_1})$ and we write it into the form $Op_\hbar(A^1)(-\eta)$. This means that we have defined formally for $0 \leq t \leq \eta$:

$$\mathrm{Op}_\hbar(A^1(t)) := U^{-t} Op_\hbar(Q_{\gamma_1}) U^t.$$

Using Egorov theorem for fixed time $\eta$ and the corresponding asymptotic expansion (see section B.3.2 for explicit formulas of the asymptotic expansion), we prove that, up to a $O_{L^2}(\hbar^\infty)$, $\mathrm{Op}_\hbar(Q_{\gamma_1})$ is equal to $\mathrm{Op}_\hbar(A^1(\eta))(-\eta)$, where $A^1(\eta)$ is in $S^{-\infty,0}$, given by the asymptotic expansion of the Egorov theorem and supported in $g^{-\eta}T^*\Omega_{\gamma_1} \cap \mathcal{E}^\theta$. We can continue this procedure formally. At the second step, we have

$$\mathrm{Op}_\hbar(Q_{\gamma_1})\mathrm{Op}_\hbar(Q_{\gamma_2})(-\eta) = U^\eta \mathrm{Op}_\hbar(A^1(\eta))\mathrm{Op}_\hbar(Q_{\gamma_2})U^{-\eta}.$$

We want this quantity to be of the form $\mathrm{Op}_\hbar(A^2(\eta))(-2\eta)$. This means that we have defined formally for $0 \leq t \leq \eta$:

$$\mathrm{Op}_\hbar(A^2(t)) := U^{-t}\mathrm{Op}_\hbar(A^1(\eta))\mathrm{Op}_\hbar(Q_{\gamma_2})U^t.$$

Using rules of pseudodifferential operators (see section B.3.1 and B.3.2), we can obtain a formal asymptotic expansion for $A^2(\eta)$ (see next section) starting from the expansion of $A^1(\eta)$. One can easily check that this formal expansion is supported in $g^{-\eta}T^*\Omega_{\gamma_2} \cap g^{-2\eta}T^*\Omega_{\gamma_1} \cap \mathcal{E}^\theta$. Following the previous method, we will construct a formal expansion of $A^j(t)$ (for $0 \leq t \leq \eta$) starting from the expansion of $A^{j-1}(\eta)$ (see next section). To do this, we will write at each step $1 \leq j \leq l$,

$$\mathrm{Op}_\hbar(A^j(t)) := U^{-t}\mathrm{Op}_\hbar(A^{j-1}(\eta))\mathrm{Op}_\hbar(Q_{\gamma_j})U^t. \tag{70}$$

With this definition, we will have:

$$\mathrm{Op}_\hbar(A^j(\eta))(-j\eta) := \left(\mathrm{Op}_\hbar(A^{j-1}(\eta))\mathrm{Op}_\hbar(Q_{\gamma_j})\right)(-(j-1)\eta).$$

Using again rules of pseudodifferential calculus (see section B.3.1 and B.3.2), we can obtain a formal asymptotic expansion for $A^j(t)$ (see next section) starting from the expansion of $A^{j-1}(\eta)$. One can easily check that this formal expansion is supported in $g^{-t}\left(T^*\Omega_{\gamma_j} \cap \cdots \cap g^{-j\eta}T^*\Omega_{\gamma_1}\right) \cap \mathcal{E}^\theta$. In the next section, we will use the induction formula (70) to deduce the $\hbar$-expansion of $A^j(t)$ from the expansion for the composition of $\mathrm{Op}_\hbar(A^{j-1}(\eta))$ and $\mathrm{Op}_\hbar(Q_{\gamma_j})$ and from the expansion for the Egorov theorem for times $0 \leq t \leq \eta$. At each step $1 \leq j \leq l$ of the induction, we will have to prove that $A^j$ stays in a good class of symbols to be able to continue the induction.

7.1.2. *Definition of the order N expansion.* We fix a large integer $N$ (to be determined). We study the previous construction by induction up to $O(\hbar^N)$. From this point, we truncate $A^j(t)$ at the order $N$ of its expansion. First, we see how we construct the symbols $A^j(t)$ by induction. To do this, we use the formulas for the asymptotic expansions for the composition of pseudodifferential operators and for the Egorov theorem (see section B.3.1 and B.3.2). Suppose that

$$A^{j-1}(\eta) = \sum_{p=0}^{N} \hbar^p A_p^{j-1}(\eta)$$

is well defined, we have to define the expansion of $A^j(t)$ from the asymptotic expansion of $A^{j-1}(\eta)$, for $0 \leq t \leq \eta$. First, we define:

$$\overline{A}^j := \sum_{p=0}^{N} \hbar^p \overline{A}_p^j, \qquad \text{where } \overline{A}_p^j := \sum_{q=0}^{p} \left( A_{p-q}^{j-1}(\eta) \sharp_M Q_{\gamma_j} \right)_q . \tag{71}$$

The symbol $\sharp_M$ represents an analogue on a manifold of the Moyal product (see appendix B.3.1): $(a\sharp_M b)_p$ is the order $p$ term in the expansion of the symbol of $\mathrm{Op}_\hbar(a)\mathrm{Op}_\hbar(b)$. Recall from the appendix that $(A_{p-q}^{j-1}\sharp_M Q_{\gamma_j})_q$ is a linear combination (that depends on the local coordinates and on the $(Q_i)_i$) of the derivatives of order less than $q$ of $A_{p-q}^{j-1}(\eta)$. Then, we define $A^j(t)$ using the exact asymptotic expansion given by proposition B.4 in appendix B.3.2. To do this, we define:

$$A_0^j(t) := \overline{A}_0^j(g^t)$$

and for $1 \leq p \leq N$,

$$A_p^j(t) := \overline{A}_p^j(g^t) + \sum_{q=0}^{p-1} \int_0^t \left\{ H, A_q^j(t-s) \right\}_M^{(p,q)} (g^s(\rho))\, ds,$$

where $H(\rho) = \frac{\|\xi\|_x^2}{2}$ is the Hamiltonian. This expression for the order $p$ term comes from proposition B.4 from the appendix where an exact expression for $\mathrm{Op}_\hbar(a)(t)$ is given. Recall also from proposition B.4 that for each $0 \leq q \leq p-1$, $\{H, A_q^j(t-s)\}_M^{(p,q)}$ is a linear combination of the derivatives of order less than $p+1-q$ of $A_q^j(t-s)$ that depends on the choice of the coordinates on the manifold. This construction is the precise way we want to define the asymptotic expansion of the symbol $A^j(t)$ in theorem 7.1. If we want the theorem to be valid, we have to check now that the remainders we forget at each step are negligible (with an arbitrary high order in $\hbar$). To do this, we will first have to control at each step $j$ the derivatives of $A_j(t)$ (see next section).

*Remark.* The support of $A_p^j(t)$ is included in $g^{-t}\left(T^*\Omega_{\gamma_j} \cap \cdots \cap g^{-(j-1)\eta} T^*\Omega_{\gamma_1}\right) \cap \mathcal{E}^\theta$ regarding the previous construction.

7.2. **Estimates of the derivatives.** The goal of the first part of this section is to prove the following lemma.

**Lemma 7.2.** *Let $N$ be a fixed integer. Fix also two integers $0 \leq p \leq N$ and $m \leq 2(N-p+1)$. Then, there exists a constant $C(m,p)$ such that for all $j \geq 1$ and for all $\rho$ in the set $g^{-t}\left(T^*\Omega_{\gamma_j} \cap \cdots \cap g^{-(j-1)\eta} T^*\Omega_{\gamma_1}\right) \cap \mathcal{E}^\theta$:*

$$\forall 0 \leq t \leq \eta,\ |d^m A_p^j(t,\rho)| \leq C(m,p) j^{m+p^2} |dg^{t+(j-1)\eta}(\rho)|^m.$$

*If $\rho$ is not in this set, the bound is trivially 0 by construction. Here the constant $C(m,p)$ depends only on $m$, $p$, the atlas we chose for the manifold and the size of the $(\Omega_\gamma)_\gamma$.*

Once this lemma will be proved, we will check that it also tells us that the $A_p^j$'s are in a nice class of symbols.

7.2.1. *Proof of lemma 7.2.* To make all the previous pseudodifferential arguments work, we will have to obtain estimates on the $m$-differential forms $d^m A_p^j$, for each $m \leq 2(N+1-p)$. If we have estimates on these derivatives, we will then check that all the asymptotic expansions given by the pseudodifferential theory are valid. To do these estimates, we will have to understand the number of derivatives that appear when we repeat the induction formula (70). The spirit of this proof is the same as in [5] 9section 3.4) when they iterate the WKB expansion $\mathcal{K}|\log\hbar|$ times. We define a vector $\mathbf{A}^j$ with entries $\mathbf{A}^j_{(p,m)} := d^m A_p^j$ (where $0 \leq p \leq N$ and $0 \leq m \leq 2(N-p+1)$). Precisely, we order it by the following way, for $0 \leq t \leq \eta$ and $\rho \in T^*M$:

$$\mathbf{A}^j = \mathbf{A}^j(t,\rho) := \begin{cases} (A_0^j, dA_0^j, \cdots, d^{2(N+1)}A_0^j, \\ A_1^j, dA_1^j, \cdots, d^{2N}A_1^j, \\ \cdots, \\ A_N^j, dA_N^j, d^2A_N^j). \end{cases}$$

Using induction formulas of the previous section, we will rewrite the link between $\mathbf{A}^j$ and $\mathbf{A}^{j-1}$ in the following form:

$$(\mathbf{I} - \mathbf{M}_1)\mathbf{A}^j = (\mathbf{M}_{0,0}^j + \mathbf{M}_{0,1}^j + \mathbf{M}_{0,2}^j)\mathbf{A}^{j-1},$$

where the operators $\mathbf{M}$ are defined in the following of the section. We will then inverse this formula to get $\mathbf{A}^j = f(\mathbf{A}^{j-1})$. Before giving the precise definition of each matrix, we define the operator, for each integer $l$:

$$G^t(d^l a)(\rho) := d^l a(g^t\rho).$$

It defines a $l$-form defined on $(T_{g^t\rho}T^*M)^l$. We write then:

$$d^m(G^t a)(\rho) = \sum_{l\leq m} G^t(d^l a)(\rho).\theta_{m,l}(t,\rho),$$

where the linear form $\theta_{m,l}(t,\rho)$ sends $(T_\rho T^*M)^m$ on $(T_{g^t\rho}T^*M)^l$. We can write the explicit form of $\theta_{m,m}$:

$$\theta_{m,m}(t,\rho) := \left(d_\rho g^t\right)^{\otimes m}.$$

Using these notations, we can rewrite the formulas for the $\mathbf{A}^j_{(p,m)}$ ($0 \leq p \leq N$ and $m \leq 2(N+1-p)$):

$$\mathbf{A}^j_{(p,m)}(t,\rho) := d^m\left(G^t\left(\overline{A}_p^j\right)(\rho)\right) + \sum_{q=0}^{p-1}\int_0^t d^m\left(G^s\left\{H, \mathbf{A}^j_{q,0}(t-s)\right\}_M^{(p,q)}(\rho)\right)ds,$$

where $\overline{A}_p^j$ depends on $\mathbf{A}^{j-1}(\eta,.)$. Using this, we can define $\mathbf{M}_1$:

$$\left(\mathbf{M}_1\mathbf{A}^j\right)_{(p,m)}(t,\rho) := \sum_{q=0}^{p-1}\int_0^t d^m\left(G^s\left\{H, \mathbf{A}^j_{q,0}(t-s)\right\}_M^{(p,q)}(\rho)\right)ds.$$

