



**HAL**  
open science

# Limit theorems for additive functionals of a Markov chain

Milton Jara, Tomasz Komorowski, Stefano Olla

► **To cite this version:**

Milton Jara, Tomasz Komorowski, Stefano Olla. Limit theorems for additive functionals of a Markov chain. 2008. hal-00315784v2

**HAL Id: hal-00315784**

**<https://hal.science/hal-00315784v2>**

Preprint submitted on 18 Sep 2008 (v2), last revised 15 Dec 2009 (v5)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# LIMIT THEOREMS FOR ADDITIVE FUNCTIONALS OF A MARKOV CHAIN

M. JARA, T.KOMOROWSKI, AND S. OLLA

ABSTRACT. Consider a Markov chain  $\{X_n\}_{n \geq 0}$  with an ergodic probability measure  $\pi$ . Let  $\Psi$  a function on the state space of the chain, with  $\alpha$ -tails with respect to  $\pi$ ,  $\alpha \in (0, 2)$ . We find sufficient conditions on the probability transition to prove convergence in law of  $N^{1/\alpha} \sum_n^N \Psi(X_n)$  to a  $\alpha$ -stable law. “Martingale approximation” approach and “coupling” approach give two different sets of conditions. We extend these results to continuous time Markov jump processes  $X_t$ , whose skeleton chain satisfies our assumptions. If waiting time between jumps has finite expectation, we prove convergence of  $N^{-1/\alpha} \int_0^{Nt} V(X_s) ds$  to a stable process. In the case of waiting times with infinite average, we prove convergence to a Mittag-Leffler process.

## 1. INTRODUCTION

Superdiffusive transport of energy is generically observed in a certain class of one-dimensional systems. Numerically in chains of anharmonic oscillators of the Fermi-Pasta-Ulam type, and experimentally in carbon nanotubes (cf. [19] for a physical review). The nature of the stochastic processes describing these emerging macroscopic behaviours is object of a vivid debate in the physical literature, and very few mathematical results are present for deterministic microscopic models.

In [1] the macroscopic behaviour of the energy in a chain of harmonic oscillators with the hamiltonian dynamics perturbed by a stochastic terms conserving energy and momentum is studied. The macroscopic equation obtained there, in a proper kinetic limit, is a linear Boltzmann equation

$$\partial_t u(t, x, k) + \omega'(k) \partial_x u(t, x, k) = \int R(k, k') (u(t, x, k') - u(t, x, k)) dk', \quad (1.1)$$

where  $u(t, x, k)$  is the density at time  $t$  of energy of waves of Fourier’s mode  $k \in [0, 1]$ , and the velocity  $\omega'(k)$  is the derivative of the dispersion relation of the lattice. It turns out that the kernel  $R(k, k')$  is positive and symmetric, consequently (1.1) has an easy probabilistic interpretation as a forward equation for the evolution of the density for a Markov process

---

*Date:* September 18, 2008, version 2.

M.J. was supported by the Belgian Interuniversity Attraction Poles Program P6/02, through the network NOSY (Nonlinear systems, stochastic processes and statistical mechanics). M.J. would like to thank the hospitality of Université Paris-Dauphine and Maria Curie-Skłodowska University, where part of this work was accomplished. Work of T. K. has been partially supported by Polish MNiSW grant N 20104531 and by EC FP6 Marie Curie ToK programme SPADE2, MTKD-CT-2004-014508 and Polish MNiSW SPB-M. S.O. research was supported by French ANR LHMSHE n.BLAN07-2184264.

$(Y(t), K(t))$  on  $\mathbb{R} \times [0, 1]$ . In fact here  $K(t)$  is an autonomous jump process on  $[0, 1]$  with jump rate  $R(k, k')$ , and  $Y(t) = \int_0^t \omega'(K(s)) ds$  is a simple additive functional of  $K(t)$ . The conservation of momentum in the microscopic model imposes a very slow jump rate for small  $k$ :  $R(k, k') \sim k^2$  as  $k \sim 0$ . Since the velocity  $\omega'(k)$  remains of order 1, the behaviour of  $Y(t)$  is superdiffusive.

The above example motivated us in studying the question of the limiting behavior of an additive functional over a Markov chain  $\{X_n, n \geq 0\}$  taking values in a general Polish metric space  $(E, d)$ . Suppose that  $\pi$  is a stationary and ergodic probability measure for this chain. Consider a function  $\Psi : E \rightarrow \mathbb{R}$  and  $S_N := \sum_{n=0}^{N-1} \Psi(X_n)$ . If  $\Psi$  is centered with respect to  $\pi$ , and possesses a second moment one expects that the central limit theorem holds for  $N^{-1/2} S_N$ , as  $N \rightarrow +\infty$ . This, of course requires some assumptions on the rate of the decay of correlation of the chain, as well as hypotheses about its dynamics. When the dynamics of the chain has the spectral gap property, it has been observed by Gordin (see [13]), that the central limit theorem is valid under the above conditions. Different versions of the central limit theorem under various conditions on the dynamics of the chain have been shown, see e.g. [17, 6, 12, 7, 20], (cf. the review paper [22] for a more detailed list).

In this paper we are concerned with the limiting behavior of functionals formed over functions  $\Psi$  with heavy tails that satisfy the power law decay, i.e.  $\pi(\Psi > \lambda) \sim c_*^+ \lambda^{-\alpha}$  and  $\pi(\Psi < -\lambda) \sim c_*^- \lambda^{-\alpha}$  for  $\lambda \gg 1$  with  $\alpha \in (0, 2)$ . We prove sufficient conditions under which the laws of the functionals of the form  $N^{-1/\alpha} S_N$  converge weakly to  $\alpha$ -stable laws, as  $N \rightarrow +\infty$ . We approach this problem with two different techniques, obtaining different sets of conditions on the Markov chain.

The first approach is by martingale approximations, see Theorems 2.3 and 2.4. The results are obtained by decomposing the sequence  $\{S_N, N \geq 1\}$  into a martingale part, for which one can use well known stable laws, see e.g. [8, 4], and an negligible remainder term. This approach, that has been proven very powerful in the normal diffusive case (cf. the seminal paper [17]), works for a general class of probability transitions matrix, but unfortunately requires here the assumption of the existence of a spectral gap.

The second approach is based on a coupling technique, inspired by [5]. The coupling argument gives simpler proof, although under more restrictive assumptions of the form of the probability transition. We point out, however, that such assumptions are of local nature, in the sense that they involve only the behavior of the process around the singularity. In particular, the spectral gap condition (which is a global condition) can be relaxed in this coupling approach, to a moment bound for some regeneration times associated to the process (cf. Theorem 2.7).

Next we apply these results to a continuous time Markov jump process  $\{X_t, t \geq 0\}$  whose skeleton chain satisfies the assumptions made in the aforementioned theorems. We prove that in case the mean waiting time  $t(x)$  has a finite moment with respect to the invariant measure  $\pi$  and the tails of  $V(x)t(x)$  obey the power laws, as above, then the scaled functional of the form  $N^{-1/\alpha} S(Nt)$ , where  $S(t) := \int_0^t V(X_s) ds$ , converges in the sense of convergence of its finite dimensional distributions to the law of a stable process, see Theorem 2.8. In case the expectation of  $t(x)$  is infinite the limit may cease to be

Markovian and its one dimensional observables satisfy the Volterra type heat equation (cf. Theorem 2.10). This type of processes are sometimes called *fractional diffusions*, see e.g. [11].

Finally, we apply the obtained results to deal with the limiting behavior of the solution of the linear Boltzmann equation (1.1). We prove that the long time, large scale limit of solutions of such an equation tends to a fractional heat equation corresponding to a stable process with exponent  $\alpha = 3/2$ . Both approaches (martingale approximation and coupling) apply to this example.

*Note added to second version:* After submission of the first version of the present paper, we received a preprint by Mellet et al. [21] that contains a completely analytical proof of the convergence of the solution of a linear Boltzmann equation to a fractional diffusion. The conditions assumed in [21] imply the same spectral gap condition as in our Theorem 2.4, consequently the corresponding result in [21] is related to our Theorem 2.8.

## 2. PRELIMINARIES AND STATEMENTS OF THE MAIN RESULTS

**2.1. Some preliminaries on stable distributions.** In this paper we shall consider two types of stable laws. When  $\alpha \in (0, 1)$  we say that  $X$  is distributed according to a stable law of type *I* if its characteristic function is of the form  $\mathbb{E}e^{i\xi X} = e^{\psi(\xi)}$ , where the Levy exponent equals

$$\psi(\xi) := \alpha \int_{\mathbb{R}} (e^{i\lambda\xi} - 1) |\lambda|^{-1-\alpha} c_*(\lambda) d\lambda \tag{2.1}$$

and

$$c_*(\lambda) := \begin{cases} c_*^-, & \text{when } \lambda < 0, \\ c_*^+, & \text{when } \lambda > 0, \end{cases} \tag{2.2}$$

where  $c_*^-, c_*^+ \geq 0$  and  $c_*^- + c_*^+ > 0$ . On the other hand the stable law is of type *II* if  $\alpha \in (1, 2)$  and its Levy exponent equals

$$\psi(\xi) := \alpha \int_{\mathbb{R}} (e^{i\lambda\xi} - 1 - i\lambda\xi) |\lambda|^{-1-\alpha} c_*(\lambda) d\lambda. \tag{2.3}$$

We say that  $\{Z(t), t \geq 0\}$  is a stable process of type *I* (or *II*) if  $Z(0) = 0$  and it is a process with independent increments such that  $Z(1)$  is distributed according to a stable law of type *I* (*II*).

Suppose that  $\alpha \in (0, 1)$  and  $c_*^- = 0, c_*^+ > 0$ . A process  $\{Z(t), t \geq 0\}$  corresponding to this value of parameter has a.s. increasing trajectories. It is a *stable subordinator*, see e.g. [23], Example 24.12. Since the trajectory of the process is in fact a.s. strictly increasing, see [23] Theorem 21.3 p. 136, its right-continuous inverse  $Z^{-1}(t) := \inf\{s : Z(s) > t\}$  (called the first passage time) has a.s. continuous trajectories. Suppose that  $\{B(t), t \geq 0\}$  is a Brownian motion with diffusion coefficient  $\sigma > 0$ . The process  $\{\zeta_t := B(Z^{-1}(t)), t \geq 0\}$  is called a *Mittag-Leffler process*. One can show, see Theorem 2 of [11], that its one-point statistics  $u(t, x) := \mathbb{E}u_0(x + \zeta_t)$  satisfies *fractional kinetic equation*, i.e. the Volterra-type equation

$$u(t, x) = u_0(x) + \frac{\sigma^2}{2c_*^+\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\partial_x^2 u)(s, x) ds. \tag{2.4}$$

When  $u_0(x) = \delta_0(x)$  its Fourier transform in  $x$  is therefore given by

$$\hat{u}(t, \xi) = \int_{\mathbb{R}} e^{i\xi x} u(t, x) dx = E_{\alpha}(-\sigma^2 t^{\alpha} \xi^2 / (2c_*^+)),$$

where

$$E_{\alpha}(z) := \sum_{m=0}^{+\infty} \frac{z^m}{\Gamma(1 + m\alpha)}$$

is a *Mittag-Leffler function*.

**2.2. A Markov chain.** Let  $(E, d)$  be a Polish metric space,  $\mathcal{E}$  its Borel  $\sigma$ -algebra. Assume that  $\{X_n, n \geq 0\}$  is a Markov chain with the state space  $E$  and  $\pi$  - the law of  $X_0$  - is an *invariant* and *ergodic* measure for the chain.

