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# HALF DELOCALIZATION OF SEMICLASSICAL MEASURES FOR ANOSOV SURFACES

GABRIEL RIVIÈRE

ABSTRACT. We study the asymptotic properties of eigenfunctions of the Laplacian in the case of a compact Riemannian surface with Anosov geodesic flow. We show that the Kolmogorov-Sinai entropy of a semiclassical measure  $\mu$  for the geodesic flow  $g^t$  is bounded from below by half of the Ruelle upper bound, i.e.

$$h_{KS}(\mu, g) \geq \frac{1}{2} \left| \int_{S^*M} \log J^u(\rho) d\mu(\rho) \right|.$$

## 1. INTRODUCTION

In quantum mechanics, the semiclassical principle asserts that in the high energy limit, one should observe classical phenomena. Our main concern will be the study of this property when the classical system is said to be chaotic.

Let  $M$  be a compact  $C^\infty$  Riemannian surface. For all  $x \in M$ ,  $T_x^*M$  is endowed with a norm  $\|\cdot\|_x$  given by the metric over  $M$ . The geodesic flow  $g^t$  over  $T^*M$  is defined as the Hamiltonian flow corresponding to the Hamiltonian  $H(x, \xi) := \frac{\|\xi\|_x^2}{2}$ . This last quantity corresponds to the classical kinetic energy in the case of the absence of potential. As any observable, this quantity can be quantized via pseudodifferential calculus and the quantum operator corresponding to  $H$  is  $-\frac{\hbar^2 \Delta}{2}$  where  $\hbar$  is proportional to the Planck constant and  $\Delta$  is the Laplace Beltrami operator acting on  $L^2(M)$ .

Our main result concerns the influence of the classical Hamiltonian behavior on the spectral asymptotic properties of  $\Delta$ . More precisely, our main interest is the study of the measure  $|\psi_{\hbar}(x)|^2 dx$  where  $\psi_{\hbar}$  is an eigenfunction of  $\Delta$  associated to the eigenvalue  $-\hbar^{-2}$ :

$$-\frac{\hbar^2 \Delta}{2} \psi_{\hbar} = \psi_{\hbar}.$$

As  $M$  is a compact Riemannian manifold, the eigenvalues form a discrete subsequence  $-\hbar^{-2}$  that tends to infinity. One natural question is to study the (weak) limit of the probability measure  $|\psi_{\hbar}(x)|^2 dx$  as  $\hbar$  tends to 0. This means studying the asymptotic behavior of the probability to find a particle in  $x$  when the system is in the state  $\psi_{\hbar}$ . Using the semiclassical principle, one could expect to see a certain kind of behavior due to the classical properties. In order to study the influence of the Hamiltonian flow, we first need to lift this measure to the cotangent bundle. This can be achieved thanks to pseudodifferential calculus. In fact there exists a procedure of quantization that gives us an operator  $\text{Op}_{\hbar}(a)$  on the phase space  $L^2(M)$  for any observable  $a(x, \xi)$  in a certain class of symbols. Then a natural way to lift the previous measure is to define the following quantity:

$$\mu_{\hbar}(a) = \int_{T^*M} a(x, \xi) d\mu_{\hbar}(x, \xi) := \langle \psi_{\hbar}, \text{Op}_{\hbar}(a) \psi_{\hbar} \rangle_{L^2(M)}.$$

This formula gives a distribution  $\mu_{\hbar}$  on the space  $T^*M$  and describe now the distribution in position and velocity.

Let  $(\psi_{\hbar_k})$  be a sequence of orthonormal eigenfunctions of the Laplacian corresponding to the eigenvalues  $-\hbar_k^{-2}$  such that the corresponding sequence of distributions  $\mu_k$  on  $T^*M$  converges as  $k$  tends to infinity to a limit  $\mu$ . Such a limit is called a semiclassical measure. Using standard

facts of pseudodifferential calculus, it can be shown that  $\mu$  is a probability measure that does not depend on the choice of the quantization  $\text{Op}_\hbar$  and that is carried on the unit energy layer

$$S^*M := \left\{ (x, \xi) : H(x, \xi) = \frac{1}{2} \right\}$$

Moreover, another result from semiclassical analysis, known as the Egorov property, states that for any fixed  $t$ :

$$(1) \quad \forall a \in \mathcal{C}_c^\infty(T^*M), \quad U^{-t} \text{Op}_\hbar(a) U^t = \text{Op}_\hbar(a \circ g^t) + O_t(\hbar),$$

where  $U^t$  denotes the quantum propagator  $e^{\frac{it\Delta}{2}}$ . Precisely, it says that for fixed times, the quantum evolution is related to the classical evolution under the geodesic flow. From this, it can be deduced that  $\mu$  is invariant under the geodesic flow. One natural question to ask is what measures supported on  $S^*M$  are in fact semiclassical measures. The corresponding question in quantum chaos is: when the classical behavior is said to be chaotic, what is the set of semiclassical measures? A first result in this direction has been found by Shnirelman [22], Zelditch [25], Colin de Verdière [9]:

**Theorem 1.1.** *Let  $(\psi_k)$  be an orthonormal basis of  $L^2(M)$  composed of eigenfunctions of the Laplacian. Moreover, suppose the geodesic flow on  $S^*M$  is ergodic with respect to Liouville measure. Then, there exists a subsequence  $(\mu_{k_p})_p$  of density one that converges to the Liouville measure on  $S^*M$  as  $p$  tends to infinity.*

By 'density one', we mean that  $\frac{1}{n} \#\{p : 1 \leq k_p \leq n\}$  tends to one as  $n$  tends to infinity. This theorem states that, in the case of an ergodic geodesic flow, almost all eigenfunctions concentrate on the Liouville measure in the high energy limit. This phenomenon is called quantum ergodicity and has many extensions. The Quantum Unique Ergodicity Conjecture states that the set of semiclassical measures should be reduced to the Liouville measure in the case of Anosov geodesic flow [20]. This question still remains widely open. In fact, in the case of negative curvature, there are many measures invariant under the geodesic flow: for example, there exists an infinity of closed geodesics (each of them carrying naturally an invariant measure). In recent papers, Lindenstrauss proved a particular form of the conjecture, the Arithmetic Quantum Unique Ergodicity [18]. Precisely, he proved that for a sequence of Hecke eigenfunctions of the Laplacian on an arithmetic surface,  $|\psi|^2 dx$  converges to the Lebesgue measure on the surface. This result is actually the best-known positive result towards the conjecture.

In order to understand the phenomenon of quantum chaos, many people started to study toy models as the cat map (a typical hyperbolic automorphism of  $\mathbb{T}^2$ ). These dynamical systems provide systems with similar dynamical properties to the geodesic flow on a manifold of negative curvature. Moreover, they can be quantized using Weyl formalism and the question of Quantum Ergodicity naturally arises. For example, Bouzouina and de Bièvre proved the Quantum Ergodicity property for the quantized cat map [6]. However, de Bièvre, Faure and Nonnenmacher proved that in this case, the Quantum Unique Ergodicity is too optimistic [13]. In fact, they constructed a sequence of eigenfunctions that converges to  $\frac{1}{2}(\delta_0 + \text{Leb})$ , where  $\delta_0$  is the Dirac measure on 0 and  $\text{Leb}$  is the Lebesgue measure on  $\mathbb{T}^2$ . Faure and Nonnenmacher also proved that if we split the semiclassical measure into its pure point, Lebesgue and singular continuous components,  $\mu = \mu_{\text{pp}} + \mu_{\text{Leb}} + \mu_{\text{sc}}$ , then  $\mu_{\text{pp}}(\mathbb{T}^2) \leq \mu_{\text{Leb}}(\mathbb{T}^2)$  and in particular  $\mu_{\text{pp}}(\mathbb{T}^2) \leq 1/2$  [14]. As in the case of geodesic flow, there is an arithmetic point of view on this problem. Recently, Kelmer proved that in the case of  $\mathbb{T}^{2d}$  ( $d \geq 2$ , for a generic family of symplectic matrices), either there exists isotropic submanifold invariant under the  $2d$  cat map or one has Arithmetic Quantum Unique Ergodicity [16]. Moreover, in the first case, he showed that we can construct semiclassical measure equal to Lebesgue on the isotropic submanifold.

**1.1. Statement of the main result.** In recent papers [2], [5], Anantharaman and Nonnenmacher got concerned with the study of the localization of eigenfunctions on  $M$  as in the case of the toy models. They tried to understand it via the Kolmogorov-Sinai entropy. This paper is in the same spirit and our main result gives an information on the set of semiclassical measures in the case of

an Anosov geodesic flow (which is the typical chaotic behavior) on a surface  $M$ . More precisely, we give an information on the localization (or complexity) of a semiclassical measure:

**Theorem 1.2.** *Let  $M$  be a  $C^\infty$  riemannian surface and  $\mu$  a semiclassical measure. Suppose the geodesic flow  $(g^t)_t$  has the Anosov property. Then,*

$$(2) \quad h_{KS}(\mu, g) \geq \frac{1}{2} \left| \int_{S^*M} \log J^u(\rho) d\mu(\rho) \right|,$$

where  $h_{KS}$  is the Kolmogorov-Sinai entropy and  $J^u(\rho)$  is the unstable Jacobian at the point  $\rho$ .

In order to comment this result, let us recall a few facts about the Kolmogorov-Sinai (also called metric) entropy. It is a positive number associated to a flow  $g$  and a  $g$ -invariant measure  $\mu$ , that estimates the complexity of  $\mu$  with respect to this flow. For example, a measure carried by a closed geodesic will have entropy zero. In particular, this theorem shows that a semiclassical measure can not only be carried by a closed geodesic (it was already contained in [2]). Moreover, this lower bound seems to be the optimal result we can prove using this method and only the dynamical properties of  $M$ . In fact, in the case of the toy models some of the counterexamples that have been constructed (see [13], [16]) have entropy equal to  $\frac{1}{2} \left| \int_{S^*M} \log J^u(\rho) d\mu(\rho) \right|$ . Recall also that a standard theorem of dynamical systems due to Ruelle [21] asserts that, for any invariant measure  $\mu$  under the geodesic flow:

$$(3) \quad h_{KS}(\mu, g) \leq \left| \int_{S^*M} \log J^u(\rho) d\mu(\rho) \right|$$

with equality if and only if  $\mu$  is the Liouville measure in the case of an Anosov flow [17].

The lower bound of theorem 1.2 was conjectured to hold for any semiclassical measure in any dimension by Anantharaman [2]. In fact, Anantharaman proved that in any dimension, the entropy of a semiclassical measure should be bounded from below by a (not really explicit) positive constant [2]. Then, Anantharaman and Nonnenmacher showed that inequality (2) holds in the case of the Walsh Baker's map [4] and in the case of constant negative curvature in all dimension [5]. In the general case, Anantharaman, Koch and Nonnenmacher [3] proved a lower bound using the same method:

$$h_{KS}(\mu, g) \geq \left| \int_{S^*M} \log J^u(\rho) d\mu(\rho) \right| - \frac{(d-1)\lambda_{\max}}{2}.$$

where  $\lambda_{\max} := \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \sup_{\rho \in S^*M} |d_\rho g^t|$  is the maximal expansion rate of the geodesic flow (in particular if this quantity is very large, the previous inequality can be trivial). However, they conjectured inequality (2) should hold in the general case [5], [3]. Now let us discuss briefly the main ideas of our proof of theorem 1.2.

**1.2. Heuristic of the proof.** The procedure developed in [3] uses a result known as the entropic uncertainty principle [19]. To use this principle in the semiclassical limit, we need to understand the precise link between the classical evolution and the quantum one for large times. Typically, we have to understand Egorov theorem (1) for large range of times of order  $t \sim |\log \hbar|$  (i.e. have an uniform remainder term of (1) for a large range of times). For a general symbol  $a$  in  $C_c^\infty(T^*M)$ , we can only expect to have an uniform Egorov property for times  $t$  in the range of times  $[-\frac{1}{2}|\log \hbar|/\lambda_{\max}, \frac{1}{2}|\log \hbar|/\lambda_{\max}]$  [7]. However, if we only consider this range of times, we do not take into account that the unstable jacobian can be very different between two points of  $S^*M$ . In this paper, we would like to say that the range of times for which the Egorov property holds depends also on the support of the symbol  $a(x, \xi)$  we consider. For particular families of symbol of small support (that depends on  $\hbar$ ), we show that we have a 'local' Egorov theorem with an allowed range of times that depends on our symbol (see (64) for example). To make this heuristic idea work, we first try to reparametrize the flow [10] in order to have an uniform expansion rate on the manifold. We define  $\bar{g}^\tau(\rho) := g^t(\rho)$  where

$$(4) \quad \tau := - \int_0^t \log J^u(g^s \rho) ds.$$

For simplicity of our discussion, we suppose that  $\log J^u$  is negative. This new flow  $\bar{g}$  has the same trajectories as  $g$ . However, the 'velocity of motion' along the trajectory at  $\rho$  is  $|\log J^u(\rho)|$  greater for  $\bar{g}$  than for  $g$ . We underline here that the unstable direction is of dimension 1 (as  $M$  is a surface) and it is crucial because it implies that  $\log J^u$  exactly measures the expansion rate in the unstable direction at each point<sup>1</sup>. As a consequence, this new flow  $\bar{g}$  has an uniform expansion rate. Once this reparametrization is done, we use the following formula to recover  $t$  knowing  $\tau$ :

$$(5) \quad t_\tau(\rho) = \inf \left\{ s > 0 : - \int_0^s \log J^u(g^{s'} \rho) ds' \geq \tau \right\}.$$

The number  $t_\tau(\rho)$  can be thought as a stopping time corresponding to  $\rho$ . We consider now  $\tau = \frac{1}{2} |\log \hbar|$ . For a given symbol  $a(x, \xi)$  localized near a point  $\rho$ ,  $t_{\frac{1}{2} |\log \hbar|}(\rho)$  is exactly the range of times for which we can expect Egorov to hold. This new flow seems in a way more adapted to our problem. Moreover, we can define a  $\bar{g}$ -invariant measure  $\bar{\mu}$  corresponding to  $\mu$  [10]. The measure  $\bar{\mu}$  is absolutely continuous with respect to  $\mu$  such that  $\frac{d\bar{\mu}}{d\mu}(\rho) = \log J^u(\rho) / \int_{S^*M} \log J^u(\rho) d\mu(\rho)$ . We can apply the classical result of Abramov:

$$h_{KS}(\mu, g) = \left| \int_{S^*M} \log J^u(\rho) d\mu(\rho) \right| h_{KS}(\bar{\mu}, \bar{g}).$$

To prove theorem 1.2, we would have to show that  $h_{KS}(\bar{\mu}, \bar{g}) \geq 1/2$ . However, the flow  $\bar{g}$  has no reason to be an Hamiltonian flow to which corresponds a quantum propagator  $\bar{U}$ . As a consequence, there is no particular reason that this inequality should be a consequence of [5]. In the quantum case, there is also no obvious reparametrization we can make as in the classical case. However, we will reparametrize the quantum propagator by introducing a small parameter  $\eta$  and looking at its iterates  $(U^{k\eta})_k$ . To have an artificial reparametrization, we will introduce a suspension of the geodesic flow. Then, in this setting, we will define discrete analogues of the previous quantities (4) and (5) that will be precised in the paper. It will allow us to prove a lower bound on the entropy of a certain reparametrized flow and then using Abramov theorem [1] deduce the expected lower bound on the entropy of a semiclassical measure.

Finally, we would like to underline that in a recent paper [15], Gutkin also used a version of the Abramov theorem to prove an analogue of theorem 1.2 in the case of toy models with an unstable direction of dimension 1.

**1.3. Organization of the paper.** In section 2, we briefly recall properties we will need about entropy in the classical and quantum settings. In particular, we recall the version of Abramov theorem we will need. In section 3, we describe the assumptions we make on the manifold  $M$  and introduce some notations. In section 4, we draw a precise outline of the proof of theorem 1.2 and state some results that we will prove in the following sections. Sections 5 and 6 are devoted to the detailed proofs of the results we admitted in section 4. Precisely, section 5 is devoted to the details of the first part of the proof (section 4.2) and section 6 to the second part (section 4.3). Finally, sections 7 and 8 are devoted to results of semiclassical analysis that are quite technical and that we will use at different points of the paper (in particular in section 6).

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<sup>1</sup>In fact, the crucial point is that at each point  $\rho$  of  $S^*M$ , the expansion rate is the same in any direction, i.e.  $dg_{|E^u(g^1 \rho)}^{-1}$  is of the form  $J^u(\rho)^{\frac{1}{d-1}} v_\rho$  where  $d$  is the dimension of the manifold  $M$  and  $v_\rho$  is an isometry. The proof of theorem 1.2 can be immediately adapted to Anosov manifolds of higher dimensions satisfying this isotropic expansion property (for example manifolds of constant negative curvature).

## 2. CLASSICAL AND QUANTUM ENTROPY

**2.1. Kolmogorov-Sinai entropy.** Let us recall a few facts about Kolmogorov-Sinai (or metric) entropy that can be found for example in [24]. Let  $(X, \mathcal{B}, \mu)$  be a measurable probability space and  $P := (P_\alpha)_{\alpha \in I}$  a finite measurable partition of  $X$ , i.e. a finite collection of measurable subsets that forms a partition. Each  $P_\alpha$  is called an atom of the partition. Assuming  $0 \log 0 = 0$ , one defines the entropy of the partition as:

$$(6) \quad H(\mu, P) := - \sum_{\alpha \in I} \mu(P_\alpha) \log \mu(P_\alpha) \geq 0.$$

Given two measurable partitions  $P := (P_\alpha)_{\alpha \in I}$  and  $Q := (Q_\beta)_{\beta \in K}$ , one says that  $P$  is a refinement of  $Q$  if every element of  $Q$  can be written as the union of elements of  $P$  and it can be shown that  $H(\mu, Q) \leq H(\mu, P)$ . Otherwise, one denotes  $P \vee Q := (P_\alpha \cap Q_\beta)_{\alpha \in I, \beta \in K}$  their join (which is still a partition) and one has  $H(\mu, P \vee Q) \leq H(\mu, P) + H(\mu, Q)$  (subadditivity property). Let  $T$  be a measure preserving transformation of  $X$ . The  $n$ -refined partition  $\bigvee_{i=0}^{n-1} T^{-i} P$  of  $P$  with respect to  $T$  is then the partition made of the atoms  $(P_{\alpha_0} \cap \dots \cap T^{-(n-1)} P_{\alpha_{n-1}})_{\alpha \in \{1, \dots, K\}^n}$ . We define the entropy with respect to this refined partition:

$$(7) \quad H_n(\mu, T, P) = - \sum_{|\alpha|=n} \mu(P_{\alpha_0} \cap \dots \cap T^{-(n-1)} P_{\alpha_{n-1}}) \log \mu(P_{\alpha_0} \cap \dots \cap T^{-(n-1)} P_{\alpha_{n-1}}).$$

Using the subadditivity property of entropy, we have for any integers  $n$  and  $m$ :

$$(8) \quad H_{n+m}(\mu, T, P) \leq H_n(\mu, T, P) + H_m(T^n \sharp \mu, T, P) = H_n(\mu, T, P) + H_m(\mu, T, P).$$

For the last equality, it is important to underline that we really use the  $T$ -invariance of the measure  $\mu$ . A classical argument for subadditive sequences allows us to define the following quantity:

$$(9) \quad h_{KS}(\mu, T, P) := \lim_{n \rightarrow \infty} \frac{H_n(\mu, T, P)}{n}.$$

It is called the Kolmogorov Sinai entropy of  $(T, \mu)$  with respect to the partition  $P$ . The Kolmogorov Sinai entropy  $h_{KS}(\mu, T)$  of  $(\mu, T)$  is then defined as the supremum of  $h_{KS}(\mu, T, P)$  over all partitions  $P$  of  $X$ . Finally, it can be denoted that this quantity can be infinite (not in our case thanks to Ruelle inequality (3) for instance). Note also that if, for all index  $(\alpha_0, \dots, \alpha_{n-1})$ ,  $\mu(P_{\alpha_0} \cap \dots \cap T^{-(n-1)} P_{\alpha_{n-1}}) \leq C e^{-\beta n}$  with  $C$  positive constant, then  $h_{KS}(\mu, T) \geq \beta$ : the metric entropy measures the exponential decrease of the atoms of the refined partition.

**2.2. Quantum entropy.** One can defined a quantum counterpart to the metric entropy. Let  $\mathcal{H}$  be an Hilbert space. We call a partition of identity  $(\tau_\alpha)_{\alpha \in I}$  a family of operators that satisfies the following relation:

$$(10) \quad \sum_{\alpha \in I} \tau_\alpha^* \tau_\alpha = \text{Id}_{\mathcal{H}}.$$

Then, one defines the quantum entropy of a normalized vector  $\psi$  as:

$$(11) \quad h_\tau(\psi) := - \sum_{\alpha \in I} \|\tau_\alpha \psi\|^2 \log \|\tau_\alpha \psi\|^2.$$

Finally, one has the following generalization of a theorem from [5] (the proof immediately generalizes to this case), known as the entropic uncertainty principle [19]:

**Theorem 2.1.** *Let  $O_\beta$  be a family of bounded operators and  $U$  an unitary operator of an Hilbert space  $(\mathcal{H}, \|\cdot\|)$ . Let  $\delta'$  be a positive number. Given  $\tau_\alpha$  and  $\pi_\beta$  two partitions of identity and  $\psi$  a normalized vector in  $\mathcal{H}$  such that*

$$\|(Id - O_\beta) \pi_\beta \psi\| \leq \delta'.$$

*Suppose both partitions are of cardinal less than  $\mathcal{N}$ , then:*

$$h_\tau(U\psi) + h_\pi(\psi) \geq -2 \log(c_O(U) + \mathcal{N} \delta'),$$

*where  $c_O(U) = \max_{\alpha, \beta} (\|\tau_\alpha U \pi_\beta^* O_\beta\|)$ , with  $\|\tau_\alpha U \pi_\beta^* O_\beta\|$  the operator norm in  $\mathcal{H}$ .*

**2.3. Entropy of a special flow.** In the previous papers of Anantharaman, Koch and Nonnenmacher (see [3] for example), the main difficulty that was faced to prove main inequality (2) was that the value of  $\log J^u(\rho)$  could change a lot depending on the point of the energy layer they looked at. As was mentioned (see section 1.2), we will try to adapt their proof and take into account the changes of the value of  $\log J^u(\rho)$ . To do this, we will, in a certain way, reparametrize the geodesic flow. Before explaining precisely this strategy, let us recall a classical fact of dynamical system for reparametrization of measure preserving transformations known as the Abramov theorem.

First, let us define a special flow (see [1], [10]). Let  $(X, \mathcal{B}, \mu)$  be a probability space,  $T$  an automorphism of  $X$  and  $f$  a measurable function such that  $f(x) > a > 0$  for all  $x$  in  $X$ . The function  $f$  is called a roof function. We are interested in the set:

$$(12) \quad \overline{X} := \{(x, s) : x \in X, 0 \leq s < f(x)\}.$$

$\overline{X}$  is equipped with the  $\sigma$ -algebra by restriction of the  $\sigma$ -algebra on the cartesian product  $X \times \mathbb{R}$ . For  $A$  measurable, one defines  $\overline{\mu}(A) := \frac{1}{\int_X f d\mu} \int \int_A d\mu(x) ds$  and  $\overline{\mu}(\overline{X}) = 1$ .

**Definition 2.2.** The special flow under the automorphism  $T$ , constructed by the function  $f$  is the flow  $(\overline{T}^t)$  that acts on  $\overline{X}$  in the following way, for  $t \geq 0$ :

$$(13) \quad \overline{T}^t(x, s) := \left( T^n x, s + t - \sum_{k=0}^{n-1} f(T^k x) \right),$$

where  $n$  is the only integer such that  $\sum_{k=0}^{n-1} f(T^k x) \leq s + t < \sum_{k=0}^n f(T^k x)$ .

For  $t < 0$ , one puts, if  $s + t > 0$ :

$$\overline{T}^t(x, s) := (x, s + t),$$

and otherwise,

$$\overline{T}^t(x, s) := \left( T^{-n} x, s + t + \sum_{k=-n}^{-1} f(T^k x) \right),$$

where  $n$  is the only integer such that  $-\sum_{k=-n}^{-1} f(T^k x) \leq s + t < -\sum_{k=-n+1}^{-1} f(T^k x)$ .

*Remark.* A suspension semi-flow can also be defined from an endomorphism.