Using our previous decomposition of $d^m(T^t a)$ and the fact that $\{H, A_q^j(t-s)\}_M^{(p,q)}$ is a linear combination of the derivatives of order less than $p+1-q$ of $A_q^j(t-s)$, we find:

$$\left(\mathbf{M}_1\mathbf{A}^j\right)_{(p,m)}(t,\rho) = \sum_{q=0}^{p-1}\sum_{l\leq m+p+1-q}\int_0^t G^s\left(\mathbf{A}^j_{q,l}(t-s)\right)(\rho).\overline{\theta}_{m,q,l}(t,s,\rho)ds,$$

where $\overline{\theta}_{m,q,l}$ is a linear form that sends $(T_\rho T^*M)^m$ on $(T_{g^{t-s}\rho}T^*M)^k$. The important fact to underline about this matrix $\mathbf{M}_1$ is that it relates $A_p$ to $(A_r)_{0\leq r\leq p-1}$. In particular, it is a nilpotent operator-valued matrix of order $N+1$. We can also give the explicit expression for $\mathbf{M}_{0,0}^j$:

$$\left(\mathbf{M}_{0,0}^j\mathbf{A}^{j-1}\right)_{(p,m)}(t,\rho) := G^t\left(Q_{\gamma_j}\mathbf{A}^{j-1}_{(p,m)}(\eta)\right)(\rho).\theta_{m,m}(t,\rho).$$

In particular, $\mathbf{M}_{0,0}^j$ is a diagonal matrix. Finally the last matrices are defined as:

$$\left(\mathbf{M}_{0,1}^j\mathbf{A}^{j-1}\right)_{(p,m)}(t) := \sum_{l<m} G^t\left(d^l\left(Q_{\gamma_j}\mathbf{A}^{j-1}_{(p,0)}(\eta)\right)\right).\theta_{m,l}(t) = \sum_{l<m}\sum_{k\leq l} G^t\left(\mathbf{A}^{j-1}_{(p,k)}(\eta)\right).\overline{\theta}^{j,1}_{m,l}(t)$$

(it corresponds to the term of order 0 in the 'Moyal product' on a manifold) and using the higher order term of the Moyal product formula on a manifold:

$$\left(\mathbf{M}^j_{0,2}\mathbf{A}^{j-1}\right)_{(p,m)}(t) := \sum_{q=1}^{p}\sum_{l\leq m}\sum_{k\leq l} G^t\left(\mathbf{A}^{j-1}_{(p-q,k+q)}(\eta)\right).\overline{\theta}^{j,2}_{m,l,q}(t).$$

Regarding these formulas, it can first be underlined that $\mathbf{M}^j_{0,1}$ relates $\mathbf{A}^j_{(p,m)}$ with $\mathbf{A}^{j-1}_{(p,k)}$ where $k<m$. As in the case of $\mathbf{M}_1$, it can be underlined that the matrix $\mathbf{M}^j_{0,2}$ relates $A_p$ to $(A_r)_{0\leq r\leq p-1}$ and in particular, it behaves like a 'nilpotent' operator-valued matrix of order $N+1$. In both cases, it results then that we have a nilpotent matrix and that the $\overline{\theta}^{j,\cdot}$ are multilinear forms depending on $j$. Outline that all the multilinear forms used to define the different matrices are uniformly bounded with respect to $\rho$, $t\in[0,\eta]$ and $j$. We will just have to check that these bounds do not accumulate too much when we iterate $j$ times the induction formula.

*Remark.* The crucial point of the previous discussion is not really the exact form of the matrices $\mathbf{M}$. The important facts are that $\mathbf{M}_1$ relates $A_p$ to $(A_r)_{0\leq r\leq p-1}$ (it is in fact nilpotent of index $N$), $\mathbf{M}^j_{0,0}$ is a diagonal matrix, $\mathbf{M}^j_{0,1}$ relates $\mathbf{A}^j_{(p,m)}$ with $\mathbf{A}^{j-1}_{(p,k)}$ (where $k<m$) and $\mathbf{M}^j_{0,2}$ relates $A_p$ to $(A_r)_{0\leq r\leq p-1}$.

Using this remark, we can inverse the induction formula to get an expression of $\mathbf{A}^j$ in function of $\mathbf{A}^{j-1}$:

$$\mathbf{A}^j := \left(\sum_{k=0}^{N}[\mathbf{M}_1]^k\right)\left(\mathbf{M}^j_{0,0}+\mathbf{M}^j_{0,1}+\mathbf{M}^j_{0,2}\right)\mathbf{A}^{j-1}.$$

Here, we put [.] around the matrix to distinguish the power of a matrix and an index. Iterating the previous expression, one then has:

$$\mathbf{A}^j := \sum_{k_2,\cdots,k_j=0}^{N-1}\sum_{\alpha_2,\cdots,\alpha_j=0}^{2}[\mathbf{M}_1]^{k_j}\mathbf{M}^j_{0,\alpha_j}[\mathbf{M}_1]^{k_{j-1}}\mathbf{M}^{j-1}_{0,\alpha_{j-1}}\cdots[\mathbf{M}_1]^{k_2}\mathbf{M}^2_{0,\alpha_2}\mathbf{A}^1.$$

From this expression, one can estimate how many terms contributes to the definition of $\mathbf{A}^j_{(p,m)}$. In fact, using the previous remark, the product of matrices that contributes to the expression of $\mathbf{A}^j_{(p,m)}$ can only be non zero if $\sum_{j'}k_{j'}+|\{j':\alpha_{j'}=2\}|\leq p$ and $|\{j':\alpha_{j'}=1\}|\leq m+p(\sum_{j'}k_{j'}+|\{j':\alpha_{j'}=2\}|)$. As a consequence, for large $j$, to be non zero, a string of matrices need to be made of at most $(N+1)^2$ matrices of the form $\mathbf{M}^j_{0,\alpha}$ and $\mathbf{M}_1$ (for $\alpha\in\{1,2\}$). Moreover, the number of string that contributes to $\mathbf{A}^j_{(p,m)}$ is of size $O(j^{m+p^2})$ when $j$ becomes large.

Then, to estimate the norm of the derivatives of $A^j$, we should look how the different matrices used to define the derivatives act. First we study the action of the diagonal matrix. As $0\leq Q_{\gamma_j}\leq 1$, one has that, for $0\leq t\leq\eta$ and for any $\rho\in g^{-t}\left(T^*\Omega_{\gamma_j}\cap\cdots\cap g^{-(j-1)\eta}T^*\Omega_{\gamma_1}\right)\cap\mathcal{E}^\theta$ (otherwise the following quantity is clearly equal to 0):

$$|\mathbf{M}^j_{0,0}\mathbf{A}^{j-1}_{(p,m)}(t,\rho)|\leq|dg^t(\rho)|^m|\mathbf{A}^{j-1}_{(p,m)}(\eta,g^t(\rho))|.$$

We note that we can iterate this bound without getting an exponentially big term in $j$. In particular, for any $j$ and $j'$ in $\mathbb{N}$, we have for any $0\leq t\leq\eta$:

$$|\mathbf{M}^{j+j'}_{0,0}\cdots\mathbf{M}^j_{0,0}\mathbf{A}^{j-1}_{(p,m)}(t,\rho)|\leq|dg^{t+j'\eta}(\rho)|^m|\mathbf{A}^{j-1}_{(p,m)}(\eta,g^{t+j'\eta}(\rho))|.$$

Now, using the fact that all the multilinear forms we considered are uniformly bounded, we get that there exists a constant $C(m,p)$ such that:

$$\sup_{0\leq t\leq\eta}\|\mathbf{M}_1\mathbf{A}^j_{(p,m)}(t)\|_{L^\infty}\leq C(m,p)\max_{m'\leq m+2}\max_{q\leq p-1}\sup_{0\leq t\leq\eta}\|\mathbf{A}^j_{(q,m')}(t)\|_{L^\infty}$$

and, for $\alpha\in\{1,2\}$:

$$\sup_{0\leq t\leq\eta}\|\mathbf{M}^j_{0,\alpha}\mathbf{A}^{j-1}_{(p,m)}(t)\|_{L^\infty}\leq C(m,p)\max_{m'\leq m}\max_{q\leq p}\|\mathbf{A}^{j-1}_{(q,m')}\|_{L^\infty}.$$

The only thing we need to know is that the constant depends only on $m$, $p$, the coordinate maps and the partition. The difference with the action of the diagonal matrix is that we have constant prefactor that can accumulate and become large. However as was already mentioned, the number of such matrices that contribute to the estimate is at most of order $N^2$. So the constant factor $C(m,p)$ does not create a factor which makes the derivatives explodes when we iterate the induction. Then, proceeding like in [5] (end of section 3.4), we can use all the previous bounds on the action of the matrices and on the number of matrices who contributes to $(p,m)$ term of the vector $\mathbf{A}^j$ and the multiplicativity property of $dg^t(\rho)$. It gives that for any $\rho \in g^{-t}\left(T^*\Omega_{\gamma_j} \cap \cdots \cap g^{-(j-1)\eta}T^*\Omega_{\gamma_1}\right) \cap \mathcal{E}^\theta$:

$$|\mathbf{A}^j_{(p,m)}(t,\rho)| \leq \tilde{C}(p,m) j^{m+p^2} |dg^{t+(j-1)\eta}(\rho)|^m \|\mathbf{A}^1(\eta)\|. \square$$

7.2.2. *Class of symbol of each term of the expansion.* Using the previous lemma, we want to show that $A^j_p(t)$ is an element of $S^{-\infty,p\epsilon}_\nu$. Let $\rho$ be an element of $g^{-t}\left(T^*\Omega_{\gamma_j} \cap \cdots \cap g^{-(j-1)\eta}T^*\Omega_{\gamma_1}\right) \cap \mathcal{E}^\theta$. Using the fact that $E^u$ is of dimension 1, we get that for any positive $t$, $|d_\rho g^t| \leq J^u_t(\rho)^{-1}$, where $J^u_t(\rho) := \det\left(dg^{-t}_{|E^u(g^t\rho)}\right)$. Then we can write the multiplicativity of the determinant and get

$$J^u_{t+(j-1)\eta}(\rho) = J^u_t(\rho) J^u_\eta(g^t\rho) J^u_\eta(g^{t+\eta}\rho) \cdots J^u_\eta(g^{t+(j-2)\eta}\rho).$$

*Remark.* Before continuing the estimate, let us underline some property of the Jacobian. Suppose $S$ is a positive integer and $1/\eta$ also (large enough to be in our setting). We have for all $0 \leq k \leq 1/\eta - 1$:

$$J^u(g^{k\eta}\rho) J^u(g^{1+k\eta}\rho) \cdots J^u(g^{S-1+k\eta}\rho) = J^u_\eta(g^{k\eta}\rho) J^u_\eta(g^{(k+1)\eta}\rho) \cdots J^u_\eta(g^{S+(k-1)\eta}\rho),$$

where $J^u(\rho)$ is the unstable Jacobian in time 1 that appears in the main theorem 1.3. We make the product over $k$ of all these equalities and we get:

$$J^u(\rho)^\eta J^u(g^\eta\rho)^\eta \cdots J^u(g^{S-\eta}\rho)^\eta \leq C(\eta) J^u_\eta(g\rho) J^u_\eta(g^{1+\eta}\rho) \cdots J^u_\eta(g^{S-\eta}\rho),$$

where $C(\eta)$ only depends on $\eta$ and does not depend on $S$.