Suppose that  $\Psi : E \rightarrow \mathbb{R}$  is a measurable function over  $(E, \mathcal{E})$  such that there exist  $\alpha \in (0, 2)$  and two non-negative constant  $c_*^+, c_*^-$  satisfying

$$\begin{aligned} \lim_{\lambda \rightarrow +\infty} \lambda^{\alpha} \pi(\Psi \geq \lambda) &= c_*^+ \\ \lim_{\lambda \rightarrow +\infty} \lambda^{\alpha} \pi(\Psi \leq -\lambda) &= c_*^- \end{aligned} \tag{2.5}$$

The above assumption guarantees that  $\Psi \in L^{\beta}(\pi)$  for any  $\beta < \alpha$ .

In the case  $\alpha \in (1, 2)$  we will always assume that  $\int \Psi d\pi = 0$ . We are interested in the asymptotic behavior of  $S_N := \sum_{n=1}^N \Psi(X_n)$ . We are looking for sufficient conditions on  $P$  such that the law  $N^{-1/\alpha} S_N$  converges to a  $\alpha$ -stable law.

We have two different approach (by martingale approximation and by coupling) with two separate set of conditions.

**2.3. The martingale approach result.** We suppose that the chain satisfies:

**Condition 2.1.** *Spectral gap condition:*

$$\sup[\|Pf\|_{L^2(\pi)} : f \perp 1, \|f\|_{L^2(\pi)} = 1] = a < 1. \tag{2.6}$$

Since  $P$  is also a contraction in  $L^1(\pi)$  and  $L^{\infty}(\pi)$  we conclude, via Riesz-Thorin interpolation theorem, that for any  $p \in [1, +\infty)$ :

$$\|Pf\|_{L^p(\pi)} \leq a^{|2/p-1|} \|f\|_{L^p(\pi)}, \tag{2.7}$$

for all  $f \in L^p(\pi)$ , such that  $\int f d\pi = 0$ .

In addition we assume that the tails of  $\Psi$  under the invariant measure do not differ very much from those with respect to the transition probabilities. Namely, we suppose that:

**Condition 2.2.** *there exist a measurable family of Borel measures  $Q(x, dy)$  and a measurable, non-negative function  $p(x, y)$  such that*

$$P(x, dy) = p(x, y)\pi(dy) + Q(x, dy), \quad \text{for all } x \in E, \tag{2.8}$$

$$C(2) := \iint p^2(x, y)\pi(dx)\pi(dy) < +\infty \tag{2.9}$$

and

$$\int Q^2(x, |\Psi| \geq \lambda) \pi(dx) \leq C \pi^2(|\Psi| \geq \lambda), \quad \forall \lambda \geq 0. \quad (2.10)$$

A simple consequence of (2.8) and the fact that  $\pi$  is invariant is that

$$\int p(x, y) \pi(dy) \leq 1 \quad \text{and} \quad \int p(y, x) \pi(dy) \leq 1, \quad \forall x \in E. \quad (2.11)$$

**Theorem 2.3.** *If  $\alpha \in (0, 1)$  and conditions 2.1 and 2.2 are satisfied, the law of  $N^{-1/\alpha} S_N$  converges weakly, as  $N \rightarrow +\infty$ , to a stable law of type I.*

If  $\alpha \in (1, 2)$  then, in particular  $\Psi$  possesses the first absolute moment.

**Theorem 2.4.** *Assume  $\alpha \in (1, 2)$ , and conditions 2.1 - 2.2 are satisfied. Furthermore, assume that for some  $\alpha' > \alpha$  we have*

$$\|P\Psi\|_{L^{\alpha'}(\pi)} < +\infty. \quad (2.12)$$

*Then, the law of  $N^{-1/\alpha} S_N$  converges weakly, as  $N \rightarrow +\infty$ , to a stable law of type II.*

#### 2.4. The coupling approach results.

**Condition 2.5.** *There exists a measurable function  $\theta : E \rightarrow [0, 1]$ , a probability  $q$  and a transition probability  $Q_1(x, dy)$ , such that*

$$P(x, dy) = \theta(x)q(dy) + (1 - \theta(x))Q_1(x, dy).$$

*Furthermore we assume that*

$$\bar{\theta} := \int \theta(x) \pi(dx) > 0, \quad (2.13)$$

*and that the tails of distribution of  $\Psi$  with respect to  $Q_1(x, dy)$  are uniformly lighter than its tails with respect to  $q$ :*

$$\lim_{\lambda \rightarrow \infty} \sup_{x \in E} \frac{Q_1(x, |\Psi| \geq \lambda)}{q(|\Psi| \geq \lambda)} = 0 \quad (2.14)$$

Clearly, because of (2.14), the function  $\Psi$  satisfies condition (2.5) also with respect to the measure  $q$ , but with different constants:

$$\begin{aligned} \lim_{\lambda \rightarrow +\infty} \lambda^\alpha q(\Psi > \lambda) c_*^+ \bar{\theta}^{-1} \\ \lim_{\lambda \rightarrow +\infty} \lambda^\alpha q(\Psi < -\lambda) c_*^- \bar{\theta}^{-1} \end{aligned} \quad (2.15)$$

The purpose of condition 2.5 is that it permits to define a Markov Chain  $\{(X_n, \delta_n), n \geq 1\}$  on  $E \times \{0, 1\}$  such that

$$\begin{aligned} \mathbb{P}(\delta_{n+1} = 0 | X_n = x, \delta_n = \epsilon) &= \theta(x) \\ \mathbb{P}(\delta_{n+1} = 1 | X_n = x, \delta_n = \epsilon) &= 1 - \theta(x) \\ \mathbb{P}(X_{n+1} \in A | \delta_{n+1} = 0, X_n = x, \delta_n = \epsilon) &= q(A) \\ \mathbb{P}(X_{n+1} \in A | \delta_{n+1} = 1, X_n = x, \delta_n = \epsilon) &= Q_1(x, A) \end{aligned} \quad (2.16)$$

We call this Markov chain the *basic coupling*. It is clear that the marginal chain  $\{X_n, n \geq 1\}$  has probability transition  $P$ . The dynamics of  $\{(X_n, \delta_n), n \geq 1\}$  is easy to understand. When  $X_n = x$ , we choose  $X_{n+1}$  according to the distribution  $q(dy)$  with probability  $\theta(x)$ , and according to the distribution  $Q_1(x, dy)$  with probability  $1 - \theta(x)$ .

Let  $\kappa_n$  be the  $n$ -th zero in the sequence  $\{\delta_n, n \geq 0\}$ . In a more precise way, define  $\kappa_0 = 0$  and, for  $i \geq 1$ ,

$$\kappa_i = \inf\{n > \kappa_{i-1}; \delta_n = 0\}.$$

Notice that the sequence  $\{\kappa_{i+1} - \kappa_i; i \geq 1\}$  is i.i.d. and  $\mathbb{E}(\kappa_{i+1} - \kappa_i)\bar{\theta}^{-1}$ . We call the sequence  $\{\kappa_n, n \geq 1\}$  the *regeneration times*.

Observe that, for any  $i \geq 1$ , the distribution of  $X_{\kappa_i}$  is given by  $q(dy)$ . In particular,  $X_{\kappa_i}$  is independent of  $\{X_0, \dots, X_{\kappa_i-1}\}$ . Therefore, the blocks

$$\{(X_{\kappa_i}, \delta_{\kappa_i}), \dots, (X_{\kappa_{i+1}-1}, \delta_{\kappa_{i+1}-1})\}$$

are independent. The dynamics for each one of these blocks is easy to understand. Start a Markov chain  $\{X_j^1, j \geq 1\}$  with initial distribution  $q(dy)$  and transition probability  $Q_1(x, dy)$ . At each step  $j$ , we stop the chain with probability  $\theta(X_j^1)$ . Each one of the blocks, except for the first one, has a distribution  $\{(X_1^1, 0), (X_2^1, 1), \dots, (X_{\kappa}^1, 1)\}$ , where  $\kappa$  is the stopping time. The first block is constructed in the same way, but starting from  $X_0^1 = X_0$  instead of with the law  $q(dy)$ . Now we are ready to state our last condition:

**Condition 2.6.**

$$\sum_{n=1}^{\infty} n^{1+\alpha} \mathbb{P}(\kappa \geq n) < +\infty.$$

**Theorem 2.7.** *Under conditions 2.5 and 2.6, the law of  $N^{-1/\alpha} S_N$  converges to a  $\alpha$ -stable law.*

**2.5. An additive functional of a continuous time jump process.** Suppose that  $\{\tau_n, n \geq 0\}$  is a sequence of i.i.d. random variables, independent of  $\mathcal{F} := \sigma(X_0, X_1, \dots)$  and such that  $\tau_0$  has exponential distribution with parameter 1. Suppose that  $t : E \rightarrow (0, +\infty)$  is a measurable function such that  $t(x) \geq t_* > 0, x \in E$ . Let

$$t_N := \sum_{n=0}^N t(X_n) \tau_n. \quad (2.17)$$

One can define a compound Poisson process  $X_t = X_n, t \in [t_N, t_{N+1})$ . It is Markovian, see e.g Section 2 of Appendix 1, pp. 314-321, of [16] with the generator

$$Lf(x) = t^{-1}(x) \int [f(y) - f(x)] P(x, dy), \quad f \in B_b(E). \quad (2.18)$$

Here  $B_b(E)$  is the space of bounded and Borel measurable functions on  $E$ . Let

$$\bar{t} := \int t d\pi \quad (2.19)$$



being finite, or not. Suppose  $V : E \rightarrow \mathbb{R}$  is measurable and  $\Psi(x) := V(x)t(x)$  satisfies condition (2.5). We shall be concerned with the limit of scaled processes

$$Y_N(t) := \frac{1}{N^{1/\alpha}} \int_0^{Nt} V(X(s))ds, \quad t \geq 0, \quad (2.20)$$

as  $N \rightarrow +\infty$ .

2.5.1. *The case when  $\bar{t} < +\infty$ .* Then,  $\bar{t}^{-1}t(x)\pi(dx)$  is an ergodic, invariant probability measure for  $\{X_t, t \geq 0\}$ . Our result can be formulated as follows.

**Theorem 2.8.** *i) Suppose that  $\alpha \in (0, 1)$  and the assumptions of either Theorem 2.3, or of Theorem 2.7, hold. Then, the finite distributions of processes  $\{Y_N(t), t \geq 0\}$  converge, as  $N \rightarrow +\infty$ , to the respective distributions of a stable process of type I.*

*ii) In case  $\alpha \in (1, 2)$  and the assumptions of either Theorem 2.4, or of Theorem 2.7, hold. Then, the convergence of finite dimensional distributions takes place to a stable process of type II.*

2.5.2. *The case when  $\bar{t} = +\infty$ .* We consider only the special situation when  $\Psi$  is centered, belongs to  $L^2(\pi)$  and  $t : E \rightarrow [0, +\infty)$  is a measurable function bounded from below, i.e.  $t(x) \geq t_* > 0, x \in E$ . We also assume that

$$\pi(t > \lambda)\lambda^{-\alpha} (c_*^+ + o(1)), \quad \text{as } \lambda \rightarrow +\infty \quad (2.21)$$

for some  $\alpha \in (0, 1)$  and parameter  $c_*^+ > 0$ .