It can be shown that this special flow preserves the measure  $\overline{\mu}$  if  $T$  preserves  $\mu$  [10]. Finally, we can state Abramov theorem for special flows [1]:

**Theorem 2.3.** *With the previous notations, one has, for all  $t \in \mathbb{R}$ :*

$$(14) \quad h_{KS}(\overline{T}^t, \overline{\mu}) = \frac{|t|}{\int_X f d\mu} h_{KS}(T, \mu).$$

### 3. CLASSICAL SETTING OF THE PAPER

Before starting the main lines of the proof, we want to describe the classical setting for our surface  $M$  and introduce notations that will be useful in the paper. We suppose the geodesic flow over  $T^*M$  to have the Anosov property. This means that for any  $\lambda > 0$ , the geodesic flow  $g^t$  is Anosov on the energy layer  $\mathcal{E}(\lambda) := H^{-1}(\lambda) \subset T^*M$  and in particular, the following decomposition holds for all  $\rho \in \mathcal{E}(\lambda)$ :

$$T_\rho \mathcal{E}(\lambda) = E^u(\rho) \oplus E^s(\rho) \oplus \mathbb{R}X_H(\rho),$$

where  $X_H$  is the Hamiltonian vector field associated to  $H$ ,  $E^u$  the unstable space and  $E^s$  the stable space [8]. It can be denoted that in the setting of this article, they are all one dimensional spaces. The unstable Jacobian  $J^u(\rho)$  at the point  $\rho$  is defined as the Jacobian of the restriction of  $g^{-1}$  to the unstable subspace  $E^u(g^1 \rho)$ :

$$J^u(\rho) := \det \left( dg_{|E^u(g^1 \rho)}^{-1} \right).$$

For  $\theta$  small positive number ( $\theta$  will be fixed all along the paper), one defines  $\mathcal{E}^\theta := H^{-1}([1/2 - \theta, 1/2 + \theta])$ . As the geodesic flow is Anosov, we can suppose there exist  $s_0 < 1$  and  $t_0 > 0$  (see the remark below) such that for all  $\rho \in \mathcal{E}^\theta$ ,  $t_0 \leq J^u(\rho) \leq s_0$ . We denote  $a_0 := -\log s_0$  and  $b_0 := -\log t_0$ .

*Remark.* In fact, in the general setting of an Anosov flow, we can only suppose that there exists  $k_0 \in \mathbb{N}$  such that  $\det \left( dg_{|E^u(g^{k_0}\rho)}^{-k_0} \right) < 1$  for all  $\rho \in \mathcal{E}^\theta$ . So, to be in the correct setting, we should take  $g^{k_0}$  instead of  $g$  in the paper. In fact, as  $h_{KS}(\mu, g^{k_0}) = k_0 h_{KS}(\mu, g)$  and

$$-\int_{S^*M} \log \det \left( dg_{|E^u(g^{k_0}\rho)}^{-k_0} \right) d\mu(\rho) = -k_0 \int_{S^*M} \log \det \left( dg_{|E^u(g\rho)}^{-1} \right) d\mu(\rho),$$

theorem 1.2 follows for  $k_0 = 1$  from the case  $k_0$  large. However, in order to avoid too many notations, we will suppose  $k_0 = 1$ .

Let  $\epsilon$  and  $\eta$  be small positive constants lower than the injectivity radius of the manifold. We choose  $\eta$  small enough to have  $(2 + \frac{b_0}{a_0})b_0\eta \leq \frac{\epsilon}{2}$  (this property will only be used in the proof of proposition 5.3). We denote  $f$  the function  $-\log J^u$  and remark that there exists  $\epsilon > 0$  such that if  $d(\rho, \rho') \leq \epsilon$  with  $\rho, \rho' \in \mathcal{E}^\theta$ , then  $|f(\rho) - f(\rho')| \leq a_0\epsilon$ .

Consider a partition  $M = \bigsqcup_{i=1}^K O_i$  of diameter smaller than  $\delta$ . Let  $(\Omega_i)_{i=1}^K$  be a finite open cover of  $M$  such that for all  $1 \leq i \leq K$ ,  $O_i \subsetneq \Omega_i$ . For  $\gamma \in \{1, \dots, K\}^2$ , define an open subset of  $T^*M$ :

$$U_\gamma := (T^*\Omega_{\gamma_0} \cap g^{-\eta}T^*\Omega_{\gamma_1}) \cap \mathcal{E}^\theta.$$

We choose the partition  $(O_i)_i$  and the open cover  $(\Omega_i)_{i=1}^K$  of  $M$  such that  $(U_\gamma)_{\gamma \in \{1, \dots, K\}^2}$  is a finite open cover of diameter smaller than  $\epsilon$  of  $\mathcal{E}^\theta$ . Then, we define the following quantity, called the discrete Jacobian in time  $\eta$ :

$$(15) \quad J_\eta^u(\gamma) := \sup \{J^u(\rho) : \rho \in U_\gamma\},$$

if the previous set is non empty,  $e^{-\Lambda}$  otherwise where  $\Lambda$  is a very large constant. Outline that  $J_\eta^u(\gamma)$  depends on  $\eta$  as  $U_\gamma$  depends on  $\eta$ . The definition can seem quite asymmetric as we consider the supremum of  $J^u(\rho)$  and not  $J_\eta^u(\rho)$ . However, this choice makes things easier for our analysis. For simplicity of notations, we also define:

$$(16) \quad f(\gamma) := -\eta \log J_\eta^u(\gamma) \leq \eta b_0 \leq \frac{\epsilon}{2},$$

where the upper bound follows from the previous hypothesis. Moreover, we have for all  $\rho \in U_\gamma$ ,

$$(17) \quad |f(\gamma) + \eta \log J^u(\rho)| \leq a_0\eta\epsilon.$$

*Remark.* This last inequality shows that even if our choice for  $J_\eta^u(\gamma)$  seems quite asymmetric, it allows to have an explicit bound in  $\eta$  for quantity (17) and it will be quite useful. With a more symmetric choice, we would not have been able to get an explicit bound in  $\eta$  for (17).

#### 4. OUTLINE OF THE PROOF

Let  $(\psi_{\hbar_k})$  be a sequence of orthonormal eigenfunctions of the Laplacian corresponding to the eigenvalues  $-\hbar_k^{-2}$  such that the corresponding sequence of distributions  $\mu_k$  on  $T^*M$  converges as  $k$  tends to infinity to the semiclassical measure  $\mu$ . For simplicity of notations and to fit semiclassical analysis notations, we will denote  $\hbar$  tends to 0 the fact that  $k$  tends to infinity and  $\psi_\hbar$  and  $\hbar^{-2}$  the corresponding eigenvector and eigenvalue. To prove our main theorem, we will in particular give a symbolic interpretation of a semiclassical measure and apply the previous results on special flows to this measure.

Let  $\epsilon' > 4\epsilon$  be a positive number, where  $\epsilon$  was defined in 3. The link between the two quantities  $\epsilon$  and  $\epsilon'$  will only be used in section 7 to define  $\nu$ . In the following of the paper, the Ehrenfest time  $n_E(\hbar)$  will be the quantity:

$$(18) \quad n_E(\hbar) := [(1 - \epsilon')|\log \hbar|].$$

We also consider a smaller non integer time:

$$(19) \quad T_E(\hbar) := (1 - \epsilon)n_E(\hbar).$$



**4.1. Quantum partitions of identity.** In order to find a lower bound on the metric entropy of the semiclassical measure  $\mu$ , we would like to apply the entropic uncertainty principle (theorem 2.1) and see what informations it will give (when  $\hbar$  tends to 0) on the metric entropy of the semiclassical measure  $\mu$ . To do this, we define quantum partitions of identity corresponding to a given partition of the manifold.

*4.1.1. Partitions of identity.* In section 3, we considered a partition of small diameter  $(O_i)_i$  of  $M$ . We also defined  $(\Omega_i)_i$  a corresponding finite open cover of small diameter of  $M$ . By convolution of the characteristic functions  $\mathbf{1}_{O_i}$ , we obtain  $\mathcal{P} = (P_i)_{i=1,\dots,K}$  a smooth partition of unity on  $M$  i.e. for all  $x \in M$ :

$$\sum_{i=1}^K P_i^2(x) = 1.$$

We assume that for all  $1 \leq i \leq K$ ,  $P_i$  is an element of  $\mathcal{C}_c^\infty(\Omega_i)$ . To this classical partition corresponds a quantum partition of identity of  $L^2(M)$ . In fact, if  $P_i$  denotes the multiplication operator by  $P_i(x)$  on  $L^2(M)$ , then one has:

$$(20) \quad \sum_{i=1}^K P_i^* P_i = \text{Id}_{L^2(M)}.$$

*4.1.2. Refinement of the quantum partition under the Schrödinger flow.* Like in the classical setting of entropy (7), we would like to make a refinement of the quantum partition. To do this refinement, we use the Schrödinger propagation operator  $U^t = e^{\frac{it\hbar\Delta}{2}}$ . We define  $A(t) := U^{-t} A U^t$ , where  $A$  is an operator on  $L^2(M)$ . To fit as much as possible with the metric entropy (see definition (7) and Egorov property (1)), we define the following operators:

$$(21) \quad \tau_\alpha = P_{\alpha_k}(k\eta) \cdots P_{\alpha_1}(\eta) P_{\alpha_0}$$

and

$$(22) \quad \pi_\beta = P_{\beta_{-k}}(-k\eta) \cdots P_{\beta_{-2}}(-2\eta) P_{\beta_0} P_{\beta_{-1}}(-\eta),$$

where  $\alpha = (\alpha_0, \dots, \alpha_k)$  and  $\beta = (\beta_{-k}, \dots, \beta_0)$  are finite sequences of symbols such that  $\alpha_j \in [1, K]$  and  $\beta_{-j} \in [1, K]$ . We can remark that the definition of  $\pi_\beta$  is the analogue for negative times of the definition of  $\tau_\alpha$ . The only difference is that we switch the two first terms  $\beta_0$  and  $\beta_{-1}$ . The reason of this choice will appear later in the application of the quantum uncertainty principle (see equality (38) in section 5.3). One can see that for fixed  $k$ , using the Egorov property (1):

$$(23) \quad \|P_{\alpha_k}(k\eta) \cdots P_{\alpha_1}(\eta) P_{\alpha_0} \psi_\hbar\|^2 \rightarrow \mu(P_{\alpha_k}^2 \circ g^{k\eta} \times \cdots \times P_{\alpha_1}^2 \circ g^\eta \times P_{\alpha_0}^2) \text{ as } \hbar \text{ tends to } 0.$$

This last quantity is the one used to compute  $h_{KS}(\mu, g^\eta)$  (with the notable difference that the  $P_j$  are here smooth functions instead of characteristic functions: see (7)). As was discussed in the heuristic of the proof 1.2, we will have to understand for which range of times  $k\eta$ , the Egorov property can be applied. In particular, we will study for which range of times, the operator  $\tau_\alpha$  is a pseudodifferential operator of symbol  $P_{\alpha_k} \circ g^{k\eta} \times \cdots \times P_{\alpha_1} \circ g^\eta \times P_{\alpha_0}$  (see (23)). In [5] and [3], they only considered  $k\eta \leq |\log \hbar| / \lambda_{\max}$  where  $\lambda_{\max} := \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \sup_{\rho \in S^*M} |d_\rho g^t|$ . This choice was not optimal and in the following, we try to define sequences  $\alpha$  for which we can say that  $\tau_\alpha$  is a pseudodifferential operator.

*4.1.3. Index family adapted to the variation of the unstable Jacobian.* Let  $\alpha = (\alpha_0, \alpha_1, \dots)$  be a sequence (finite or infinite) of elements of  $\{1, \dots, K\}$  whose length is larger than 1. We define the following quantity (see (16)):

$$f(\alpha) := f(\alpha_0, \alpha_1).$$

We also define a natural shift on these sequences (again finite or infinite and larger than 1 in length):

$$\sigma((\alpha_0, \alpha_1, \dots)) := (\alpha_1, \dots).$$

In the paper, we will try to use the symbol  $x$  for infinite sequences and  $\alpha$  for finite ones. For negative times, we define the analogous functions, for  $\beta := (\dots, \beta_{-1}, \beta_0)$ :

$$\bar{f}(\beta) := \bar{f}(\beta_{-1}, \beta_0)$$

and the backward shift

$$T(\beta) := (\cdots, \beta_{-1}).$$

Let  $\alpha$  and  $\beta$  be as previously (finite or infinite). We define:

$$\beta.\alpha := (\cdots, \beta_{-1}, \beta_0, \alpha_0, \alpha_1, \cdots).$$

The same obviously works for any sequences of the form  $(\cdots, \beta_{p-1}, \beta_p)$  and  $(\alpha_q, \alpha_{q+1}, \cdots)$ . Then, as described in section 5, index families depending on the value of the unstable Jacobian can be defined as follows:

$$(24) \quad I^\eta(\hbar) := I^\eta(T_E(\hbar)) = \left\{ (\alpha_0, \cdots, \alpha_k) : k \geq 3, \sum_{i=1}^{k-2} f(\sigma^i \alpha) \leq T_E(\hbar) < \sum_{i=1}^{k-1} f(\sigma^i \alpha) \right\},$$

$$(25) \quad K^\eta(\hbar) := K^\eta(T_E(\hbar)) = \left\{ (\beta_{-k}, \cdots, \beta_0) : k \geq 3, \sum_{i=1}^{k-2} \bar{f}(T^i \beta) \leq T_E(\hbar) < \sum_{i=1}^{k-1} \bar{f}(T^i \beta) \right\}.$$

These sets define the maximal sequences for which we can expect to have Egorov property for the corresponding  $\tau_\alpha$ . The sums used to define these sets are in a way a discrete analogue of the integral in the inversion formula (5) defined in the introduction<sup>2</sup>. The sums used to define the allowed sequences are in fact Riemann sums (with small parameter  $\eta$ ) corresponding to the integral (4). We can think of the time  $|\alpha|\eta$  as a stopping time for which property (23) will hold (for a symbol  $a$  corresponding to  $\alpha$ ).

A good way of thinking of these families of words is by keeping in mind picture 1. On this figure, we draw the case  $K = 4$ . The biggest square has sides of length 1. Each square represents an element of  $I^\eta(\hbar)$  and each square with sides of length  $1/2^k$  represents a sequence of length  $k + 1$  (for  $k \geq 0$ ). If we denote  $C(\alpha)$  the square that represents  $\alpha$ , then we can represent the sequences  $\alpha.\gamma$  for each  $\gamma$  in  $\{1, \cdots, 4\}$  by subdividing the square  $C(\alpha)$  in 4 squares of same size. Finally, by definition of  $I^\eta(\hbar)$ , we can remark that if  $\alpha.\gamma$  is represented in the subdivision (for  $\gamma$  in  $\{1, \cdots, 4\}$ ), then  $\alpha.\gamma'$  is represented in the subdivision for each  $\gamma'$  in  $\{1, \cdots, 4\}$ . Families of operators can

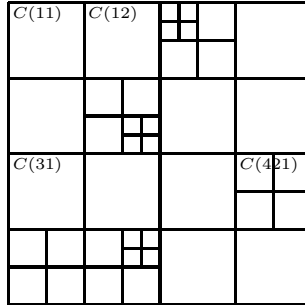


FIGURE 1. *Refinement of variable size*

be associated to these families of index:  $(\tau_\alpha)_{\alpha \in I^\eta(\hbar)}$  and  $(\pi_\beta)_{\beta \in K^\eta(\hbar)}$ . One can show that these partitions form quantum partitions of identity (see section 5):

**Proposition 4.1.**

$$\sum_{\alpha \in I^\eta(\hbar)} \tau_\alpha^* \tau_\alpha = Id_{L^2(M)} \quad \text{and} \quad \sum_{\beta \in K^\eta(\hbar)} \pi_\beta^* \pi_\beta = Id_{L^2(M)}.$$

<sup>2</sup>In the higher dimension case mentioned in the footnote of section 1.2, we should take  $(d-1)T_E(\hbar)$  (where  $d$  is the dimension of  $M$ ) instead of  $T_E(\hbar)$  in the definition of  $I^\eta(\hbar)$  and  $K^\eta(\hbar)$ .

**4.2. Symbolic interpretation of semiclassical measures.** Now that we have defined these partitions of variable size, we want to show that they are adapted to compute the entropy of a certain measure with respect to some reparametrized flow associated to the geodesic flow. To do this, we start by giving a symbolic interpretation of the quantum partitions. Denote  $\Sigma := \{1, \dots, K\}^{\mathbb{N}}$  and  $\mathcal{C}_i$  the subset of sequences  $(x_n)_{n \in \mathbb{N}}$  such that  $x_0 = i$ . Define also:

$$[\alpha_0, \dots, \alpha_k] := \mathcal{C}_{\alpha_0} \cap \dots \cap \sigma^{-k} \mathcal{C}_{\alpha_k},$$

where  $\sigma$  is the shift  $\sigma((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$  (it fits the notations of the previous section). The set  $\Sigma$  is then endowed with the probability measure (not necessarily  $\sigma$ -invariant):

$$\mu_{\hbar}^{\Sigma}([\alpha_0, \dots, \alpha_k]) = \mu_{\hbar}^{\Sigma}(\mathcal{C}_{\alpha_0} \cap \dots \cap \sigma^{-k} \mathcal{C}_{\alpha_k}) = \|P_{\alpha_k}(k\eta) \dots P_{\alpha_0} \psi_{\hbar}\|^2.$$

Using the property (10), it is clear that this definition assures the compatibility conditions to define a probability measure:

$$\sum_{\alpha_{k+1}} \mu_{\hbar}^{\Sigma}([\alpha_0, \dots, \alpha_{k+1}]) = \mu_{\hbar}^{\Sigma}([\alpha_0, \dots, \alpha_k]).$$

Then, we can define the special flow, in the sense of Abramov (section 2.3), associated to this probability measure. To do this, the suspension set (12) is defined as:

$$(26) \quad \bar{\Sigma} := \{(x, s) : \alpha \in \Sigma, 0 \leq s < f(x)\}.$$

Recall that the roof function  $f$  is defined as  $f(x) := f(x_0, x_1)$ . We define a probability measure  $\bar{\mu}_{\hbar}^{\Sigma}$  on  $\bar{\Sigma}$ :

$$(27) \quad \bar{\mu}_{\hbar}^{\Sigma} = \mu_{\hbar}^{\Sigma} \times \frac{dt}{\sum_{\alpha \in \{1, \dots, K\}^2} f(\alpha) \|P_{\alpha} \psi_{\hbar}\|^2} = \mu_{\hbar}^{\Sigma} \times \frac{dt}{\sum_{\alpha \in \{1, \dots, K\}^2} f(\alpha) \mu_{\hbar}^{\Sigma}([\alpha])}.$$

The semi-flow (13) associated to  $\sigma$  is for time  $s$ :

$$(28) \quad \bar{\sigma}^s(x, t) := \left( \sigma^{n-1}(x), s + t - \sum_{j=0}^{n-2} f(\sigma^j x) \right),$$

where  $n$  is the only integer such that  $\sum_{j=0}^{n-2} f(\sigma^j x) \leq s + t < \sum_{j=0}^{n-1} f(\sigma^j x)$ . In the following, we will only consider time 1 of the flow and its iterates and we will denote  $\bar{\sigma} := \bar{\sigma}^1$ .

*Remark.* It can be underlined that the same procedure holds for the partition  $(\pi_{\beta})$ . The only differences are that we have to consider  $\Sigma_- := \{1, \dots, K\}^{-\mathbb{N}}$ ,  $T((x_n)_{n \leq 0}) = (x_{n-1})_{n \leq 0}$  and that the corresponding measure is, for  $k \geq 1$ :

$$\mu_{\hbar}^{\Sigma_-}([\beta_{-k}, \dots, \beta_0]) = \mu_{\hbar}^{\Sigma_-}(T^{-k} \mathcal{C}_{\beta_{-k}} \cap \dots \cap \mathcal{C}_{\beta_0}) = \|P_{\beta_{-k}}(-k\eta) \dots P_{\beta_0} P_{\beta_{-1}}(-\eta) \psi_{\hbar}\|^2.$$

For  $k = 0$ , one should take the only possibility to assure the compatibility condition:

$$\mu_{\hbar}^{\Sigma_-}([\beta_0]) = \sum_{j=1}^K \mu_{\hbar}^{\Sigma_-}([\beta_{-1}, \beta_0]).$$

The definition is quite different from the positive case but in the semiclassical limit, it will not change anything as  $P_{\beta_0}$  and  $P_{\beta_{-1}}(-\eta)$  commute.

Now let  $\alpha$  be an element of  $I^n(\hbar)$ . Define:

$$(29) \quad \tilde{\mathcal{C}}_{\alpha} := \mathcal{C}_{\alpha_0} \cap \dots \cap \sigma^{-k} \mathcal{C}_{\alpha_k}.$$

This new family of subsets forms a partition of  $\Sigma$ . Then, a partition  $\bar{\mathcal{C}}_{\hbar}$  of  $\bar{\Sigma}$  can be defined starting from the partition  $\tilde{\mathcal{C}}$  and  $[0, f(\alpha)[$ . An atom of this suspension partition is an element of the form  $\bar{\mathcal{C}}_{\alpha} = \tilde{\mathcal{C}}_{\alpha} \times [0, f(\alpha)[$ . For  $\bar{\Sigma}^-$  (the suspension set corresponding to  $\Sigma_-$ ), we define an

analogous partition  $\overline{\mathcal{C}}_{\hbar}^-$ . Finally, with this interpretation, equality (46) from section 5.3 (which is just a careful adaptation of the uncertainty principle) can be read as follows:

$$(30) \quad H\left(\overline{\mu}_{\hbar}^{\Sigma}, \overline{\mathcal{C}}_{\hbar}\right) + H\left(\overline{\mu}_{\hbar}^{\Sigma-}, \overline{\mathcal{C}}_{\hbar}^-\right) \geq (1 - \epsilon')(1 - \epsilon)|\log \hbar| + C,$$

where  $H$  is defined by (6). To fit as much as possible with the setting of the classical metric entropy, we expect  $\overline{\mathcal{C}}_{\hbar}$  to be a refined partition under the special flow. It is not exactly the case but the following lemma can be shown (see proposition 5.3):

**Lemma 4.2.** *There exists an explicit partition  $\overline{\mathcal{C}}$  of  $\overline{\Sigma}$ , independent of  $\hbar$  such that  $\bigvee_{i=0}^{n_E(\hbar)-1} \overline{\sigma}^{-i} \overline{\mathcal{C}}$  is a refinement of the partition  $\overline{\mathcal{C}}_{\hbar}$ . Moreover, let  $n$  be a fixed positive integer. Then, an atom of the refined partition  $\bigvee_{i=0}^{n-1} \overline{\sigma}^{-i} \overline{\mathcal{C}}$  is of the form  $\tilde{\mathcal{C}}_{\alpha} \times B(\alpha)$ , where  $\alpha = (\alpha_0, \dots, \alpha_k)$  is a  $k+1$ -uple such that  $(\alpha_0, \dots, \alpha_k)$  verifies  $n(1 - \epsilon) \leq \sum_{j=0}^{k-1} f(\sigma^j \alpha) \leq n(1 + \epsilon)$  and  $B(\alpha)$  is a subinterval of  $[0, f(\alpha)[$ .*

Then applying basic properties of  $H$  (see section 2.1), one finds that:

$$(31) \quad H\left(\overline{\mu}_{\hbar}^{\Sigma}, \overline{\mathcal{C}}_{\hbar}\right) \leq H\left(\overline{\mu}_{\hbar}^{\Sigma}, \bigvee_{i=0}^{n_E(\hbar)-1} \overline{\sigma}^{-i} \overline{\mathcal{C}}\right) = H_{n_E(\hbar)}\left(\overline{\mu}_{\hbar}^{\Sigma}, \overline{\sigma}, \overline{\mathcal{C}}\right).$$

*Remark.* This lemma will be proved in section 5.

To conclude this symbolic interpretation of quantum entropy, with natural notations, inequality (30) together with (31) give the following proposition:

**Proposition 4.3.** *With the previous notations, one has the following inequality:*

$$(32) \quad \frac{1}{n_E(\hbar)} \left( H_{n_E(\hbar)}\left(\overline{\mu}_{\hbar}^{\Sigma}, \overline{\sigma}, \overline{\mathcal{C}}\right) + H_{n_E(\hbar)}\left(\overline{\mu}_{\hbar}^{\Sigma-}, \overline{T}, \overline{\mathcal{C}}_-\right) \right) \geq (1 - \epsilon) + \frac{C}{n_E(\hbar)}.$$

The quantum entropic uncertainty principle gives an information on the entropy of a special flow. Now, we would like to let  $\hbar$  tends to 0 to find a lower on the metric entropy of a limit measure (that we will precise in section 4.3) with respect to  $\overline{\sigma}$ . However, both  $n_E(\hbar)$  and  $\mu_{\hbar}$  depend on  $\hbar$  and we have to be careful before passing to the semiclassical limit.

**4.3. Subadditivity of the entropy.** The Egorov property (1) implies that  $\mu_{\hbar}^{\Sigma}$  tends to a measure  $\mu^{\Sigma}$  on  $\Sigma$  (as  $\hbar$  tends to 0) defined as follows:

$$(33) \quad \mu^{\Sigma}([\alpha_0, \dots, \alpha_k]) = \mu(P_{\alpha_k}^2 \circ g^{k\eta} \times \dots \times P_{\alpha_0}^2).$$

Using the property of partition, this defines a probability measure on  $\Sigma$ . To this probability measure corresponds a probability measure  $\overline{\mu}^{\Sigma}$  on the suspension set  $\overline{\Sigma}$ . It is an immediate corollary that  $\overline{\mu}^{\Sigma}$  is the limit of the probability measure  $\overline{\mu}_{\hbar}^{\Sigma}$ . Moreover, using Egorov one more time, one can check that the measure  $\mu^{\Sigma}$  is  $\sigma$ -invariant and using results about special flows [10],  $\overline{\mu}^{\Sigma}$  is  $\overline{\sigma}$ -invariant. The same works for  $\mu_{\hbar}^{\Sigma-}$  and  $\overline{\mu}_{\hbar}^{\Sigma-}$ .

*Remark.* In the following, we will often prove properties in the case of  $\Sigma$ . The proofs are the same in the case of  $\Sigma_-$ .