Finally, using previous remark and inequality (19), the following estimate holds:

$$|d_\rho g^{t+(j-1)\eta}| \leq C(\eta) e^{j\epsilon\eta a_0} J^u_\eta(\gamma_j,\gamma_{j-1})^{-\eta} J^u_\eta(\gamma_{j-1},\gamma_{j-2})^{-\eta} \cdots J^u_\eta(\gamma_2,\gamma_1)^{-\eta}$$

with $C(\eta)$ independent of $j$. Then:

$$|d_\rho g^{t+(j-1)\eta}| \leq C(\eta) e^{l(\gamma)\epsilon\eta a_0} e^{t(\gamma)},$$

where $t(\gamma) = \sum_{j=0}^{l-1} f(\gamma_{j+1},\gamma_j)$. As $t(\gamma) \leq n_E(\hbar)/2$, this last quantity is bounded by $\hbar^{\frac{\epsilon'-1}{2}-\epsilon}$ (as $l(\gamma)a_0\eta \leq n_E(\hbar)/2$). Using lemma 7.2, we want to estimate the $m$ derivatives of the symbol $A^j_p$. To do this, we can fix a large $\tilde{N}$ such that $m \leq 2(\tilde{N}-N)$ and then we have ($p < N,\ 0 \leq t \leq \eta$):

$$|d^m A^j_p(t,x)| \leq \tilde{C}(m,p)\hbar^{-p\epsilon}\hbar^{m(\frac{\epsilon'-1}{2}-2\epsilon)}. \tag{72}$$

We used the fact that $j = O(|\log\hbar|)$. Here appears the fact that we only apply the backward quantum evolution for times $l$. In fact, as we want our symbols to be in the class $S^{-\infty,\cdot}_\nu$, we need derivatives to lose at most a factor $\hbar^{-1/2}$ (this would have not been the case if we had considered times of size $n_E(\hbar)$ instead of size $n_E(\hbar)/2$). The previous estimate (72) is uniform for all the $\gamma$ in the allowed set of theorem 7.1.
Finally, to summarize this section, we can write the following proposition:

**Proposition 7.3.** *Let $p$ and $m$ be elements of $\mathbb{N}$. There exists $C(m,p,(Q_i)_i,\eta)$ (depending on $m$, $p$, $\eta$, $(Q_i)_{i=1}^K$ and the coordinate charts) such that for all $\gamma = (\gamma_0,\cdots,\gamma_l)$ such that*

$$\sum_{j=0}^{l-1} f(\gamma_{j+1},\gamma_j) \leq \frac{n_E(\hbar)}{2},$$

*for all $0 \leq j \leq l$ and for all $0 \leq t \leq \eta$:*

$$|d^m A^{\gamma_1,\cdots,\gamma_j}_p(t,x)| \leq C(m,p,(Q_i)_i,\eta)\hbar^{-p\epsilon}\hbar^{m(\frac{\epsilon'-1}{2}-2\epsilon)}.$$

*Then, as the $A_p^j$ are compactly supported, $A_p^j$ is in class $S_\nu^{-\infty,p\epsilon}$, where $\nu = \frac{1-\epsilon'+4\epsilon}{2}$.*

So, our formal construction allows us to define a family of symbol $A_p^j$ and each of them belongs to $S_\nu^{-\infty,p\epsilon}$. Moreover the constants implied in the bounds of the derivatives are uniform with respect to the allowed sequences. We underline that the same proof would show that the intermediate symbols $\overline{A}_p^j$ (71) are also in the same class of symbols.

7.3. **Estimate of the remainder terms.** We are now able to conclude the proof of theorem 7.1 starting from the family we have just constructed. We have to verify that the remainder is of small order in $\hbar$. Fix a large integer $N$ and denote $A_{(N)}^j := \sum_{p=0}^{N} \hbar^p A_p^j(\eta)$. We want to estimate:

$$R_N^j = \|\mathrm{Op}_\hbar(Q_{\gamma_1}) \cdots \mathrm{Op}_\hbar(Q_{\gamma_j})(-(j-1)\eta) - \mathrm{Op}_\hbar(A_{(N)}^j)(-j\eta)\|_{L^2}.$$

Using the induction formula (70), we write:

$$R_N^j \leq \|U^{-\eta}\mathrm{Op}_\hbar(A_{(N)}^{j-1})\mathrm{Op}_\hbar(Q_{\gamma_j})U^\eta - \mathrm{Op}_\hbar(A_{(N)}^j)\|_{L^2} + R_N^{j-1},$$

where $R_N^{j-1} = \|\mathrm{Op}_\hbar(Q_{\gamma_1}) \cdots \mathrm{Op}_\hbar(Q_{\gamma_{j-1}})(-(j-2)\eta) - \mathrm{Op}_\hbar(A_{(N)}^{j-1})\|_{L^2}$. Then, we can use section B.3.1 and the estimates (81), to write the estimate:

$$\|U^{-\eta}\mathrm{Op}_\hbar(A_{(N)}^{j-1})\mathrm{Op}_\hbar(Q_{\gamma_j})U^\eta - U^{-\eta}\mathrm{Op}_\hbar(\overline{A}_{(N)}^j)U^\eta\|_{L^2} \leq C_N\hbar^{(N+1)(1-\nu)-(C+C')\nu}.$$

We also use section B.3.2, to write the exact formula (as $\overline{A}^j = A^{j-1}\sharp_M Q_{\gamma_j}$):

$$U^{-\eta}\mathrm{Op}_\hbar(\overline{A}_{(N)}^j)U^\eta - \mathrm{Op}_\hbar(A_{(N)}^j) = \hbar^{N+1}\int_0^\eta U^{-s}\mathrm{Op}_\hbar\left(R^{(N)}(\overline{A}_{(N)}^j)(\eta-s)\right)U^s ds,$$

An estimate on the norm of $\mathrm{Op}_\hbar(R^{(N)}(\overline{A}_{(N)}^j)(s))$ will be given in section B.3.2, depending on the derivatives of the $\overline{A}_p^j$'s (for $p \leq N$). So, up to a $O_{L^2}(\hbar^\infty)$, one has:

$$R_N^j \leq R_N^{j-1} + \hbar^{N+1}\eta \sup_s \left\|\mathrm{Op}_\hbar\left(R^{(N)}(\overline{A}_{(N)}^j)(s)\right)\right\|_{L^2} + C_N\hbar^{(N+1)(1-\nu)-(C+C')\nu}.$$

Using estimates (84) from section B.3.2 and the last remark of the previous section, one can check that:

$$R_N^j \leq R_N^{j-1} + C_N\hbar^{(N+1)(1-\nu)-(C+C')\nu},$$

where $C + C'$ depends only on the dimension of the manifold. By induction on $j$ (which is $O(|\log\hbar|)$), we prove that, for large enough $N$, $\mathrm{Op}_\hbar(Q_{\gamma_1}) \cdots \mathrm{Op}_\hbar(Q_{\gamma_j})(-(j-1)\eta)$ is well approximated by $\mathrm{Op}_\hbar(A_{(N)}^j)(-j\eta)$. In fact, $\mathrm{Op}_\hbar(A_{(N)}^j)$ is in $\Psi_\nu^{-\infty,0}$. As a consequence for large $N$, $\hbar^{(N+1)(1-\nu)-\tilde{C}\nu}$ is arbitrarily small compared to $\|\mathrm{Op}_\hbar(A_{(N)}^j)(-j\eta)\|$ (which is bounded). This concludes the proof of theorem 7.1.□

## Appendix A. The case of surfaces of nonpositive curvature

In this appendix, we consider a surface of nonpositive sectional curvature $M$ and show how we can adapt the proof of the main inequality (5) in this slightly different case (theorem 1.2). Before starting the proof, we recall briefly a few facts about surfaces of nonpositive curvature. More details can be found for example in [29] (chapter 3). We recall that this proof is now specific to dimension 2 and to nonpositive curvature: on this form, it cannot be extended to any manifold without conjugate points. We will mention the points of the proof which use the dimension 2 and the nonpositivity assumptions.

A.1. **Surfaces of nonpositive curvature.** We define $\pi : S^*M \to M$ the canonical projection $\pi(x,\xi) := x$. The vertical subspace $V_\rho$ at the point $\rho = (x,\xi)$ is the kernel of the application $d_\rho\pi$. We underline that it is in fact the tangent space in $\rho$ of the 1-dimensional submanifold $S^*_xM$. In the case of a surface, it has dimension 1. We can also define the horizontal subspace in $\rho$. Precisely, for $Z \in T_\rho S^*M$, we consider a smooth curve $c(t) = (a(t), b(t))$, $t \in (-\epsilon,\epsilon)$, in $S^*M$ such that $c(0) = \rho$ and $c'(0) = Z$. Then, we define the horizontal space $\mathcal{H}_\rho$ as the kernel of the application $\mathbb{K}_\rho(Z) = \nabla_{a'(0)}b(0) = \nabla_{d_\rho\pi(Z)}b(0)$, where $\nabla$ is the Levi-Civita connection. This subspace contains $X_H(\rho)$ the vector field tangent to the Hamiltonian flow. For a surface, this subspace is of dimension 2. We know that we can use these two subspaces to split the tangent space $T_\rho S^*M = \mathcal{H}_\rho \oplus V_\rho$ (it is the usual way to split the tangent space in order to define the Sasaki metric on $S^*M$ [29]). Using this decomposition, we would like to recall an important link between the linearization of the geodesic flow and the Jacobi fields on $M$. To do this, we underline that to each point $\rho$ in $S^*M$ corresponds a unique unit speed geodesic $\gamma_\rho$. Then we define a Jacobi field in $\rho$ (or along $\gamma_\rho$) as a solution of the differential equation:

$$\mathbb{J}"(t) + R(\gamma'_\rho(t), \mathbb{J}(t))\gamma'_\rho(t) = 0,$$

where $R(X,Y)Z$ is the curvature tensor applied to the vector fields $X$, $Y$ and $Z$ and $\mathbb{J}'(t) = \nabla_{\gamma'_\rho(t)}\mathbb{J}(t)$. We recall that we can interpret Jacobi fields as geodesic variation vector fields [16]. Precisely, consider a $\mathcal{C}^\infty$ family of curves $c_s : [a,b] \to M$, $s$ in $(-\epsilon,\epsilon)$. We say that it is a $\mathcal{C}^\infty$ variation of $c = c_0$. It defines a corresponding variation vector field $Y(t) = \frac{\partial}{\partial s}(c_s(t))_{|s=0}$ that gives the initial velocity of $s \mapsto c_s(t)$. If we suppose now that $c$ is a geodesic of $M$, then a $\mathcal{C}^2$ vector field $Y(t)$ on $c$ is a Jacobi vector field if and only if $Y(t)$ is the the variation vector field of a geodesic variation of $c$ (i.e. $\forall s \in (-\epsilon,\epsilon)$, $c_s$ is a geodesic of $M$). For instance, $\gamma'_\rho(t)$ and $t\gamma'_\rho(t)$ are Jacobi vector fields along $\gamma_\rho$.
Consider now a vector $(V,W)$ in $T_\rho S^*M$ given in the coordinates $\mathcal{H}_\rho \oplus V_\rho$. Using the canonical identification given by $d_\rho\pi$ and $\mathbb{K}_\rho$, there exists a unique Jacobi field $\mathbb{J}_{V,W}(t)$ in $\rho$ whose initial conditions are $\mathbb{J}_{V,W}(0) = V$ and $\mathbb{J}'_{V,W}(0) = W$, such that

$$d_\rho g^t(V,W) = (\mathbb{J}_{V,W}(t), \mathbb{J}'_{V,W}(t))$$

in coordinates $\mathcal{H}_{g^t\rho} \oplus V_{g^t\rho}$ [29] (lemma 1.4). We define $N_\rho$ the subspace of $T_\rho S^*M$ of vectors orthogonal to $X_H(\rho)$ and $H_\rho$ the intersection of this subspace with $\mathcal{H}_\rho$. Using the previous property about Jacobi fields, we know that the subbundle $\mathcal{N}$ perpendicular to the Hamiltonian vector field is invariant by $g^t$ and that we have the following splitting [29] (lemma 1.5):

$$T_\rho S^*M = \mathbb{R}X_H(\rho) \oplus H_\rho \oplus V_\rho.$$

Obviously, these properties can be extended to any energy layer $\mathcal{E}(\lambda)$ for any positive $\lambda$. Following [29] (lemma 3.1), we can make the following construction of two particular Jacobi fields along $\gamma_\rho$. We denote $(\gamma'_\rho(t), e(t))$ an orthonormal basis defined along $\gamma_\rho(t)$. Given a positive $T$ and because there are no conjugate points on the manifold $M$, there exists a unique Jacobi field $\mathbb{J}_T(t)$ such that $\mathbb{J}_T(0) = e(0)$ and $\mathbb{J}_T(T) = 0$. Moreover, $\mathbb{J}_T(t)$ is perpendicular to $\gamma_\rho(t)$ for all $t$ in $\mathbb{R}$ [29] (page 50). As a consequence, $\mathbb{J}_T(t)$ can be identified with its coordinate along $e(t)$ (as $T_{\gamma_\rho(t)}M$ is of dimension 2). A result due to Hopf (lemma 3.1 in [29]) tells us that the limits

$$\lim_{T\to+\infty} \mathbb{J}_T(t) \text{ and } \lim_{T\to-\infty} \mathbb{J}_T(t)$$

exist. They are denoted $\mathbb{J}^s_\rho(t)$ and $\mathbb{J}^u_\rho(t)$ (respectively the stable and the unstable Jacobi field). They satisfy the simplified one dimensional Jacobi equation:

$$\mathbb{J}"(t) + K(t)\mathbb{J}(t) = 0,$$

where $K(t) = K(\gamma_\rho(t))$ is the sectional curvature at $\gamma_\rho(t)$. They are never vanishing Jacobi fields with $\mathbb{J}^*_\rho(0) = e(0)$ and for all $t$ in $\mathbb{R}$, they are perpendicular to $\gamma'_\rho(t)$. Moreover, we have $\|\mathbb{J}^{*'}_\rho(t)\| \leq \sqrt{K_0}\|\mathbb{J}^*_\rho(t)\|$ for every $t$ in $\mathbb{R}$ (where $-K_0$ is some negative lower bound on the curvature). Using the previous link between geodesic flow and Jacobi fields, we can lift these subspaces to invariant subspaces $E^s(\rho)$ and $E^u(\rho)$ called the Green stable and unstable subspaces. These subspaces have dimension 1 (in the case of surfaces) and are included in $N_\rho$. A basis of $E^s(g^t\rho)$

is given by $(\mathbb{J}^s_\rho(t), \mathbb{J}^{s'}_\rho(t))$ in coordinates $H_{g^t\rho} \oplus V_{g^t\rho}$. We can underline that both subspaces are uniformly transverse to $V_\rho$ and that it can happen that they are equal to each other (which was not the case in the Anosov setting). In the case of nonpositive curvature, these subspaces depend continuously in $\rho$ and are integrable as in the Anosov case [16].
In the case where the Green subspaces attached to $\rho$ are linearly independent, a splitting of $N_\rho$ is given by $E^u(\rho) \oplus E^s(\rho)$ and the splitting holds for all the trajectory. For the opposite case, we know that the Green subspaces attached to $\rho$ (and hence to a geodesic $\gamma_\rho$) are linearly dependent if and only if the sectional curvature is vanishing at every point of the geodesic $\gamma_\rho$ [29]. As a consequence, we cannot use the same kind of splitting. However, there exists a splitting of $N_\rho$ that we can use in both cases, precisely $E^u(\rho) \oplus V_\rho$. We would like to mention that the one dimensional Jacobi equation defined earlier gives rise to the Riccati equation:

$$U'(t) + U^2(t) + K(t) = 0,$$

where $U(t) = \mathbb{J}'(t)\mathbb{J}(t)^{-1}$ for non vanishing $\mathbb{J}$. Then we define the corresponding unstable Riccati solution associated to the unstable Jacobi field as $U^u_\rho(t) := \mathbb{J}^{u'}_\rho(t)(\mathbb{J}^u_\rho(t))^{-1}$. It is a nonnegative quantity and it describes the growth of the unstable Jacobi field (in dimension 2) as follows:

$$\|\mathbb{J}^u_\rho(t)\| = \|\mathbb{J}^u_\rho(0)\| e^{\int_0^t U^u_\rho(s)ds}.$$

The same works for the stable Jacobi field. Both quantities are continuous[4] with respect to $\rho$. We underline that, we can use the previous results to obtain the bound $\|d_\rho g^t_{|E^u(\rho)}\| \leq \sqrt{1+K_0} e^{\int_0^t U^u_\rho(s)ds}$. So the unstable Riccati solution describe the infinitesimal growth of the geodesic flow along the unstable direction, whereas $J^u(\rho)^{-1}$ used in the previous sections described the growth at time 1. More precisely, as for the unstable Jacobian, Freire and Mañé showed that the unstable Riccati solutions are related to the Lyapunov exponents. In fact, they proved that the Ruelle bound for the entropy of a $g$-invariant measure $\mu$ in the case of nonpositive curvature (precisely for manifolds without conjugate points) [20] is:

$$h_{KS}(\mu, g) \leq \int_{S^*M} U^u(\rho) d\mu(\rho).$$

A last point we would like to recall is a result due to Green [21] and to Eberlein in the general case [15]. It asserts that for any positive $c$ there exists a positive $T = T(c)$ such that for any $\rho$ in $S^*M$ and for any nontrivial Jacobi field $\mathbb{J}(t)$ along $\gamma_\rho$ such that $\mathbb{J}(0) = 0$ and $\|\mathbb{J}'(0)\| \geq 1$, for all $t$ larger than $T$, we have $\|\mathbb{J}(t)\| \geq c$ (proposition 3.1 [29]). This property of uniform divergence only holds in dimension 2 and as it is crucially used in the following, our proof only works for surfaces of nonpositive curvature. In larger dimensions, the same result holds but without any uniformity in $\rho$. Finally, all these properties allow to prove the following lemma:

**Lemma A.1.** *Let $v = (0, V)$ be a unit vertical vector at $\rho$. Then for any $c > 0$, there exists $T = T(c) > 0$ (independent of $\rho$ and of $v$) such that for any $t \geq T$, $\|d_\rho g^t v\| \geq c$.*

We underline that, for $t \geq T$, the angle between $E^u(g^t\rho)$ and $d_\rho g^t v$ is bounded by some $\kappa(c)$ with $\kappa(c)$ arbitrarly small as $c$ tends to infinity.

A.2. **Modifications of the technical assumptions.** Now that our new classical setting is well defined, we explain which modifications should be performed to prove the main inequality 2 for surfaces of nonpositive curvature. We define:

$$f(\rho) := U^u(\rho). \tag{73}$$

Recall that this quantity is nonnegative and continuous with respect to $\rho$ (as $M$ is of nonpositive curvature [29]). We also fix $\epsilon_0$ a small positive number and define:

$$f_0(\rho) := \sup\{\epsilon_0, f(\rho)\}.$$

The function $f_0$ will be used to ensure that the stopping time is finite. The main difference with the case of an Anosov flow is that the function that will appear in the main estimate (75) will be

[4]The continuity in $\rho$ is a crucial property that we will use in our proof. We underline that it is not true if we only suppose the surface to be without conjugate points [6].

slightly different and will be played by the function $f$ defined by equation (73). We define $f(\gamma)$ and $f_0(\gamma)$ as in section 3 i.e.

$$f_0(\gamma) := \eta \inf\{f_0(\rho) : \rho \in U_\gamma\} \text{ and } f(\gamma) := \eta \inf\{f(\rho) : \rho \in U_\gamma\}.$$

The role of $a_0$ will be played by $\epsilon_0$. In particular, we can rewrite the continuity of $f$ (inequality (19)) under the following slightly diferent form using again the small parameter $\epsilon$:

$$\forall \rho \in U_\gamma, \ \left| \int_0^\eta U_\rho^u(s)ds - f(\gamma) \right| \leq \eta\epsilon_0\epsilon. \tag{74}$$

We make the extra assumption that the small parameter $\epsilon$ used for the continuity is smaller than $\epsilon_0$.