Suppose that  $\{K_N, N \geq 1\}$  is an arbitrary sequence of positive numbers that increases monotonically to infinity. Consider the joint partial sum process  $(T_{K_N}(\cdot), B_{K_N}(\cdot))$ , where

$$\begin{aligned} T_{K_N}(t) &:= \frac{1}{K_N^{1/\alpha}} \sum_{n=0}^{[K_N t]} t(X_n)\tau_n, \\ B_{K_N}(t) &:= \frac{1}{K_N^{1/2}} \sum_{n=0}^{[K_N t]} \Psi(X_n)\tau_n, \quad t \geq 0. \end{aligned} \quad (2.22)$$

**Theorem 2.9.** *With the assumptions made as in Theorem 2.8, the laws of the joint process  $(B_{K_N}(\cdot), T_{K_N}(\cdot))$  converge weakly, as  $N \rightarrow +\infty$ , over  $D([0, +\infty), \mathbb{R}^2)$  to the joint law of  $(B(\cdot), T(\cdot))$ , where  $B(\cdot)$  is a Brownian motion and  $T(\cdot)$  is independent of it  $\alpha$  stable process with  $c_*^- = 0$  (subordinator).*

Let

$$Y_N(t) := \frac{1}{N^{\alpha/2}} \int_0^{Nt} V(X(s))ds, \quad t \geq 0, \quad (2.23)$$

As a corollary of Theorem 2.9 we shall show the following result.

**Theorem 2.10.** *Under the same assumptions made in Theorem 2.8, the processes  $\{Y_N(t), t \geq 0\}$  converge weakly in  $C[0, +\infty)$ , as  $N \rightarrow +\infty$ , to the limit that is a Mittag-Leffler process corresponding to a subordinator  $\{T(t), t \geq 0\}$  and Brownian motion  $\{B(t), t \geq 0\}$ .*



## 3. APPLICATION: SUPERDIFFUSION OF ENERGY IN A LATTICE DYNAMICS

In [1] it is proven that the Wigner distribution associated to the energy of a system of interacting oscillators with momentum and energy conserving noise, converges, in an appropriate kinetic limit to the solution  $u(t, x, k)$  of the linear kinetic equation

$$\begin{cases} \partial_t u(t, x, k) + \omega'(k) \partial_x u(t, x, k) = \mathcal{L}u(t, x, k), \\ u(0, x, k) = u_0(x, k), \end{cases} \quad (3.1)$$

where  $(t, x, k) \in [0, +\infty) \times \mathbb{R}^d \times \mathbb{T}^d$  and the initial condition  $u_0(\cdot, \cdot)$  is a function of class  $C^{1,0}(\mathbb{R}^d \times \mathbb{T}^d)$ . Here  $\mathbb{T}$  is the one dimensional circle, understood as the interval  $[-1/2, 1/2]$  with identified endpoints, and  $\mathbb{T}^d$  is the  $d$ -dimensional torus. The function  $\omega(k)$  is the dispersion relation of the lattice and it is assumed that  $\omega(-k) = \omega(k)$  and  $\omega(k) \sim |k|$  for  $|k| \sim 0$  (*acoustic dispersion*). The scattering operator  $\mathcal{L}$ , acting in (3.1) on variable  $k$ , is usually an integral operator that is a generator of a certain jump process.

In case  $d = 1$  the scattering operator is given by

$$\mathcal{L}f(k) = \int_{\mathbb{T}} R(k, k') [f(k') - f(k)] dk', \quad (3.2)$$

with the scattering kernel

$$\begin{aligned} R(k, k') &= \frac{4}{3} [2 \sin^2(2\pi k) \sin^2(\pi k') + 2 \sin^2(2\pi k') \sin^2(\pi k) \\ &\quad - \sin^2(2\pi k) \sin^2(2\pi k')] \end{aligned} \quad (3.3)$$

We shall assume that the dispersion relation is a function  $\omega : \mathbb{T} \rightarrow [0, +\infty)$  that satisfies,  $\omega \in C^1(\mathbb{T} \setminus \{0\})$  and

$$c_l |\sin(\pi k)| \leq \omega(k) \leq c_u |\sin(\pi k)|, \quad k \in \mathbb{T} \quad (3.4)$$

for some  $0 < c_l \leq c_u < +\infty$ , while

$$\lim_{k \rightarrow \pm 0} \omega'(k) = \pm c_\omega. \quad (3.5)$$

In the typical case of a simple one dimensional lattice, this is  $\omega(k) = c |\sin(\pi k)|$ .

The total scattering cross section is given by

$$R(k) = \int_{\mathbb{T}} R(k, k') dk' = \frac{4}{3} \sin^2(\pi k) (1 + 2 \cos^2(\pi k)) \quad (3.6)$$

We define  $t(k) = R(k)^{-1}$ , since these are the expected waiting times of the scattering process.

Let  $\{X_n, n \geq 0\}$  be a Markov chain on  $\mathbb{T}$  whose transition probability equals

$$P(k, dk') := t(k) R(k, k') dk'.$$

Suppose that  $\{\tau_n, n \geq 0\}$  is an i.i.d. sequence of random variables such that  $\tau_0$  is exponentially distributed with intensity 1. Let  $t_n := t(X_n) \tau_n$ ,  $n \geq 0$ . One can represent then the solution of (3.1) with the formula

$$u(t, x, k) = \mathbb{E} u_0(x(t), k(t)), \quad (3.7)$$

where

$$\begin{aligned} x(t) &= x + \int_0^t \omega'(k(s)) ds, \\ k(t) &= X_n, \quad t \in [t_n, t_{n+1}) \end{aligned}$$

and  $k(0) = X_0 = k$ . We shall be concerned in determining the weak limit of the finite dimensional distribution of the scaled process  $\{N^{-1/\alpha}x(Nt), t \geq 0\}$ , as  $N \rightarrow +\infty$ , for an appropriate scaling exponent  $\alpha$ .

It is straightforward to verify that

$$\pi(dk) = \frac{t^{-1}(k)}{\bar{R}} dk = \frac{R(k)}{\bar{R}} dk \quad (3.8)$$

where  $\bar{R} := \int_{\mathbb{T}} R(k) dk$  is a stationary and reversible measure for the chain. Then,  $P(k, dk') = p(k, k')\pi(dk')$ , where

$$p(k, k') = \bar{R} t(k)R(k, k')t(k').$$

and, after straightforward calculations, we obtain

$$\begin{aligned} p(k, k') &= 6[\cos^2(\pi k) + \cos^2(\pi k') - 2\cos^2(\pi k)\cos^2(\pi k')] \\ &\quad \times [(1 + 2\cos^2(\pi k))(1 + 2\cos^2(\pi k'))]^{-1} \\ &= 6\{[|\cos(\pi k)| - |\cos(\pi k')|]^2 + 2|\cos(\pi k)\cos(\pi k')|[1 - |\cos(\pi k)\cos(\pi k')|]\} \\ &\quad \times [(1 + 2\cos^2(\pi k))(1 + 2\cos^2(\pi k'))]^{-1}. \end{aligned} \quad (3.9)$$

We apply Theorem 2.8 and probabilistic representation (3.7) to describe the asymptotic behavior for long times and large spatial scales of solutions of the kinetic equation (3.1). We start our verification of the hypotheses of the theorem finding the tails of

$$\Psi(k) = \omega'(k)t(k) \quad (3.10)$$

under measure  $\pi$ . Since  $\omega'(k)$  is both bounded away from zero and also bounded from above, the tails of  $\Psi(k)$ , under  $\pi$ , are the same as those of  $t(k)$ . Note that

$$\pi(t(k) \geq \lambda) = C_R \lambda^{-3/2}(1 + O(1)) \quad \text{for } \lambda \gg 1 \quad (3.11)$$

and some  $C_R > 0$ . This verifies (2.5). Since the density of  $\pi$  with respect to the Lebesgue measure is even and  $\Psi$  is odd,  $\pi$  has null  $\pi$ -average.

To verify the remaining hypotheses of Theorem 2.8 note that we can decompose  $P(k, dk')$  as in (2.8) with  $p(k, k')$  given by (3.9) and  $Q(k, dk') \equiv 0$ . Since  $p(k, k')$  is bounded, conditions 2.2 and (2.12) are obviously satisfied. Operator  $P$  is a contraction on  $L^2(\pi)$  and, by the Hilbert-Schmidt theorem, see e.g. Theorem 4, p. 247 of [18], is symmetric and compact. In consequence, its spectrum is contained in  $[-1/2, 1/2]$  and is discrete, except for a possible accumulation point at 0.

**Lemma 3.1.** *Point 1 is a simple eigenvalue of both  $P$  and  $P^2$ .*

*Proof.* Suppose

$$Pf = f. \quad (3.12)$$

We claim that  $f$  is either everywhere positive, or everywhere negative. Let  $f^+, f^-$  be the positive and negative parts of  $f$ . Suppose also that  $f^+$  is non zero on a set of positive  $\pi$  measure. Then  $f = f^+ - f^-$  and  $Pf = Pf^+ - Pf^-$ . Thus  $f^+ = (Pf)^+ \leq Pf^+$ . Yet

$$\int f^+ d\pi \leq \int Pf^+ d\pi = \int f^+ d\pi,$$

thus  $Pf^+ = f^+$ . Likewise,  $Pf^- = f^-$ . Since for each  $k$  we have  $p(k, k') > 0$ , except for a set of  $k'$  of measure  $\pi$  zero, we conclude that  $f^+ > 0$   $\pi$  a.e., hence  $f^- \equiv 0$ .

Now, we know that  $P1 = 1$ . We claim that any other  $f \not\equiv 0$  that satisfies (3.12) belongs to  $\text{span}\{1\}$ . Otherwise  $f - c1$  for some  $c$  would suffer change of sign. But this contradicts our conclusion reached above so the lemma holds for  $P$ . The argument for  $P^2$  is analogous.  $\square$

As a corollary of the above lemma we conclude that condition 2.1 holds. Applying part ii) of Theorem 2.8 to  $N^{-2/3} \int_0^{Nt} \omega'(k(s)) ds$  we conclude that its finite dimensional distributions converge in law to an  $\alpha$ -stable Levy process for  $\alpha = 3/2$ . The following result summarizes our considerations in terms of the convergence of solution  $u(t, x, k)$  of the kinetic equation (3.1).