As  $n_E(\hbar)$  and  $\mu_{\hbar}$  depend both on  $\hbar$ , we cannot let  $\hbar$  tend to 0 if we want to keep an information about the metric entropy. In fact, the left quantity in (32) does not tend a priori to the Kolmogorov-Sinai entropy. We want to proceed as in the classical case (see (8)) and prove a subadditivity property. This will allow to replace  $n_E(\hbar)$  by a fixed  $n_0$  (see below) in the left hand side of (32). This is done with the following theorem that will be proved in section 6:

**Theorem 4.4.** *Let  $\overline{\mathcal{C}}$  be the partition of lemma (4.2). There exists a function  $R(n_0, \hbar)$  on  $\mathbb{N} \times (0, 1]$  such that*

$$\forall n_0 \in \mathbb{N}, \quad \lim_{\hbar \rightarrow 0} |R(n_0, \hbar)| = 0.$$

Moreover, for any  $\hbar \in (0, 1]$  and any  $n_0, m \in \mathbb{N}$  such that  $n_0 + m \leq n_E(\hbar)$ , one has:

$$H_{n_0+m}\left(\overline{\mu}_{\hbar}^{\Sigma}, \overline{\sigma}, \overline{\mathcal{C}}\right) \leq H_{n_0}\left(\overline{\mu}_{\hbar}^{\Sigma}, \overline{\sigma}, \overline{\mathcal{C}}\right) + H_m\left(\overline{\mu}_{\hbar}^{\Sigma}, \overline{\sigma}, \overline{\mathcal{C}}\right) + R(n_0, \hbar).$$

The same holds for  $\Sigma_-$ .

This theorem says that the entropy satisfies almost the subadditivity property (see (8)) for time lower than the Ehrenfest time. It is an analogue of a theorem from [5] (proposition 2.8) except that we have taken into account the fact that the unstable jacobian varies on the surface and that we can make our semiclassical analysis for larger time than in [5]. The proof of this theorem is the object of section 6 and 7 (where a semiclassical analysis for 'local Ehrenfest time' is performed). Then, one can apply the standard argument for subadditive sequences. Let  $n_0$  be a fixed integer in  $\mathbb{N}$  and write the euclidian division  $n_E(\hbar) = qn_0 + r$  with  $r < n_0$ . The previous theorem then implies:

$$\frac{H_{n_E(\hbar)}(\bar{\mu}_{\hbar}^{\Sigma}, \bar{\sigma}, \bar{\mathcal{C}})}{n_E(\hbar)} \leq \frac{H_{n_0}(\bar{\mu}_{\hbar}^{\Sigma}, \bar{\sigma}, \bar{\mathcal{C}})}{n_0} + \frac{H_r(\bar{\mu}_{\hbar}^{\Sigma}, \bar{\sigma}, \bar{\mathcal{C}})}{n_E(\hbar)} + \frac{R(n_0, \hbar)}{n_0}.$$

As  $r$  stays uniformly bounded in  $n_0$ , the inequality (32) becomes:

$$(34) \quad \frac{1}{n_0} \left( H_{n_0}(\bar{\mu}_{\hbar}^{\Sigma}, \bar{\sigma}, \bar{\mathcal{C}}) + H_{n_0}(\bar{\mu}_{\hbar}^{\Sigma-}, \bar{\sigma}_-, \bar{\mathcal{C}}_-) \right) \geq (1 - \epsilon) + \frac{C(n_0)}{n_E(\hbar)} - 2 \frac{R(n_0, \hbar)}{n_0}.$$

**4.4. Application of the Abramov theorem.** Using inequality (34), we can conclude using Abramov theorem (14). Making  $\hbar$  tend to 0, one finds that (as was mentioned at the beginning of 4.3):

$$\frac{1}{n_0} \left( H_{n_0}(\bar{\mu}^{\Sigma}, \bar{\sigma}, \bar{\mathcal{C}}) + H_{n_0}(\bar{\mu}^{\Sigma-}, \bar{T}, \bar{\mathcal{C}}_-) \right) \geq (1 - \epsilon).$$

The Abramov theorem holds for automorphisms so one can look at the natural extension  $\Sigma' = \{1, \dots, K\}^{\mathbb{Z}}$  of  $\Sigma$  and the same inequality holds:

$$(35) \quad \frac{1}{n_0} \left( H_{n_0}(\bar{\mu}^{\Sigma'}, \bar{\sigma}', \bar{\mathcal{C}}') + H_{n_0}(\bar{\mu}^{\Sigma'}, \bar{\sigma}'^{-1}, \bar{\mathcal{C}}'_-) \right) \geq (1 - \epsilon),$$

where  $\sigma'$  is the two-sided shift on  $\Sigma'$  and  $\bar{\mathcal{C}}'$  is the lift of the partition  $\bar{\mathcal{C}}$  to the natural extension. In view of section 5, we have an exact expression for  $\mathcal{C}$  in terms of the functions  $(P_i)_i$  (see proposition 5.3). The measure  $\bar{\mu}^{\Sigma'}$  is  $\bar{\sigma}'$ -invariant as  $\mu^{\Sigma'}$  is  $\sigma'$ -invariant [10]. In this previous inequality, there is still one notable difference with the metric entropy: we consider smooth partitions of identity  $(P_i)_i$  (as it was necessary to make the semiclassical analysis). To return to the classical case, the procedure of [5] can be adapted using the exact form of the partition  $\bar{\mathcal{C}}$  (see proposition 5.3). Recall that each  $P_i$  is an element of  $\mathcal{C}_c^{\infty}(\Omega_i)$  and that we considered a partition  $M = \bigsqcup_i O_i$  of small diameter  $\delta$ , where each  $O_i \subsetneq \Omega_i$  (see section 3). We suppose it is small enough so that  $\mu$  does not charge the boundary of the  $O_i$ . By convolution of the  $\mathbf{1}_{O_i}$ , we obtained the smooth partition  $(P_i)_i$  of identity of diameter smaller than  $2\delta$ . The previous inequality does not depend on the derivatives of the  $P_i$ . Regarding also the form of the partition  $\bar{\mathcal{C}}$  (see lemma 4.2), we can replace the smooth functions  $P_i$  by the characteristic functions  $\mathbf{1}_{O_i}$  in inequality (35). One can let  $n_0$  tend to infinity and find:

$$2h_{KS}(\bar{\mu}^{\Sigma'}, \bar{\sigma}') \geq h_{KS}(\bar{\mu}^{\Sigma'}, \bar{\sigma}', \bar{\mathcal{C}}') + h_{KS}(\bar{\mu}^{\Sigma'}, \bar{\sigma}'^{-1}, \bar{\mathcal{C}}'_-) \geq (1 - \epsilon).$$

We used the fact that  $h_{KS}(\mu, T^{-1}) = h_{KS}(\mu, T)$  for the metric entropy of a dynamical system  $(X, \mu, T)$ . Then, using Abramov theorem (14), the previous inequality implies:

$$\eta h_{KS}(\mu, g) = h(\mu, g^n) = h_{KS}(\mu^{\Sigma'}, \sigma') \geq -\frac{1}{2}(1 - \epsilon) \sum_{\gamma \in \{1, \dots, K\}^2} \eta \log J^u(\gamma) \mu^{\Sigma'}([\gamma]).$$

After division by  $\eta$  and letting the diameter of the partition and  $\epsilon$  tend to 0, one gets:

$$h_{KS}(\mu, g) \geq \frac{1}{2} \left| \int_{S^*M} \log J^u(\rho) d\mu(\rho) \right|. \square$$

## 5. PARTITIONS OF VARIABLE SIZE

In this section, we define precisely the index families  $I^\eta$  and  $K^\eta$  depending on the unstable jacobian used in section 4. These families are used to construct quantum partitions of identity and partitions adapted to the special flow (see section 5.2). In the last section, we apply the uncertainty principle to eigenfunctions of the Laplacian for these quantum partitions of variable size.

**5.1. Stopping time.** Let  $t$  be a real positive number that will be greater than 1 in all the paper. Define index families as follows (see section 4.1.3 for definitions of  $f$ ,  $\sigma$ ,  $\bar{f}$  and  $T$ ):

$$I^\eta(t) := \left\{ \alpha = (\alpha_0, \dots, \alpha_k) : k \geq 3, \sum_{i=1}^{k-2} f(\sigma^i \alpha) \leq t < \sum_{i=1}^{k-1} f(\sigma^i \alpha) \right\},$$

$$K^\eta(t) := \left\{ \beta = (\beta_{-k}, \dots, \beta_0) : k \geq 3, \sum_{i=1}^{k-2} \bar{f}(T^i \beta) \leq t < \sum_{i=1}^{k-1} \bar{f}(T^i \beta) \right\}.$$

Let  $x$  be an element of  $\{1, \dots, K\}^{\mathbb{N}}$ . We denote  $k_t(x)$  the unique integer  $k$  such that

$$\sum_{i=1}^{k-2} f(\sigma^i x) \leq t < \sum_{i=1}^{k-1} f(\sigma^i x).$$

In the probability language,  $k_t$  is a stopping time in the sense that the property  $\{k_t(x) \leq k\}$  depends only on the  $k+1$  first letters of  $x$ . For a finite word  $\alpha = (\alpha_0, \dots, \alpha_k)$ , we say that  $k = k_t(\alpha)$  if  $\alpha$  satisfies the previous inequality. With these notations,  $I^\eta(t) := \{\alpha : |\alpha| = k_t(\alpha) + 1\}$ . The same holds for  $K^\eta(t)$ .

*Remark.* This stopping time  $k_t(\alpha)$  for  $t \sim \frac{n_E(\hbar)}{2}$  will be the time for which we will later try to make the Egorov property work. Precisely, we will prove an Egorov property for some symbols corresponding to the sequence  $\alpha$  (see (64) for example).

**5.2. Partitions associated.**

**5.2.1. Partitions of identity.** Let  $\alpha = (\alpha_0, \dots, \alpha_k)$  be a finite sequence. Recall that we denoted  $\tau_\alpha := P_{\alpha_k}(k\eta) \cdots P_{\alpha_0}$ , where  $A(s) := U^{-s} A U^s$ . In [5] and [3], they used quantum partitions of identity by considering  $(\tau_\alpha)_{|\alpha|=k}$ . In our paper, we consider a slightly different partition that is more adapted to the variations of the unstable jacobian:

**Lemma 5.1.** *Let  $t$  be in  $[1, +\infty[$ . The family  $(\tau_\alpha)_{\alpha \in I^\eta(t)}$  is a partition of identity:*

$$\sum_{\alpha \in I^\eta(t)} \tau_\alpha^* \tau_\alpha = \text{Id}_{L^2(M)}.$$

*Proof.* We define for each  $1 \leq l \leq N$  (where  $N+1$  is the size of the longest word of  $I^\eta(t)$ ):

$$I_l^\eta(t) := \{\alpha = (\alpha_0, \dots, \alpha_l) : \exists \gamma = (\gamma_{l+1}, \dots, \gamma_k), N \geq k > l \text{ s.t. } \alpha.\gamma \in I^\eta(t)\}.$$

For  $l = N$ , this set is empty. We want to show that for each  $2 \leq l \leq N$ , we have:

$$(36) \quad \sum_{\alpha \in I^\eta(t), |\alpha|=l+1} \tau_\alpha^* \tau_\alpha + \sum_{\alpha \in I_l^\eta(t)} \tau_\alpha^* \tau_\alpha = \sum_{\alpha \in I_{l-1}^\eta(t)} \tau_\alpha^* \tau_\alpha.$$

To prove this equality we use the fact that  $\sum_{\gamma=1}^K P_\gamma(l)^* P_\gamma(l) = \text{Id}_{L^2(M)}$  to write:

$$\sum_{\alpha \in I_{l-1}^\eta(t)} \tau_\alpha^* \tau_\alpha = \sum_{\gamma=1}^K \sum_{\alpha \in I_{l-1}^\eta(t)} \tau_{\alpha.\gamma}^* \tau_{\alpha.\gamma}.$$

We split then this sum in two parts to find equality (36). To conclude the proof, we write:

$$\sum_{\alpha \in I^\eta(t)} \tau_\alpha^* \tau_\alpha = \sum_{k=2}^N \sum_{\alpha \in I^\eta(t), |\alpha|=k+1} \tau_\alpha^* \tau_\alpha$$

As  $t \geq 1 > b_0\eta \geq \max_\gamma f(\gamma)$ , the set  $I_1^\eta(t)$  is equal to  $\{1, \dots, K\}^2$ . By induction from  $N$  to 1 using equality (36) at each step, we find then:

$$\sum_{\alpha \in I^\eta(t)} \tau_\alpha^* \tau_\alpha = \text{Id}_{L^2(M)}.$$

□

*Remark.* A step of the induction can be easily understood by looking at figure 2 where each square represents an index over which the sum is made (as it was explained for figure 1).

Following the same procedure, let  $\beta$  be an element of  $K^\eta(t)$  and denote  $\pi_\beta = P_{\beta-k}(-k\eta) \cdots P_{\beta_0} P_{\beta-1}(-\eta)$ . These operators follows the relation:  $\sum_{\beta \in K} \pi_\beta^* \pi_\beta = \text{Id}_{L^2(M)}$ . As was mentioned in section 4.1.2, because of a technical reason that will appear in the application of the entropic uncertainty principle (see (38)), the two definitions are slightly different.

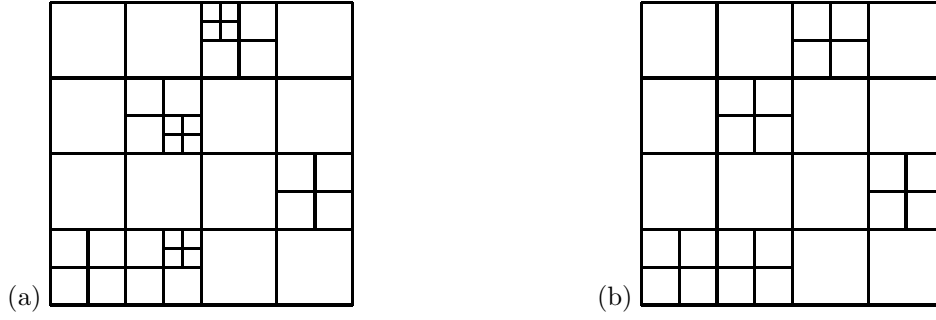


FIGURE 2. A step of the induction

5.2.2. *Partitions of  $\{1, \dots, K\}^\mathbb{N}$  associated.* In this section, we would like to consider some partitions of  $\Sigma := \{1, \dots, K\}^\mathbb{N}$  and of  $\bar{\Sigma}$  (see (26)) associated to the family  $I^\eta(1)$ . Recall that:

$$I^\eta(1) := \left\{ \alpha = (\alpha_0, \dots, \alpha_k) : k \geq 3, \sum_{i=1}^{k-2} f(\sigma^i \alpha) \leq 1 < \sum_{i=1}^{k-1} f(\sigma^i \alpha) \right\}.$$

For  $\alpha \in I^\eta(1)$ , it can be easily remarked that  $\sum_{j=0}^{k-1} f(\sigma^j \alpha) > 1$ . It means that there exists an unique integer  $k' \leq k$  such that:

$$\sum_{j=0}^{k'-2} f(\sigma^j \alpha) \leq 1 < \sum_{j=0}^{k'-1} f(\sigma^j \alpha).$$

In the following,  $k$  and  $k'$  will be often denoted  $k(\alpha) = k_1(\alpha)$  and  $k'(\alpha)$  to remember the dependence in  $\alpha$ . The following lemma can be easily shown:

**Lemma 5.2.** *Let  $\alpha \in I^\eta(1)$ . One has  $|k(\alpha) - k'(\alpha)| \leq \frac{b_0}{a_0} + 1$ .*

*Proof.* Suppose  $k' + 1 < k$  (otherwise it is trivial). Write:

$$\sum_{j=1}^{k-2} f(\sigma^j \alpha) - \sum_{j=0}^{k'-1} f(\sigma^j \alpha) \leq 1 - 1 \text{ implies } \sum_{j=k'}^{k-2} f(\sigma^j \alpha) \leq f(\alpha).$$

And finally, one finds  $(k - 2 - k' + 1)a_0\eta \leq b_0\eta$ . □

Let  $\alpha$  be an element of  $I^n(1)$ . We make a partition of the interval  $[0, f(\alpha)[$  under a form that will be useful (as it is adapted to the dynamics of the special flow). Motivated by the definition of a special flow, let us divide it as follows for  $k = k(\alpha)$  and  $k' = k'(\alpha)$ :

$$I_{k'-2}(\alpha) = [0, \sum_{j=0}^{k'-1} f(\sigma^j \alpha) - 1[, \dots, I_{p-2}(\alpha) = [\sum_{j=0}^{p-2} f(\sigma^j \alpha) - 1, \sum_{j=0}^{p-1} f(\sigma^j \alpha) - 1[, \dots$$

$$I_{k-2}(\alpha) = [\sum_{j=0}^{k-2} f(\sigma^j \alpha) - 1, f(\alpha) [,$$

where  $k'(\alpha) \leq p \leq k(\alpha)$ . If  $k(\alpha) = k'(\alpha)$ , one puts  $I_{k'-2}(\alpha) = I_{k-2}(\alpha) = [0, f(\alpha)[$ . A new

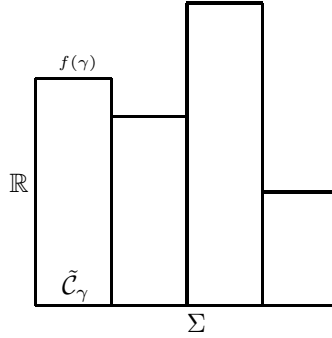


FIGURE 3. The suspension set  $\bar{\Sigma}$

partition  $\tilde{\mathcal{C}}$  of  $\Sigma$  can be defined. It is composed of the following atoms:

$$\tilde{\mathcal{C}}_\gamma := \mathcal{C}_{\gamma_0} \cap \dots \cap \sigma^{-k} \mathcal{C}_{\gamma_k},$$

where  $\gamma$  be an element of  $I^n(1)$ . A partition  $\bar{\mathcal{C}}$  of  $\bar{\Sigma}$  can be constructed starting from the partition  $\tilde{\mathcal{C}}$  and the partition of  $[0, f(\gamma)[$ . In fact, let  $\gamma$  be in  $I^n(1)$  and  $k'(\gamma) \leq p \leq k(\gamma)$ . An atom of the partition  $\bar{\mathcal{C}}$  is defined as  $\bar{\mathcal{C}}_{\gamma,p} = \tilde{\mathcal{C}}_\gamma \times I_{p-2}(\gamma)$ . The choice of these specific intervals allows to know the exact action of  $\bar{\sigma}$  on each atom of the partition: for  $(x, t) \in \bar{\mathcal{C}}_{\gamma,p}$ ,  $\bar{\sigma}(x, t) = (\sigma^{p-1}(x), 1 + t - \sum_{j=0}^{p-2} f(\sigma^j x))$ . Figure 3 represents the form of the partition we considered. We remark that the roof function used to define the suspension set is 'locally' constant (for words with the same starting letters).

5.2.3. *Partitions adapted to the special flow.* In this section, lemma 4.2 is shown and proves in particular that the previous partitions are well adapted to the special flow on  $\bar{\Sigma}$ . Lemma 4.2 can be written precisely:

**Proposition 5.3.** *Let  $n$  be a positive integer. Let  $(\gamma_i, p_i)_{0 \leq i \leq n-1}$  be a family of couples such that  $\gamma_i \in I^n(1)$  and  $k'(\gamma_i) \leq p_i \leq k(\gamma_i)$ . There exists  $\alpha'$  in  $I^n(n(1-\epsilon))$  such that:*

$$\bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(n-1)} \bar{\mathcal{C}}_{\gamma_{n-1}, p_{n-1}} \subset \tilde{\mathcal{C}}_{\alpha'} \times [0, f(\gamma_0)[.$$

Moreover, an atom of the refined partition  $\bigvee_{i=0}^{n-1} \bar{\sigma}^{-i} \bar{\mathcal{C}}$  can be written as follows:

$$(37) \quad \bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(n-1)} \bar{\mathcal{C}}_{\gamma_{n-1}, p_{n-1}} = \tilde{\mathcal{C}}_\alpha \times B(\gamma),$$

where  $\alpha = (\alpha_0, \dots, \alpha_k)$  is a  $k+1$ -uple and  $B(\gamma)$  a subinterval of  $[0, f(\gamma_0)[$  (possibly empty).

Finally,  $(\alpha_0, \dots, \alpha_k)$  satisfies  $\sum_{j=0}^{k-1} f(\sigma^j \alpha) \leq n(1+\epsilon)$ .

Before entering the proof, a simple reformulation of the first part of the proposition is that  $\bigvee_{i=0}^{n-1} \bar{\sigma}^{-i} \bar{\mathcal{C}}$  is a refinement of the partition  $\left( \tilde{\mathcal{C}}_\alpha \times [0, f(\alpha)[ \right)_{\alpha \in I^n(n(1-\epsilon))}$ .



*Proof.* We begin by proving the second part of the proposition. First suppose the considered atom is a non empty atom of  $\bigvee_{i=0}^{n-1} \bar{\sigma}^{-i} \bar{\mathcal{C}}$  (otherwise the result is trivial by taking  $B(\gamma)$  empty).

Let  $(x, t)$  be an element of  $\bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(n-1)} \bar{\mathcal{C}}_{\gamma_{n-1}, p_{n-1}}$ . We denote  $k_j = k(\gamma_j)$ . The sequence  $x$  is of the form  $(\gamma_0^0, \dots, \gamma_0^{k_0}, x')$  and  $t$  belongs to  $I_{p_0-2}(\gamma_0)$ . We recall that for  $(x, t) \in \bar{\mathcal{C}}_{\gamma_0, p_0}$ :

$$\bar{\sigma}(x, t) = \left( \sigma^{p_0-1}(x), 1 + t - \sum_{j=0}^{p_0-2} f(\sigma^j x) \right).$$

Necessarily, one has  $\gamma_1 = (\gamma_0^{p_0-1}, \dots, \gamma_0^{k_0}, \gamma_1^{k_0-p_0+2}, \dots, \gamma_1^{k_1})$ . Proceeding by induction, one finds that  $x = (\gamma_0^0, \dots, \gamma_0^{k_0}, \gamma_1^{k_0-p_0+2}, \dots, \gamma_{n-1}^{k_{n-1}}, x')$ . Define then  $\alpha = (\gamma_0^0, \dots, \gamma_0^{k_0}, \gamma_1^{k_0-p_0+2}, \dots, \gamma_{n-1}^{k_{n-1}})$  and:

$$B(\gamma) := \left\{ t \in [0, f(\gamma_0)[ : \exists x \text{ st } (x, t) \in \bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(n-1)} \bar{\mathcal{C}}_{\gamma_{n-1}, p_{n-1}} \right\}.$$

The first inclusion  $\bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(n-1)} \bar{\mathcal{C}}_{\gamma_{n-1}, p_{n-1}} \subset \tilde{\mathcal{C}}_\alpha \times B(\gamma)$  is clear.

Now we will prove the converse inclusion. Consider  $(x, t)$  an element of  $\bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(n-1)} \bar{\mathcal{C}}_{\gamma_{n-1}, p_{n-1}}$ .

The only thing to prove is that  $(X, t) = ((\gamma_0^0, \dots, \gamma_0^{k_0}, \gamma_1^{k_0-p_0+2}, \dots, \gamma_{n-1}^{k_{n-1}}, x'), t)$  is still an element of  $\bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(n-1)} \bar{\mathcal{C}}_{\gamma_{n-1}, p_{n-1}}$ , for every  $x'$  in  $\{1, \dots, K\}^{\mathbb{N}}$ . We proceed by induction and suppose  $(X, t)$  belongs to  $\bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(j-1)} \bar{\mathcal{C}}_{\gamma_{j-1}, p_{j-1}}$  for some  $j < n$ . We have to verify that  $\bar{\sigma}^j(X, t)$  belongs to  $\bar{\mathcal{C}}_{\gamma_j, p_j}$ . As  $(X, t)$  belongs to  $\bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(j-1)} \bar{\mathcal{C}}_{\gamma_{j-1}, p_{j-1}}$ , we have:

$$\bar{\sigma}^j(X, t) = \left( \sigma^{p_0+\dots+p_{j-1}-j}(X), j + t - \sum_{i=0}^{p_0+\dots+p_{j-1}-j-1} f(\sigma^i X) \right).$$

It has already been mentioned that for all  $i$ ,  $(\gamma_i^0, \dots, \gamma_i^{k_i-p_i+1}) = (\gamma_{i-1}^{p_{i-1}-1}, \dots, \gamma_{i-1}^{k_{i-1}})$  (as the considered atom is not empty). It follows that  $\sigma^{p_0+\dots+p_{j-1}-j}(X)$  belongs to  $\tilde{\mathcal{C}}_{\gamma_j}$ . We know that  $\bar{\sigma}^j(x, t)$  is an element of  $\bar{\mathcal{C}}_{\gamma_j, p_j}$  and as a consequence:

$$j + t - \sum_{i=0}^{p_0+\dots+p_{j-1}-j-1} f(\sigma^i X) = j + t - \sum_{i=0}^{p_0+\dots+p_{j-1}-j-1} f(\sigma^i x) \in I_{p_j-2}(\gamma_j).$$

By induction, equality (37) is true. For each  $0 \leq j \leq n-1$ ,  $t$  belongs to  $B(\gamma)$  implies that:

$$t \in I_{p_j-2}(\gamma_j) - j + \sum_{i=0}^{p_0+\dots+p_{j-1}-j-1} f(\sigma^i \alpha).$$

The set  $B(\gamma)$  is then defined as the intersection of  $n$  subintervals of  $[0, f(\gamma_0)[$  and is in fact a subinterval of  $[0, f(\gamma_0)[$ .