A.3. **How do we modify the proof?** First we underline that we can extend theorem 5.6 to surfaces of nonpositive curvature in the following way:

**Theorem A.2.** *Let $M$ be a surface of nonpositive sectional curvature. For every $\mathcal{K} > 0$ ($\mathcal{K} \leq C_\delta$), there exists $\hbar_\mathcal{K}$ and $C_\mathcal{K}$ such that uniformly for all $\hbar \leq \hbar_\mathcal{K}$, for all $k \leq \mathcal{K}|\log\hbar|$, for all $\alpha = (\alpha_0, \cdots, \alpha_k)$,*

$$\|P_{\alpha_k} U^\eta P_{\alpha_{k-1}} \cdots U^\eta P_{\alpha_0} Op(\chi^{(k)})\|_{L^2(M)} \leq C_\mathcal{K} \hbar^{-\frac{1}{2}} e^{2k\eta\epsilon_0\epsilon} \exp\left( -\frac{1}{2} \sum_{j=0}^{k-1} f(\sigma^j \alpha) \right), \tag{75}$$

*using the previous notations from section A.2.*

We underline that the parameter $\epsilon_0\epsilon$ is a small parameter that we choose to be proportional to $\epsilon_0$ for simplicity (as $k\eta\epsilon_0$ will be bounded by $n_E(\hbar)$ in the following). In [5], the proof of this theorem was based on a WKB expansion. The same proof works: we just mention certain points that need attention. We refer the reader to [5] for the detailed proof. The first point we underline is that they used the fact that an Anosov manifold has no conjugate points. This property remains true in the case of manifolds of nonpositive curvature [16]. The second point is that they used that under the flow a vector $V$ in the vertical space $V_\rho$ tends exponentially fast to the unstable direction. In fact, one only needs the fact that $d_\rho g^t V_\rho$ converges uniformly to $E^u(\rho)$. This modification requires a multiplicative correction factor of order $Ce^{k\eta\delta}$ (with $\delta$ arbitrarly small that we will choose equal to $\epsilon_0\epsilon$) that depends on the length of the word $\alpha$ (with $C$ independent of $\rho$ and $\alpha$). This property holds as was mentioned by lemma A.1 and works only in dimension 2 for nonpositively curved manifolds. A last point that should be adapted is that the contraction of $g^{-1}$ along $E^u(\rho)$ can be understood in term of the unstable Riccati solution. Using section A.1, we know that for positive $t$:

$$|d_\rho g^{-t}_{|E^u(\rho)}| \leq \sqrt{1+K_0} \|\mathbb{J}^u_\rho(-t)\|,$$

and that this quantity behaves like $e^{\int_0^{-t} U^u_\rho(s)ds}$ where $U^u_\rho(s)$ is the unstable Riccati solution in $\rho$. As the manifold $M$ is compact and of nonpositive curvature, we can use inequality (74) to replace $\int_0^{-k\eta} U^u_\rho(s)ds$ by $-\sum_j f(\sigma^j\alpha)$. This modification also introduces a multiplicative correction factor $e^{k\eta\epsilon_0\epsilon}$ that depends on the length of the word $\alpha$.
In theorem 5.6, the functions $f$ and $f_0$ coincide as they are both bounded from below by the same constant. The fact that $f_0(\gamma)$ is positive (bounded from below by $\eta\epsilon_0$) allows to define a stopping time by using the sets, for any real number $t$ ($f_0 \geq \epsilon_0$ ensures that we consider finite sequences):

$$I^\eta(t) := \left\{ \alpha = (\alpha_0, \cdots, \alpha_k) : k \geq 3, \sum_{i=1}^{k-2} f_0\left(\sigma^i\alpha\right) \leq t < \sum_{i=1}^{k-1} f_0\left(\sigma^i\alpha\right) \right\},$$

As the upper bound in previous theorem can be of order $\hbar^{-\frac{1}{2}}$ (if $f$ is equal to 0 for all the sequence), the lower bound in the uncertainty principle can be empty if we proceed as in the case of an Anosov

surface (see section 5.3). To solve this complication, we introduce a quantum pressure as in [5] and [3]:

$$p^c_{\tau,t}(\psi_\hbar) := h^c_\pi(\psi_\hbar) - 2\sum_{\alpha\in I^\eta(t)} c_\alpha \|\tau_\alpha\psi\|^2 \log W_\alpha \tag{76}$$

(see section 5.3 for definition of $h^c$). In the previous expression, we defined the weights

$$W_\alpha := \exp\left(\frac{1}{2}\sum_{j=1}^{k-1} f(\sigma^j\alpha)\right).$$

We underline that this quantity is larger than 1 and that the function that appears is $f$ and not $f_0$ (see theorem A.2). In the Anosov, this complication was not necessary as all $W_\alpha$ were of order $\hbar^{-\frac{1}{2}}$ and the introduction of this correction term was not useful. According to corollary 2.5 from [5], we can rewrite corollary 5.5 as an uncertainty principle for quantum pressures associated to quantum partitions of identity:

**Proposition A.3.** *With obvious adaptation of notations from section 5.3, we have:*

$$p^c_{\tau,T_E(\hbar)}(\psi_\hbar) + p^c_{\pi,T_E(\hbar)}(\psi_\hbar) \geq -2\log\left(\tilde{c}_\chi(U^{-\eta}) + \hbar^{L-K_0}\right) - \log\left(\max_\gamma c_\gamma\right),$$

*where*

$$\tilde{c}_\chi(U^{-\eta}) := \max_{\gamma\in\{1,\cdots,K\}^2}\max_{\alpha\in I_\hbar(\gamma),\beta\in K_\hbar(\gamma)}\left(W^\alpha\overline{W}^\beta\|\tilde{\tau}_\alpha U^{-\eta}\tilde{\pi}^*_\beta Op(\chi^{(k')})\|\right).$$

Thanks to theorem A.2 and as $(k+k')\epsilon_0\eta \leq 2|\log\hbar|$, we know that the lower bound is of order $-\log C_{\mathcal{K}} - \frac{1}{2}(1+\epsilon'+4\epsilon)n_E(\hbar)$. Then, as in section 4.2 and 5.2, we can make a symbolic interpretation of this quantum pressure. To do this, we introduce for each atom $A$ of the partition $\vee_{j=0}^{n_E(\hbar)-1}\overline{\sigma}^{-j}\overline{\mathcal{C}}$ (which corresponds to a family $(\gamma_0,\cdots,\gamma_{n_E(\hbar)-1})$) the weights:

$$W_A := \prod_{j=0}^{n_E(\hbar)-1} W_{\gamma_j}.$$

We also define the refined pressure at time $n_E(\hbar)$:

$$p_{n_E(\hbar)}\left(\overline{\mu}^{\overline{\Sigma}}_\hbar,\overline{\sigma},\overline{\mathcal{C}}\right) := H_{n_E(\hbar)}\left(\overline{\mu}^{\overline{\Sigma}}_\hbar,\overline{\sigma},\overline{\mathcal{C}}\right) - 2\sum_{A\in\vee_{j=0}^{n_E(\hbar)-1}\overline{\sigma}^{-j}\overline{\mathcal{C}}}\overline{\mu}^{\overline{\Sigma}}_\hbar(A)\log W_A.$$

Then, following proposition 5.3 and with obvious adaptation of the notations for the backward side, we can rewrite the result of proposition 4.3 as follows:

$$-2\frac{\log C_{\mathcal{K}}}{n_E(\hbar)} - 2\epsilon - (1+\epsilon'+4\epsilon) \leq \frac{1}{n_E(\hbar)}\left(p_{n_E(\hbar)}\left(\overline{\mu}^{\overline{\Sigma}}_\hbar,\overline{\sigma},\overline{\mathcal{C}}\right) + p_{n_E(\hbar)}\left(\overline{\mu}^{\overline{\Sigma}^-}_\hbar,\overline{T},\overline{\mathcal{C}}_-\right)\right).$$

The aditional $\epsilon$ that appears is due to the fact that to each atom $A$ corresponds an unique $\alpha'$ (see proposition 5.3) and $W_A$ is not exactly equal to its corresponding weight $W_{\alpha'}$. Mimicking the proof for Anosov surfaces, we need to prove the following subadditivity property:

**Theorem A.4.** *Let $\overline{C}$ be the partition of lemma 4.2. There exists a function $R(n_0,\hbar)$ on $\mathbb{N}\times(0,1]$ and $R$ independent of $n_0$ such that*

$$\forall n_0\in\mathbb{N},\qquad \limsup_{\hbar\to 0}|R(n_0,\hbar)| = R.$$

*Moreover, for any $\hbar\in(0,1]$ and any $n_0,m\in\mathbb{N}$ such that $n_0+m\leq n_E(\hbar)$, one has:*

$$p_{n_0+m}\left(\overline{\mu}^{\overline{\Sigma}}_\hbar,\overline{\sigma},\overline{\mathcal{C}}\right) \leq p_{n_0}\left(\overline{\mu}^{\overline{\Sigma}}_\hbar,\overline{\sigma},\overline{\mathcal{C}}\right) + p_m\left(\overline{\mu}^{\overline{\Sigma}}_\hbar,\overline{\sigma},\overline{\mathcal{C}}\right) + R(n_0,\hbar).$$

As in theorem 4.4, this result came from the almost invariance of the measure $\overline{\mu_\hbar^\Sigma}$ under $\overline{\sigma}$ (proposition 6.2). In fact, compared to the Anosov case, the corection term in the pressure also follows a subadditivity property: we can mimick the proof from [5] (section 4.2) based on the invariance of the measure for logarithmic times. So, as in section 6, we only have to verify the almost invariance of the measure for logarithmic times. It requires to verify that the proof of theorem 7.1 still works in the case of surfaces of nonpositive curvature. The key point of the proof of theorem 7.1 is that in the allowed range of times $|d_\rho g^t|$ is bounded by some $\hbar^{-\nu}$ (with $\nu < 1/2$). We know that to each $\rho$ we can associate a word $\alpha$ of length $k$ (see section 7.2.2 for notations). The range of times we will consider will be $0 \leq t \leq k\eta$. To prove previous property in the case of surfaces of nonpositive curvature, we use the splitting of $T_\rho S^*M$ given by $\mathbb{R}X_H(\rho) \oplus E^u(\rho) \oplus V_\rho$. These three subspaces are uniformly transverse so we only have to give an estimate of $\|d_\rho g^t_{E \to T^*_{g^t\rho}M}\|$ when $E$ is one of them. In the case where $E = \mathbb{R}X_H(\rho)$, it is bounded by 1 and in the case where $E = E^u(\rho)$, it is bounded by $\sqrt{1+K_0}e^{\int_0^t U^u_\rho(s)ds}$. In the last case, lemma A.1 tells us that the spaces $d_\rho g^t V_\rho$ and $E^u(g^t\rho)$ become uniformly close (in direction) to each other. Then, we consider $e_0$ a unit vector in $V_\rho$ and for $0 \leq p \leq k-1$, we define the $e_{p\eta}$ as the unit vector $\frac{d_\rho g^{p\eta} e_0}{\|d_\rho g^{p\eta} e_0\|}$. We can write:

$$\|d_\rho g^{k\eta} e_0\| = |\langle d_\rho g^{k\eta} e_0, e_{k\eta}\rangle| = |\langle d_{g^{(k-1)\eta}\rho} g^\eta e_{(k-1)\eta}, e_{k\eta}\rangle \cdots \langle d_\rho g^\eta e_0, e_\eta\rangle|.$$