**Theorem 3.2.** *The finite dimensional distributions of scaled processes  $\{N^{-2/3}x(Nt), t \geq 0\}$  converge weakly to those of a stable process of type II. In addition, for any  $t > 0$ ,  $x \in \mathbb{R}$  we have*

$$\lim_{N \rightarrow +\infty} \int_{\mathbb{T}} |u(Nt, N^{2/3}x, k) - \bar{u}(t, x)|^2 dk = 0,$$

where  $u(t, x, k)$  satisfies (3.1) with the initial condition  $u_0(N^{-2/3}x, k)$ , such that  $u_0$  is compactly supported, and  $\bar{u}(t, x)$  is the solution of

$$\begin{cases} \partial_t \bar{u}(t, x) = -(-\partial_x^2)^{3/4} \bar{u}(t, x), \\ \bar{u}(0, x) = \int_{\mathbb{T}} u_0(x, k) dk. \end{cases} \quad (3.13)$$

*Proof.* To abbreviate the notation denote  $Y_N(t) := x + N^{-2/3} \int_0^{Nt} \omega'(k(s)) ds$ . Using probabilistic representation for a solution of (3.1) we can write

$$\begin{aligned} u(Nt, N^{3/2}x, k) &= \mathbb{E}_k u_0(Y_N(t), k(Nt)) \\ &= \sum_{\eta \in \mathbb{Z}} \int_{\mathbb{R}} \hat{u}_0(\xi, \eta) \mathbb{E}_k \exp\{i\xi Y_N(t) + i\eta k(Nt)\} d\xi. \end{aligned} \quad (3.14)$$

Here  $\hat{u}_0(\xi, \eta)$  is the Fourier transform of  $u_0(x, k)$  and  $\mathbb{E}_k$  is the expectation with respect to the path measure corresponding to the momentum process  $\{k(t), t \geq 0\}$  that satisfies  $k(0) = k$ . Since the dynamics of the momentum process is reversible with respect to the normalized Lebesgue measure  $m$  on the torus and 0 is a simple eigenvalue for the generator  $\mathcal{L}$  we have  $\|P^t f\|_{L^2(m)} \rightarrow 0$ , as  $t \rightarrow +\infty$ , provided  $\int_{\mathbb{T}} f dk = 0$ . Suppose that  $\{a_N, N \geq 1\}$  is

an increasing sequence of positive numbers tending to infinity and such that  $a_N N^{-3/2} \rightarrow 0$ . A simple calculation shows that for any  $\xi, \eta \in \mathbb{R}$  and  $e_\xi(x) := e^{ix\xi}$  we have

$$|\mathbb{E}_k[e_\xi(Y_N(t))e_\eta(k(Nt))] - \mathbb{E}_k[e_\xi(Y_N(t - ta_N/N))e_\eta(k(Nt))]| \rightarrow 0, \quad \text{as } N \rightarrow +\infty. \quad (3.15)$$

Using Markov property we can write that the second term under the absolute value in the formula above equals

$$\mathbb{E}_k[e_\xi(Y_N(t - ta_N/N))P^{a_N t}e_\eta(k((N - a_N)t))].$$

Let  $\tilde{e}_\eta(k) := e_\eta(k) - \bar{e}_\eta$ , where  $\bar{e}_\eta := \int_{\mathbb{T}} e_\eta(k) dk$ . By Cauchy-Schwartz inequality we obtain

$$\begin{aligned} & \left| \mathbb{E}_k[e_\xi(Y_N(t - ta_N/N))P^{a_N t}e_\eta(k((N - a_N)t))] - \mathbb{E}_k e_\xi(Y_N(t - ta_N/N))\bar{e}_\eta \right| \\ & \leq \left\{ \mathbb{E}_k \left| P^{a_N t} \tilde{e}_\eta(k((N - a_N)t)) \right|^2 \right\}^{1/2}. \end{aligned} \quad (3.16)$$

The right hand side of (3.16) tends to 0 in the  $L^2$  sense with respect to  $k \in \mathbb{T}$ , as  $N \rightarrow +\infty$ .

From (3.16) we conclude that

$$\begin{aligned} & \left| \sum_{\eta \in \mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{T}} \hat{u}_0(\xi, \eta) \mathbb{E}_k [e_\xi(Y_N(t - ta_N/N))P^{a_N t}e_\eta(k((N - a_N)t))] d\xi dk \right. \\ & \left. - \sum_{\eta \in \mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{T}} \hat{u}_0(\xi, \eta) \mathbb{E}_k e_\xi(Y_N(t - ta_N/N))\bar{e}_\eta d\xi dk \right| \rightarrow 0, \end{aligned} \quad (3.17)$$

as  $N \rightarrow +\infty$ . Combining this with (3.15) we conclude the proof of the theorem.  $\square$

**3.1. Proof of Theorem 3.2: the coupling method.** In this section we prove Theorem 3.2 by using the coupling approach of section 4. Define the functions

$$q_0(k) := \sin^2(2\pi k) = 4[\sin^2(\pi k) - \sin^4(\pi k)],$$

$$q_1(k) := \frac{4}{3} \sin^4(\pi k),$$

which are densities with respect to the Lebesgue measure in  $\mathbb{T}$ . A simple computation shows that  $R(k, k') = 2^{-4}[q_0(k)q_1(k') + q_1(k)q_0(k')]$  and therefore  $R(k) = 2^{-4}[q_0(k) + q_1(k)]$ . Therefore, the transition probability  $P(k, dk')$  can be written as

$$P(k, dk') = \frac{q_1(k)}{q_0(k) + q_1(k)} q_0(k') dk' + \frac{q_0(k)}{q_0(k) + q_1(k)} q_1(k') dk'.$$

In particular, in the notation of section 4, this model satisfies condition 2.5 with  $q(dk')q_0(k')dk'$ ,  $\theta = q_1/(q_0 + q_1)$  and  $Q_1(k, dk')q_1(k')dk'$ . Notice that the behavior around 0 of the measures  $\pi$  and  $q$  is the same. Therefore  $q(\Psi(k) \geq \lambda) \sim c\lambda^{-3/2}$  for  $\lambda \gg 1$ . We conclude therefore that the function  $\Psi(k)$ , given by (3.10), satisfies (2.15). Notice as well that  $Q_1$  does not depend on  $k$  and  $Q_1(k', t(k) \geq \lambda) \sim c\lambda^{-5/2}$  for  $\lambda \gg 1$ . Due to this last observation, condition 2.14 is satisfied.

We are only left to check condition 2.6. But this one is also simple, once we observe that the sequence  $\{\delta_n, n \geq 0\}$  is a Markov chain with transition probabilities

$$\begin{aligned} P(\delta_{n+1} = 1 | \delta_n = 0) &= P(\delta_{n+1} = 0 | \delta_n = 1) \\ &= \int_{-1}^1 \frac{q_0(k)q_1(k)dk}{q_0(k) + q_1(k)} \\ P(\delta_{n+1} = 1 | \delta_n = 1) &= \int_{-1/2}^{1/2} \frac{q_1^2(k)dk}{q_0(k) + q_1(k)} \\ P(\delta_{n+1} = 0 | \delta_n = 0) &= \int_{-1/2}^{1/2} \frac{q_0^2(k)dk}{q_0(k) + q_1(k)}. \end{aligned}$$

We conclude that the regeneration time  $\kappa$  satisfies  $\mathbb{E}[\exp\{\gamma\kappa\}] < +\infty$  for  $\gamma$  small enough, and therefore condition 2.6 as well.

#### 4. PROOF BY COUPLING

Because of its simplicity, we expose first the proof of Theorem 2.7, based on the basic coupling. This coupling permits to decomposed the initial Markov chain into independent blocks. Let us define

$$\varphi_i = \sum_{j=\kappa_i}^{\kappa_{i+1}-1} \Psi(X_j),$$

$$\mathcal{M}(N) = \sup\{i \geq 0; \kappa_i \leq N\}.$$

Note that  $\mathcal{M}(N) < +\infty$  a.s. An alternative way of defining  $\mathcal{M}(N)$  is demanding the inequality  $\kappa_{\mathcal{M}(N)} \leq N < \kappa_{\mathcal{M}(N)+1}$  to be satisfied. Then, we have

$$S_N = \sum_{i=0}^{\mathcal{M}(N)} \varphi_i + R_N, \tag{4.1}$$

where

$$R_N := \sum_{j=\kappa_{\mathcal{M}(N)+1}}^N \Psi(X_j).$$

In (4.1) we have decomposed  $S_N$  into a random sum of i.i.d. random variables  $\{\varphi_i, i \geq 1\}$  and two boundary terms:  $\varphi_0$  and  $R_N$ . Notice also that  $\kappa_N - \kappa_1$  is a sum of i.i.d. random variables. Consequently, the law of large numbers gives

$$\frac{\kappa_N}{N} \rightarrow \bar{\kappa} = \mathbb{E}(\kappa_2 - \kappa_1), \quad \text{and} \quad \frac{\mathcal{M}(N)}{N} \rightarrow \bar{\kappa}^{-1} = \bar{\theta}, \tag{4.2}$$

a.s., as  $N \rightarrow +\infty$ .

The idea now is that under conditions 2.5 – 2.6, the random variable  $\varphi_i$  is equal to  $\Psi(X_{\kappa_i})$  plus a term with lighter tails. Before stating this result, we need a simple lemma:

**Lemma 4.1.** *Let  $\zeta$  be a random variable such that*

$$\lim_{x \rightarrow \infty} x^\alpha \mathbb{P}(\zeta > x) = c^+, \quad \lim_{x \rightarrow \infty} x^\alpha \mathbb{P}(\zeta < -x) = c^-.$$

*Let  $\xi$  be such that  $\lim_{x \rightarrow \infty} \mathbb{P}(|\xi| > x) / \mathbb{P}(|\zeta| > x) = 0$ . Then,*

$$\lim_{x \rightarrow \infty} x^\alpha \mathbb{P}(\zeta + \xi > x) = c^+, \quad \lim_{x \rightarrow \infty} x^\alpha \mathbb{P}(\zeta + \xi < -x) = c^-.$$

*Proof.* Without loss of generality, we just consider the first limit, the second one follows considering  $-\zeta, -\xi$ . We will prove that the  $\liminf_{x \rightarrow \infty}$  of the previous expression is bigger than  $c_+$  and the  $\limsup$  is smaller than  $c_+$ . We start with the upper bound: for any  $\epsilon > 0$ ,

$$\begin{aligned} x^\alpha \mathbb{P}(\zeta + \xi > x) &\leq x^\alpha \mathbb{P}(\zeta > (1 - \epsilon)x) + x^\alpha \mathbb{P}(\xi > \epsilon x) = \\ &\leq \frac{c_+}{(1 - \epsilon)^\alpha} + \frac{\mathbb{P}(|\xi| > \epsilon x)}{\mathbb{P}(\zeta > \epsilon x)} \frac{c_+}{\epsilon^\alpha}. \end{aligned}$$

Now take above the limit as  $x \rightarrow +\infty$  to get

$$\limsup_{x \rightarrow +\infty} x^\alpha \mathbb{P}(\zeta + \xi > x) \leq \frac{c_+}{(1 - \epsilon)^\alpha}.$$

Since  $\epsilon$  is arbitrary, we have proved the upper bound. The lower bound is very similar:

$$\begin{aligned} \mathbb{P}(\zeta + \xi > x) &= \mathbb{P}(\zeta + \xi > x, \xi > -\epsilon x) + \mathbb{P}(\zeta + \xi > x, \xi \leq -\epsilon x) \\ &\geq \mathbb{P}(\zeta > (1 + \epsilon)x, \xi > -\epsilon x) = \\ &\geq \mathbb{P}(\zeta > (1 + \epsilon)x) - \mathbb{P}(\zeta > (1 + \epsilon)x, \xi \leq -\epsilon x) \\ &\geq \mathbb{P}(\zeta > (1 + \epsilon)x) - \mathbb{P}(\xi < -\epsilon x). \end{aligned}$$

Starting from this last expression, the same computations done for the upper bound show that

$$\liminf_{x \rightarrow +\infty} x^\alpha \mathbb{P}(\zeta + \xi > x) \geq \frac{c_+}{(1 + \epsilon)^\alpha}.$$

Since  $\epsilon > 0$  is arbitrary, the lemma is proved.  $\square$

**Lemma 4.2.** *Let  $\Psi$  satisfy (2.5) with constants  $c_*^+, c_*^-$  together with conditions 2.5 - 2.6. Then, the law of each  $\varphi_i$  satisfies*

$$\begin{aligned} \lim_{\lambda \rightarrow +\infty} \lambda^\alpha \mathbb{P}(\varphi_i > \lambda) c_*^+ \bar{\theta}^{-1} \\ \lim_{\lambda \rightarrow +\infty} \lambda^\alpha \mathbb{P}(\varphi_i < -\lambda) c_*^- \bar{\theta}^{-1} \end{aligned} \tag{4.3}$$