It remains now to prove upper and lower bounds on  $\sum_{j=0}^{k-1} f(\sigma^j \alpha)$ . Recall that:

$$\alpha = (\gamma_0^0, \dots, \gamma_0^{k_0}, \gamma_1^{k_0-p_0+2}, \dots, \gamma_1^{k_1}, \dots, \gamma_{n-1}^{k_{n-1}}).$$

As  $0 \leq f(\gamma) \leq \frac{\epsilon}{2}$  for all  $\gamma$  (finite or infinite subsequence: see inequality (16)), we have then:

$$\sum_{j=0}^{k-1} f(\sigma^j \alpha) \leq \sum_{l=0}^{n-2} \sum_{j=0}^{k_l-2} f(\sigma^j \gamma_l) + \sum_{j=0}^{k_{n-1}-1} f(\sigma^j \gamma_{n-1}) \leq n(1 + \epsilon).$$

For the lower bound, the same kind of procedure works with a little more care. For  $\gamma_0$ :

$$\sum_{j=1}^{k_0-1} f(\sigma^j \alpha) = \sum_{j=1}^{k_0-1} f(\sigma^j \gamma_0) > 1 > 1 - \epsilon.$$

and for  $1 \leq l \leq n-1$ , one has using lemma 5.2:

$$\sum_{j=k_{l-1}-p_{l-1}+1}^{k_l-1} f(\sigma^j \gamma_l) > 1 - (k_{l-1} - p_{l-1} + 1)b_0\eta > 1 - \left(2 + \frac{b_0}{a_0}\right)b_0\eta > 1 - \epsilon,$$

where the relations between  $\epsilon$ ,  $\eta$ ,  $a_0$  and  $b_0$  are defined in section 3. A lower bound on  $\sum_{j=1}^{k-1} f(\sigma^j \alpha)$  is  $n(1 - \epsilon)$ . Considering a word  $\alpha'$  starting like  $\alpha$  but of smaller size, the first part of the proposition is proved as:

$$I^n(n(1 - \epsilon)) := \left\{ (\alpha'_0, \dots, \alpha'_k) : k \geq 2, \sum_{j=1}^{k-2} f(\sigma^j \alpha') \leq n(1 - \epsilon) < \sum_{j=1}^{k-1} f(\sigma^j \alpha') \right\}.$$

□

*Remark.* As a final comment on this section, we underline again that all the proofs have been done in the case of  $\{1, \dots, K\}^{\mathbb{N}}$ . The reader can check that the same proofs work for  $\{1, \dots, K\}^{-\mathbb{N}}$ .

**5.3. Uncertainty principle for eigenfunctions of the Laplacian.** In the previous section 5.2, we have seen that the partitions of variable size are well adapted to the reparametrized flow (used in the Abramov theorem). In section 4, we used also the fact that we have a lower bound (30) on the entropy of  $\overline{\mu}_{\hbar}^{\Sigma}$  with respect to the suspension application  $\overline{\sigma}$ . The goal of this last section is to prove this lower bound (precisely proposition 5.7). To do this, we use the entropic uncertainty principle (theorem 2.1).

**5.3.1. Energy cutoff.** Before applying the uncertainty principle, we proceed to sharp energy cutoffs so as to get precise lower bounds on the quantum entropy (as it was done in [2], [5] and [3]). These cutoffs are made in our microlocal analysis in order to get as good exponential decrease as possible of the norm of the refined quantum partition. This cutoff in energy is possible because even if the distributions  $\mu_{\hbar}$  are defined on  $T^*M$ , they concentrate on the energy layer  $S^*M$ . The following energy localization is made in a way to compactify the phase space and in order to preserve the semiclassical measure.

Let  $\delta$  be a positive number less than 1 and  $\chi_{\delta}(t)$  in  $\mathcal{C}^{\infty}(\mathbb{R}, [0, 1])$ . Moreover,  $\chi_{\delta}(t) = 1$  for  $|t| \leq e^{-\delta/2}$  and  $\chi_{\delta}(t) = 0$  for  $|t| \geq 1$ . As in [5], the sharp  $\hbar$ -dependent cutoffs are then defined in the following way:

$$\forall \hbar \in (0, 1), \forall n \in \mathbb{N}, \forall \rho \in T^*M, \quad \chi^{(n)}(\rho, \hbar) := \chi_{\delta}(e^{-n\delta} \hbar^{-1+\delta} (H(\rho) - 1/2)).$$

For  $n$  fixed, the cutoff  $\chi^{(n)}$  is localized in an energy interval of length  $2e^{n\delta} \hbar^{1-\delta}$  centered around the energy layer  $\mathcal{E}$ . In this paper, indices  $n$  will satisfy  $2e^{n\delta} \hbar^{1-\delta} \ll 1$ . It implies that the widest cutoff is supported in an energy interval of microscopic length and that  $n \leq K_{\delta} |\log \hbar|$ , where  $K_{\delta} \leq \delta^{-1}$ . Using then a non standard pseudodifferential calculus (see [5] for a brief reminder of the procedure from [23]), one can quantize these cutoffs into pseudodifferential operators. We will denote  $\text{Op}(\chi^{(n)})$  the quantization of  $\chi^{(n)}$ . The main properties of this quantization are recalled in section 8.2. In particular, the quantization of these cutoffs preserve the eigenfunctions of the Laplacian:

$$\|\psi_{\hbar} - \text{Op}(\chi^{(n)})\psi_{\hbar}\| = O(\hbar^{\infty})\|\psi_{\hbar}\|.$$

**5.3.2. Applying the entropic uncertainty principle.** To get bound on the entropy of the suspension measure, the entropic uncertainty principle should not be applied to the family of operators  $(\tau_{\alpha})_{\alpha \in I^n(\hbar)}$  directly but it will be applied several times to get terms of the form  $c_{\alpha} \|\tau_{\alpha}^* \psi_{\hbar}\|^2$  (see (30)) instead of one of the form  $\|\tau_{\alpha}^* \psi_{\hbar}\|^2$  in the formula of the quantum entropy (11). We remind that the  $c_{\alpha}$  correspond to the Lebesgues part of the suspension measure  $\overline{\mu}_{\hbar}^{\Sigma}$ . They are defined by

$$c_{\alpha} = \frac{f(\alpha)}{\sum_{\gamma \in \{1, \dots, K\}^2} f(\gamma) \overline{\mu}_{\hbar}^{\Sigma}([\gamma])}.$$

The previous goal can be achieved after defining new families of quantum partitions. Let  $\gamma = (\gamma_0, \gamma_1)$  be an element of  $\{1, \dots, K\}^2$ . Denote then:

$$I_{\hbar}(\gamma) := \{(\alpha') : \gamma.\alpha' \in I^\eta(\hbar)\},$$

$$K_{\hbar}(\gamma) := \{(\beta') : \beta'.\gamma \in K^\eta(\hbar)\}.$$

The following partitions of identity can be associated to them, for  $\alpha' \in I_{\hbar}(\gamma)$  and  $\beta' \in K_{\hbar}(\gamma)$ :

$$\tilde{\tau}_{\alpha'} = P_{\alpha'_n}(n\eta) \cdots P_{\alpha'_2}(2\eta),$$

$$\tilde{\pi}_{\beta'} = P_{\beta'_{-n}}(-n\eta) \cdots P_{\beta'_{-2}}(-2\eta).$$

For analogous reasons as the case of  $I^\eta(\hbar)$ , the families  $(\tilde{\tau}_{\alpha'})_{\alpha' \in I_{\hbar}(\gamma)}$  and  $(\tilde{\pi}_{\beta'})_{\beta' \in K_{\hbar}(\gamma)}$  form quantum partitions of identity for every  $\gamma = (\gamma_0, \gamma_1) \in \{1, \dots, K\}^2$ .

Given these new quantum partitions of identity, the entropic principle should be applied for given initial conditions in times 0 and 1. Let  $\|\psi_{\hbar}\| = 1$  be a fixed element of the sequence of eigenfunctions of the Laplacian defined earlier, associated to the eigenvalue  $\frac{1}{\hbar^2}$ .

Let  $\gamma = (\gamma_0, \gamma_1)$  be an element of  $\{1, \dots, K\}^2$ . Define:

$$P_{\gamma} := P_{\gamma_1} P_{\gamma_0}(-\eta).$$

We underline that for  $\alpha' \in I_{\hbar}(\gamma)$  and  $\beta' \in K_{\hbar}(\gamma)$ :

$$(38) \quad \tilde{\tau}_{\alpha'} U^{-\eta} P_{\gamma} = \tau_{\gamma.\alpha'} U^{-\eta} \text{ and } \tilde{\pi}_{\beta'} P_{\gamma} = \pi_{\beta'.\gamma},$$

where  $\gamma.\alpha' \in I^\eta(\hbar)$  and  $\beta'.\gamma \in K^\eta(\hbar)$  by definition. In equality (38) appears the fact that the definitions of  $\tau$  and  $\pi$  are slightly different (see (21) and (22)). It is due to the fact that we want to compose  $\tilde{\tau}$  and  $\tilde{\pi}$  with the same operator  $P_{\gamma}$ . Suppose that  $\|P_{\gamma}\psi_{\hbar}\|$  is not equal to 0 (otherwise the obtained result is trivial). We apply the quantum uncertainty principle (2.1) to the partitions of identity:  $(\tilde{\tau}_{\alpha'})_{\alpha' \in I_{\hbar}(\gamma)}$  and  $(\tilde{\pi}_{\beta'})_{\beta' \in K_{\hbar}(\gamma)}$ . These two partitions are of cardinality  $\mathcal{N} \simeq \hbar^{-K_0}$  where  $K_0$  is some fixed positive number (depending on the cardinality of the partition  $K$ , on  $a_0$ , on  $b_0$  and  $\eta$ ). We choose  $\text{Op}(\chi^{(k')})$  for the family of bounded operators  $O_{\beta'}$  (where  $k'$  is the length of  $\beta'$ ) and  $\delta' = \|P_{\gamma}\psi_{\hbar}\|^{-1} \hbar^L$  (see corollary 8.2) such that  $\hbar^{L-K_0} \ll \hbar^{1/2(1-\epsilon')(1-\epsilon)}$ . The isometry chosen is  $U^{-\eta}$  and the normalized vector is  $\tilde{\psi}_{\hbar} := \frac{P_{\gamma}\psi_{\hbar}}{\|P_{\gamma}\psi_{\hbar}\|}$ . Applying the entropic uncertainty principle (2.1), one gets:

**Corollary 5.4.**

$$h_{\tilde{\tau}}(U^{-\eta}\tilde{\psi}_{\hbar}) + h_{\tilde{\pi}}(\tilde{\psi}_{\hbar}) \geq -2 \log (c_{\chi}^{\gamma}(U^{-\eta}) + \hbar^{L-K_0} \|P_{\gamma}\psi_{\hbar}\|^{-1}),$$

$$\text{where } c_{\chi}^{\gamma}(U^{-\eta}) = \max_{\alpha' \in I_{\hbar}(\gamma), \beta' \in K_{\hbar}(\gamma)} \left( \|\tilde{\tau}_{\alpha'} U^{-\eta} \tilde{\pi}_{\beta'}^* \text{Op}(\chi^{(k')})\| \right).$$

First, remark that the quantity  $c_{\chi}^{\gamma}(U^{-\eta})$  can be easily replaced by

$$(39) \quad c_{\chi}(U^{-\eta}) := \max_{\gamma \in \{1, \dots, K\}^2} \max_{\alpha \in I_{\hbar}(\gamma), \beta \in K_{\hbar}(\gamma)} \left( \|\tilde{\tau}_{\alpha} U^{-\eta} \tilde{\pi}_{\beta}^* \text{Op}(\chi^{(k')})\| \right),$$

which is independent of  $\gamma$ . Then, it can be noted that the quantity  $h_{\tilde{\tau}}(U^{-\eta}\tilde{\psi}_{\hbar})$  can be written:

$$h_{\tilde{\tau}}(U^{-\eta}\tilde{\psi}_{\hbar}) = - \sum_{\alpha' \in I_{\hbar}(\gamma)} \|\tilde{\tau}_{\alpha'} U^{-\eta} \tilde{\psi}_{\hbar}\|^2 \log \|\tilde{\tau}_{\alpha'} U^{-\eta} P_{\gamma} \psi_{\hbar}\|^2 + \sum_{\alpha' \in I_{\hbar}(\gamma)} \|\tilde{\tau}_{\alpha'} U^{-\eta} \tilde{\psi}_{\hbar}\|^2 \log \|P_{\gamma} \psi_{\hbar}\|^2.$$

Using the fact that  $\psi_{\hbar}$  is an eigenvector of  $U^\eta$  and that  $(\tilde{\tau}_{\alpha'})_{\alpha' \in I_{\hbar}(\gamma)}$  is a partition of identity, one has:

$$h_{\tilde{\tau}}(U^{-\eta}\tilde{\psi}_{\hbar}) = - \frac{1}{\|P_{\gamma}\psi_{\hbar}\|^2} \sum_{\alpha' \in I_{\hbar}(\gamma)} \|\tau_{\gamma.\alpha'} \psi_{\hbar}\|^2 \log \|\tau_{\gamma.\alpha'} \psi_{\hbar}\|^2 + \log \|P_{\gamma}\psi_{\hbar}\|^2.$$

The same holds for  $h_{\tilde{\pi}}(\tilde{\psi}_{\hbar})$  (using here equality (38)):

$$h_{\tilde{\pi}}(\tilde{\psi}_{\hbar}) = - \frac{1}{\|P_{\gamma}\psi_{\hbar}\|^2} \sum_{\beta' \in K_{\hbar}(\gamma)} \|\pi_{\beta'.\gamma} \psi_{\hbar}\|^2 \log \|\pi_{\beta'.\gamma} \psi_{\hbar}\|^2 + \log \|P_{\gamma}\psi_{\hbar}\|^2.$$

Then, the quantity

$$(40) \quad - \sum_{\alpha' \in I_{\hbar}(\gamma)} \|\tau_{\gamma, \alpha'} \psi_{\hbar}\|^2 \log \|\tau_{\gamma, \alpha'} \psi_{\hbar}\|^2 - \sum_{\beta' \in K_{\hbar}(\gamma)} \|\pi_{\beta', \gamma} \psi_{\hbar}\|^2 \log \|\pi_{\beta', \gamma} \psi_{\hbar}\|^2$$

is bounded from below by:

$$(41) \quad -2\|P_{\gamma} \psi_{\hbar}\|^2 \log (c_{\chi}(U^{-\eta})\|P_{\gamma} \psi_{\hbar}\| + \hbar^{L-K_0}) \geq -2\|P_{\gamma} \psi_{\hbar}\|^2 \log (c_{\chi}(U^{-\eta}) + \hbar^{L-K_0})$$

as  $\|P_{\gamma} \psi_{\hbar}\| \leq 1$ . As was already mentioned, this lower bound is trivial in the case where  $\|P_{\gamma} \psi_{\hbar}\|$  is equal to 0. Using the fact that:

$$(42) \quad c_{\gamma} = \frac{f(\gamma)}{\sum_{\gamma' \in \{1, \dots, K\}^2} f(\gamma') \|P_{\gamma'} \psi_{\hbar}\|^2},$$

one easily checks that  $\sum_{\gamma \in \{1, \dots, K\}^2} c_{\gamma} \|P_{\gamma} \psi_{\hbar}\|^2 = 1$ . If we multiply (40) and (41) by  $c_{\gamma}$  and make the sum over all  $\gamma$  in  $\{1, \dots, K\}^2$ , we find:

$$- \sum_{\alpha \in I^{\eta}(\hbar)} c_{\alpha} \|\tau_{\alpha} \psi_{\hbar}\|^2 \log \|\tau_{\alpha} \psi_{\hbar}\|^2 - \sum_{\beta \in K^{\eta}(\hbar)} c_{\beta} \|\pi_{\beta} \psi_{\hbar}\|^2 \log \|\pi_{\beta} \psi_{\hbar}\|^2 \geq -2 \log (c_{\chi}(U^{-\eta}) + \hbar^{L-K_0}).$$

Finally, we have the following property:

**Corollary 5.5.** *Define:*

$$(43) \quad h_{\tau}^c(\psi_{\hbar}) := - \sum_{\alpha \in I^{\eta}(\hbar)} c_{\alpha} \|\tau_{\alpha} \psi_{\hbar}\|^2 \log (c_{\alpha} \|\tau_{\alpha} \psi_{\hbar}\|^2), \quad h_{\pi}^c(\psi_{\hbar}) := - \sum_{\beta \in K^{\eta}(\hbar)} c_{\beta} \|\pi_{\beta} \psi_{\hbar}\|^2 \log (c_{\beta} \|\pi_{\beta} \psi_{\hbar}\|^2)$$

where  $c_{\alpha} = c_{\alpha_0, \alpha_1}$  and  $c_{\beta} = c_{\beta_{-1}, \beta_0}$ . One has:

$$(44) \quad h_{\tau}^c(\psi_{\hbar}) + h_{\pi}^c(\psi_{\hbar}) \geq -2 \log (c_{\chi}(U^{-\eta}) + \hbar^{L-K_0}) - \log \left( \max_{\gamma} c_{\gamma} \right).$$

To prove this corollary, it has been used that  $\sum_{\alpha \in I^{\eta}(\hbar)} c_{\alpha} \|\tau_{\alpha} \psi_{\hbar}\|^2 = 1$  and  $\sum_{\beta \in K^{\eta}(\hbar)} c_{\beta} \|\pi_{\beta} \psi_{\hbar}\|^2 = 1$ .

Precisely it was used to take into account the introduction of the weight  $c_{\alpha}$  in the expression (43) of  $h^c$ . This new quantity is slightly different from the one used in the definition of the quantum entropy (see (11)). In fact, we have introduced some weights  $c_{\gamma}$  in each term of the sum. However, if we think of the definition of  $\overline{\mu}_{\hbar}^{\Sigma}$  (which is of the form  $C\mu_{\hbar}^{\Sigma} \times \text{Leb}$ : see (27)), this quantity is more adapted to compute the classical entropy of  $\overline{\mu}_{\hbar}^{\Sigma}$  than the usual quantum entropy.

**5.3.3. Exponential decrease of the atoms of the quantum partition.** Now that we have lower bound (44), we give an estimate on the exponential decrease of the atoms of the quantum partition. As in [2], [5], [3], one has<sup>3</sup>:

**Theorem 5.6.** [2] [5] [3] *For every  $\mathcal{K} > 0$  ( $\mathcal{K} \leq C_{\delta}$ ), there exists  $\hbar_{\mathcal{K}}$  and  $C_{\mathcal{K}}$  such that uniformly for all  $\hbar \leq \hbar_{\mathcal{K}}$ , for all  $k + k' \leq \mathcal{K} \lfloor \log \hbar \rfloor$ ,*

$$(45) \quad \begin{aligned} & \|P_{\alpha_k} U^{\eta} P_{\alpha_{k-1}} \dots U^{\eta} P_{\alpha_0} U^{3\eta} P_{\alpha'_k} U^{\eta} \dots P_{\alpha'_0} Op(\chi^{(k')})\|_{L^2(M)} \\ & \leq C_{\mathcal{K}} \hbar^{-\frac{1}{2}} \exp \left( -\frac{1}{2} \left( \sum_{j=0}^{k-1} f(\sigma^j \alpha) + \sum_{j=0}^{k'-1} f(\sigma^j \alpha') \right) \right). \end{aligned}$$

<sup>3</sup>In the higher dimension case mentioned in the footnote of section 1.2, we should replace  $\hbar^{-\frac{1}{2}}$  (where  $d$  is the dimension of  $M$ ) by  $\hbar^{-\frac{d-1}{2}}$  in inequality (45).

Outline that the crucial role of the sharp energy cutoff appears in particular to prove this theorem. In fact, without the cutoff, the previous norm operator could have only be bounded by 1 and the entropic uncertainty principle would have been empty. The previous inequality (45) allows to give an estimate on the quantity (39) (as it allows us to bound  $c_\chi(U^{-\eta})$ ). In fact, one has, for each  $\gamma \in \{1, \dots, K\}^2$ :

$$\|\tilde{\tau}_\alpha U^{-\eta} \tilde{\pi}_\beta^* \text{Op}(\chi^{(k')})\| = \|P_{\alpha_k} U^\eta P_{\alpha_{k-1}} \dots U^\eta P_{\alpha_2} U^{3\eta} P_{\beta_{-2}} U^\eta \dots P_{\beta_{-k'}} \text{Op}(\chi^{(k')})\|,$$

where  $(\alpha_2, \dots, \alpha_k) \in I_\hbar(\gamma)$  and  $(\beta_{-k'}, \dots, \beta_{-2}) \in K_\hbar(\gamma)$ . Using the definition of the sets  $I^\eta(\hbar)$  (24) and  $K^\eta(\hbar)$  (25), one has  $k + k' \leq \frac{2}{a_0\eta} |\log \hbar|$ . Using theorem (5.6) with  $\mathcal{K} = \frac{2}{a_0\eta}$ , one has:

$$\|\tilde{\tau}_\alpha U^{-\eta} \tilde{\pi}_\beta^* \text{Op}(\chi^{(k')})\| \leq C_{\mathcal{K}} \hbar^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \left( \sum_{j=2}^{k-1} f(\sigma^j \alpha) + \sum_{j=2}^{k'-1} \bar{f}(T^j \beta) \right)\right),$$

where  $C_{\mathcal{K}}$  does not depend on  $\hbar$ . Using again the definition of the sets  $I^\eta(\hbar)$  (24) and  $K^\eta(\hbar)$  (25), one has:

$$c_\chi(U^{-\eta}) = \max_{\gamma \in \{1, \dots, K\}^2} \max_{\alpha \in I_\hbar(\gamma), \beta \in K_\hbar(\gamma)} \left( \|\tilde{\tau}_\alpha U^{-\eta} \tilde{\pi}_\beta^* \text{Op}(\chi^{(k')})\| \right) \leq \tilde{C}_{\mathcal{K}} \hbar^{\frac{1}{2}(1-\epsilon')(1-\epsilon)},$$

where  $\tilde{C}_{\mathcal{K}}$  does not depend on  $\hbar$ . The main inequality (44) for the quantum entropy can be rewritten and the discussion of this section can be summarized finally by the following proposition:

**Proposition 5.7.** *Define:*

$$h_\tau^c(\psi_\hbar) := - \sum_{\alpha \in I^\eta(\hbar)} c_\alpha \|\tau_\alpha \psi_\hbar\|^2 \log(c_\alpha \|\tau_\alpha \psi_\hbar\|^2), \quad h_\pi^c(\psi_\hbar) := - \sum_{\beta \in K^\eta(\hbar)} c_\beta \|\pi_\beta \psi_\hbar\|^2 \log(c_\beta \|\pi_\beta \psi_\hbar\|^2)$$

where  $c_\alpha = c_{\alpha_0, \alpha_1}$ ,  $c_\beta = c_{\beta_{-1}, \beta_0}$  and  $c_\gamma = \frac{f(\gamma)}{\sum_{\gamma \in \{1, \dots, K\}^2} f(\gamma) \|P_\gamma \psi_\hbar\|^2}$ . One has:

$$(46) \quad h_\tau^c(\psi_\hbar) + h_\pi^c(\psi_\hbar) \geq (1 - \epsilon')(1 - \epsilon) |\log \hbar| + C,$$

where  $C \in \mathbb{R}$  does not depend on  $\hbar$ .

This lower bound is the one we used in section 4 to get lower bound on the classical entropy. In fact, this last inequality gives a lower bound on  $h_\tau^c$  and  $h_\pi^c$  and we already explained in section 4.2 that these quantities are in fact the classical entropy of the measure  $\bar{\mu}_\hbar^\Sigma$  for a specific partition (see (29) and (30)).

## 6. SUBADDITIVITY OF THE QUANTUM ENTROPY

As was mentioned in section 4 and proved in section 5, the uncertainty principle gives (if we forget the backward side  $\Sigma_-$ ):

$$\frac{1}{n_E(\hbar)} H_{n_E(\hbar)} \left( \bar{\mu}_\hbar^\Sigma, \bar{\sigma}, \bar{\mathcal{C}} \right) \geq \frac{1}{2} (1 - \epsilon).$$

To prove our main theorem 1.2, we need to show that this lower bound holds for a fixed  $n_0$  on the quantity  $\frac{1}{n_0} H_{n_0} \left( \bar{\mu}_\hbar^\Sigma, \bar{\sigma}, \bar{\mathcal{C}} \right)$  (as we need to let  $\hbar$  tend to 0 independently of  $n$  to recover the semiclassical measure  $\bar{\mu}^\Sigma$ : see section 4.3). To do this we want to reproduce the classical argument for the existence of the metric entropy (see (8)), i.e. we need to prove a subadditivity property for logarithmic time:

**Theorem 6.1.** *Let  $\bar{\mathcal{C}}$  be the partition of proposition 5.3. There exists a function  $R(n_0, \hbar)$  on  $\mathbb{N} \times (0, 1]$  such that*

$$\forall n_0 \in \mathbb{N}, \quad \lim_{\hbar \rightarrow 0} |R(n_0, \hbar)| = 0.$$

Moreover, for any  $\hbar \in (0, 1]$ , any  $n_0, m \in \mathbb{N}$  such that  $n_0 + m \leq T_E(\hbar)$ , one has:

$$H_{n_0+m} \left( \bar{\mu}_\hbar^\Sigma, \bar{\sigma}, \bar{\mathcal{C}} \right) \leq H_{n_0} \left( \bar{\mu}_\hbar^\Sigma, \bar{\sigma}, \bar{\mathcal{C}} \right) + H_m \left( \bar{\mu}_\hbar^\Sigma, \bar{\sigma}, \bar{\mathcal{C}} \right) + R(n_0, \hbar).$$

A key point to prove the subadditivity property in the case of the metric entropy is that the measure is invariant under the dynamics (see (8)). In our case, invariance of the semiclassical measure under the geodesic flow is a consequence of the Egorov property (1): to prove that subadditivity almost holds (in the sense of the previous theorem), we will have to prove an Egorov property for logarithmic times. We will see that with our choice of 'local' Ehrenfest time, this will be possible and the previous theorem then holds. The proof of this theorem is the subject of this section (and it also uses results from section 7).