We also define the corresponding sequence $e^u_{p\eta} := \frac{d_\rho g^{p\eta} e^u_0}{\|d_\rho g^{p\eta} e^u_0\|}$ of unit unstable vectors, where $e^u_0 := \frac{(J^u_\rho(0), J^{u'}_\rho(0))}{\|(J^u_\rho(0), J^{u'}_\rho(0))\|}$. From lemma A.1, we know that $e_{p\eta}$ becomes uniformly close (in $\rho$) to $e^u_{p\eta}$. So, up to an error term of order $Ce^{k\eta\delta}$ (with $C$ uniform in $\rho$ and $\delta$ arbitrarly small), we have:

$$\|d_\rho g^{k\eta} e_0\| \leq Ce^{k\eta\epsilon_0\epsilon} |\langle d_{g^{(k-1)\eta}\rho} g^\eta e^u_{(k-1)\eta}, e^u_{k\eta}\rangle \cdots \langle d_\rho g^\eta e^u_0, e^u_\eta\rangle| = Ce^{k\eta\delta} \|d_\rho g^{k\eta}_{|E^u(\rho)}\|.$$

Finally, taking $\delta = \epsilon_0\epsilon$, we have that $\|d_\rho g^{k\eta}\|$ is bounded by $Ce^{k\eta\epsilon_0\epsilon} e^{\int_0^{k\eta} U^u_\rho(s)ds}$ (with $C$ uniform in $\rho$). For the allowed words, $e^{k\eta\epsilon_0\epsilon}$ is of order $\hbar^{-\epsilon}$ (as $k\eta\epsilon_0 \leq 1/2n_E(\hbar)$). To conclude, we can estimate:

$$\left| \int_0^{k\eta} U^u_\rho(s)ds - \sum_{j=0}^{k-1} f(\sigma^j\alpha) \right| \leq \sum_{j=0}^{k-1} \left| \int_{j\eta}^{(j+1)\eta} U^u_\rho(s)ds - f(\sigma^j\alpha) \right|.$$

To bound this sum, we can use the continuity of $U^u$ (see inequality (74)) to show that this quantity is bounded by $\epsilon|\log\hbar|$. By definition of the allowed words $\alpha$, we know that $\sum_{j=0}^{k-1} f(\sigma^j\alpha) \leq 1/2n_E(\hbar)$. This allows to conclude that $|d_\rho g^t|$ is bounded by some $C\hbar^{-\nu}$ (with $C$ independent of $\rho$ and $\nu < 1/2$).

Now, we can reproduce the subadditivity argument from the outline of the proof, let $\hbar$ tends to 0 and substitute the smooth partitions by true partitions (see section 4.4 for details). We have finally:

$$-3\epsilon - 1 - \epsilon' - \frac{R}{n_0} \leq \frac{1}{n_0}\left(p_{n_0}\left(\overline{\mu^\Sigma}, \overline{\sigma}, \overline{\mathcal{C}}\right) + p_{n_0}\left(\overline{\mu^\Sigma}_-, \overline{T}, \overline{\mathcal{C}}_-\right)\right).$$

We can use the multiplicative property of $W_A$ to write:

$$\sum_{A \in \vee_{j=0}^{n_0-1} \overline{\sigma}^{-j}\overline{\mathcal{C}}} \overline{\mu^\Sigma}(A) \log W_A = \sum_{A_0, \cdots, A_{n_0-1} \in \overline{\mathcal{C}}} \overline{\mu^\Sigma}(A_0 \cap \cdots \cap \overline{\sigma}^{-(n_0-1)} A_{n_0-1}) \sum_{j=0}^{n_0-1} \log W_{A_j}.$$

After simplification and using the fact that $\overline{\mathcal{C}}$ is a partition, we can let $n_0$ tends to $\infty$ and apply the Abramov theorem as in the outline of the proof (see section 4.4). We find (as $h_{KS}(\mu, g) = h_{KS}(\mu, g^{-1})$):

$$\sum_{\gamma \in \{1,\cdots,K\}^2} f_0(\gamma)\, \mu^\Sigma([\gamma]) \left( -3\epsilon - \frac{1}{2}(1+\epsilon') + 2\sum_{A \in \overline{\mathcal{C}}} \overline{\mu^\Sigma}(A) \log W_A \right) \leq \eta h_{KS}(\mu, g).$$

After simplification, we find that:

$$-\sum_{\gamma\in\{1,\cdots,K\}^2} f_0(\gamma)\,\mu^{\Sigma}([\gamma])\left(3\epsilon+\frac{1}{2}(1+\epsilon')\right)+\sum_{\gamma\in I^{\eta}(1)} f_0(\gamma)\mu^{\Sigma}([\gamma])\log W_{\gamma}\leq \eta h_{KS}(\mu,g).$$

At this point of the discussion, we would like to underline that our choice of $I^{\eta}(1)$ (and of the associated partition $\overline{\mathcal{C}}$) was quite arbitrary in the setting of nonpositive curvature. In fact, as was discussed earlier, the unstable Riccati solution measures the growth along the unstable direction at an infinitesimal level whereas the unstable jacobian were describing it in time 1 (see section A.1). So the choice of time 1 was justified for the unstable jacobian while for the Riccati solution, a more appropriate choice would be $1/N_0$ for a large integer $N_0$ (such that $\epsilon' << 1/N_0 << \epsilon_0$). We can consider the family $I^{\eta}(1/N_0)$ (and the corresponding partition $\overline{\mathcal{C}}_{N_0}$). The previous inequality would have been:

$$-\sum_{\gamma\in\{1,\cdots,K\}^2} f_0(\gamma)\,\mu^{\Sigma}([\gamma])\left(3\epsilon+\frac{1}{2N_0}(1+\epsilon')\right)+\sum_{\gamma\in I^{\eta}(1/N_0)} f_0(\gamma)\mu^{\Sigma}([\gamma])\log W_{\gamma}\leq \frac{\eta}{N_0}h_{KS}(\mu,g).$$

With the notations of the outline of the proof (see section 4.4), we define for $\rho\in\mathcal{E}^{\theta}$:

$$F_0(\rho):=\sum_{\gamma\in I^{\eta}(1/N_0)} f_0(\gamma)\log W_{\gamma}\mathbf{1}_{O_{\gamma_0}}(\rho)\cdots\mathbf{1}_{O_{\gamma_k}}\circ g^{k\eta}(\rho).$$

We underline that for each $\rho$ in $\mathcal{E}^{\theta}$, there exists an unique $\gamma$ in $I^{\eta}(1/N_0)$ such that $\mathbf{1}_{O_{\gamma_0}}(\rho)\cdots\mathbf{1}_{O_{\gamma_k}}\circ g^{k\eta}(\rho)$ is non zero (it is then equal to 1). We define

$$X_0:=\left\{\rho\in\mathcal{E}^{\theta}:\forall 0\leq t\leq\frac{1}{N_0\epsilon_0},U^u(g^t\rho)>2\epsilon_0\right\}.$$

We can verify that $F_0(\rho)\geq(1/N_0)\sum_{\gamma_0,\gamma_1} f_0(\gamma)\mathbf{1}_{X_0}(\rho)\mathbf{1}_{O_{\gamma_0}}(\rho)\mathbf{1}_{O_{\gamma_1}}\circ g^{\eta}(\rho)$ for all $\rho$ in $\mathcal{E}^{\theta}$ (we underline that we used the fact that $\epsilon<\epsilon_0$, inequality (19) and that $k\eta\leq 1/(N_0\epsilon_0)$). We can substitute this inequality on the previous bound on the entropy, let the diameter of the partition (and $\theta$) tend to 0 and simplify by $\eta$. We find then:

$$-\left(\frac{1}{2N_0}(1+\epsilon')+3\epsilon\right)\int_{S^*M} f_0(\rho)d\mu(\rho)+\frac{1}{N_0}\int_{S^*M} f_0(\rho)\mathbf{1}_{X_0}(\rho)d\mu(\rho)\leq\frac{1}{N_0}h_{KS}(\mu,g).$$

Finally, we let $\eta$, $\epsilon$, $\epsilon'$ tend to 0 (in this order) and simplify by $1/N_0$. We find the following bound on the entropy:

$$-\frac{1}{2}\int_{S^*M} f_0(\rho)d\mu(\rho)+\int_{S^*M} f_0(\rho)\mathbf{1}_{X_0}(\rho)d\mu(\rho)\leq h_{KS}(\mu,g).$$

We let now $N_0$ tend to infinity and then $\epsilon_0$ tend to 0 (in this order). We find finally the expected lower bound:

$$\frac{1}{2}\int_{S^*M} U^u(\rho)d\mu(\rho)\leq h_{KS}(\mu,g).\square$$

## Appendix B. Pseudodifferential calculus on a manifold

In this appendix, a few facts about pseudodifferential calculus on a manifold and the sharp energy cutoff used in this paper are recalled. Even if most of this setting can be found in [5], it is recalled because it is extensively used in section 6.2 and 7. The results from the two first sections of this appendix can be found in more details in [31] or [5]. The results of the last section of this appendix are the extension to the case of a manifold of standard results from semiclassical analysis that can be found either in [9], [13] or [17].