*Proof.* The idea of the proof is simple.  $\varphi_i$  is the sum of a random variable with an  $\alpha$ -tail,  $\Psi(X_{\kappa_i})$ , and a finite (but random) number of random variables with lighter tails ( $\Psi(X_{\kappa_i+1}), \dots, \Psi(X_{\kappa_i+1-1})$ ). By condition 2.6, the random number can be efficiently controlled. To simplify the notation, assume that  $X_0$  is distributed according to  $q$ , so the first

block is also distributed like the other ones. Then,

$$\begin{aligned}
\mathbb{P}\left(\sum_{j=1}^{\kappa_1-1} \Psi(X_j) \geq t\right) &= \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{n-1} \Psi(X_j) \geq t, \kappa_1 = n\right) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{n-1} \Psi(X_j) \geq t, \kappa_1 \geq n\right) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{n-1} \Psi(X_j) \geq t \mid \kappa_1 \geq n\right) \mathbb{P}(\kappa_1 \geq n) \\
&\leq \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \mathbb{P}(\Psi(X_j) \geq t/(n-1) \mid \kappa_1 \geq n) \mathbb{P}(\kappa_1 \geq n) \\
&\leq \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \mathbb{P}(\Psi(X_j) \geq t/(n-1) \mid \delta_j = 1) \mathbb{P}(\kappa_1 \geq n) \\
&\leq \sum_{n=1}^{\infty} \frac{g(t/n)n^{1+\alpha}}{t^\alpha} \mathbb{P}(\kappa_1 \geq n)
\end{aligned}$$

for some bounded function  $g(x)$  that goes to 0 as  $x \rightarrow \infty$ . We have used the Markov property in the third line and condition (2.15) in the last line. We conclude that this last term is  $o(t^{-\alpha})$  by invoking Fatou's lemma and condition 2.6. The negative tails are treated in the same way. Therefore,  $\varphi_0 - \Psi(X_0)$  has lighter tails than  $\Psi(X_0)$  itself. By lemma 4.1, the sum of a random variable satisfying condition (2.15) and a random variable with lighter tails also satisfies condition (2.15) for the same constants  $c^+\bar{\theta}^{-1}$ ,  $c^-\bar{\theta}^{-1}$ .  $\square$

At this point we are only left to recall the classical limit theorem for i.i.d. random variables. It follows

$$\begin{aligned}
N^{-1/\alpha} S_N &= \left(\frac{\mathcal{M}(N)}{N}\right)^{1/\alpha} \frac{1}{\mathcal{M}(N)^{1/\alpha}} \sum_{i=0}^{\mathcal{M}(N)} \varphi_i + \frac{1}{N^{1/\alpha}} \sum_{j=\kappa_{\mathcal{M}(N)}+1}^N \Psi(X_j) \\
&= \left(\frac{\mathcal{M}(N)}{N\bar{\theta}}\right)^{1/\alpha} \frac{1}{\mathcal{M}(N)^{1/\alpha}} \sum_{i=0}^{\mathcal{M}(N)} \bar{\theta}^{1/\alpha} \varphi_i + \frac{1}{N^{1/\alpha}} \sum_{j=\kappa_{\mathcal{M}(N)}+1}^N \Psi(X_j)
\end{aligned}$$

Recall (4.2), and notice that, by (4.3),

$$\begin{aligned}
\lim_{\lambda \rightarrow +\infty} \lambda^\alpha \mathbb{P}(\bar{\theta}^{1/\alpha} \varphi_i > \lambda) &= c_*^+ \\
\lim_{\lambda \rightarrow +\infty} \lambda^\alpha \mathbb{P}(\bar{\theta}^{1/\alpha} \varphi_i < -\lambda) &= c_*^-
\end{aligned}$$

Denote  $\mathcal{K}_N := N^{-1/\alpha} \sum_{i=0}^N \bar{\theta}^{1/\alpha} \varphi_i$ . By virtue of the stable limit theorem for i.i.d. random variables, see e.g. [14] Theorem 2.6.1, p. 76, we know that the laws of  $\mathcal{K}_N$  converge to an  $\alpha$ -stable law  $\nu_*$ . Since  $\mathcal{L}_N := \bar{\theta}^{-1/\alpha} N^{-1} \mathcal{M}(N)$  converges a.s. to 1, the joint law of  $(\mathcal{K}_N, \mathcal{L}_N)$



converges  $\nu_* \otimes \delta_1$ , as  $N \rightarrow +\infty$ . According to Skorochod's representation theorem there exists a probability space and random variables  $(\bar{\mathcal{K}}_N, \bar{\mathcal{L}}_N)$  such that  $(\bar{\mathcal{K}}_N, \bar{\mathcal{L}}_N) \stackrel{d}{=} (\mathcal{K}_N, \mathcal{L}_N)$  for each  $N$  and  $(\bar{\mathcal{K}}_N, \bar{\mathcal{L}}_N) \rightarrow (Y_*, 1)$  a.s.. The above in particular implies that  $\bar{\mathcal{K}}_{N\bar{\mathcal{L}}_N}$  converges a.s. to  $Y_*$ . Since  $\bar{\mathcal{K}}_{N\bar{\mathcal{L}}_N} \stackrel{d}{=} \mathcal{K}_{N\mathcal{L}_N}$  we conclude the convergence of the laws of  $\mathcal{K}_{N\mathcal{L}_N}$  to  $\nu_*$ .

## 5. PROOF BY MARTINGALE APPROXIMATION

Below we formulate a stable limit law that substantiates applications of Propositions 6.1 and 6.5 in the proof of Theorem 7.1. The proof of this result is essentially contained in Section 4 of [8]. We present it in the Appendix for the convenience of a reader.

Suppose that  $\{Z_n : n \geq 1\}$  is a stationary sequence that is adapted with respect to the filtration  $\{\mathcal{G}_n : n \geq 0\}$  and such that for any  $f$  bounded and measurable  $\{\mathbb{E}[f(Z_n)|\mathcal{G}_{n-1}] : n \geq 1\}$  is also stationary. We assume furthermore that:

$$\mathbb{E}[Z_n|\mathcal{G}_{n-1}] = 0 \quad \text{for } n \geq 1, \quad (5.1)$$

and there exist  $\alpha \in (0, 2)$  and  $c_*^+, c_*^- \geq 0$  such that  $c_*^+ + c_*^- > 0$  and

$$\begin{aligned} \mathbb{P}[Z_1 > \lambda] &= \lambda^{-\alpha}(c_*^+ + o(1)), \quad \text{as } \lambda \rightarrow +\infty, \\ \mathbb{P}[Z_1 < -\lambda] &= \lambda^{-\alpha}(c_*^- + o(1)), \quad \text{as } \lambda \rightarrow +\infty. \end{aligned} \quad (5.2)$$

In addition, for any  $g \in C_0^\infty(\mathbb{R} \setminus \{0\})$  we have

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left[ \sum_{n=1}^{\lfloor Nt \rfloor} \mathbb{E} \left[ g \left( \frac{Z_n}{N^{1/\alpha}} \right) | \mathcal{G}_{n-1} \right] - \alpha t \int_{\mathbb{R}} g(\lambda) \frac{c_*(\lambda) d\lambda}{|\lambda|^{1+\alpha}} \right] = 0 \quad (5.3)$$

and

$$\lim_{N \rightarrow +\infty} N \mathbb{E} \left\{ \mathbb{E} \left[ g \left( \frac{Z_1}{N^{1/\alpha}} \right) | \mathcal{G}_0 \right] \right\}^2 = 0. \quad (5.4)$$

Here  $c_*(\cdot)$  appearing in (5.3) is given by (2.2). Let  $M_N := \sum_{n=1}^N Z_n$ ,  $N \geq 1$  and  $M_0 := 0$ .

**Theorem 5.1.** *i) Suppose that  $\alpha \in (1, 2)$  and that assumptions (5.1)-(5.4) hold. We have then  $N^{-1/\alpha} M_{\lfloor N \cdot \rfloor} \Rightarrow Z(\cdot)$ , as  $N \rightarrow +\infty$ , weakly in  $D[0, +\infty)$ , where  $\{Z(t), t \geq 0\}$  is an  $\alpha$ -stable process with the characteristic functional*

$$\mathbb{E} e^{i\xi Z(t)} = \exp \left\{ t \int_{-\infty}^{+\infty} (e^{i\xi\lambda} - 1 - i\xi\lambda) |\lambda|^{-1-\alpha} c_*(\lambda) d\lambda \right\}, \quad \xi \in \mathbb{R}, t \geq 0. \quad (5.5)$$

*ii) Suppose that  $\alpha \in (0, 1)$  and that assumptions (5.2)-(5.4) hold. We have then  $N^{-1/\alpha} M_{\lfloor N \cdot \rfloor} \Rightarrow Z(\cdot)$ , as  $N \rightarrow +\infty$ , weakly in  $D[0, +\infty)$ , where  $\{Z(t), t \geq 0\}$  is an  $\alpha$ -stable process with the characteristic functional*

$$\mathbb{E} e^{i\xi Z(t)} = \exp \left\{ t \int_{-\infty}^{+\infty} (e^{i\xi\lambda} - 1) |\lambda|^{-1-\alpha} c_*(\lambda) d\lambda \right\}, \quad \xi \in \mathbb{R}, t \geq 0. \quad (5.6)$$

## 6. THE PROOFS OF THEOREMS 2.3 AND 2.4

**6.1. The proof of Theorem 2.4.** Let  $\chi \in L^\beta(\pi)$ ,  $\beta \in (1, \alpha)$  be the unique, zero mean, solution of the equation

$$\chi - P\chi = \Psi. \quad (6.1)$$

Since  $\Psi \in L^\beta(\pi)$  for  $\beta \in (0, \alpha)$  is of zero mean, the solution to (6.1) exists in  $L^\beta(\pi)$  and is given by  $\chi = \sum_{n \geq 0} P^n \Psi$ . This follows from the fact that  $\|P^n \Psi\|_{L^\beta} \leq a^{2/\alpha-1} \|\Psi\|_{L^\beta}$ ,  $n \geq 0$ , (see (2.7)) so the series defining  $\chi$  geometrically converges. Uniqueness follows from the spectral gap. Note also that from (2.12) it follows that in fact  $P\chi = (I - P)^{-1}(P\Psi) \in L^{\alpha'}(\pi)$ . Thus, in particular

$$\pi(|P\chi| > \lambda) \leq \frac{\|P\chi\|_{L^{\alpha'}(\pi)}^{\alpha'}}{\lambda^{\alpha'}}. \quad (6.2)$$

and consequently  $\chi$  satisfies the same tail condition as  $\Psi$  (cf. (2.5)).

Then, by using (6.1), we can write

$$S_N = \sum_{n=1}^N \Psi(X_n) = \sum_{n=1}^N Z_n + P\chi(X_0) - P\chi(X_N) \quad (6.3)$$

with  $Z_n = \chi(X_n) - P\chi(X_{n-1})$ . According to part i) of Theorem 5.1, we only need to demonstrate the following.