Let  $n_0$  and  $m$  be two positive integers such that  $m + n_0 \leq T_E(\hbar)$ . One has:

$$H\left(\bigvee_{i=0}^{n+n_0-1} \bar{\sigma}^{-i} \bar{\mathcal{C}}, \bar{\mu}_\hbar^\Sigma\right) = H\left(\bigvee_{i=0}^{n-1} \bar{\sigma}^{-i} \bar{\mathcal{C}} \vee \bigvee_{i=n}^{n_0+n-1} \bar{\sigma}^{-i} \bar{\mathcal{C}}, \bar{\mu}_\hbar^\Sigma\right).$$

Using classical properties of the metric entropy, one has (see section 2.1):

$$H_{n+n_0}\left(\bar{\sigma}, \bar{\mu}_\hbar^\Sigma, \bar{\mathcal{C}}\right) \leq H_n\left(\bar{\sigma}, \bar{\mu}_\hbar^\Sigma, \bar{\mathcal{C}}\right) + H_{n_0}\left(\bar{\sigma}, \bar{\sigma}^{\#} \bar{\mu}_\hbar^\Sigma, \bar{\mathcal{C}}\right).$$

Using proposition 6.2 and the continuity of the function  $x \log x$  on  $[0, 1]$ , there exists a function  $R(n_0, \hbar)$  with the properties of theorem 6.1 such that  $H_{n_0}\left(\bar{\sigma}, \bar{\sigma}^{\#} \bar{\mu}_\hbar^\Sigma, \bar{\mathcal{C}}\right) = H_{n_0}\left(\bar{\sigma}, \bar{\mu}_\hbar^\Sigma, \bar{\mathcal{C}}\right) + R(n_0, \hbar)$  and thus:

$$(47) \quad H_{n+n_0}\left(\bar{\sigma}, \bar{\mu}_\hbar^\Sigma, \bar{\mathcal{C}}\right) \leq H_n\left(\bar{\sigma}, \bar{\mu}_\hbar^\Sigma, \bar{\mathcal{C}}\right) + H_{n_0}\left(\bar{\sigma}, \bar{\mu}_\hbar^\Sigma, \bar{\mathcal{C}}\right) + R(n_0, \hbar). \square$$

So the crucial point to prove this theorem is to show that the measure of the atoms of the refined partition is almost invariant under  $\bar{\sigma}$  (proposition 6.2). In the following of this section,  $A$  is defined as:

$$A = \bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(n_0-1)} \bar{\mathcal{C}}_{\gamma_{n_0-1}, p_{n_0-1}}.$$

*Remark.* In this section, only the case of  $\{1, \dots, K\}^{\mathbb{N}}$  is treated. The proof of the backward case  $\{1, \dots, K\}^{-\mathbb{N}}$  works in the same way.

**6.1. Pseudo-invariance of the measure of the atoms of the partitions.** From this point, our main goal is to show the pseudo invariance of the atoms of the refined partition. More precisely:

**Proposition 6.2.** *Let  $m, n_0$  be two positive integers such that  $m + n_0 \leq T_E(\hbar)$ . Consider an atom of the refined partition  $A = \bar{\mathcal{C}}_{\gamma_0, p_0} \cap \dots \cap \bar{\sigma}^{-(n_0-1)} \bar{\mathcal{C}}_{\gamma_{n_0-1}, p_{n_0-1}}$ . One has:*

$$\bar{\mu}_\hbar^\Sigma(\bar{\sigma}^{-m} A) = \bar{\mu}_\hbar^\Sigma(A) + O(\hbar^{(1-2\nu)/6}),$$

with a uniform constant in  $n_0$  and  $m$  in the allowed interval. The constant  $\nu < 1/2$  is the one defined by theorem 7.1.

This result says that the measure  $\bar{\mu}_\hbar^\Sigma$  is almost  $\bar{\sigma}$  invariant for logarithmic times. As a consequence, the classical argument (see (8)) for subadditivity of the entropy can be applied as long as we consider times where the pseudo invariance holds (see (47)).

Let  $A$  be as in the proposition. From proposition 5.3, there exists  $(\alpha_0, \dots, \alpha_k)$  and  $B(\gamma)$  such that:

$$A = (\mathcal{C}_{\alpha_0} \cap \dots \cap \sigma^{-k} \mathcal{C}_{\alpha_k}) \times B(\gamma).$$

Still from proposition 5.3, one knows that  $B(\gamma)$  is a subinterval of  $[0, f(\gamma_0)[$ . Moreover, the following property on  $\alpha$  holds:

$$(48) \quad n_0(1 - \epsilon) \leq \sum_{j=0}^{k-1} f(\sigma^j \alpha) \leq n_0(1 + \epsilon).$$

The plan of the proof of proposition 6.2 is the following. First, we will give an exact expression in terms of  $\alpha$  and  $B(\gamma)$  of  $\bar{\mu}_\hbar^\Sigma(\bar{\sigma}^{-m} A)$ . Then, we will see how to prove the proposition making the simplifying assumption that all operators  $(P_i(k\eta))_{i,k}$  commute. Finally, we will estimate the error term due to the fact that operators do not exactly commute.

6.1.1. *Computation of  $\overline{\mu}_h^\Sigma(\overline{\sigma}^{-m}A)$ .* As a first step of the proof, we want to give a precise formula for the measure of  $\overline{\sigma}^{-m}A$ . To do this, we have to determine the shape of the set  $\overline{\sigma}^{-m}A$ . Let us then define:

$$\overline{\Sigma}_p^m := \left\{ (x, t) \in \overline{\Sigma} : \sum_{j=0}^{p-2} f(\sigma^j x) \leq m + t < \sum_{j=0}^{p-1} f(\sigma^j x) \right\}.$$

One has  $\overline{\Sigma} = \bigsqcup_{p \geq 1} \overline{\Sigma}_p^m$  and as a consequence:

$$\begin{aligned} \overline{\sigma}^{-m}A &= \bigsqcup_{p \geq 1} \left( \overline{\Sigma}_p^m \cap \overline{\sigma}^{-m}A \right) \\ &= \bigsqcup_{p \geq 1} \left\{ (x, t) \in \overline{\Sigma}_p^m : m + t - \sum_{j=0}^{p-2} f(\sigma^j x) \in B(\gamma), (x_{p-1}, \dots, x_{p+k-1}) = \alpha \right\}. \end{aligned}$$

Note that  $t \in B(\gamma) - m + \sum_{j=0}^{p-2} f(\sigma^j x)$  together with  $(x_{p-1}, \dots, x_{p+k-1}) = \alpha$  imply that  $\sum_{j=0}^{p-2} f(\sigma^j x) \leq m + t < \sum_{j=0}^{p-1} f(\sigma^j x)$ . It allows to rewrite:

$$\overline{\sigma}^{-m}A = \bigsqcup_{p \geq 1} \left\{ (x, t) \in \Sigma \times \mathbb{R}_+ : 0 \leq t < f(x), t \in B(\gamma) - m + \sum_{j=0}^{p-2} f(\sigma^j x), (x_{p-1}, \dots, x_{p+k-1}) = \alpha \right\}.$$

Finally, one can write the measure of this suspension set:

$$\overline{\mu}_h^\Sigma(\overline{\sigma}^{-m}A) = \sum_{p \geq 1} \sum_{\substack{|\beta| = p+k \\ (\beta_{p-1}, \dots, \beta_{p+k-1}) = \alpha}} c_{\beta, \alpha}(m) \|P_{\beta_{k+p-1}}((k+p-1)\eta) P_{\beta_{k+p-2}}((k+p-2)\eta) \cdots P_{\beta_0} \psi_h\|^2,$$

where

$$c_{\beta, \alpha}(m) = \lambda \text{Leb} \left( B(\gamma) \cap \left[ m - \sum_{j=0}^{p-2} f(\sigma^j \beta), m - \sum_{j=1}^{p-2} f(\sigma^j \beta) \right] \right)$$

with  $\lambda^{-1} := \sum_{\gamma' \in \{1, \dots, K\}^2} f(\gamma') \mu_h^\Sigma([\gamma'])$  the normalization constant of the measure. Outline that

the previous sum is finite with at most  $2b_0/a_0$  non zeros terms in each string  $\beta$  (as  $c_{\bullet, \alpha}(m)$  is zero except a finite number of times). For simplicity of the following of the proof, we reindex the previous expressions:

$$(49) \quad \overline{\mu}_h^\Sigma(\overline{\sigma}^{-m}A) = \sum_{p \geq 1} \sum_{\substack{|\beta| = p+k \\ (\beta_0, \dots, \beta_k) = \alpha}} c_{\beta, \alpha}(m) \|P_{\beta_k}((k+p-1)\eta) P_{\beta_{k-1}}((k+p-2)\eta) \cdots P_{\beta_{-p+1}} \psi_h\|^2,$$

where  $c_{\beta, \alpha}(m) = \lambda \text{Leb} \left( B(\gamma) \cap \left[ m - \sum_{j=0}^{p-2} f(\sigma^j \beta), m - \sum_{j=1}^{p-2} f(\sigma^j \beta) \right] \right)$  with  $\lambda$  defined as previously. Then, to prove proposition 6.2, we have to show that the previous quantity (49) is equal to:

$$\lambda \text{Leb}(B(\gamma)) \|P_{\alpha_k}(k\eta) \cdots P_{\alpha_0} \psi_h\|_{L^2}^2 + O_{L^2}(\hbar^{(1-2\nu)/6}).$$

6.1.2. *If everything commutes...* We will now use our explicit expression for  $\overline{\mu}_h^\Sigma(\overline{\sigma}^{-m}A)$  (see (49)) and verify it is equal to  $\overline{\mu}_h^\Sigma(A)$  under the simplifying assumption that all the involved pseudo-differential operators commute. In the next section, we will then give an estimate of the error term due to the fact that the operators do not exactly commute. In order to prove the pseudo invariance, denote:

$$K_m(\alpha) := \{ \beta = (\beta_{-p+1}, \dots, \beta_k) : (\beta_0, \dots, \beta_k) = \alpha, c_{\beta, \alpha}(m) \neq 0 \}$$

and

$$K_m^{(q)}(\alpha) := \{ (\beta_{-q+1}, \dots, \beta_k) : \exists \gamma = (\gamma_{-p+1}, \dots, \gamma_{-q}) \text{ s.t. } q < p, \gamma \cdot \beta \in K_m(\alpha) \}.$$

With these notations, we can write (49) as follows:

$$(50) \quad \overline{\mu}_{\hbar}^{\Sigma}(\overline{\sigma}^{-m}A) = \sum_{\beta \in K_m(\alpha)} c_{\beta,\alpha}(m) \|\tau_{\beta}\psi_{\hbar}\|^2 = \sum_{p=3}^N \sum_{\beta \in K_m(\alpha): |\beta|=k+p} c_{\beta,\alpha}(m) \|\tau_{\beta}\psi_{\hbar}\|^2.$$

Recall that by definition (see (21))  $\tau_{\beta} := P_{\beta_k}((k+p-1)\eta)P_{\beta_{k-1}}((k+p-2)\eta) \cdots P_{\beta_{-p+1}}$ . For simplicity of notations, let us denote  $B(\gamma) = [a, b[$  (where  $a$  and  $b$  obviously depend on  $\gamma$ ). A last notation we define is for  $\beta$  such that  $|\beta| = k+q$  and  $\sigma^{q-1}\beta = \alpha$ :

$$(51) \quad \overline{c}_{\beta,\alpha}(m) := \lambda \text{Leb} \left( \left[ a, b[\cap \left[ a, m - \sum_{j=1}^{q-2} f(\sigma^j\beta)[ \right) \right), \right.$$

where  $\lambda$  is the normalization constant of the measure previously defined. Suppose now that all the operators  $(P_i(k\eta))_{i,k}$  commute. We have the following lemma:

**Lemma 6.3.** *If all the operators  $(P_i(k\eta))_{i,k}$  commute, then one has, for  $2 \leq q \leq N$ :*

$$\sum_{\beta \in K_m(\alpha): |\beta|=k+q} c_{\beta,\alpha}(m) \|\tau_{\beta}\psi_{\hbar}\|^2 + \sum_{\beta \in K_m^{(q)}(\alpha)} \overline{c}_{\beta,\alpha}(m) \|\tau_{\beta}\psi_{\hbar}\|^2 = \sum_{\beta \in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m) \|\tau_{\beta}\psi_{\hbar}\|^2.$$

*Proof.* Let  $2 \leq q \leq N$ . Consider  $\beta$  an element of  $K_m^{(q-1)}(\alpha)$ . Using the property of partition of identity, we have:

$$\sum_{\beta \in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m) \|\tau_{\beta}\psi_{\hbar}\|^2 = \sum_{\gamma=1}^K \sum_{\beta \in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m) \|P_{\gamma}(-\eta)\tau_{\beta}\psi_{\hbar}\|^2.$$

For each  $1 \leq \gamma \leq K$ , we have the following property for  $\overline{c}_{\gamma,\beta,\alpha}(m)$  (as  $f \geq 0$ ):

$$\overline{c}_{\beta,\alpha}(m) = \overline{c}_{\gamma,\beta,\alpha}(m) + c_{\gamma,\beta,\alpha}(m).$$

We can write then:

$$\sum_{\beta \in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m) \|\tau_{\beta}\psi_{\hbar}\|^2 = \sum_{\gamma=1}^K \sum_{\beta \in K_m^{(q-1)}(\alpha)} (\overline{c}_{\gamma,\beta,\alpha}(m) + c_{\gamma,\beta,\alpha}(m)) \|P_{\gamma}(-\eta)\tau_{\beta}\psi_{\hbar}\|^2.$$

Notice that we have assumed the operators commute, we have:

$$(52) \quad P_{\gamma}(-\eta)P_{\beta_k}((k+q-2)\eta) \cdots P_{\beta_{-q+2}}\psi_{\hbar} = P_{\beta_k}((k+q-1)\eta) \cdots P_{\beta_{-q+2}}P_{\gamma}(-\eta)\psi_{\hbar}.$$

As a consequence, we have:

$$\sum_{\beta \in K_m^{(q-1)}(\alpha)} \overline{c}_{\beta,\alpha}(m) \|\tau_{\beta}\psi_{\hbar}\|^2 = \sum_{\gamma=1}^K \sum_{\beta \in K_m^{(q-1)}(\alpha)} (\overline{c}_{\gamma,\beta,\alpha}(m) + c_{\gamma,\beta,\alpha}(m)) \|\tau_{\beta}P_{\gamma}(-\eta)\psi_{\hbar}\|^2.$$

By definition of the different sets  $K_m$  and as  $\psi_{\hbar}$  is an eigenfunction of the Laplacian, this last equality allows to conclude the proof of the lemma.  $\square$

Proceeding then by induction from  $N$  to 1 (see equality (50)) and using the previous lemma at each step, we can conclude that if all the operators commute:

$$\overline{\mu}_{\hbar}^{\Sigma}(\overline{\sigma}^{-m}A) = \overline{\mu}_{\hbar}^{\Sigma}(A).$$

**6.1.3. Estimates of the error terms.** Regarding to the previous section, we have to see what is exactly the error term we forgot at each step of the recurrence and we have to verify that it is bounded by some positive power of  $\hbar$ . Precisely, we have to understand what is the error term in equation (52) if we do not suppose anymore that all the operators commute. Precisely, the error term we have to take into account in (52) is:

$$R_{\beta,\gamma,\hbar} = \sum_{j=-q+2}^k P_{\beta_k}((k+q-2)\eta) \cdots P_{\beta_{j+1}}((j+q-1)\eta) R^j(\beta,\gamma) P_{\beta_{j-1}}((j+q-3)\eta) \cdots P_{\beta_{-q+2}}\psi_{\hbar},$$



where  $R^j(\beta, \gamma) = [P_\gamma(-\eta), P_{\beta_j}((j+q-2)\eta)]$  is the bracket of the two operators. We denote  $R_{\beta, \gamma, \hbar}^j$  each term of the previous sum. The error term we forgot at each step  $q$  of the induction in the previous section is then:

$$(53) \quad E(\hbar, q) := \sum_{\gamma=1}^K \sum_{\beta \in K_m^{(q-1)}(\alpha)} (\langle R_{\beta, \gamma, \hbar}, P_\gamma(-\eta)\tau_\beta\psi_\hbar \rangle + \langle \tau_\beta P_\gamma(-\eta)\psi_\hbar, R_{\beta, \gamma, \hbar} \rangle).$$

So for each step  $q$  of the induction to prove the pseudo invariance of the measure, a first error term we have to estimate is of the form:

$$(54) \quad \sum_{j=-q+2}^k \sum_{\gamma=1}^K \sum_{\beta \in K_m^{(q-1)}(\alpha)} \bar{c}_{\beta, \alpha}(m) \langle R_{\beta, \gamma, \hbar}^j, P_\gamma(-\eta)\tau_\beta\psi_\hbar \rangle.$$

Using Cauchy Schwarz inequality twice and the fact that  $0 \leq \bar{c}_{\beta, \alpha}(m) \leq \text{Leb}(B(\gamma)) \leq b_0\eta$ , this last quantity is bounded by

$$(55) \quad \lambda b_0\eta \left( \sum_{j=-q+2}^k \sum_{\gamma=1}^K \sum_{\beta \in K_m^{(q-1)}(\alpha)} \|R_{\beta, \gamma, \hbar}^j\|^2 \right)^{\frac{1}{2}} \left( \sum_{j=-q+2}^k \sum_{\gamma=1}^K \sum_{\beta \in K_m^{(q-1)}(\alpha)} \|P_\gamma(-\eta)\tau_\beta\psi_\hbar\|^2 \right)^{\frac{1}{2}}.$$

The last term of the product is bounded as:

$$\sum_{j=-q+2}^k \sum_{\gamma=1}^K \sum_{\beta \in K_m^{(q-1)}(\alpha)} \|P_\gamma(-\eta)\tau_\beta\psi_\hbar\|^2 \leq (k+q)K \sum_{|\beta|=k+q-1} \|\tau_\beta\psi_\hbar\|^2 = (k+q)K = O(|\log \hbar|).$$

As a consequence, the error term (55) is bounded by:

$$C |\log \hbar| \left( \sum_{j=-q+2}^k \sum_{\gamma=1}^K \sum_{\beta \in K_m^{(q-1)}(\alpha)} \|R_{\beta, \gamma, \hbar}^j\|^2 \right)^{\frac{1}{2}},$$

where  $C$  is some positive uniform constant (depending only on the partition and on  $\eta$ ). We extend now the definition of  $R^j(\beta, \gamma)$  previously defined for  $\beta$  in  $K_m^{(q-1)}(\alpha)$  (as  $[P_\gamma(-\eta), P_{\beta_j}((j+q-2)\eta)]$ ) to any word  $\beta$  of length  $k+q-1$ . If  $j+q-1$  letters of  $\beta$  are also the  $j+q-1$  first letters of a word  $\beta'$  in  $K_m^{(q-1)}(\alpha)$ , we take  $R^j(\beta, \gamma) := [P_\gamma(-\eta), P_{\beta_j}((j+q-2)\eta)]$ . Otherwise, we take  $R^j(\beta, \gamma) := \hbar \text{Id}_{L^2(M)}$ . We define then for any sequence of length  $k+q-1$ :

$$R_{\beta, \gamma, \hbar}^j = P_{\beta_k}((k+q-2)\eta) \cdots P_{\beta_{j+1}}((j+q-1)\eta) R^j(\beta, \gamma) P_{\beta_{j-1}}((j+q-3)\eta) \cdots P_{\beta_{-q+2}}\psi_\hbar.$$

In theorem 6.4 from the section 6.2, we will prove in particular that for every  $\beta$  of size  $q+k-1$  and for each  $-q+2 \leq j \leq k$ :

$$(56) \quad \|R^j(\beta, \gamma) P_{\beta_{j-1}}((j+q-3)\eta) \cdots P_{\beta_{-q+2}}\psi_\hbar\|_{L^2(M)} \leq C \hbar^{1-2\nu} \|P_{\beta_{j-1}}((j+q-3)\eta) \cdots P_{\beta_{-q+2}}\psi_\hbar\|_{L^2(M)},$$

where  $C$  is a uniform constant for  $n_0$  and  $m$  positive integers such that  $n_0+m \leq T_E(\hbar)$  and  $\nu < 1/2$  (defined in section 7). The theorem 6.4 can be applied as  $\sum_{j=0}^{k+q-2} \bar{f}(T^j\beta) \leq (n_0+m)(1+\epsilon) \leq n_E(\hbar)$  (see (48) and (51)). Using bound (56) and the property of partition of identity, we have:

$$\sum_{|\beta|=k+q-1} \|R_{\beta, \gamma, \hbar}^j\|^2 = O(\hbar^{2(1-2\nu)}).$$

The error term (55) (and as a consequence (54)) is then bounded by:

$$\tilde{C} |\log \hbar| \left( \sum_{j=-q+2}^k \sum_{\gamma=1}^K \sum_{|\beta|=k+q-1} \|R_{\beta, \gamma, \hbar}^j\|^2 \right)^{\frac{1}{2}} = O(\hbar^{\frac{1-2\nu}{4}}).$$

Looking at equation (53), we see that the other error term for the step  $q$  of the induction can be estimated with the same method and is also a  $O(\hbar^{\frac{1-2\nu}{4}})$ . As the number  $N$  of steps in the

induction is a  $O(|\log \hbar|)$ , the error term we forgot in the previous section (due to the fact that the operators do not commute) is a  $O(\hbar^{\frac{1-2\nu}{6}})$ . This concludes the proof of proposition 6.2.  $\square$

**6.2. Commutation of pseudodifferential operators.** In order to complete the proof of the pseudo invariance of the measure (proposition 6.2), we need to prove inequality (56). It will be a consequence of (58) below. Once we have proved this inequality, the subadditivity property will be completely proved. The exact property we need is stated by the following theorem:

**Theorem 6.4.** *Let  $(\gamma_0, \dots, \gamma_k)$  be such that*

$$(57) \quad \sum_{j=0}^{k-1} f(\sigma^j \gamma) \leq n_E(\hbar).$$

*One has:*

$$(58) \quad \left\| [P_{\gamma_k}(k\eta), P_{\gamma_0}] P_{\gamma_{k-1}}((k-1)\eta) \cdots P_{\gamma_1}(\eta) \psi_{\hbar} \right\|_{L^2} \leq C \hbar^{1-2\nu} \left\| P_{\gamma_{k-1}}((k-1)\eta) \cdots P_{\gamma_1}(\eta) \psi_{\hbar} \right\|_{L^2},$$

where  $\nu < 1/2$  is defined in section 7,  $C$  is a constant depending on the partition and uniform in all  $\gamma$  satisfying (57).

Let  $\gamma$  be a finite sequence as in the previous theorem. Denote  $t(\gamma) = \sum_{j=0}^{k(\gamma)-1} f(\sigma^j \gamma)$ . This quantity is less than  $n_E(\hbar)$  in the setting of theorem 6.4. There exists a unique integer  $l(\gamma) < k(\gamma)$  such that:

$$\sum_{j=0}^{l(\gamma)-2} f(\sigma^j \gamma) \leq \frac{t(\gamma)}{2} < \sum_{j=0}^{l(\gamma)-1} f(\sigma^j \gamma).$$

In the following, the dependence of  $l$  and  $k$  in  $\gamma$  will be often omitted for simplicity of notations and will be recalled only when it is necessary. This definition allows to write the quantity we want to bound

$$\left\| [P_{\gamma_k}(k\eta), P_{\gamma_0}] P_{\gamma_{k-1}}((k-1)\eta) \cdots P_{\gamma_1}(\eta) \psi_{\hbar} \right\|_{L^2}$$

in the following way:

$$(59) \quad \left\| [P_{\gamma_k}((k-l+1)\eta), P_{\gamma_0}((-l+1)\eta)] P_{\gamma_{k-1}}((k-l)\eta) \cdots P_{\gamma_l}(\eta) P_{\gamma_{l-1}} \cdots P_{\gamma_1}((-l+2)\eta) \psi_{\hbar} \right\|_{L^2}.$$

The reason why we choose to write the quantity we want to bound in (58) in the previous form instead of its original form is for pseudodifferential technical reasons that will appear and be mentioned in the proof. To prove the bound in theorem 6.4, a class of symbols taken from [11] will be used (see (73) for a definition) and results about them are recalled in section 8. Before starting the proof, using proposition 8.3, we can restrict ourselves to observables carried on a thin energy strip around the energy layer  $\mathcal{E}^\theta$ . It means that the quantity we want to bound is the following norm:

$$(60) \quad \left\| \left[ \hat{P}_{\gamma_k}((k-l+1)\eta), \hat{P}_{\gamma_0}((-l+1)\eta) \right] \hat{P}_{\gamma_{k-1}}((k-l)\eta) \cdots \hat{P}_{\gamma_l}(\eta) \hat{P}_{\gamma_{l-1}} \cdots \hat{P}_{\gamma_1}((-l+2)\eta) \psi_{\hbar} \right\|_{L^2},$$

where  $\hat{P}_i$  is now equal to  $\text{Op}_{\hbar}(P_i^f)$ , where  $P_i^f$  is compactly supported in  $T^*\Omega_i \cap \mathcal{E}^\theta$  (see proposition 8.3).