B.1. **Pseudodifferential calculus on a manifold.** In this section, some facts of pseudodifferential calculus that can be found in [13] or in [17] are recalled. Recall that we define on $\mathbb{R}^{2d}$ the following class of symbols:

$$S^k(\langle\xi\rangle^m) := \left\{ a \in C^\infty(\mathbb{R}^{2d}) : |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta}\hbar^{-k}\langle\xi\rangle^{m-|\beta|} \right\}.$$

Let $M$ be a smooth Riemannian $d$-manifold without boundary. Consider a smooth atlas $(f_l, V_l)$ of $M$, where each $f_l$ is a smooth diffeomorphism from $V_l \subset M$ to a bounded open set $W_l \subset \mathbb{R}^d$. To each $f_l$ correspond a pull back $f_l^* : C^\infty(W_l) \to C^\infty(V_l)$ and a canonical map $\tilde{f}_l$ from $T^*V_l$ to $T^*W_l$:

$$\tilde{f}_l : (x,\xi) \mapsto \left(f_l(x), (Df_l(x)^{-1})^T\xi\right).$$

Consider now a smooth locally finite partition of identity $(\phi_l)$ adapted to the previous atlas $(f_l, V_l)$. That means $\sum_l \phi_l = 1$ and $\phi_l \in C^\infty(V_l)$. Then, any observable $a$ in $C^\infty(T^*M)$ can be decomposed as follows: $a = \sum_l a_l$, where $a_l = a\phi_l$. Each $a_l$ belongs to $C^\infty(T^*V_l)$ and can be pushed to a function $\tilde{a}_l = (\tilde{f}_l^{-1})^* a_l \in C^\infty(T^*W_l)$. As in [17], define the class of symbols of order $m$ and index $k$:

$$S^{m,k}(T^*M) := \left\{ a \in C^\infty(T^*M) : |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta}\hbar^{-k}\langle\xi\rangle^{m-|\beta|} \right\}. \tag{77}$$

Then, for $a \in S^{m,k}(T^*M)$ and for each $l$, one can associate to the symbol $\tilde{a}_l \in S^k(\langle\xi\rangle^m)$ the standard Weyl quantization:

$$\mathrm{Op}_\hbar^w(\tilde{a}_l)u(x) := \frac{1}{(2\pi\hbar)^d}\int_{R^{2d}} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}\tilde{a}_l\left(\frac{x+y}{2},\xi;\hbar\right)u(y)dyd\xi,$$

where $u \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz class. Consider now a smooth cutoff $\psi_l \in C_c^\infty(V_l)$ such that $\psi_l = 1$ close to the support of $\phi_l$. A quantization of $a \in S^{m,k}$ is then defined in the following way:

$$\mathrm{Op}_\hbar(a)(u) := \sum_l \psi_l \times \left(f_l^* \mathrm{Op}_\hbar^w(\tilde{a}_l)(f_l^{-1})^*\right)(\psi_l \times u), \tag{78}$$

where $u \in C^\infty(M)$. According to the appendix of [17], the quantization procedure $\mathrm{Op}_\hbar$ sends $S^{m,k}(T^*M)$ onto the space of pseudodifferential operators of order $m$ and of index $k$, denoted $\Psi^{m,k}(M)$. It can be shown that the dependence in the cutoffs $\phi_l$ and $\psi_l$ only appears at order 2 in $\hbar$ and the principal symbol map $\sigma_0 : \Psi^{m,k}(M) \to S^{m,k}/S^{m,k-1}(T^*M)$ is then intrinsically defined. All the rules (for example the composition of operators, the Egorov theorem) that holds in the case of $\mathbb{R}^{2d}$ still holds in the case of $\Psi^{m,k}(M)$. Because our study concerns behavior of quantum evolution for logarithmic times in $\hbar$, a larger class of symbols should be introduced as in [13] or [17], for $0 \leq \nu < 1/2$:

$$S_\nu^{m,k}(T^*M) := \left\{ a \in C^\infty(T^*M) : |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta}\hbar^{-k-\nu|\alpha+\beta|}\langle\xi\rangle^{m-|\beta|} \right\}. \tag{79}$$

Results of [13] can be applied to this new class of symbols. For example, a symbol of $S_\nu^{0,0}$ gives a bounded operator on $L^2(M)$ (with norm independent of $\hbar$).
As was explained, one needs to quantize the sharp energy cutoff $\chi^{(\cdot)}$ (see section 5.3.1) to get sharp bounds in 5.6. As $\chi^{(0)}$ localize in a strip of size $\hbar^{1-\delta}$ with delta close to 0, the $m$-th derivatives transversally to $\mathcal{E}$ grows like $\hbar^{m(\delta-1)}$. As $\delta$ is close to 0, $\chi^{(0)}$ does not belongs to the previous class of symbols that allows $\nu < 1/2$. However, as the variations only appears in one direction, it is possible to define a new pseudodifferential calculus for these symbols. The procedure taken from [31] is briefly recalled in [5] (section 5) and introduces a class of anisotropic symbols $S_{\mathcal{E},\nu'}^{-\infty,0}$ (where $\mathcal{E} := S^*M$ and $\nu' < 1$) for which a quantization procedure $\mathrm{Op}_{\mathcal{E},\nu'}$ can be defined. In the next section, we recall briefly a few results about the quantization $\mathrm{Op}_{\mathcal{E},\nu'}(\chi^{(n)})$ of the symbol $\chi^{(\cdot)}$.

B.2. **Energy cutoff.** Let $\chi^{(\cdot)}$ be as in section 5.3.1. Consider $\delta > 0$ and $K_\delta$ associated to it (see section 5.3.1). Taking $\nu' = 1-\delta$, it can be checked that the cutoffs defined in section 5.3.1 belongs to the class $S_{\mathcal{E},\nu'}^{-\infty,0}$ defined in [5]. A pseudodifferential operator corresponding to it can be defined following the nonstandard procedure mentioned above. Using results from [5] (section 5), one has

$\|\mathrm{Op}_{\mathcal{E},\nu'}(\chi^{(n)})\| = 1 + O(\hbar^{\nu'/2})$ for all $n \leq K_\delta|\log\hbar|$. For simplicity of notations, in the paper $\mathrm{Op}(\chi^{(n)}) := \mathrm{Op}_{\mathcal{E},\nu'}(\chi^{(n)})$. In [5], it is also proved that:

**Proposition B.1.** [5] *For $\hbar$ small enough and any $n \in \mathbb{N}$ such that $0 \leq n \leq K_\delta|\log\hbar|$ and for any $\psi_\hbar = -\hbar^2\Delta\psi_\hbar$ eigenstate, one has:*

$$\|\psi_\hbar - Op(\chi^{(n)})\psi_\hbar\| = O(\hbar^\infty)\|\psi_\hbar\|.$$

*Moreover for any sequence $\alpha$ and $\beta$ of length $n$ less than $K_\delta|\log\hbar|$, one has:*

$$\left\|\left(1 - Op\left(\chi^{(n)}\right)\right)\tau_\alpha Op\left(\chi^{(0)}\right)\right\| = O(\hbar^\infty) \qquad \left\|\left(1 - Op\left(\chi^{(n)}\right)\right)\pi_\beta Op\left(\chi^{(0)}\right)\right\| = O(\hbar^\infty)$$

*where $\tau$ and $\pi$ are given by (23) and (24).*

This proposition tells that the quantization of this energy cutoff exactly have the expected property, meaning that it preserves the eigenfunction of the Laplacian. So, in the paper, introducing the energy cutoff $\mathrm{Op}(\chi^{(n)})$ does not change the semiclassical limit. Moreover this proposition implies the following corollary that allows to apply theorem 2.1 in section 5.3.2:

**Corollary B.2.** [5] *For any fixed $L > 0$, there exists $\hbar_L$ such that for any $\hbar \leq \hbar_L$, any $n \leq K_\delta|\log\hbar|$ and any sequence of length $n$, the Laplacian eigenstate verify*

$$\left\|\left(1 - Op\left(\chi^{(n)}\right)\right)\pi_\beta\psi_\hbar\right\| \leq \hbar^L\|\psi_\hbar\|.$$

A last property of the quantization of this cutoff that we can quote from [5] is that we can restrict ourselves to study observables carried in a thin neighborhood around $S^*M = H^{-1}(1/2)$:

**Proposition B.3.** [5] *For $\hbar$ small enough and any $n \in \mathbb{N}$ such that $0 \leq n \leq K_\delta|\log\hbar|/2$, one has:*

$$\forall|\gamma| = n,\ \|\tau_\gamma Op(\chi) - \tau_\gamma^f Op(\chi)\| = O(\hbar^\infty),$$

*where $P^f_{\gamma_j} = Op_\hbar(P_{\gamma_j}f)$, $f$ is a smooth compactly supported function in a thin neighborhood of $\mathcal{E}$ and $\tau^f_\gamma = P^f_{\gamma_{n-1}}((n-1)\eta)\cdots P^f_{\gamma_0}$.*

B.3. **$\hbar$-expansion for pseudodifferential operators on a manifold.** The goal of this last section is to explain how the usual $\hbar$-expansion of order $N$ for composition of pseudodifferential operators and Egorov theorem can be extended in the case of pseudodifferential calculus on a manifold. Except for terms of order 0, the $\hbar$-expansion will depend on the partition of identity in section B.1. In fact, on a manifold, the formulas for the terms of order larger than 1 on the $\hbar$-expansion will depend on the local coordinates. For simplicity and as it is the case of all the symbols we consider (thanks to the energy cutoff: for example, see proposition B.3), we now restrict ourselves to symbols supported in $\mathcal{E}^\theta = H^{-1}([1/2-\theta, 1/2+\theta])$. The symbols are now elements of $S^{-\infty,0}_\nu(T^*M)$.

B.3.1. *Composition of pseudodifferential operators on a manifold.* First, recall that the usual semiclassical theory on $\mathbb{R}^d$ (see [13] or [17]) tells that the composition of two elements $\mathrm{Op}^w_\hbar(a)$ and $\mathrm{Op}^w_\hbar(b)$ in $\Psi^{-\infty,k}_\nu(\mathbb{R}^d)$ is still in $\Psi^{-\infty,k}_\nu(\mathbb{R}^d)$ and that its symbol is of support equal to $\mathrm{supp}(a) \cap \mathrm{supp}(b)$. More precisely, it says that $\mathrm{Op}^w_\hbar(a) \circ \mathrm{Op}^w_\hbar(b) = \mathrm{Op}^w_\hbar(a\sharp b)$, where $a\sharp b$ is in $S^{-\infty,k}_\nu$ and its asymptotic expansion in power of $\hbar$ is given by the Moyal product:

$$a\sharp b(x,\xi) \sim \sum_k \frac{1}{k!}\left(\frac{i\hbar}{2}\omega(D_x, D_\xi, D_y, D_\eta)\right)^k a(x,\xi)b(y,\eta)|_{x=y,\xi=\eta}, \tag{80}$$

where $\omega$ is the standard symplectic form. Outline that it is clear that each element of the sum is supported in $\mathrm{supp}(a)\cap\mathrm{supp}(b)$. As quantization on a manifold is constructed from quantization on $\mathbb{R}^{2d}$ (see definition (78)), one can prove an analogue of this asymptotic expansion in the case of a manifold $M$ (except that it will not be intrinsically defined). Precisely, let $a$ and $b$ be two symbols in $S^{-\infty,0}_\nu(T^*M)$. For a choice of quantization $\mathrm{Op}_\hbar$ (that depends on the coordinates maps), one

has $\mathrm{Op}_\hbar(a)\circ\mathrm{Op}_\hbar(b)$ is a pseudodifferential operator in $\Psi_\nu^{-\infty,0}(M)$. Its symbol is denoted $a\sharp_M b$ and its asymptotic expansion is of the following form:

$$a\sharp_M b\sim\sum_{p\geq 0}\hbar^p(a\sharp_M b)^p.$$

In the previous asymptotic expansion, $(a\sharp_M b)^p$ is a linear combination (that depends on the cutoffs and the local coordinates) of elements of the form $\partial^\gamma a\partial^{\gamma'}b$ with $|\gamma|\leq p$ and $|\gamma'|\leq p$. As a consequence, $(a\sharp_M b)^p$ is an element of $S_\nu^{m,2p\nu}(T^*M)$.