**Proposition 6.1.** *For any  $g \in C_0^\infty(\mathbb{R} \setminus \{0\})$*

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left| \sum_{n=1}^N \mathbb{E} \left[ g \left( \frac{Z_n}{N^{1/\alpha}} \right) \middle| \mathcal{G}_{n-1} \right] - \int_{\mathbb{R}} g(\lambda) \frac{c_*(\lambda) d\lambda}{|\lambda|^{1+\alpha}} \right| = 0 \quad (6.4)$$

and

$$\lim_{N \rightarrow +\infty} N \mathbb{E} \left\{ \mathbb{E} \left[ g \left( \frac{Z_1}{N^{1/\alpha}} \right) \middle| \mathcal{G}_0 \right] \right\}^2 = 0. \quad (6.5)$$

More explicitly we have

$$\mathbb{E} \left[ g \left( \frac{Z_n}{N^{1/\alpha}} \right) \middle| \mathcal{G}_{n-1} \right] = \int g(N^{-1/\alpha} [\chi(y) - P\chi(X_{n-1})]) P(X_{n-1}, dy)$$

and using stationarity of  $\pi$  we can bound the left hand side of (6.4) by

$$\begin{aligned} & \mathbb{E} \left| \sum_{n=1}^N \int \left[ g(N^{-1/\alpha} [\chi(y) - P\chi(X_{n-1})]) - g \left( \frac{\chi(y)}{N^{1/\alpha}} \right) \right] P(X_{n-1}, dy) \right| \\ & + \mathbb{E} \left| \sum_{n=1}^N \int g \left( \frac{\chi(y)}{N^{1/\alpha}} \right) P(X_{n-1}, dy) - N \int g \left( \frac{\chi(y)}{N^{1/\alpha}} \right) \pi(dy) \right| \\ & + \left| N \int g \left( \frac{\chi(y)}{N^{1/\alpha}} \right) \pi(dy) - \alpha \int_{\mathbb{R}} g(\lambda) \frac{c_*(\lambda) d\lambda}{|\lambda|^{1+\alpha}} \right| \end{aligned} \quad (6.6)$$

so (6.4) is a consequence of the following three lemmas, each taking care of the respective term of (6.6):

**Lemma 6.2.**

$$\lim_{N \rightarrow \infty} N \iint |g(N^{-1/\alpha}[\chi(y) - P\chi(x)]) - g(N^{-1/\alpha}\chi(y))| P(x, dy)\pi(dx) = 0 \quad (6.7)$$

**Lemma 6.3.**

$$\mathbb{E} \left| \sum_{n=1}^N \mathbb{E} [g(N^{-1/\alpha}\chi(X_n)) | \mathcal{G}_{n-1}] - N \int g(N^{-1/\alpha}\chi(y)) \pi(dy) \right| = 0. \quad (6.8)$$

**Lemma 6.4.**

$$\lim_{N \rightarrow \infty} \left| N \int g(N^{-1/\alpha}\chi(y)) \pi(dy) - \alpha \int_{\mathbb{R}} g(\lambda) \frac{c_*(\lambda)d\lambda}{|\lambda|^{1+\alpha}} \right| = 0, \quad (6.9)$$

where  $c_*(\cdot)$  is given by (2.2).

*Proof of lemma 6.2.*

Suppose that  $\text{supp } g \subset [-M, M] \setminus [-m, m]$  for some  $0 < m < M < +\infty$  and  $\theta > 0$ . Denote

$$A_{N,\theta} = \{(x, y) : |\chi(y) - \theta P\chi(x)| > N^{1/\alpha}m\}$$

The left hand side of (6.7) can be bounded from above by

$$\begin{aligned} N^{1-1/\alpha} \int_0^1 d\theta \iint |g'(N^{-1/\alpha}[\chi(y) - \theta P\chi(x)])P\chi(x)| P(x, dy)\pi(dx) \\ \leq CN^{1-1/\alpha} \int_0^1 d\theta \iint_{A_{N,\theta}} |P\chi(x)| P(x, dy)\pi(dx) \\ \leq CN^{1-1/\alpha} \int_0^1 d\theta \left( \iint_{A_{N,\theta}} P(x, dy)\pi(dx) \right)^{1-1/\alpha'} \|P\chi\|_{L^{\alpha'}} \end{aligned}$$

From the tail behavior of  $\chi$  and of  $P\chi$ , see (6.2) and the remark below that estimate, it is easy to see that for any  $\theta \in [0, 1]$

$$\begin{aligned} \iint_{A_{N,\theta}} P(x, dy)\pi(dx) &\leq \mathbb{P}[|\chi(X_1)| \geq (mN^{1/\alpha})/2] + \mathbb{P}[|P\chi(X_0)| \geq (mN^{1/\alpha})/2] \\ &\leq C[(Nm^\alpha)^{-1} + (Nm^\alpha)^{-\alpha'/\alpha}] = \frac{C}{N}(1 + o(1)), \end{aligned}$$

as  $N \gg 1$ . Since  $\alpha' > \alpha$  we obtain (6.7).  $\square$

*Proof of lemma 6.3.* To simplify the notation we assume that  $\text{supp } g \subset [m, M]$  for  $0 < m < M < +\infty$ . Denote  $B_{N,\lambda} = \{y : \chi(y) \geq N^{1/\alpha}\lambda\}$ . We can rewrite the left hand

side of (6.8) as

$$\mathbb{E} \left| \int_0^\infty g'(\lambda) \sum_{n=1}^N G_N(X_{n-1}, \lambda) d\lambda \right| \quad (6.10)$$

where

$$G_N(x, \lambda) = P(x, B_{N,\lambda}) - \pi(B_{N,\lambda}).$$

Notice that  $\int G_N(y, \lambda) \pi(dy) = 0$  and

$$\begin{aligned} & \int G_N^2(y, \lambda) \pi(dy) = \int P^2(y, B_{N,\lambda}) \pi(dy) - \pi^2(B_{N,\lambda}) \\ & \leq 2 \int \left( \int_{B_{N,\lambda}} p(x, y) \pi(dx) \right)^2 \pi(dy) + 2 \int Q^2(x, B_{N,\lambda}) \pi(dy) - \pi^2(B_{N,\lambda}) \end{aligned} \quad (6.11)$$

To estimate the first term on the utmost right hand side we use Cauchy Schwartz inequality, while for the second one we apply condition (2.10). For  $\lambda \geq m$  we can bound the expression on the right hand side of (6.11) by

$$\begin{aligned} & \frac{1}{2} \pi(B_{N,m}) \int \int_{B_{N,m}} p^2(x, y) \pi(dx) \pi(dy) + C \pi^2(B_{N,m}) \\ & \leq \frac{1}{N} o(1), \quad \text{as } N \rightarrow \infty, \end{aligned} \quad (6.12)$$

by virtue of (2.5) and (2.9). Thus, we have shown that

$$N \sup_{\lambda \geq m} \int G_N^2(y, \lambda) \pi(dy) \rightarrow 0, \quad (6.13)$$

as  $N \rightarrow \infty$ . We will show now that (6.13) and the spectral gap together imply that

$$\sup_{\lambda \geq m} \mathbb{E} \left| \sum_{n=1}^N G_N(X_{n-1}, \lambda) \right|^2 \rightarrow 0, \quad (6.14)$$

as  $N \rightarrow \infty$ . Since  $\text{supp } g' \subset [m, M]$  expression in (6.10) can be then estimated by

$$\sup_{\lambda \geq m} \mathbb{E} \left| \sum_{n=1}^N G_N(X_{n-1}, \lambda) \right| \times \int_0^\infty |g'(\lambda)| d\lambda \rightarrow 0,$$

as  $N \rightarrow +\infty$  and the conclusion of the lemma follows.

To prove (6.13) let  $u_N(\cdot, \lambda) = (I - P)^{-1} G_N(\cdot, \lambda)$ . By the spectral gap condition (2.7) we have

$$\int u_N^2(y, \lambda) \pi(dy) \leq \frac{1}{1-a} \int G_N^2(y, \lambda) \pi(dy) \quad (6.15)$$

We can rewrite then

$$\sum_{n=1}^N G_N(X_{n-1}, \lambda) = u_N(X_0) - u_N(X_N) + \sum_{n=1}^{N-1} Z_n,$$

where  $Z_n = u_N(X_n) - Pu_N(X_{n-1})$ ,  $n \geq 1$  is a stationary sequence of martingale differences with respect to the natural filtration corresponding to  $\{X_n, n \geq 0\}$ . Consequently,

$$\mathbb{E} \left| \sum_{n=1}^N G_N(X_{n-1}, \lambda) \right|^2 \leq CN \int u_N^2(y, \lambda) \pi(dy) \rightarrow 0$$

and (6.14) follows from (6.13) and (6.15).  $\square$

*Proof of Lemma 6.4.* To avoid long notation we again assume that  $\text{supp } g \subset [m, M]$  for  $0 < m < M < +\infty$ . The proof in case  $g \subset [-M, -m]$  is virtually the same. Note that

$$\begin{aligned} & N \int g \left( \frac{\chi(y)}{N^{1/\alpha}} \right) \pi(dy) \\ &= N \int \int_0^{+\infty} N^{-1/\alpha} g' \left( \frac{\lambda}{N^{1/\alpha}} \right) 1_{[0, \chi(y)]}(\lambda) \pi(dy) d\lambda \\ &= N \int_0^{+\infty} N^{-1/\alpha} g' \left( \frac{\lambda}{N^{1/\alpha}} \right) \pi(\chi > \lambda) d\lambda \\ &= N \int_0^{+\infty} g'(\lambda) \pi(\chi \geq N^{1/\alpha} \lambda) d\lambda. \end{aligned}$$

Thanks to (2.5) the last expression tends however, as  $N \rightarrow +\infty$ , to

$$\int_0^{+\infty} g'(\lambda) \frac{c_*^+ d\lambda}{\lambda^\alpha} = \alpha \int_{\mathbb{R}} g(\lambda) \frac{c_*(\lambda) d\lambda}{|\lambda|^{\alpha+1}}.$$

$\square$

**6.2. The proof of Theorem 2.3.** The proof of this part relies on part ii) of Theorem 5.1. The following analogue of Proposition 6.1 can be established.

**Proposition 6.5.** *Suppose that  $\alpha \in (0, 1)$ . Then, for any  $g \in C_0^\infty(\mathbb{R} \setminus \{0\})$*

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left| \sum_{n=1}^N \mathbb{E} \left[ g \left( \frac{\Psi(X_n)}{N^{1/\alpha}} \right) \mid \mathcal{G}_{n-1} \right] - \int_{\mathbb{R}} g(\lambda) \frac{C_*(\lambda) d\lambda}{|\lambda|^{1+\alpha}} \right| = 0 \quad (6.16)$$

and

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left\{ \mathbb{E} \left[ g \left( \frac{\Psi(X_1)}{N^{1/\alpha}} \right) \mid \mathcal{G}_0 \right] \right\}^2 = 0. \quad (6.17)$$

The proof of this proposition is a simplified version of the argument used for the proof of Proposition 6.1. We can repeat word by word the argument used.