**6.2.1. Defining cutoffs.** If we consider quantity (60), we can see that because we consider large times  $k\eta$ , we can not estimate directly the norm of the bracket  $\left[ \hat{P}_{\gamma_k}((k-l+1)\eta), \hat{P}_{\gamma_0}((-l+1)\eta) \right]$ . However, the quantity we are really interested in is the norm of this bracket on the image of  $\hat{P}_{\gamma_{k-1}}((k-l)\eta) \cdots \hat{P}_{\gamma_l}(\eta) \hat{P}_{\gamma_{l-1}} \cdots \hat{P}_{\gamma_1}((-l+2)\eta)$ . So we will introduce some cutoff operators to localize the bracket we want to estimate on the image of  $\hat{P}_{\gamma_{k-1}}((k-l)\eta) \cdots \hat{P}_{\gamma_l}(\eta) \hat{P}_{\gamma_{l-1}} \cdots \hat{P}_{\gamma_1}((-l+2)\eta)$ . Then, as was discussed in section 1.2, we will have to verify that it defines a particular family of operators for which the Egorov theorem can be applied for large times.

First, we introduce a new family of functions  $(Q_i)_{i=1}^K$  such that for each  $1 \leq i \leq K$ ,

$Q_i$  belongs to  $\mathcal{C}^\infty(T^*\Omega_i \cap \mathcal{E}^\theta)$ ,  $0 \leq Q_i \leq 1$  and  $Q_i \equiv 1$  on  $\text{supp}P_i^f$ . We then define two cutoffs associated to the strings  $(\gamma_1, \dots, \gamma_{l-1})$  and  $(\gamma_l, \dots, \gamma_{k-1})$ :

$$(61) \quad Q_{\gamma_{k-1}, \dots, \gamma_l} := Q_{\gamma_l} \circ g^{-(k-l)\eta} \dots Q_{\gamma_{k-1}} \circ g^{-\eta}$$

and

$$(62) \quad \tilde{Q}_{\gamma_{l-1}, \dots, \gamma_1} := Q_{\gamma_1} \circ g^\eta \dots Q_{\gamma_{l-1}} \circ g^{(l-1)\eta}.$$

The first point of our discussion will be to prove that Egorov theorem can be applied for large times to the pseudodifferential operators corresponding to these two symbols.

We prove the Egorov property for  $Q_{\gamma_{k-1}, \dots, \gamma_l}$  for example (the proof works in the same way for the other one). Recall that one has the exact equality, for a symbol  $a$ :

$$(63) \quad U^{-t} \text{Op}_{\hbar}(a) U^t - \text{Op}_{\hbar}(a(t)) = \int_0^t U^{-s} (\text{Diff}^{t-s}) U^s ds,$$

where  $a(t) := a \circ g^t$  and  $\text{Diff}^t := \frac{i}{\hbar} [-\hbar^2 \Delta, \text{Op}_{\hbar}(a(t))] - \text{Op}_{\hbar}(\{H, a(t)\})$ . Here, we will consider  $a := Q_{\gamma_{k-1}, \dots, \gamma_l}$ . One has, for  $0 \leq t \leq (k-l+1)\eta$ :

$$Q_{\gamma_{k-1}, \dots, \gamma_l}(t) := Q_{\gamma_{k-1}, \dots, \gamma_l} \circ g^t = Q_{\gamma_l} \circ g^{-(k-l)\eta+t} \dots Q_{\gamma_{k-1}} \circ g^{-\eta+t}.$$

There exists an unique integer  $1 \leq j \leq (k-l)$  such that  $t - j\eta$  is negative and  $t - (j-1)\eta$  is nonnegative. This allows us to rewrite:

$$Q_{\gamma_{k-1}, \dots, \gamma_l}(t) = \left( Q_{\gamma_l} \circ g^{-(k-l-j)\eta} \dots Q_{\gamma_{k-j}} \right) \circ g^{-j\eta+t} \left( Q_{\gamma_{k-j+1}} \dots Q_{\gamma_{k-1}} \circ g^{(j-2)\eta} \right) \circ g^{-(j-1)\eta+t}.$$

Using the last part of theorem 7.1 and the corresponding remark, we know that  $Q_{\gamma_l} \circ g^{-(k-l-j)\eta} \dots Q_{\gamma_j}$  and  $Q_{\gamma_{j-1}} \dots Q_{\gamma_{k-1}} \circ g^{(j-2)\eta}$  are symbols of the class  $S_{\nu}^{-\infty, 0}$  (see the appendix for a definition of this class of symbols), where  $\nu := \frac{1-\epsilon'+4\epsilon}{2}$ . Moreover the constants in the bounds of the derivatives are uniform for the words  $\gamma$  in the allowed set (see theorem 7.1 and proposition 7.3). As  $-\eta \leq t - j\eta < 0 \leq t - (j-1)\eta \leq \eta$  and as the class  $S_{\nu}^{-\infty, 0}$  is stable by product, we have then that  $Q_{\gamma_{k-1}, \dots, \gamma_{k-l}}(t)$  is in the class  $S_{\nu}^{-\infty, 0}$ , for  $0 \leq t \leq (k-l+1)\eta$ , with uniform bounds in  $t$  and  $\gamma$  in the allowed set. As, in [5], we can verify that  $\text{Diff}^{Q_{\gamma_{k-1}, \dots, \gamma_l}^t}$  is in  $S_{\nu}^{-\infty, \nu-1}$  and then apply the Calderon-Vaillancourt theorem for  $\Psi_{\nu}^{-\infty, \nu-1}$ . As a consequence, there exists a constant  $C$  depending only on the family  $Q_i$  and on the derivatives of  $g^s$  (for  $-\eta \leq s \leq \eta$ ) such that:

$$(64) \quad \forall 0 \leq t \leq (k-l+1)\eta, \quad \|\text{Op}_{\hbar}(Q_{\gamma_{k-1}, \dots, \gamma_l})(t) - \text{Op}_{\hbar}(Q_{\gamma_{k-1}, \dots, \gamma_l}(t))\|_{L^2} \leq C\hbar^{\frac{1-\nu}{2}}.$$

This exactly means that the Egorov theorem can be applied for positive times to  $Q_{\gamma_{k-1}, \dots, \gamma_l}$ . As we mentioned it in the heuristic of the proof (section 1.2), taking into account the support of the symbol, we can prove a 'local' Egorov property for a range of time that depends on the support of our symbol. Precisely, we have shown that the Egorov property holds until the stopping time defined in section 5.1.

**6.2.2. Proof of theorem 6.4.** Before proving theorem 6.4, we define (in order to have simpler expressions):

$$\psi_{\hbar}^{\gamma} := \hat{P}_{\gamma_{k-1}}((k-l)\eta) \dots \hat{P}_{\gamma_1}((-l+2)\eta) \psi_{\hbar}.$$

To prove theorem 6.4, we need to bound quantity (60) and precisely to estimate (60), we have to estimate:

$$(65) \quad (60) = \left\| \left[ \hat{P}_{\gamma_0}((-l+1)\eta), \hat{P}_{\gamma_k}((k-l+1)\eta) \right] \psi_{\hbar}^{\gamma} \right\|_{L^2}.$$

Now we want to introduce our cutoff operators  $\text{Op}_{\hbar}(Q_{\bullet})$  in the previous expression:

$$\begin{aligned} \hat{P}_{\gamma_0}((-l+1)\eta) \hat{P}_{\gamma_k}((k-l+1)\eta) &= \hat{P}_{\gamma_0}((-l+1)\eta) \left( \text{Id} - \left( \hat{P}_{\gamma_k} \text{Op}_{\hbar}(Q_{\gamma_{k-1}, \dots, \gamma_l}) \right) ((k-l+1)\eta) \right) \\ &\quad + \left( \hat{P}_{\gamma_k} \text{Op}_{\hbar}(Q_{\gamma_{k-1}, \dots, \gamma_l}) \right) ((k-l+1)\eta). \end{aligned}$$

We will first estimate the norm

$$\left\| \hat{P}_{\gamma_0}((-l+1)\eta) \left( \text{Id} - \left( \hat{P}_{\gamma_k} \text{Op}_{\hbar}(Q_{\gamma_{k-1}, \dots, \gamma_l}) \right) ((k-l+1)\eta) \right) \psi_{\hbar}^{\gamma} \right\|_{L^2}.$$

To do this, we first outline that  $\hat{P}_{\gamma_k}$  is in  $\Psi^{-\infty,0}(M)$  and  $\text{Op}_{\hbar}(Q_{\gamma_{k-1}, \dots, \gamma_l})$  is in  $\Psi_{\nu}^{-\infty,0}(M)$ . Using the standard rules for a product, we know that the previous expression can be transformed as follows:

$$\left\| \hat{P}_{\gamma_0}((-l+1)\eta) \left( Id - \text{Op}_{\hbar}(P_{\gamma_k}^f Q_{\gamma_{k-1}, \dots, \gamma_l})((k-l+1)\eta) \right) \psi_{\hbar}^{\gamma} \right\|_{L^2} + R_{\gamma}^1(\hbar),$$

where  $\|R_{\gamma}^1(\hbar)\|_{L^2} \leq C\hbar^{1-2\nu}\|\psi_{\hbar}^{\gamma}\|_{L^2}$  (where  $C$  is independent of  $k-l$  as the bounds implied in the derivatives in theorem 7.1 are uniform for words  $\gamma$  in the allowed set: see proposition 7.3). We can apply the strategy of the previous section to prove an Egorov property for the operator  $\text{Op}_{\hbar}(P_{\gamma_k}^f Q_{\gamma_{k-1}, \dots, \gamma_l})$ . So, up to a  $O_{L^2}(\hbar^{\frac{1-\nu}{2}})$ ,  $\text{Op}_{\hbar}(P_{\gamma_k}^f Q_{\gamma_{k-1}, \dots, \gamma_l})((k-l+1)\eta)$  is equal to the pseudodifferential operator in  $\Psi_{\nu}^{-\infty,0}$ :

$$\text{Op}_{\hbar} \left( (P_{\gamma_k}^f Q_{\gamma_{k-1}, \dots, \gamma_l}) \circ g^{(k-l+1)\eta} \right)$$

supported in  $g^{-\eta}T^*\Omega_{\gamma_l} \cap \dots \cap g^{-(k-l+1)\eta}T^*\Omega_{\gamma_k} \cap \mathcal{E}^{\theta}$ . Using then theorem 7.1, the following holds:

$$\left( Id - \text{Op}_{\hbar} \left( (P_{\gamma_k}^f Q_{\gamma_{k-1}, \dots, \gamma_l}) \circ g^{(k-l+1)\eta} \right) \right) \hat{P}_{\gamma_{k-1}}((k-l)\eta) \dots \hat{P}_{\gamma_1}((-l+2)\eta)\psi_{\hbar} = O_{L^2}(\hbar^{\infty}).$$

Even if the proof of this fact is rather technical, it is intuitively quite clear. In fact, if we suppose that the standard pseudodifferential rules (Egorov, composition) apply,  $\hat{P}_{\gamma_{k-1}}((k-l)\eta) \dots \hat{P}_{\gamma_1}((-l+2)\eta)$  is a pseudodifferential operator compactly supported in  $g^{(l-2)\eta}T^*\Omega_{\gamma_1} \cap \dots \cap g^{(l-k)\eta}T^*\Omega_{\gamma_{k-1}} \cap \mathcal{E}^{\theta}$ . On this set, by definition of the cutoff operators ( $Q_i \equiv 1$  on  $\text{supp}(P_i)$ ),  $(1 - (P_{\gamma_k}^f Q_{\gamma_{k-1}, \dots, \gamma_l}) \circ g^{(k-l+1)\eta})$  is equal to 0. As a consequence, we consider the product of two pseudodifferential operators of disjoint supports: it is  $O_{L^2}(\hbar^{\infty})$ . Theorem 7.1 proves the fact we need:  $\hat{P}_{\gamma_{k-1}}((k-l)\eta) \dots \hat{P}_{\gamma_1}((-l+2)\eta)$  is a pseudodifferential operator of well determined support. To conclude the previous lines of the proof, we have:

$$(66) \quad \left\| \hat{P}_{\gamma_0}((-l+1)\eta) \left( Id - \left( \hat{P}_{\gamma_k} \text{Op}_{\hbar}(Q_{\gamma_{k-1}, \dots, \gamma_l}) \right) ((k-l+1)\eta) \right) \psi_{\hbar}^{\gamma} \right\|_{L^2} \leq \tilde{C}\hbar^{1-2\nu}\|\psi_{\hbar}^{\gamma}\|_{L^2}.$$

Performing this procedure for the other operators, we finally obtain that the only quantity we need to bound to prove theorem 6.4 is the following quantity:

$$(67) \quad \left\| \left[ \left( \hat{P}_{\gamma_k} \text{Op}_{\hbar}(Q_{\gamma_{k-1}, \dots, \gamma_l}) \right) ((k-l+1)\eta), \left( \hat{P}_{\gamma_0} \text{Op}_{\hbar}(\tilde{Q}_{\gamma_{l-1}, \dots, \gamma_1}) \right) ((-l+1)\eta) \right] \right\|_{L^2}.$$

Using the property of the product on  $\Psi_{\nu}^{-\infty,0}$ , we know that, up to a  $O_{L^2}(\hbar^{1-2\nu})$ , the previous quantity is equal to

$$\left\| \left[ \text{Op}_{\hbar}(P_{\gamma_k}^f Q_{\gamma_{k-1}, \dots, \gamma_l})((k-l+1)\eta), \text{Op}_{\hbar}(P_{\gamma_0}^f \tilde{Q}_{\gamma_{l-1}, \dots, \gamma_1})((-l+1)\eta) \right] \right\|_{L^2}.$$

Using the same method that in the previous section (which uses theorem 7.1), we can prove an Egorov property for the two pseudodifferential operators that are in the previous bracket and show that, up to a  $O_{L^2}(\hbar^{1-2\nu})$ , the quantity (67) is equal to:

$$\left\| \left[ \text{Op}_{\hbar}((P_{\gamma_k}^f Q_{\gamma_{k-1}, \dots, \gamma_l}) \circ g^{(k-l+1)\eta}), \text{Op}_{\hbar}((P_{\gamma_0}^f \tilde{Q}_{\gamma_{l-1}, \dots, \gamma_1}) \circ g^{(-l+1)\eta}) \right] \right\|_{L^2}.$$

Using the pseudodifferential rules in  $\Psi_{\nu}^{-\infty,0}(M)$  (proceeding as in the previous section, the two symbols stay in the good class of symbol using theorem 7.1), we know that the previous bracket is in  $\Psi_{\nu}^{-\infty,2\nu-1}$ . Using the Calderon-Vaillancourt theorem, we know that quantity (67) is a  $O_{L^2}(\hbar^{1-2\nu})$ , where the constant depends only on the partition. This concludes the proof of theorem 6.4.  $\square$

## 7. PRODUCTS OF MANY EVOLVED PSEUDODIFFERENTIAL OPERATORS

The goal of this section is to prove a property used in the proof of theorem 6.4. Precisely, the following theorem states that the product of a large number of evolved pseudodifferential operators remains in a good class of pseudodifferential operators provided the range of times is smaller than the 'local' Ehrenfest time. First, recall that using proposition 8.3, we can restrict ourselves to observables carried on a thin energy strip around the energy layer  $\mathcal{E}^{\theta}$ . We underline that we do not suppose anymore that this thin energy strip is of size  $\hbar^{1-\delta}$ : we only need to have a small macroscopic neighborhood of the unit energy layer. Moreover, the class of symbols we will

consider will be the class  $S_\nu^{-\infty,0}$  (see (73) for a precise definition) with  $\nu := \frac{1-\epsilon'+4\epsilon}{2}$  ( $< 1/2$ , see section 4).

**Theorem 7.1.** *Let  $(Q_i)_{i=1}^K$  be a family of smooth functions on  $T^*M$  such that for each  $1 \leq i \leq K$ ,  $Q_i$  belongs to  $\mathcal{C}^\infty(T^*\Omega_i \cap \mathcal{E}^\theta)$  and  $0 \leq Q_i \leq 1$ . Consider a family of indices  $(\gamma_1, \dots, \gamma_l)$  such that:*

$$\sum_{j=1}^{l-1} f(\gamma_{j+1}, \gamma_j) \leq \frac{n_E(\hbar)}{2}.$$

Then, for any  $1 \leq j \leq l$ , one has:

$$\text{Op}_\hbar(Q_{\gamma_1})\text{Op}_\hbar(Q_{\gamma_2})(-\eta) \cdots \text{Op}_\hbar(Q_{\gamma_j})(-(j-1)\eta) = \text{Op}_\hbar(A^{\gamma_1, \dots, \gamma_j})(-j\eta) + O_{L^2}(\hbar^\infty),$$

where  $A^{\gamma_1, \dots, \gamma_j}$  is in the class  $S_\nu^{-\infty,0}$ . Precisely, one has the following asymptotic expansion:

$$A^{\gamma_1, \dots, \gamma_j} \sim \sum_{p \geq 0} \hbar^p A_p^{\gamma_1, \dots, \gamma_j},$$

where  $A_p^{\gamma_1, \dots, \gamma_j}$  is in the class  $S_\nu^{-\infty, p\epsilon}$  (with the symbols semi norm uniform for  $\gamma$  in the allowed set of sequences and  $1 \leq j \leq l$ : see proposition 7.3) and compactly supported in  $g^{-\eta}T^*\Omega_{\gamma_j} \cap \cdots g^{-j\eta}T^*\Omega_{\gamma_1} \cap \mathcal{E}^\theta$ . Finally the principal symbol  $A_0^{\gamma_1, \dots, \gamma_j}$  is given by the following formula:

$$A_0^{\gamma_1, \dots, \gamma_j} = Q_{\gamma_j} \circ g^\eta \cdots Q_{\gamma_2} \circ g^{(j-1)\eta} Q_{\gamma_1} \circ g^{j\eta}.$$

*Remark.* First note that this theorem holds in particular for the smooth partition of identity we considered previously on the paper. Note also that the result can be rephrased by saying that  $\text{Op}_\hbar(Q_{\gamma_1})(j\eta)\text{Op}_\hbar(Q_{\gamma_2})((j-1)\eta) \cdots \text{Op}_\hbar(Q_{\gamma_j})(\eta)$  is, up to a  $O_{L^2}(\hbar^\infty)$ , a pseudodifferential operator of the class  $\Psi_\nu^{-\infty,0}$  and of well determined support. Under the assumption  $\sum_{j=1}^{l-1} f(\gamma_j, \gamma_{j+1}) \leq \frac{n_E(\hbar)}{2}$ , we would have proved that  $\text{Op}_\hbar(Q_{\gamma_1})(-j\eta)\text{Op}_\hbar(Q_{\gamma_2})(-(j-1)\eta) \cdots \text{Op}_\hbar(Q_{\gamma_j})(-\eta)$  is, up to a  $O_{L^2}(\hbar^\infty)$ , a pseudodifferential operator of the class  $\Psi_\nu^{-\infty,0}$  and of well determined support. These are exactly the properties we used in section 6.2.

The plan of the proof is the following. First, we will construct formally  $A^{\gamma_1, \dots, \gamma_j}$  and its asymptotic expansion in powers of  $\hbar$ . Then, we will check that these different symbols are in a good class. Finally, we will check that these operators approximate the product we considered. For simplicity of notations, we will forget (for a time) the dependence on  $\gamma$  and denote the previous symbol  $A^j$  for  $l \geq j \geq 1$ .

**7.1. Definition of  $A_{\gamma_1, \dots, \gamma_l}$ .** In this section, we construct formally the symbol  $A^j$ . The way to do it is by induction on  $j$ . First, we will see how to define formally  $A^j$  from  $A^{j-1}$ . Then, using the formulas of the previous section, we will construct the formal order  $N$  expansion associated to this  $A^j$ . We only construct what the order  $N$  expansion should be regarding to the previous formulas. We will check in the next section that this expansion has the properties of theorem 7.1 and in the final section that all the remainder terms are negligible.

**7.1.1. Definition at each step.** To construct  $A^j$ , we proceed by induction and at the first step, we consider  $\text{Op}_\hbar(Q_{\gamma_1})$  and we write it into the form  $\text{Op}_\hbar(A^1)(-\eta)$ . This means that we have defined formally:

$$\text{Op}_\hbar(A^1(\eta)) := U^{-\eta} \text{Op}_\hbar(Q_{\gamma_1}) U^\eta.$$

We also define formally for  $0 \leq t \leq \eta$ :

$$\text{Op}_\hbar(A^1(t)) := U^{-t} \text{Op}_\hbar(Q_{\gamma_1}) U^t.$$

Using Egorov theorem for fixed time  $\eta$  and the corresponding asymptotic expansion (see section 8.3.2 for explicit formulas of the asymptotic expansion), we prove that, up to a  $O_{L^2}(\hbar^\infty)$ ,  $\text{Op}_\hbar(Q_{\gamma_1})$  is equal to  $\text{Op}_\hbar(A^1(\eta))(-\eta)$ , where  $A^1(\eta)$  is in  $S^{-\infty,0}$ , given by the asymptotic expansion of the Egorov theorem and supported in  $g^{-\eta}T^*\Omega_{\gamma_1} \cap \mathcal{E}^\theta$ . We can continue this procedure formally. At the second step, we have

$$\text{Op}_\hbar(Q_{\gamma_1})\text{Op}_\hbar(Q_{\gamma_2})(-\eta) = U^\eta \text{Op}_\hbar(A^1(\eta)) \text{Op}_\hbar(Q_{\gamma_2}) U^{-\eta}.$$

We want this quantity to be of the form  $\text{Op}_{\hbar}(A^2(\eta))(-2\eta)$ . This means that we have defined formally:

$$\text{Op}_{\hbar}(A^2(\eta)) := U^{-\eta} \text{Op}_{\hbar}(A^1(\eta)) \text{Op}_{\hbar}(Q_{\gamma_2}) U^{\eta}.$$

Using rules of pseudodifferential operators (see section 8.3.1 and 8.3.2), we can obtain a formal asymptotic expansion for  $A^2(\eta)$  (see next section) starting from the expansion of  $A^1(\eta)$ . One can easily check that this formal expansion is supported in  $g^{-\eta} T^* \Omega_{\gamma_2} \cap g^{-2\eta} T^* \Omega_{\gamma_1} \cap \mathcal{E}^{\theta}$ . We defined again  $\text{Op}_{\hbar}(A^2(t)) := U^{-t} \text{Op}_{\hbar}(A^1(\eta)) \text{Op}_{\hbar}(Q_{\gamma_2}) U^t$ .

Following the previous method, we will construct a formal expansion of  $A^j(t)$  (for  $0 \leq t \leq \eta$ ) starting from the expansion of  $A^{j-1}(\eta)$  (see next section). To do this, we will write at each step  $1 \leq j \leq l$ ,

$$(68) \quad \text{Op}_{\hbar}(A^j(t)) := U^{-t} \text{Op}_{\hbar}(A^{j-1}(\eta)) \text{Op}_{\hbar}(Q_{\gamma_j}) U^t.$$

With this definition, we will have:

$$\text{Op}_{\hbar}(A^j(\eta))(-j\eta) := (\text{Op}_{\hbar}(A^{j-1}(\eta)) \text{Op}_{\hbar}(Q_{\gamma_j}))(-j\eta).$$

Using again rules of pseudodifferential calculus (see section 8.3.1 and 8.3.2), we can obtain a formal asymptotic expansion for  $A^j(t)$  (see next section) starting from the expansion of  $A^{j-1}(\eta)$ . One can easily check that this formal expansion is supported in  $g^{-t} (T^* \Omega_{\gamma_j} \cap \dots \cap g^{-j\eta} T^* \Omega_{\gamma_1}) \cap \mathcal{E}^{\theta}$ .

In the next section, we will use the induction formula (68) to deduce the  $\hbar$ -expansion of  $A^j(t)$  from the expansion for the composition of  $\text{Op}_{\hbar}(A^{j-1}(\eta))$  and  $\text{Op}_{\hbar}(Q_{\gamma_j})$  and from the expansion for the Egorov theorem for times  $0 \leq t \leq \eta$ . At each step  $1 \leq j \leq l$  of the induction, we will have to prove that  $A^j$  stays in a good class of symbols to be able to continue the induction.