*Remark.* We know that we have an asymptotic expansion so by definition and using Calderon-Vaillancourt theorem, we know that each remainder is bounded in norm by a constant which is a small power of $\hbar$ (in fact $C\hbar^{(N+1)(1-2\nu)}$ for the remainder of order $N$). In our analysis, we need to know precisely how these bounds depend on $a$ and $b$ as we have to make large product of pseudodifferential operators (see section 7) and to use the composition formula to get Egorov theorem (see next section). The following lines explain how the remainder in the asymptotic expansion in powers of $\hbar$ is bounded by the derivatives of $a$ and $b$.

In the appendix of [9], they defined the remainder of the order $N$ expansion, in the case of $\mathbb{R}^{2d}$:

$$\hbar^{N+1}R_{N+1}(a,b,\hbar):=a\sharp b-\sum_{p=0}^{N}\hbar^p(a\sharp b)_p$$

and, using a stationary phase argument, they get the following estimates on the remainder, for all $\gamma$ and all $N$:

$$|\partial_z^\gamma R_{N+1}(a,b,z,\hbar)|\leq\rho_d K_d^{N+|\gamma|}(N!)^{-1}\sup_{(*)}|\partial_u^{(\alpha,\beta)+\mu}a(u+z)||\partial_v^{(\beta,\alpha)+\nu}b(v+z)|,$$

where $(*)$ means:

$$u,v\in\mathbb{R}^{2d}\times\mathbb{R}^{2d},\ |\mu|+|\nu|\leq 4d+|\gamma|,\ |(\alpha,\beta)|=N+1,\ \alpha,\beta\in\mathbb{N}^d.$$

Applying Calderon-Vailancourt theorem (see [13] or [17]), one knows that there exist a constant $C$ and a constant $M$ (depending only on $d$), such that for a symbol $a$ in $S_\nu^0(1)$:

$$\|\mathrm{Op}_\hbar^w(a)\|_{L^2}\leq C\sup_{|\alpha|\leq M}\hbar^{\frac{|\alpha|}{2}}\|\partial^\alpha a\|_\infty.$$

Combining this result with the previous estimates on the $R^{(N+1)}$, one finds that:

$$\|\mathrm{Op}_\hbar^w(R_{N+1}(a,b,z,\hbar))\|_{L^2}\leq C(d,N)\sup_{(*)}\hbar^{\frac{|\alpha|}{2}}\|\partial^{\beta+\beta'}a\|_\infty\|\partial^{\gamma+\gamma'}b\|_\infty,\tag{81}$$

where $(*)$ means:

$$|\alpha|\leq C',\ |\beta|\leq N+1,\ |\gamma|\leq N+1\text{ and }|\beta'|+|\gamma'|\leq C+|\alpha|.$$

The constants $C$ and $C'$ depend only on the dimension $d$. The same kind of estimates holds on the remainder in the asymptotic expansion for change of variables. As the asymptotic expansion for composition of pseudodifferential operators is obtained from the composition and variable change rules on $\mathbb{R}^d$ [17], the previous estimates (81) hold for semiclassical analysis on a manifold.

B.3.2. *Egorov expansion on a manifold.* In this section, we want to recall how we prove an Egorov property with an expansion of any order. We follow the proof from [9]. First, for the order 0 term, we write the following exact expression for a symbol $a$:

$$U^{-t}\mathrm{Op}_\hbar(a)U^t-\mathrm{Op}_\hbar(a(t))=\hbar\int_0^t U^{-s}(\mathrm{Diff}a^{t-s})U^s ds,\tag{82}$$

where $a(t):=a\circ g^t$, $H(\rho)=\frac{\|\xi\|_x^2}{2}$ is the Hamiltonian and

$$\mathrm{Diff}a^t:=\frac{1}{\hbar}\left(\frac{\imath}{\hbar}[-\hbar^2\Delta,\mathrm{Op}_\hbar(a(t))]-\mathrm{Op}_\hbar(\{H,a(t)\})\right).$$

It can be also written with the notations of the previous section:

$$\text{Diff}a^t := \imath(R_2(H, a(t), \hbar) - R_2(a(t), H, \hbar)),$$

where $R_i$ is now the remainder of the order $i$ expansion in the composition of two pseudodifferential operators on a manifold. We proceed then by induction to recover the terms of higher order. For these higher order terms, we will see terms depending on the local coordinates appear in the expansion and we will obtain expressions as in [9] for the higher order terms of the expansion that will be different from the case of $\mathbb{R}^d$ [9]. However, we do not need to have an exact expression for each term of the expansion: we only need to know on how many derivatives the order $p$ term depends and how the remainder can be bounded at each step. To obtain, the $\hbar$ formal term of the Egorov expansion, we first outline that $\text{Diff}a^t$ is a pseudodifferential operator whose asymptotic expansion is given by the composition rules on a manifold (see previous section). Then, we can apply the same procedure as in equation (82) to get the exact expression:

$$\text{Op}_\hbar(a)(t) = \text{Op}_\hbar(a^{(1)}(t)) + \hbar^2 \int_0^t U^{-s} \text{Op}_\hbar \left( R^{(2)}(t-s) \right) U^s ds.$$

In this equation, we have, according to the composition rules on a manifold:

$$a^{(1)}(t) := a \circ g^t + \imath\hbar \int_0^t ((H \sharp_M a(t-s))_2 - (a(t-s) \sharp_M H)_2) \circ g^s ds.$$

According to the rules for composition of pseudodifferential operators, $(H \sharp_M a(t-s))_2$ is a linear combination of derivatives of $H(x,\xi)a(t-s)(y,\eta)$ of order less than 4 with at most 2 derivatives of $a$ in each term. We denote the previous formula in a more compact way:

$$a^{(1)}(t) := a_0(t) + \hbar a_1(t),$$

where $a_1(t,\rho) := \displaystyle\int_0^t \{H, a_0(t-s)\}_M^{(1,0)} (g^s(\rho))\, ds$. As was mentioned, this generalized 'bracket' is a differential operator of order less than 4 applied to the product $H(x,\xi)a(t-s)(y,\eta)$ (with at most 2 derivatives for $a_{t-s}$). The remainder $R^{(2)}$ is a linear combination of terms of the form $R_3(H, a_0(t), \hbar)$ and $R_2(H, a_1(t), \hbar)$ (where $R_i$ was defined in the previous section as the remainder in the composition of two pseudodifferential operators). Proceeding by induction and using the same procedure as for the term of order 1, we can show that for any order $N$:

$$\text{Op}_\hbar(a)(t) = \text{Op}_\hbar(a^{(N)}(t)) + \hbar^{N+1} \int_0^t U^{-s} \text{Op}_\hbar \left( R^{(N+1)}(t-s) \right) U^s ds.$$

In the previous formula, $a^{(N)}(t)$ is defined as follows:

$$a^{(N)}(t) := \sum_{p=0}^{N} \hbar^p a_p(t) \text{ where } \ a_0(t) := a \circ g^t$$

and for $1 \leq p \leq N$:

$$a_p(t,\rho) := \sum_{j=0}^{p-1} \int_0^t \{H, a_j(t-s)\}_M^{(p,j)} (g^s(\rho))\, ds,$$

where $\{.,.\}_M^{(p,j)}$ is a generalized 'bracket' of order $(p,j)$ depending on the local coordinates on the manifold (it is the analogue of formula given by theorem 1.2 in [9]). We do not need to have an exact expression for these brackets: we only need to know on how many derivatives it depends. From the previous section, we know how the order $p$ term in the expansion of $a \sharp_M b$ depends linearly on products of the $p$ derivatives of $a$ and $b$. The term $\{H, a_0(t-s)\}^{(p,0)}$ comes from the order $p+1$ term of the asymptotic expansion of $\text{Diff}a^t$. According to the rules of composition of pseudodifferential operators on a manifold, it is a linear combination of product of derivatives of $a$ of order less than $p+1$ and of derivatives of $H$ of order less than $p+1$. More generally, $\{H, a_j(t-s)\}_M^{(p,j)}$ is a linear combination of product of derivatives of $a_j(t)$ of order less than $p-j+1$ and of derivatives of $H$ of order less than $p-j+1$. For the remainder term $R^{(N+1)}(s)$ of order $N$, using the formulas for the composition of pseudodifferential operators, it can be shown

that it is a linear combination of terms of the form $R_{N+2-j}(H, a_j(t), \hbar)$ for $0 \leq j \leq N$ (where $R_i$ was defined in the previous section as the remainder in the composition of two pseudodifferential operators). Recall that the estimates for the norms of the $\mathrm{Op}_\hbar(R_i)$ still hold on a manifold with slight modifications: it depends on estimates on the derivatives of $H$ and $a_j$. The previous discussion can be summarized in the following proposition:

**Proposition B.4** (Egorov expansion on a manifold)**.** *Let $a$ be a symbol in $S_\nu^{-\infty,0}$. One has the exact expression for every $N \geq 0$:*

$$Op_\hbar(a)(t) = Op_\hbar(a^{(N)}(t)) + \hbar^{N+1} \int_0^t U^{-s} Op_\hbar \left( R^{(N+1)}(a)(t-s) \right) U^s ds. \tag{83}$$

*In the previous formula, one has:*

$$a^{(N)}(t) := \sum_{p=0}^{N} \hbar^p a_p(t) \text{ where } a_0(t) := a \circ g^t$$

*and for $1 \leq p \leq N$:*

$$a_p(t, \rho) := \sum_{j=0}^{p-1} \int_0^t \{H, a_j(t-s)\}_M^{(p,j)} (g^s(\rho))\, ds.$$

*For each $0 \leq j \leq p-1$, $\{H, a_j(t-s)\}_M^{(p,j)}$ is a linear combination of the $p+1-j$ derivatives of $a_j(t-s)$ that depends on the choice of coordinates on the manifold. Finally, the norm of $Op_\hbar(R^{(N+1)}(a)(t))$ satisfies the following bound:*

$$\| Op_\hbar(R^{(N+1)}(a)(t)) \|_{L^2} \leq C(d, N) \sup_{(*)} \hbar^{\frac{|\alpha|}{2}} \| \partial^{\beta+\beta'} (a_j(s)) \|_\infty, \tag{84}$$

*where $(*)$ means:*

$$j \leq N+1,\ 0 \leq s \leq \eta,\ |\alpha| \leq C',\ |\beta| \leq N+2-j \text{ and } |\beta'| \leq C + |\alpha|.$$

*The constants $C$ and $C'$ depend only on the dimension of the manifold.*

*Remark.* Theorem 1.2 in [9] gives an exact expression of each term of this exact expansion in the case of $\mathbb{R}^{2d}$. In our analysis, we do not need an exact expression but only to know the dependence on the derivatives of $a$. In fact, an exact expression would be very hard to explicit as it would depend on the choice of the atlas on the manifold. Finally, outline that equation (83) is an exact expression.

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