## 7. THE PROOFS OF THEOREMS 2.8 AND 2.10

**7.1. The case when the first moment of the mean of jump times is finite.** Suppose that we are given a sequence of i.i.d. nonnegative random variables  $\{\rho_n, n \geq 0\}$  independent of  $\{X_n, n \geq 0\}$  and such that  $A_\alpha := \int_0^{+\infty} \rho^\alpha \varphi(d\rho) < +\infty$ , where  $\varphi(\cdot)$  is the distribuant of  $\rho_0$  and  $\alpha \in (0, 2)$ . We consider a slightly more general situation than the one presented in Theorem 2.4 by allowing

$$S_N(t) := \sum_{n=0}^{[Nt]} \Psi(X_n) \rho_n. \quad (7.1)$$

Observe that, if  $X_n$  has law  $\pi$ ,  $\Psi$  satisfies the tail conditions (2.5), and  $\rho_n$  is independent of  $X_n$ , then

$$\lambda^\alpha \mathbb{P}(\Psi(X_n) \rho_n > \lambda) \int_0^\infty \lambda^\alpha \pi(\Psi > \lambda \rho^{-1}) \varphi(d\rho) \xrightarrow{\lambda \rightarrow +\infty} c_*^+ A_\alpha$$

Consider then the Markov chain  $\{(X_n, \rho_n), n \geq 0\}$  on  $E \times R_+$ . This Markov chain satisfies all conditions used in the previous sections, with stationary ergodic measure given by  $\pi(dy) \otimes \varphi(d\rho)$ . Then with the same argument as used in section 6 we get:

**Theorem 7.1.** *i) Under the assumptions of Theorem 2.4 we have  $N^{-1/\alpha} S_N(\cdot) \xrightarrow{f.d.} Z(\cdot)$ , as  $N \rightarrow +\infty$ , where  $\{Z(t), t \geq 0\}$  is an  $\alpha$ -stable process of type II with the parameters of the corresponding Levy measure, cf (2.2), given by*

$$C_*(\lambda) := \begin{cases} \alpha A_\alpha c_*^-, & \text{when } \lambda < 0, \\ \alpha A_\alpha c_*^+, & \text{when } \lambda > 0. \end{cases} \quad (7.2)$$

Here  $\xrightarrow{f.d.}$  denotes the convergence in the sense of finite dimensional distributions.

*ii) Under the assumptions of Theorem 2.3 we have  $N^{-1/\alpha} S_N(\cdot) \xrightarrow{f.d.} Z(\cdot)$ , as  $N \rightarrow +\infty$ . Here  $\{Z(t), t \geq 0\}$  is an  $\alpha$ -stable process of type I with the respective parameters given by (7.2)*

Let us consider now the process  $Y_N(t)$  defined by (2.20). We only show that one dimensional distributions of  $Y_N(t)$  converge weakly to the respective distribution of a suitable stable process  $\{Z(t), t \geq 0\}$ . The proof of convergence of finite dimensional distributions can be done in the same way.

Given  $t > 0$  define  $n(t)$  as the positive integer, such that

$$t_{n(t)} \leq t < t_{n(t)+1},$$

where  $t_N$  is given by (2.17). Let

$$s(t) := t/\bar{t},$$

$$B_N(t) := N^{-1/\alpha} \sum_{k=0}^{[Nt]} \Psi(X_k) \tau_k, \quad t \geq 0,$$

where, as we recall,  $\Psi(x) := V(x)t(x)$ ,  $x \in E$  and  $\{\tau_k, k \geq 0\}$  is a sequence of i.i.d. variables distributed according to an exponential distribution with parameter 1. Using the ergodic theorem one can easily conclude that

$$s_N(t) := \frac{n(Nt)}{N} \rightarrow s(t), \quad \text{as } N \rightarrow +\infty \quad (7.3)$$

a.s. uniformly on intervals of the form  $[t_0, T]$ , where  $0 < t_0 < T$ . We have

$$Y_N(t) = \frac{1}{N^{1/\alpha}} \sum_{k=0}^{n(Nt)-1} \Psi(X_k)\tau_k + \frac{Nt - t_{n(Nt)}}{N^{1/\alpha}} V(X_k).$$

Note that

$$\frac{1}{N^{1/\alpha}} \sum_{k=0}^{n(Nt)} \Psi(X_k)\tau_k = B_N(s_N(t)).$$

**Lemma 7.2.** *For any  $t > 0$  and  $\varepsilon > 0$  fixed we have*

$$\lim_{N \rightarrow +\infty} \mathbb{P}[|Y_N(t) - B_N(s_N(t))| > \varepsilon] = 0. \quad (7.4)$$

*Proof.* Let  $\sigma > 0$  be arbitrary. We can write that

$$\begin{aligned} \mathbb{P}[|Y_N(t) - B_N(s_N(t))| > \varepsilon] &\leq \mathbb{P}[|s_N(t) - s(t)| > \varepsilon] \\ &+ \mathbb{P}[|s_N(t) - s(t)| \leq \sigma, |Y_N(t) - B_N(s_N(t))| > \varepsilon]. \end{aligned} \quad (7.5)$$

The second term on the right hand side can be estimated from above by

$$\begin{aligned} &\mathbb{P}[|s_N(t) - s(t)| \leq \sigma, N^{-1/\alpha} |\Psi(X_{n(Nt)})| \tau_{n(Nt)} > \varepsilon] \\ &\leq \mathbb{P}[\sup\{|\Psi(X_k)|\tau_k : k \in [(s(t) - \sigma)N, (s(t) + \sigma)N]\} > N^{1/\alpha}\varepsilon]. \end{aligned}$$

Using stationarity of  $\{|\Psi(X_k)|\tau_k, k \geq 0\}$  the term on the right hand side equals

$$\begin{aligned} &\mathbb{P}[\sup\{|\Psi(X_k)|\tau_k : k \in [0, 2\sigma N]\} > N^{1/\alpha}\varepsilon] \\ &\leq 2\sigma N \int_0^{+\infty} e^{-\tau} \pi[|\Psi(x)| \geq \tau^{-1} N^{1/\alpha}\varepsilon] d\tau \leq \frac{C\sigma}{\varepsilon^\alpha} \end{aligned}$$

for some constant  $C > 0$ , by virtue of (2.5). From (7.5) we obtain therefore

$$\limsup_{N \rightarrow +\infty} \mathbb{P}[|Y_N(t) - B_N(s_N(t))| > \varepsilon] \leq \frac{C\sigma}{\varepsilon^\alpha}$$

for an arbitrary  $\sigma > 0$ , which in turn implies (7.4).  $\square$

It suffices therefore to prove that the laws of  $B_N(s_N(t))$  converge, as  $N \rightarrow +\infty$ , to the law of the respective stable process. According to Skorochod's embedding theorem one can find pairs of random elements  $(\tilde{B}_N(\cdot), \tilde{s}_N(t))$ ,  $N \geq 1$ , with values in  $D[0, +\infty) \times [0, +\infty)$ , such that the law of each pair is identical with that of  $(B_N(\cdot), s_N(t))$  and  $(\tilde{B}_N(\cdot), \tilde{s}_N(t))$  converges a.s., as  $N \rightarrow +\infty$ , in the Skorochod topology to  $(Z(\cdot), s(t))$ . Here,  $\{Z(t), t \geq 0\}$  is the stable process, as in Theorem 7.1. According to Proposition 3.5.3 p. 119 of [9] the



above means that for each  $T > 0$  there exist a sequence of increasing homeomorphisms  $\lambda_N : [0, T] \rightarrow [0, T]$  such that

$$\lim_{N \rightarrow +\infty} \gamma(\lambda_N) = 0, \quad (7.6)$$

where

$$\gamma(\lambda_N) := \sup_{0 < s < t < T} \left| \log \frac{\lambda_N(t) - \lambda_N(s)}{t - s} \right| = 0,$$

and

$$\sup_{t \in [0, T]} |\tilde{B}_N \circ \lambda_N(t) - Z(t)| = 0. \quad (7.7)$$

As a consequence of (7.6) we have of course that

$$\lim_{N \rightarrow +\infty} \sup_{t \in [0, T]} |\lambda_N(t) - t| = 0. \quad (7.8)$$

Note that the law of each  $B_N(s_N(t))$  is identical with that of  $\tilde{B}_N(\tilde{s}_N(t))$ . We also have

$$\begin{aligned} & |\tilde{B}_N(\tilde{s}_N(t)) - Z(s(t))| \leq |\tilde{B}_N(\tilde{s}_N(t)) - Z \circ \lambda_N^{-1}(\tilde{s}_N(t))| \\ & + |Z \circ \lambda_N^{-1}(\tilde{s}_N(t)) - Z(s(t))|. \end{aligned}$$

The right hand side however vanishes a.s., as  $N \rightarrow +\infty$ , thanks to (7.7), (7.8) and the fact that for each fixed  $s > 0$  one has  $\mathbb{P}[Z(s-) = Z(s)] = 1$ , see e.g. Theorem 11.1, p. 59 of [23]. The above allows us to conclude that  $|\tilde{B}_N(\tilde{s}_N(t)) - Z(s(t))| \rightarrow 0$  a.s., as  $N \rightarrow +\infty$ , thus the assertion of Theorem 2.8 follows.

## 7.2. The case when the first moment of the mean of jump times is infinite.

Recall that throughout this section  $\alpha \in (0, 1)$ . To simplify the notation and avoid writing double subscripts we shall assume here that  $K_N = N$ . A reader can easily generalize the argument to an arbitrary  $\{K_N, N \geq 1\}$  as in the statement of Theorem 2.9

Recall that  $\{T_N(t), t \geq 0\}$  is the process defined by (2.22). For any  $1 < \Delta < +\infty$  consider also the processes

$$T_N^\Delta(t) := \frac{1}{N^{1/\alpha}} \sum_{n=0}^{[Nt]} t(X_n) \tau_n 1[t(X_n) \tau_n < \Delta N^{1/\alpha}], \quad (7.9)$$

and

$$\begin{aligned} \tilde{T}_N^\Delta(t) &:= \frac{1}{N^{1/\alpha}} \sum_{n=0}^{[Nt]} \{t(X_n) \tau_n 1[t(X_n) \tau_n < \Delta N^{1/\alpha}] - \\ & \mathbb{E}[t(X_n) \tau_n 1[t(X_n) \tau_n < \Delta N^{1/\alpha}] | \mathcal{G}_{n-1}]\}. \end{aligned} \quad (7.10)$$

Here  $\mathcal{G}_{-1}$  is the trivial  $\sigma$ -algebra. For a given  $L > 0$  and  $x(\cdot) \in D[0, T]$  we let also  $w''(x, \delta) := \sup_{0 \leq t_1 < t < t_2 \leq L} [\max(|x(t_2) - x(t)|, |x(t) - x(t_1)|), |t_2 - t_1| \leq \delta]$ , see p. 131 of [2].

7.2.1. *Tightness of the laws of  $\{(B_N(t), T_N(t)), t \geq 0\}$ .* We know from the discussion carried out in Section A.2 of the Appendix that each of the families of processes given by (2.22), (7.9) and (7.10) is tight in  $D[0, +\infty)$ . In consequence, see remark on p. 141 after Theorem 13.2 of [2], for an arbitrary  $L, \varepsilon, \eta > 0$  one can find  $\delta > 0$  such that

$$\begin{aligned} \limsup_{N \rightarrow +\infty} \mathbb{P}[|T_N(L) - T_N(L - \delta)| \geq \varepsilon, \text{ or } |T_N(\delta)| \geq \varepsilon] &\leq \eta, \\ \limsup_{N \rightarrow +\infty} \mathbb{P}[w''(T_N(\cdot), \delta) \geq \varepsilon, \text{ or } |T_N(\delta)| \geq \varepsilon] &\leq \eta. \end{aligned} \quad (7.11)$$

In fact, the weak limit of  $\{T_N(t), t \geq 0\}$  is the law of a stable process  $\{T(t), t \geq 0\}$  of type I with the parameter  $c_*(\lambda) = c_* := \Gamma(\alpha + 1)c_*^+$  for  $\lambda > 0$  and  $c_*(\lambda) = 0$  for  $\lambda < 0$ . It is also well known that the sequence  $\{B_N(t), t \geq 0\}$  converges weakly in  $D[0, +\infty)$  to a Brownian motion  $\{B(t), t \geq 0\}$ , described by the characteristic function  $\mathbb{E}e^{iB(t)\xi} = \exp\{-c_B t \xi^2\}$ . In particular an analogue of (7.11) holds for  $\{B_N(t), t \geq 0\}$ . It is clear therefore that the family of processes  $\{(B_N(t), T_N(t)), t \geq 0\}$  is tight.