*7.1.2. Definition of the order  $N$  expansion.* We fix a large integer  $N$  (to be determined). We study the previous construction by induction up to  $O(\hbar^N)$ . From this point, we truncate  $A^j(t)$  at the order  $N$  of its expansion. First, we see how we construct the symbols  $A^j(t)$  by induction. To do this, we use the formulas for the asymptotic expansions of the composition of pseudodifferential operators and for the Egorov theorem (see section 8.3.1 and 8.3.2). Suppose that

$$A^{j-1}(\eta) = \sum_{p=0}^N \hbar^p A_p^{j-1}(\eta)$$

is well defined, we have to define the expansion of  $A^j(t)$  from the asymptotic expansion of  $A^{j-1}(\eta)$ , for  $0 \leq t \leq \eta$ . First, we define:

$$(69) \quad \bar{A}^j := \sum_{p=0}^N \hbar^p \bar{A}_p^j, \quad \text{where } \bar{A}_p^j := \sum_{q=0}^p \left( A_{p-q}^{j-1}(\eta) \sharp_M Q_{\gamma_j} \right)_q.$$

The symbol  $\sharp_M$  represents an analogue on a manifold of the Moyal product (see appendix 8.3.1):  $(a \sharp_M b)_p$  is the order  $p$  term in the expansion of the symbol of  $\text{Op}_{\hbar}(a) \text{Op}_{\hbar}(b)$ . Recall from the appendix that  $(A_{p-q}^{j-1} \sharp_M Q_{\gamma_j})_q$  is a linear combination (that depends on the local coordinates and on the  $(Q_i)_i$ ) of the derivatives of order less than  $q$  of  $A_{p-q}^{j-1}(\eta)$ . Then, we define  $A^j(t)$  using the exact asymptotic expansion given by proposition 8.4 in appendix 8.3.2. To do this, we define:

$$A_0^j(t) := \bar{A}_0^j(g^t)$$

and for  $1 \leq p \leq N$ ,

$$A_p^j(t) := \bar{A}_p^j(g^t) + \sum_{q=0}^{p-1} \int_0^t \{H, A_q^j(t-s)\}_M^{(p,q)} (g^s(\rho)) ds,$$

where  $H(\rho) = \frac{\|\xi\|_x^2}{2}$  is the Hamiltonian. This expression for the order  $p$  term comes from proposition 8.4 from the appendix where an exact expression for  $\text{Op}_{\hbar}(a)(t)$  is given. Recall also from proposition 8.4 that for each  $0 \leq q \leq p-1$ ,  $\{H, A_q^j(t-s)\}_M^{(p,q)}$  is a linear combination of the derivatives of order less than  $p+1-q$  of  $A_q^j(t-s)$  that depends on the choice of the coordinates on the manifold. This construction is the precise way we want to define the asymptotic expansion

of the symbol  $A^j(t)$  in theorem 7.1. If we want the theorem to be valid, we have to check now that the remainders we forget at each step are negligible (with an arbitrary high order in  $\hbar$ ). To do this, we will first have to control at each step  $j$  the derivatives of  $A_j(t)$  (see next section).

*Remark.* The support of  $A_p^j(t)$  is included in  $g^{-t}(T^*\Omega_{\gamma_j} \cap \dots \cap g^{-(j-1)\eta}T^*\Omega_{\gamma_1}) \cap \mathcal{E}^\theta$  regarding the previous construction.

**7.2. Estimates of the derivatives.** The goal of the first part of this section is to prove the following lemma.

**Lemma 7.2.** *Let  $N$  be a fixed integer. Fix also two integers  $0 \leq p \leq N$  and  $m \leq 2(N - p + 1)$ . Then, there exists a constant  $C(m, p)$  such that for all  $j \geq 1$  and for all  $\rho$  in the set  $g^{-t}(T^*\Omega_{\gamma_j} \cap \dots \cap g^{-(j-1)\eta}T^*\Omega_{\gamma_1}) \cap \mathcal{E}^\theta$ :*

$$\forall 0 \leq t \leq \eta, |d^m A_p^j(t, \rho)| \leq C(m, p) j^{m+p^2} |dg^{t+(j-1)\eta}(\rho)|^m.$$

If  $\rho$  is not in this set, the bound is trivially 0 by construction. Here the constant  $C(m, p)$  depends only on  $m, p$ , the atlas we chose for the manifold and the size of the  $(\Omega_\gamma)_\gamma$ .

Once this lemma will be proved, we will check that it also tells us that the  $A_p^j$ 's are in a nice class of symbols.

**7.2.1. Proof of lemma 7.2.** To make good estimates on the derivatives of the  $A_p^j$ 's at each step, we will follow the same kind of procedure that in section 3.4 of [5]. To make all the previous pseudodifferential arguments work, we will have to obtain estimates on the  $m$ -differential forms  $d^m A_p^j$ , for each  $m \leq 2(N + 1 - p)$ . If we have estimates on these derivatives, we will then check that all the asymptotic expansions given by the pseudodifferential theory are valid. To do these estimates, we will have to understand the number of derivatives that appear when we repeat the induction formula (68). The spirit of this proof is the same as in [5] when they iterate the WKB expansion  $|\log \hbar|$  times. We define a vector  $\mathbf{A}^j$  with entries  $\mathbf{A}_{(p,m)}^j := d^m A_p^j$  (where  $0 \leq p \leq N$  and  $0 \leq m \leq 2(N - p + 1)$ ). Precisely, we order it by the following way, for  $0 \leq t \leq \eta$  and  $\rho \in T^*M$ :

$$\mathbf{A}^j = \mathbf{A}^j(t, \rho) := \begin{cases} (A_0^j, dA_0^j, \dots, d^{2(N+1)}A_0^j, \\ A_1^j, dA_1^j, \dots, d^{2N}A_1^j, \\ \dots, \\ A_N^j, dA_N^j, d^2A_N^j). \end{cases}$$

Using induction formulas of the previous section, we will rewrite the link between  $\mathbf{A}^j$  and  $\mathbf{A}^{j-1}$  in the following form:

$$(\mathbf{I} - \mathbf{M}_1)\mathbf{A}^j = (\mathbf{M}_{0,0}^j + \mathbf{M}_{0,1}^j + \mathbf{M}_{0,2}^j)\mathbf{A}^{j-1},$$

where the operators  $\mathbf{M}$  are defined in the following of the section. We will then inverse this formula to get  $\mathbf{A}^j = f(\mathbf{A}^{j-1})$ . Before giving the precise definition of each matrix, we define the operator, for each integer  $l$ :

$$G^t(d^l a)(\rho) := d^l a(g^t \rho).$$

It defines a  $l$ -form defined on  $(T_{g^t \rho} T^*M)^l$ . We write then:

$$d^m(G^t a)(\rho) = \sum_{l \leq m} G^t(d^l a)(\rho) \cdot \theta_{m,l}(t, \rho),$$

where the linear form  $\theta_{m,l}(t, \rho)$  sends  $(T_\rho T^*M)^m$  on  $(T_{g^t \rho} T^*M)^l$ . We can write the explicit form of  $\theta_{m,m}$ :

$$\theta_{m,m}(t, \rho) := (d_\rho g^t)^{\otimes m}.$$

Using these notations, we can rewrite the formulas for the  $\mathbf{A}_{(p,m)}^j$  ( $0 \leq p \leq N$  and  $m \leq 2(N + 1 - p)$ ):

$$\mathbf{A}_{(p,m)}^j(t, \rho) := d^m \left( G^t \left( \overline{A}_p^j \right) (\rho) \right) + \sum_{q=0}^{p-1} \int_0^t d^m \left( G^s \left\{ H, \mathbf{A}_{q,0}^j(t-s) \right\}_M^{(p,q)} (\rho) \right) ds,$$

where  $\bar{A}_p^j$  depends on  $\mathbf{A}^{j-1}(\eta, \cdot)$ . Using this, we can define  $\mathbf{M}_1$ :

$$(\mathbf{M}_1 \mathbf{A}^j)_{(p,m)}(t, \rho) := \sum_{q=0}^{p-1} \int_0^t d^m \left( G^s \left\{ H, \mathbf{A}_{q,0}^j(t-s) \right\}_M^{(p,q)}(\rho) \right) ds.$$

Using our previous decomposition of  $d^m(T^t a)$  and the fact that  $\{H, A_q^j(t-s)\}_M^{(p,q)}$  is a linear combination of the derivatives of order less than  $p+1-q$  of  $A_q^j(t-s)$ , we find:

$$(\mathbf{M}_1 \mathbf{A}^j)_{(p,m)}(t, \rho) = \sum_{q=0}^{p-1} \sum_{l \leq m+p+1-q} \int_0^t G^s \left( \mathbf{A}_{q,l}^j(t-s) \right) (\rho) \cdot \bar{\theta}_{m,q,l}(t, s, \rho) ds,$$

where  $\bar{\theta}_{m,q,l}$  is a linear form that sends  $(T_\rho T^* M)^m$  on  $(T_{g^{t-s}\rho} T^* M)^k$ . The important fact to underline about this matrix  $\mathbf{M}_1$  is that it relates  $A_p$  to  $(A_r)_{0 \leq r \leq p-1}$ . In particular, it is a nilpotent operator-valued matrix of order  $N+1$ . We can also give the explicit expression for  $\mathbf{M}_{0,0}^j$ :

$$\left( \mathbf{M}_{0,0}^j \mathbf{A}^{j-1} \right)_{(p,m)}(t, \rho) := G^t \left( Q_{\gamma_j} \mathbf{A}_{(p,m)}^{j-1}(\eta) \right) (\rho) \cdot \theta_{m,m}(t, \rho).$$

In particular,  $\mathbf{M}_{0,0}^j$  is a diagonal matrix. Finally the last matrices are defined as:

$$\left( \mathbf{M}_{0,1}^j \mathbf{A}^{j-1} \right)_{(p,m)}(t) := \sum_{l < m} G^t \left( d^l \left( Q_{\gamma_j} \mathbf{A}_{(p,0)}^{j-1}(\eta) \right) \right) \cdot \theta_{m,l}(t) = \sum_{l < m} \sum_{k \leq l} G^t \left( \mathbf{A}_{(p,k)}^{j-1}(\eta) \right) \cdot \bar{\theta}_{m,l}^{j,1}(t)$$

(it corresponds to the term of order 0 in the 'Moyal product' on a manifold) and using the higher order term of the Moyal product formula on a manifold:

$$\left( \mathbf{M}_{0,2}^j \mathbf{A}^{j-1} \right)_{(p,m)}(t) := \sum_{q=1}^p \sum_{l < m} \sum_{k \leq l} G^t \left( \mathbf{A}_{(p-q,k+q)}^{j-1}(\eta) \right) \cdot \bar{\theta}_{m,l,q}^{j,2}(t).$$

Regarding these formulas, it can first be underlined that  $\mathbf{M}_{0,1}^j$  relates  $\mathbf{A}_{(p,m)}^j$  with  $\mathbf{A}_{(p,k)}^{j-1}$  where  $k < m$ . As in the case of  $\mathbf{M}_1$ , it can be underlined that the matrix  $\mathbf{M}_{0,2}^j$  relates  $A_p$  to  $(A_r)_{0 \leq r \leq p-1}$  and in particular, it behaves like a 'nilpotent' operator-valued matrix of order  $N+1$ . In both cases, it results then that we have a nilpotent matrix and that the  $\bar{\theta}^{j,\cdot}$  are multilinear forms depending on  $j$ . Outline that all the multilinear forms used to define the different matrices are uniformly bounded with respect to  $\rho$ ,  $t \in [0, \eta]$  and  $j$ . We will just have to check that these bounds do not accumulate too much when we iterate  $j$  times the induction formula.

*Remark.* The crucial point of the previous discussion is not really the exact form of the matrices  $\mathbf{M}$ . The important facts are that  $\mathbf{M}_1$  relates  $A_p$  to  $(A_r)_{0 \leq r \leq p-1}$  (it is in fact nilpotent of index  $N$ ),  $\mathbf{M}_{0,0}^j$  is a diagonal matrix,  $\mathbf{M}_{0,1}^j$  relates  $\mathbf{A}_{(p,m)}^j$  with  $\mathbf{A}_{(p,k)}^{j-1}$  (where  $k < m$ ) and  $\mathbf{M}_{0,2}^j$  relates  $A_p$  to  $(A_r)_{0 \leq r \leq p-1}$ .

Using this remark, we can inverse the induction formula to get an expression of  $\mathbf{A}^j$  in function of  $\mathbf{A}^{j-1}$ :

$$\mathbf{A}^j := \left( \sum_{k=0}^N [\mathbf{M}_1]^k \right) \left( \mathbf{M}_{0,0}^j + \mathbf{M}_{0,1}^j + \mathbf{M}_{0,2}^j \right) \mathbf{A}^{j-1}.$$

Here, we put  $[\cdot]$  around the matrix to distinguish the power of a matrix and an index. Iterating the previous expression, one then has:

$$\mathbf{A}^j := \sum_{k_2, \dots, k_j=0}^{N-1} \sum_{\alpha_2, \dots, \alpha_j=0}^2 [\mathbf{M}_1]^{k_j} \mathbf{M}_{0,\alpha_j}^j [\mathbf{M}_1]^{k_{j-1}} \mathbf{M}_{0,\alpha_{j-1}}^{j-1} \dots [\mathbf{M}_1]^{k_2} \mathbf{M}_{0,\alpha_2}^2 \mathbf{A}^1.$$

From this expression, one can estimate how many terms contributes to the definition of  $\mathbf{A}_{(p,m)}^j$ . In fact, using the previous remark, the product of matrices that contributes to the expression of  $\mathbf{A}_{(p,m)}^j$  can only be non zero if  $\sum_{j'} k_{j'} + |\{j' : \alpha_{j'} = 2\}| \leq p$  and  $|\{j' : \alpha_{j'} = 1\}| \leq m + p(\sum_{j'} k_{j'} + |\{j' : \alpha_{j'} = 2\}|)$ . As a consequence, for large  $j$ , to be non zero, a string of matrices need to be



made of at most  $(N + 1)^2$  matrices of the form  $\mathbf{M}_{0,\alpha}^j$  and  $\mathbf{M}_1$  (for  $\alpha \in \{1, 2\}$ ). Moreover, the number of string that contributes to  $\mathbf{A}_{(p,m)}^j$  is of size  $O(j^{m+p^2})$  when  $j$  becomes large.

Then, to estimate the norm of the derivatives of  $A^j$ , we should look how the different matrices used to define the derivatives act. First we study the action of the diagonal matrix. As  $0 \leq Q_{\gamma_j} \leq 1$ , one has that, for  $0 \leq t \leq \eta$  and for any  $\rho \in g^{-t} (T^* \Omega_{\gamma_j} \cap \dots \cap g^{-(j-1)\eta} T^* \Omega_{\gamma_1}) \cap \mathcal{E}^\theta$  (otherwise the following quantity is clearly equal to 0):

$$|\mathbf{M}_{0,0}^j \mathbf{A}_{(p,m)}^{j-1}(t, \rho)| \leq |dg^t(\rho)|^m |\mathbf{A}_{(p,m)}^{j-1}(\eta, g^t(\rho))|.$$

We note that we can iterate this bound without getting an exponentially big term in  $j$ . In particular, for any  $j$  and  $j'$  in  $\mathbb{N}$ , we have for any  $0 \leq t \leq \eta$ :

$$|\mathbf{M}_{0,0}^{j+j'} \dots \mathbf{M}_{0,0}^j \mathbf{A}_{(p,m)}^{j-1}(t, \rho)| \leq |dg^{t+j'\eta}(\rho)|^m |\mathbf{A}_{(p,m)}^{j-1}(\eta, g^{t+j'\eta}(\rho))|.$$

Now, using the fact that all the multilinear form we considered are uniformly bounded, we get that there exists a constant  $C(m, p)$  such that:

$$\sup_{0 \leq t \leq \eta} \|\mathbf{M}_1 \mathbf{A}_{(p,m)}^j(t)\|_{L^\infty} \leq C(m, p) \max_{m' \leq m+2} \max_{q \leq p-1} \sup_{0 \leq t \leq \eta} \|\mathbf{A}_{(q,m')}^j(t)\|_{L^\infty}$$

and, for  $\alpha \in \{1, 2\}$ :

$$\sup_{0 \leq t \leq \eta} \|\mathbf{M}_{0,\alpha}^j \mathbf{A}_{(p,m)}^{j-1}(t)\|_{L^\infty} \leq C(m, p) \max_{m' \leq m} \max_{q \leq p} \|\mathbf{A}_{(q,m')}^{j-1}\|_{L^\infty}.$$

The only thing we need to know is that the constant depends only on  $m, p$  and the partition. The difference with the action of the diagonal matrix is that we have constant prefactor that can accumulate and become large. However as we already mentioned it, the number of such matrices that contribute to the estimate is at most of order  $N^2$ . So the constant factor  $C(m, p)$  does not create a factor which makes the derivatives explodes when we iterate the induction. Then, proceeding like in [5] (end of section 3.4), we can use all the previous bounds on the action of the matrices and on the number of matrices who contributes to  $(p, m)$  term of the vector  $\mathbf{A}^j$  and the multiplicativity property of  $dg^t(\rho)$ . It gives that for any  $\rho \in g^{-t} (T^* \Omega_{\gamma_j} \cap \dots \cap g^{-(j-1)\eta} T^* \Omega_{\gamma_1}) \cap \mathcal{E}^\theta$ :

$$|\mathbf{A}_{(p,m)}^j(t, \rho)| \leq \tilde{C}(p, m) j^{m+p^2} |dg^{t+(j-1)\eta}(\rho)|^m \|\mathbf{A}^1(\eta)\|. \square$$

**7.2.2. Class of symbol of each term of the expansion.** Using the previous lemma, we want to show that  $A_p^j(t)$  is an element of  $S_\nu^{-\infty, p\epsilon}$ . Let  $\rho$  be an element of  $g^{-t} (T^* \Omega_{\gamma_j} \cap \dots \cap g^{-(j-1)\eta} T^* \Omega_{\gamma_1}) \cap \mathcal{E}^\theta$ . Using the fact that  $E^u$  is of dimension 1, we get that for any positive  $t$ ,  $|d_\rho g^t| \leq J_t^u(\rho)^{-1}$ , where  $J_t^u(\rho) := \det(dg_{|E^u(g^t \rho)}^{-t})$ . Then we can write the multiplicativity of the determinant and get

$$J_{t+(j-1)\eta}^u(\rho) = J_t^u(\rho) J_\eta^u(g^t \rho) J_\eta^u(g^{t+\eta} \rho) \dots J_\eta^u(g^{t+(j-2)\eta} \rho).$$

*Remark.* Before continuing the estimate, let us underline some property of the Jacobian. Suppose  $S$  is a positive integer and  $1/\eta$  also (large enough to be in our setting). We have for all  $0 \leq k \leq 1/\eta - 1$ :

$$J^u(g^{k\eta} \rho) J^u(g^{1+k\eta} \rho) \dots J^u(g^{S-1+k\eta} \rho) = J_\eta^u(g^{k\eta} \rho) J_\eta^u(g^{(k+1)\eta} \rho) \dots J_\eta^u(g^{S+(k-1)\eta} \rho),$$

where  $J^u(\rho)$  is the unstable Jacobian in time 1 that appears in the main theorem 1.2. We make the product over  $k$  of all these equalities and we get:

$$J^u(\rho)^\eta J^u(g^\eta \rho)^\eta \dots J^u(g^{S-\eta} \rho)^\eta \leq C(\eta) J_\eta^u(g\rho) J_\eta^u(g^{1+\eta} \rho) \dots J_\eta^u(g^{S-\eta} \rho),$$

where  $C(\eta)$  only depends on  $\eta$  and does not depend on  $S$ .

Finally, using previous remark and inequality (17), the following estimate holds:

$$|d_\rho g^{t+(j-1)\eta}| \leq C(\eta) e^{j\epsilon\eta a_0} J_\eta^u(\gamma_j, \gamma_{j-1})^{-\eta} J_\eta^u(\gamma_{j-1}, \gamma_{j-2})^{-\eta} \dots J_\eta^u(\gamma_2, \gamma_1)^{-\eta}$$

with  $C(\eta)$  independent of  $j$ . Then:

$$|d_\rho g^{t+(j-1)\eta}| \leq C(\eta) e^{l(\gamma)\epsilon\eta a_0} e^{t(\gamma)},$$

where  $t(\gamma) = \sum_{j=0}^{l-1} f(\gamma_{j+1}, \gamma_j)$ . As  $t(\gamma) \leq n_E(\hbar)/2$ , this last quantity is bounded by  $\hbar^{\frac{\epsilon'-1}{2}-\epsilon}$  (as  $l(\gamma)a_0\eta \leq n_E(\hbar)/2$ ). Using lemma 7.2, we want to estimate the  $m$  derivatives of the symbol  $A_p^j$ . To do this, we can fix a large  $\tilde{N}$  such that  $m \leq 2(\tilde{N} - N)$  and then we have ( $p < N$ ,  $0 \leq t \leq \eta$ ):

$$(70) \quad |d^m A_p^j(t, x)| \leq \tilde{C}(m, p) \hbar^{-p\epsilon} \hbar^{m(\frac{\epsilon'-1}{2}-2\epsilon)}.$$

We used the fact that  $j = O(|\log \hbar|)$ . Here appears the fact that we only apply the backward quantum evolution for times  $l$ . In fact, as we want our symbols to be in the class  $S_\nu^{-\infty}$ , we need derivatives to lose at most a factor  $\hbar^{-1/2}$  (this would have not been the case if we had considered times of size  $n_E(\hbar)$  instead of size  $n_E(\hbar)/2$ ). The previous estimate (70) is uniform for all the  $\gamma$  in the allowed set of theorem 7.1.

Finally, to summarize this section, we can write the following proposition:

**Proposition 7.3.** *Let  $p$  and  $m$  be elements of  $\mathbb{N}$ . There exists  $C(m, p, (Q_i)_i, \eta)$  (depending on  $m$ ,  $p$ ,  $\eta$ ,  $(Q_i)_{i=1}^K$  and the coordinate charts) such that for all  $\gamma = (\gamma_0, \dots, \gamma_l)$  such that*

$$\sum_{j=0}^{l-1} f(\gamma_{j+1}, \gamma_j) \leq \frac{n_E(\hbar)}{2},$$

for all  $0 \leq j \leq l$  and for all  $0 \leq t \leq \eta$ :

$$|d^m A_p^{\gamma_1, \dots, \gamma_j}(t, x)| \leq C(m, p, (Q_i)_i, \eta) \hbar^{-p\epsilon} \hbar^{m(\frac{\epsilon'-1}{2}-2\epsilon)}.$$

Then, as the  $A_p^j$  are compactly supported,  $A_p^j$  is in class  $S_\nu^{-\infty, p\epsilon}$ , where  $\nu = \frac{1-\epsilon'+4\epsilon}{2}$ .

So, our formal construction allows us to define a family of symbol  $A_p^j$  and each of them belongs to  $S_\nu^{-\infty, p\epsilon}$ . Moreover the constants implied in the bounds of the derivatives are uniform with respect to the allowed sequences.

*Remark.* Outline that the same proof would show that the intermediate symbols  $\bar{A}_p^j$  (69) are also in the same class of symbols.

**7.3. Estimate of the remainder terms.** We are now able to conclude the proof of theorem 7.1 starting from the family we have just constructed. We have to verify that the remainder is of small order in  $\hbar$ . Fix a large integer  $N$  and denote  $A_{(N)}^j := \sum_{p=0}^N \hbar^p A_p^j(\eta)$ . We want to estimate:

$$R_N^j = \|\text{Op}_{\hbar}(Q_{\gamma_1}) \cdots \text{Op}_{\hbar}(Q_{\gamma_j})(-j\eta) - \text{Op}_{\hbar}(A_{(N)}^j)(-j\eta)\|_{L^2}.$$

Using the induction formula (68), we write:

$$R_N^j \leq \|U^{-\eta} \text{Op}_{\hbar}(A_{(N)}^{j-1}) \text{Op}_{\hbar}(Q_{\gamma_j}) U^\eta - \text{Op}_{\hbar}(A_{(N)}^j)\|_{L^2} + R_N^{j-1},$$

where  $R_N^{j-1} = \|\text{Op}_{\hbar}(Q_{\gamma_1}) \cdots \text{Op}_{\hbar}(Q_{\gamma_{j-1}})(-(j-2)\eta) - \text{Op}_{\hbar}(A_{(N)}^{j-1})\|_{L^2}$ . Then, we can use section 8.3.1 and the estimates (75), to write the estimate:

$$\|U^{-\eta} \text{Op}_{\hbar}(A_{(N)}^{j-1}) \text{Op}_{\hbar}(Q_{\gamma_j}) U^\eta - U^{-\eta} \text{Op}_{\hbar}(\bar{A}_{(N)}^j) U^\eta\|_{L^2} \leq C_N \hbar^{(N+1)(1-\nu)-(C+C')\nu}.$$

We also use section 8.3.2, to write the exact formula (as  $\bar{A}^j = A^{j-1} \#_M Q_{\gamma_j}$ ):

$$U^{-\eta} \text{Op}_{\hbar}(\bar{A}_{(N)}^j) U^\eta - \text{Op}_{\hbar}(A_{(N)}^j) = \hbar^{N+1} \int_0^\eta U^{-s} \text{Op}_{\hbar} \left( R^{(N)}(\bar{A}_{(N)}^j)(\eta-s) \right) U^s ds,$$

An estimate on the norm of  $\text{Op}_{\hbar}(R^{(N)}(\bar{A}_{(N)}^j)(s))$  will be given in section 8.3.2, depending on the derivatives of the  $\bar{A}_p^j$ 's (for  $p \leq N$ ). So, up to a  $O_{L^2}(\hbar^\infty)$ , one has:

$$R_N^j \leq R_N^{j-1} + \hbar^{N+1} \eta \sup_s \left\| \text{Op}_{\hbar} \left( R^{(N)}(\bar{A}_{(N)}^j)(s) \right) \right\|_{L^2} + C_N \hbar^{(N+1)(1-\nu)-(C+C')\nu}.$$

Using estimates (78) from section 8.3.2 and the last remark of the previous section, one can check that:

$$R_N^j \leq R_N^{j-1} + C_N \hbar^{(N+1)(1-\nu)-(C+C')\nu},$$

where  $C + C'$  depends only on the dimension of the manifold. By induction on  $j$  (which is  $O(|\log \hbar|)$ ), we prove that, for large enough  $N$ ,  $\text{Op}_\hbar(Q_{\gamma_1}) \cdots \text{Op}_\hbar(Q_{\gamma_j})(-(j-1)\eta)$  is well approximated by  $\text{Op}_\hbar(A_{(N)}^j)(-j\eta)$ . In fact,  $\text{Op}_\hbar(A_{(N)}^j)$  is in  $\Psi_\nu^{-\infty,0}$ . As a consequence for large  $N$ ,  $\hbar^{(N+1)(1-\nu)-\tilde{C}\nu}$  is arbitrarily small compared to  $\|\text{Op}_\hbar(A_{(N)}^j)(-j\eta)\|$  (which is bounded). This concludes the proof of theorem 7.1.  $\square$

## 8. APPENDIX: PSEUDODIFFERENTIAL CALCULUS ON A MANIFOLD

In this appendix, a few facts about pseudodifferential calculus on a manifold and the sharp energy cutoff used in this paper are recalled. Even if most of this setting can be found in [5], it is recalled because it is extensively used in section 6.2 and 7. The results from the two first sections of this appendix can be found in more details in [23] or [5]. The results of the last section of this appendix are the extension to the case of a manifold of standard results from semiclassical analysis that can be found either in [7], [11] or [12].

**8.1. Pseudodifferential calculus on a manifold.** In this section, some facts of pseudodifferential calculus that can be found in [11] or in [12] are recalled. Recall that we define on  $\mathbb{R}^{2d}$  the following class of symbols:

$$S^k(\langle \xi \rangle^m) := \left\{ a \in C^\infty(\mathbb{R}^{2d}) : |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta} \hbar^{-k} \langle \xi \rangle^{m-|\beta|} \right\}.$$

Let  $M$  be a smooth Riemannian  $d$ -manifold without boundary. Consider a smooth atlas  $(f_l, V_l)$  of  $M$ , where each  $f_l$  is a smooth diffeomorphism from  $V_l \subset M$  to a bounded open set  $W_l \subset \mathbb{R}^d$ . To each  $f_l$  correspond a pull back  $f_l^* : C^\infty(W_l) \rightarrow C^\infty(V_l)$  and a canonical map  $\tilde{f}_l$  from  $T^*V_l$  to  $T^*W_l$ :

$$\tilde{f}_l : (x, \xi) \mapsto (f_l(x), (Df_l(x))^{-1})^T \xi).$$

Consider now a smooth locally finite partition of identity  $(\phi_l)$  adapted to the previous atlas  $(f_l, V_l)$ . That means  $\sum_l \phi_l = 1$  and  $\phi_l \in C^\infty(V_l)$ . Then, any observable  $a$  in  $C^\infty(T^*M)$  can be decomposed as follows:  $a = \sum_l a_l$ , where  $a_l = a\phi_l$ . Each  $a_l$  belongs to  $C^\infty(T^*V_l)$  and can be pushed to a function  $\tilde{a}_l = (\tilde{f}_l^{-1})^* a_l \in C^\infty(T^*W_l)$ . As in [12], define the class of symbols of order  $m$  and index  $k$ :

$$(71) \quad S^{m,k}(T^*M) := \left\{ a \in C^\infty(T^*M) : |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta} \hbar^{-k} \langle \xi \rangle^{m-|\beta|} \right\}.$$

Then, for  $a \in S^{m,k}(T^*M)$  and for each  $l$ , one can associate to the symbol  $\tilde{a}_l \in S^k(\langle \xi \rangle^m)$  the standard Weyl quantization:

$$\text{Op}_\hbar^w(\tilde{a}_l)u(x) := \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar}\langle x-y, \xi \rangle} \tilde{a}_l\left(\frac{x+y}{2}, \xi; \hbar\right) u(y) dy d\xi,$$

where  $u \in \mathcal{S}(\mathbb{R}^d)$ , the Schwartz class. Consider now a smooth cutoff  $\psi_l \in C_c^\infty(V_l)$  such that  $\psi_l = 1$  close to the support of  $\phi_l$ . A quantization of  $a \in S^{m,k}$  is then defined in the following way:

$$(72) \quad \text{Op}_\hbar(a)(u) := \sum_l \psi_l \times (f_l^* \text{Op}_\hbar^w(\tilde{a}_l)(f_l^{-1})^*)(\psi_l \times u),$$

where  $u \in C^\infty(M)$ . According to the appendix of [12], the quantization procedure  $\text{Op}_\hbar$  sends  $S^{m,k}(T^*M)$  onto the space of pseudodifferential operators of order  $m$  and of index  $k$ , denoted  $\Psi^{m,k}(M)$ . It can be shown that the dependence in the cutoffs  $\phi_l$  and  $\psi_l$  only appears at order 2 in  $\hbar$  and the principal symbol map  $\sigma_0 : \Psi^{m,k}(M) \rightarrow S^{m,k}/S^{m,k-1}(T^*M)$  is then well defined. All the rules (for example the composition of operators, the Egorov theorem) that holds in the case of  $\mathbb{R}^{2d}$  still holds in the case of  $\Psi^{m,k}(M)$ . However, because our study concerns behavior of

quantum evolution for logarithmic times in  $\hbar$ , a larger class of symbols should be introduced as in [11] or [12], for  $0 \leq \nu < 1/2$ :

$$(73) \quad S_\nu^{m,k}(T^*M) := \left\{ a \in C^\infty(T^*M) : |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta} \hbar^{-k-\nu|\alpha+\beta|} \langle \xi \rangle^{m-|\beta|} \right\}.$$

Results of [11] can be applied to this new class of symbols. For example, a symbol of  $S_\nu^{0,0}$  gives a bounded operator on  $L^2(M)$  (with norm independent of  $\hbar$ ).

As was explained, one needs to quantize the sharp energy cutoff  $\chi^{(\cdot)}$  (see section 5.3.1) to get sharp bounds in 5.6. As  $\chi^{(0)}$  localize in a strip of size  $\hbar^{1-\delta}$  with delta close to 0, the  $m$ -th derivatives transversally to  $\mathcal{E}$  grows like  $\hbar^{m(\delta-1)}$ . As  $\delta$  is close to 0,  $\chi^{(0)}$  does not belongs to the previous class of symbols that allows  $\nu < 1/2$ . However, as the variations only appears in one direction, it is possible to define a new pseudodifferential calculus for these symbols. The procedure taken from [23] is briefly recalled in [5] (section 5) and introduces a class of anisotropic symbols  $S_{\mathcal{E},\nu'}^{-\infty,0}$  (where  $\mathcal{E} := S^*M$  and  $\nu' < 1$ ) for which a quantization procedure  $\text{Op}_{\mathcal{E},\nu'}$  can be defined. In the next section, we recall briefly a few results about the quantization  $\text{Op}_{\mathcal{E},\nu'}(\chi^{(n)})$  of the symbol  $\chi^{(\cdot)}$ .

**8.2. Energy cutoff.** Let  $\chi^{(\cdot)}$  be as in section 5.3.1. Consider  $\delta > 0$  and  $K_\delta$  associated to it (see section 5.3.1). Taking  $\nu' = 1 - \delta$ , it can be checked that the cutoffs defined in section 5.3.1 belongs to the class  $S_{\mathcal{E},\nu'}^{-\infty,0}$  defined in [5]. A pseudodifferential operator corresponding to it can be defined following the nonstandard procedure mentioned above. Using results from [5] (section 5), one has  $\|\text{Op}_{\mathcal{E},\nu'}(\chi^{(n)})\| = 1 + O(\hbar^{\nu'/2})$  for all  $n \leq K_\delta |\log \hbar|$ . For simplicity of notations, in the paper  $\text{Op}(\chi^{(n)}) := \text{Op}_{\mathcal{E},\nu'}(\chi^{(n)})$ . In [5], it is also proved that:

**Proposition 8.1.** [5] *For  $\hbar$  small enough and any  $n \in \mathbb{N}$  such that  $0 \leq n \leq K_\delta |\log \hbar|$  and for any  $\psi_\hbar = -\hbar^2 \Delta \psi_\hbar$  eigenstate, one has:*

$$\|\psi_\hbar - \text{Op}(\chi^{(n)})\psi_\hbar\| = O(\hbar^\infty) \|\psi_\hbar\|.$$

Moreover for any sequence  $\alpha$  and  $\beta$  of length  $n$  less than  $K_\delta |\log \hbar|$ , one has:

$$\left\| \left(1 - \text{Op}(\chi^{(n)})\right) \tau_\alpha \text{Op}(\chi^{(0)}) \right\| = O(\hbar^\infty) \quad \left\| \left(1 - \text{Op}(\chi^{(n)})\right) \pi_\beta \text{Op}(\chi^{(0)}) \right\| = O(\hbar^\infty)$$

where  $\tau$  and  $\pi$  are given by (21) and (22).

This proposition tells that the quantization of this energy cutoff exactly have the expected property, meaning that it preserves the eigenfunction of the Laplacian. So, in the paper, introducing the energy cutoff  $\text{Op}(\chi^{(n)})$  does not change the semiclassical limit. Moreover this proposition implies the following corollary that allows to apply theorem 2.1 in section 5.3.2:

**Corollary 8.2.** [5] *For any fixed  $L > 0$ , there exists  $\hbar_L$  such that for any  $\hbar \leq \hbar_L$ , any  $n \leq K_\delta |\log \hbar|$  and any sequence of length  $n$ , the Laplacian eigenstate verify*

$$\left\| \left(1 - \text{Op}(\chi^{(n)})\right) \pi_\beta \psi_\hbar \right\| \leq \hbar^L \|\psi_\hbar\|.$$

A last result about the quantization of this cutoff from [5] that we can quote is that we can restrict ourselves to study observables carried in a thin neighborhood around  $S^*M = H^{-1}(1/2)$ :

**Proposition 8.3.** [5] *For  $\hbar$  small enough and any  $n \in \mathbb{N}$  such that  $0 \leq n \leq K_\delta |\log \hbar|/2$ , one has:*

$$\forall |\gamma| = n, \quad \|\tau_\gamma \text{Op}(\chi) - \tau_\gamma^f \text{Op}(\chi)\| = O(\hbar^\infty),$$

where  $P_{\gamma_j}^f = \text{Op}_\hbar(P_{\gamma_j} f)$ ,  $f$  is a smooth compactly supported function in a thin neighborhood of  $\mathcal{E}$  and  $\tau_\gamma^f = P_{\gamma_{n-1}}^f \cdots P_{\gamma_0}^f$ .

**8.3.  $\hbar$ -expansion for pseudodifferential operators on a manifold.** The goal of this last section is to explain how the usual  $\hbar$ -expansion of order  $N$  for composition of pseudodifferential operators and Egorov theorem can be extended in the case of pseudodifferential calculus on a manifold. Except for terms of order 0, the  $\hbar$ -expansion will depend on the partition of identity in section 8.1. In fact, on a manifold, the formulas for the terms of order larger than 1 on the  $\hbar$ -expansion will depend on the local coordinates. For simplicity and as it is the case of all the symbols we consider (thanks to the energy cutoff: for example, see proposition 8.3), we now restrict ourselves to symbols supported in  $\mathcal{E}^\theta = H^{-1}([1/2 - \theta, 1/2 + \theta])$ . The symbols are now elements of  $\Psi_\nu^{-\infty,0}(M)$ .

**8.3.1. Composition of pseudodifferential operators on a manifold.** First, recall that the usual semiclassical theory on  $\mathbb{R}^d$  (see [11] or [12]) tells that the composition of two elements  $\text{Op}_\hbar^w(a)$  and  $\text{Op}_\hbar^w(b)$  in  $\Psi_\nu^{-\infty,k}(\mathbb{R}^d)$  is still in  $\Psi_\nu^{-\infty,k}(\mathbb{R}^d)$  and that its symbol is of support equal to  $\text{supp}(a) \cap \text{supp}(b)$ . More precisely, it says that  $\text{Op}_\hbar^w(a) \circ \text{Op}_\hbar^w(b) = \text{Op}_\hbar^w(a\sharp b)$ , where  $a\sharp b$  is in  $S_\nu^{-\infty,k}$  and its asymptotic expansion in power of  $\hbar$  is given by the Moyal product:

$$(74) \quad a\sharp b(x, \xi) \sim \sum_k \frac{1}{k!} \left( \frac{i\hbar}{2} \omega(D_x, D_\xi, D_y, D_\eta) \right)^k a(x, \xi) b(y, \eta)|_{x=y, \xi=\eta},$$

where  $\omega$  is the standard symplectic form. Outline that it is clear that each element of the sum is supported in  $\text{supp}(a) \cap \text{supp}(b)$ . As quantization on a manifold is constructed from quantization on  $\mathbb{R}^{2d}$  (see definition (72)), one can prove an analogue of this asymptotic expansion in the case of a manifold  $M$ . Precisely, let  $a$  and  $b$  be two symbols in  $S_\nu^{-\infty,0}(T^*M)$ . For a choice of quantization  $\text{Op}_\hbar$  (that depends on the coordinates maps), one has  $\text{Op}_\hbar(a) \circ \text{Op}_\hbar(b)$  is a pseudodifferential operator in  $\Psi_\nu^{-\infty,0}(M)$ . Its symbol is denoted  $a\sharp_M b$  and its asymptotic expansion is of the following form:

$$a\sharp_M b \sim \sum_{p \geq 0} \hbar^p (a\sharp_M b)^p.$$

In the previous asymptotic expansion,  $(a\sharp_M b)^p$  is a linear combination (that depends on the cutoffs and the local coordinates) of elements of the form  $\partial^\gamma a \partial^{\gamma'} b$  with  $|\gamma| \leq p$  and  $|\gamma'| \leq p$ . As a consequence,  $(a\sharp_M b)^p$  is an element of  $S_\nu^{m,2p\nu}(T^*M)$ .

*Remark.* We know that we have an asymptotic expansion so by definition and using Calderon-Vaillancourt theorem, we know that each remainder is bounded in norm by a constant which is a small power of  $\hbar$  (in fact  $C\hbar^{(N+1)(1-2\nu)}$  for the remainder of order  $N$ ). In our analysis, we need to know precisely how these bounds depends on  $a$  and  $b$  as we have to make large product of pseudodifferential operator (see section 7) and to use the composition formula to get Egorov theorem (see next section). The following lines explain how the remainder in the asymptotic expansion in powers of  $\hbar$  is bounded by the derivatives of  $a$  and  $b$ .

In the appendix of [7], they defined the remainder of the order  $N$  expansion, in the case of  $\mathbb{R}^{2d}$ :

$$\hbar^{N+1} R_{N+1}(a, b, \hbar) := a\sharp b - \sum_{p=0}^N \hbar^p (a\sharp b)_p$$

and, using a stationary phase argument, they get the following estimates on the remainder, for all  $\gamma$  and all  $N$ :

$$|\partial_z^\gamma R_{N+1}(a, b, z, \hbar)| \leq \rho_d K_d^{N+|\gamma|} (N!)^{-1} \sup_{(*)} |\partial_u^{(\alpha,\beta)+\mu} a(u+z)| |\partial_v^{(\beta,\alpha)+\nu} b(v+z)|,$$

where  $(*)$  means:

$$u, v \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}, \quad |\mu| + |\nu| \leq 4d + |\gamma|, \quad |(\alpha, \beta)| = N + 1, \quad \alpha, \beta \in \mathbb{N}^d.$$

Applying Calderon-Vaillancourt theorem (see [11] or [12]), one knows that there exist a constant  $C$  and a constant  $M$  (depending only on  $d$ ), such that for a symbol  $a$  in  $S_\nu^0(1)$ :

$$\|\text{Op}_\hbar^w(a)\|_{L^2} \leq C \sup_{|\alpha| \leq M} \hbar^{\frac{|\alpha|}{2}} \|\partial^\alpha a\|_\infty.$$

Combining this result with the previous estimates on the  $R^{(N+1)}$ , one finds that:

$$(75) \quad \|\text{Op}_\hbar^w(R_{N+1}(a, b, z, \hbar))\|_{L^2} \leq C(d, N) \sup_{(*)} \hbar^{\frac{|\alpha|}{2}} \|\partial^{\beta+\beta'} a\|_\infty \|\partial^{\gamma+\gamma'} b\|_\infty,$$

where  $(*)$  means:

$$|\alpha| \leq C', \quad |\beta| \leq N+1, \quad |\gamma| \leq N+1 \quad \text{and} \quad |\beta'| + |\gamma'| \leq C + |\alpha|.$$

The constants  $C$  and  $C'$  depend only on the dimension of the manifold. The same kind of estimates on the remainder in the asymptotic expansion for change of variables hold. As the asymptotic expansion for composition of pseudodifferential operators is obtained from the composition and variable change rules on  $\mathbb{R}^d$  [12], the previous estimates (75) holds for semiclassical analysis on a manifold.

**8.3.2. Egorov expansion on a manifold.** In this section, we want to recall how we prove an Egorov property with an expansion of any order. We follow the proof from [7]. First, for the order 0 term, we write the following exact expression for a symbol  $a$ :

$$(76) \quad U^{-t} \text{Op}_\hbar(a) U^t - \text{Op}_\hbar(a(t)) = \hbar \int_0^t U^{-s} (\text{Diff} a^{t-s}) U^s ds,$$

where  $a(t) := a \circ g^t$ ,  $H(\rho) = \frac{\|\xi\|_x^2}{2}$  is the Hamiltonian and

$$\text{Diff} a^t := \frac{1}{\hbar} \left( \frac{i}{\hbar} [-\hbar^2 \Delta, \text{Op}_\hbar(a(t))] - \text{Op}_\hbar(\{H, a(t)\}) \right).$$

It can be also written with the notations of the previous section:

$$\text{Diff} a^t := i(R_2(H, a(t), \hbar) - R_2(a(t), H, \hbar)),$$

where  $R_i$  is now the remainder of the order  $i$  expansion in the composition of two pseudodifferential operators on a manifold. We proceed then by induction to recover the term of higher order. For these higher terms, we will see terms depending on the local coordinates appear in the expansion and we will obtain expressions as in [7] for the higher order terms of the expansion that will be slightly different from the case of  $\mathbb{R}^d$  [7]. However, we do not need to have an exact expression for each term of the expansion: we only need to know on how many derivatives the order  $p$  term depends and how the remainder can be bounded at each step. To obtain, the  $\hbar$  formal term of the Egorov expansion, we first outline that  $\text{Diff} a^t$  is a pseudodifferential operator whose asymptotic expansion is given by the composition rules on a manifold (see previous section). Then, we can apply the same procedure as in equation (76) to get the exact expression:

$$\text{Op}_\hbar(a)(t) = \text{Op}_\hbar(a^{(1)}(t)) + \hbar^2 \int_0^t U^{-s} \text{Op}_\hbar(R^{(2)}(t-s)) U^s ds.$$

In this equation, we have, according to the composition rules on a manifold:

$$a^{(1)}(t) := a \circ g^t + i\hbar \int_0^t ((H \#_M a(t-s))_2 - (a(t-s) \#_M H)_2) \circ g^s ds.$$

According to the rules for composition of pseudodifferential operators,  $(H \#_M a(t-s))_2$  is a linear combination of derivatives of  $H(x, \xi)a(t-s)(y, \eta)$  of order less than 4 with at most 2 derivatives of  $a$  in each term. We denote the previous formula in a more compact way:

$$a^{(1)}(t) := a_0(t) + \hbar a_1(t),$$

where  $a_1(t, \rho) := \int_0^t \{H, a_0(t-s)\}_M^{(1,0)}(g^s(\rho)) ds$ . As was mentioned, this generalized 'bracket' is a differential operator of order less than 4 applied to the product  $H(x, \xi)a(t-s)(y, \eta)$  (with at most 2 derivatives for  $a_{t-s}$ ). The remainder  $R^{(2)}$  is a linear combination of terms of the form  $R_3(H, a_0(t), \hbar)$  and  $R_2(H, a_1(t), \hbar)$  (where  $R_i$  was defined in the previous section as the remainder

in the composition of two pseudodifferential operators). Proceeding by induction and using the same procedure as for the term of order 1, we can show that for any order  $N$ :

$$\text{Op}_{\hbar}(a)(t) = \text{Op}_{\hbar}(a^{(N)}(t)) + \hbar^{N+1} \int_0^t U^{-s} \text{Op}_{\hbar} \left( R^{(N+1)}(t-s) \right) U^s ds.$$

In the previous formula,  $a^{(N)}(t)$  is defined as follows:

$$a^{(N)}(t) := \sum_{p=0}^N \hbar^p a_p(t) \text{ where } a_0(t) := a \circ g^t$$

and for  $1 \leq p \leq N$ :

$$a_p(t, \rho) := \sum_{j=0}^{p-1} \int_0^t \{H, a_j(t-s)\}_M^{(p,j)} (g^s(\rho)) ds,$$

where  $\{.,.\}_M^{(p,j)}$  is a generalized 'bracket' of order  $(p, j)$  depending on the local coordinates on the manifold (it is the analogue of formula given by theorem 1.2 in [7]). We do not need to have an exact expression for these brackets: we only need to know on how many derivatives it depends. From the previous section, we know how the order  $p$  term in the expansion of  $a \sharp_M b$  depends linearly on products of the  $p$  derivatives of  $a$  and  $b$ . The term  $\{H, a_0(t-s)\}_M^{(p,0)}$  comes from the order  $p+1$  term of the asymptotic expansion of  $\text{Diff} a^t$ . According to the rules of composition of pseudodifferential operators on a manifold, it is a linear combination of product of derivatives of  $a$  of order less than  $p+1$  and of derivatives of  $H$  of order less than  $p+1$ . More generally,  $\{H, a_j(t-s)\}_M^{(p,j)}$  is a linear combination of product of derivatives of  $a_j(t)$  of order less than  $p-j+1$  and of derivatives of  $H$  of order less than  $p-j+1$ . For the remainder term  $R^{(N+1)}(s)$  of order  $N$ , using the formulas for the composition of pseudodifferential operators, it can be shown that it is a linear combination of terms of the form  $R_{N+2-j}(H, a_j(t), \hbar)$  for  $0 \leq j \leq N$  (where  $R_i$  was defined in the previous section as the remainder in the composition of two pseudodifferential operators). Recall that the estimates for the norms of the  $\text{Op}_{\hbar}(R_i)$  still holds on a manifold with slight modifications: it depends on estimates on the derivatives of  $H$  and  $a_j$ . The previous discussion can be summarized in the following proposition:

**Proposition 8.4** (Egorov expansion on a manifold). *Let  $a$  be a symbol in  $S_{\nu}^{-\infty, 0}$ . One has the exact expression for every  $N \geq 0$ :*

$$(77) \quad \text{Op}_{\hbar}(a)(t) = \text{Op}_{\hbar}(a^{(N)}(t)) + \hbar^{N+1} \int_0^t U^{-s} \text{Op}_{\hbar} \left( R^{(N+1)}(a)(t-s) \right) U^s ds.$$

In the previous formula, one has:

$$a^{(N)}(t) := \sum_{p=0}^N \hbar^p a_p(t) \text{ where } a_0(t) := a \circ g^t$$

and for  $1 \leq p \leq N$ :

$$a_p(t, \rho) := \sum_{j=0}^{p-1} \int_0^t \{H, a_j(t-s)\}_M^{(p,j)} (g^s(\rho)) ds.$$

For each  $0 \leq j \leq p-1$ ,  $\{H, a_j(t-s)\}_M^{(p,j)}$  is a linear combination of the  $p+1-j$  derivatives of  $a_j(t-s)$  that depends on the choice of the coordinates on the manifold. Finally, the norm of  $\text{Op}_{\hbar}(R^{(N+1)}(a)(t))$  satisfy the following bounds:

$$(78) \quad \|\text{Op}_{\hbar}(R^{(N+1)}(a)(t))\|_{L^2} \leq C(d, N) \sup_{(*)} \hbar^{\frac{|\alpha|}{2}} \|\partial^{\beta+\beta'} (a_j(s))\|_{\infty},$$

where  $(*)$  means:

$$j \leq N+1, \quad 0 \leq s \leq \eta, \quad |\alpha| \leq C', \quad |\beta| \leq N+2-j \text{ and } |\beta'| \leq C+|\alpha|.$$

The constants  $C$  and  $C'$  depend only on the dimension of the manifold.

*Remark.* Theorem 1.2 in [7] gives an exact expression of each term of this exact expansion in the case of  $\mathbb{R}^{2d}$ . In our analysis, we do not need an exact expression but only to know the dependence on the derivatives of  $a$ . In fact, an exact expression would be very hard to explicit as it would depend on the choice of the atlas on the manifold. Finally, outline that equation (77) is an exact expression.

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