7.2.2. *Convergence of finite dimensional distributions.* We prove that for any  $0 = t_0 < t_1 < \dots < t_N$  and  $\theta_1, \theta_2 \in \mathbb{R}$  the finite dimensional distributions of  $\{\theta_1 B_N(t) + \theta_2 T_N(t), t \geq 0\}$  at respective times converge to the corresponding finite dimensional distribution of  $\{\theta_1 B(t) + \theta_2 T(t), t \geq 0\}$ , where  $B(\cdot)$  and  $T(\cdot)$  are independent Brownian motions and the stable process, described in the previous section. Let

$$\tilde{B}_N(t) := \frac{1}{N^{1/2}} \sum_{n=0}^{[Nt]} R_0(X_{n+1}, X_n, \tau_{n+1}), \quad (7.12)$$

where  $\chi \in L^2(\pi)$  is the unique zero mean solution of (6.1) and

$$R_0(x, y, \tau) := \Psi(x)(\tau - 1) + \chi(x) - P\chi(y) \quad x, y, \in E, \tau \in \mathbb{R}.$$

Note that from (6.1) it follows that  $\|\Psi\|_{L^p(\pi)} \leq 2\|\chi\|_{L^p(\pi)}$  for all  $p \in [1, 2]$ . For each  $T > 0$  we have

$$\lim_{N \rightarrow +\infty} \sup_{t \in [0, T]} |\tilde{B}_N(t) - B_N(t)| = 0 \quad \text{a.s.} \quad (7.13)$$

Indeed,

$$\tilde{B}_N(t) - B_N(t) = N^{-1/2} [\Psi(X_{[Nt]+1})\tau_{[Nt]+1} + P\chi(X_{[Nt]+1}) - P\chi(X_0)].$$

and (7.13) is a consequence of the following elementary fact, proven in Appendix B.

**Lemma 7.3.** *Suppose that  $\{Z_n, n \geq 0\}$  is a stationary sequence such that  $\mathbb{E}|Z_0| < +\infty$ . Then*

$$\lim_{N \rightarrow +\infty} N^{-1} \max\{Z_0, \dots, Z_N\} = 0$$

*both a.s. and in the  $L^1$  sense.*

To prove (7.13) it suffices only to take  $Z_n := [\Psi(X_n)\tau_n + P\chi(X_n)]^2$ . Formula (7.13) and Lemma A.9 allow us to reduce the proof to showing that the finite dimensional distributions

of  $\{\theta_1 \tilde{B}_N(t) + \theta_2 \tilde{T}_N^\Delta(t), t \geq 0\}$  converge in law to  $\{\theta_1 B(t) + \theta_2 \tilde{T}^\Delta(t), t \geq 0\}$ , where  $\tilde{T}^\Delta(\cdot)$  is a Levy process, independent of Brownian motion  $\{B(t), t \geq 0\}$  with the exponent

$$\tilde{\psi}_\Delta(\xi) := \alpha \int_0^\Delta (e^{i\lambda\xi} - 1 - i\lambda\xi) |\lambda|^{-1-\alpha} c_* d\lambda. \quad (7.14)$$

We only prove the convergence of one dimensional marginals. The general case is not that much different. We use Theorem 1 p. 450 of [4]. Let  $Z_{n,N} := \theta_1 Z_{n,N}^{(1)} + \theta_2 Z_{n,N}^{(2)}$ , where  $Z_{0,N}^{(1)} := 0$  and

$$\begin{aligned} Z_{n,N}^{(1)} &:= \frac{1}{N^{1/2}} R_0(X_n, X_{n-1}, \tau_n), \quad n \geq 1, \\ Z_{n,N}^{(2)} &:= \frac{1}{N^{1/\alpha}} \{t(X_n) \tau_n 1[t(X_n) \tau_n < \Delta_N] \\ &\quad - \mathbb{E}[t(X_n) \tau_n 1[t(X_n) \tau_n < \Delta_N] | \mathcal{G}_{n-1}]\}, \quad n \geq 0, \end{aligned}$$

Here  $\Delta_N := \Delta N^{1/\alpha}$ . Note that  $\{Z_{n,N}, n \geq 0\}$  constitute an array of martingale differences. The following result holds.

**Proposition 7.4.** *There exists a bounded increasing function  $G_{\theta_1, \theta_2}(\cdot)$  such that*

$$\lim_{N \rightarrow +\infty} \sum_{n=1}^{[Nt]-1} \mathbb{E} [Z_{n,N}^2 1[a < Z_{n,N} < b] | \mathcal{G}_{n-1}] = t[G_{\theta_1, \theta_2}(b) - G_{\theta_1, \theta_2}(a)] \quad \text{for any } a < b \quad (7.15)$$

in probability. The function  $G_{\theta_1, \theta_2}(\cdot)$  satisfies

$$\int_{\mathbb{R}} (e^{i\xi\lambda} - 1 - i\xi\lambda) \lambda^{-2} G_{\theta_1, \theta_2}(d\lambda) = -(c_B \theta_1^2 \xi^2 + c_* |\theta_2|^\alpha |\xi|^\alpha).$$

Here  $\mathcal{G}_n, n \geq 0$  is the natural filtration corresponding to the sequence  $\{(X_n, \tau_n), n \geq 0\}$ .

According to Theorem 1 of [4], the above proposition implies that the characteristic function of the limiting process equals  $\exp\{-t(c_B \theta_1^2 \xi^2 + c_* |\theta_2|^\alpha |\xi|^\alpha)\}$ . This concludes the proof of the convergence of finite dimensional distributions.

**7.2.3. The proof of Proposition 7.4.** Let  $\phi_\Delta(x) := x 1[x < \Delta]$ . Suppose that  $g(x) = x^2 \varphi(x)$  where  $\varphi \in C_0^\infty(\mathbb{R})$  is such that  $0 \leq \varphi \leq 1$ ,  $\text{supp } \varphi \subset (a, b)$  and  $0 < a < b$ . We can expand  $g(Z_{n+1,N})$  using Taylor formula, up to a second derivative, around  $z^{(N)}(X_{n+1}, \tau_{n+1})$ , where  $z^{(N)}(x, \tau) := N^{-1/\alpha} \theta_2 \phi_{\Delta_N}(\tau t(x))$ , and obtain

$$\begin{aligned} \sum_{n=0}^{[Nt]-1} \mathbb{E} [g(Z_{n+1,N}) | \mathcal{G}_n] &= \sum_{n=0}^{[Nt]-1} \mathbb{E} [g(z^{(N)}(X_{n+1}, \tau_{n+1})) | \mathcal{G}_n] \\ &\quad + \sum_{n=0}^{[Nt]-1} \mathbb{E} \{R(X_{n+1}, X_n, \tau_{n+1}) g'(z^{(N)}(X_{n+1}, \tau_{n+1})) | \mathcal{G}_n\} \end{aligned} \quad (7.16)$$

$$+ \sum_{n=0}^{\lfloor Nt \rfloor - 1} \int_0^1 d\lambda \int_0^\lambda \mathbb{E} [R^2(X_{n+1}, X_n, \tau_{n+1}) g''(z_n^{(N)}(\lambda')) | \mathcal{G}_n] d\lambda',$$

where

$$z_n^{(N)}(\lambda') := \lambda' R(X_{n+1}, X_n, \tau_{n+1}) + z(X_{n+1}, \tau_{n+1})$$

and

$$R(x, y, \tau) := \frac{\theta_1}{N^{1/2}} R_0(x, y, \tau) - \frac{\theta_2}{N^{1/\alpha}} P(\phi_{\Delta_N}(\tau t))(y).$$

Denote the terms appearing on the right hand side of (7.16) by  $I_N$ ,  $II_N$  and  $III_N$  respectively. To calculate the limit of  $I_N$  we can use again the argument made in the proof of Proposition 6.1, cf. also Proposition 6.5, and obtain that

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left| I_N - \alpha \Gamma(\alpha) t |\theta_2|^\alpha \int_0^\Delta g(\lambda) \frac{c_*^+ d\lambda}{|\lambda|^{1+\alpha}} \right| = 0. \quad (7.17)$$

The remaining terms  $II_N$  and  $III_N$  tend to 0, as  $N \rightarrow +\infty$ , in the  $L^1$  sense. Indeed,

$$\mathbb{E} |II_N| \leq E_N^{(1)} + E_N^{(2)},$$

where

$$\begin{aligned} E_N^{(1)} &:= |\theta_1| N^{1/2} \mathbb{E} |R_0(X_1, X_0, \tau_1) g'(z(X_1, \tau_1))| \\ E_N^{(2)} &:= |\theta_2| N^{1-1/\alpha} \mathbb{E} |P(\phi_{\Delta_N}(\tau_1 t))(X_0) g'(z(X_1, \tau_1))|. \end{aligned}$$

We have

$$\begin{aligned} E_N^{(1)} &\leq C N^{1/2} \|\chi\|_{L^1(\pi)} \|g'\|_\infty \\ &\times \int_0^{+\infty} (1+\tau) e^{-\tau} \pi[|t(x)|\tau > a|\theta_2^{-1}|N^{1/\alpha}] d\tau \stackrel{(2.21)}{\leq} \frac{C}{N^{1/2}} \end{aligned}$$

for all  $N \geq 1$  and some  $C > 0$ . On the other hand,

$$\begin{aligned} \|P(\phi_{\Delta_N}(\tau t))\|_{L^2(\pi)}^2 &\leq \|\phi_{\Delta_N}(\tau t)\|_{L^2(\pi)}^2 \\ &\leq \tau^2 \int_{[\tau t(y) < \Delta_N]} t^2(y) \pi(dy) \\ &= 2\tau^2 \int_0^{\Delta_N \tau^{-1}} \lambda \pi[t(y) > \lambda] d\lambda \\ &\leq C\tau^2 \int_0^{\Delta_N \tau^{-1}} \frac{\lambda d\lambda}{1+\lambda^\alpha} \leq C' N^{2(1/\alpha-1)} \tau^{2\alpha} \end{aligned} \quad (7.18)$$

for some constants  $C, C' > 0$ . Hence,

$$\mathbb{E} \|P(\phi_{\Delta_N}(\tau_1 t))\|_{L^2(\pi)}^2 \leq C' N^{2(1/\alpha-1)} \quad (7.19)$$

and in consequence

$$\begin{aligned} E_N^{(2)} &\leq |\theta_2| N^{1-1/\alpha} \|g'\|_\infty \mathbb{E} \left[ P(\phi_{\Delta_N}(\tau_1 t))(X_0), |\theta_2| t(X_1) \tau_1 > a N^{1/\alpha} \right] \\ &\leq C N^{1-1/\alpha} \left\{ \mathbb{E} \|P(\phi_{\Delta_N}(\tau_1 t))\|_{L^2(\pi)}^2 \right\}^{1/2} \mathbb{P}^{1/2} [|\theta_2| t(X_1) \tau_1 > a N^{1/\alpha}] \leq \frac{C}{\sqrt{N}} \rightarrow 0 \end{aligned}$$

for some constant  $C > 0$ . This proves that  $\mathbb{E}|II_N| \rightarrow 0$ , as  $N \rightarrow +\infty$ .

Note also that from (7.19) it follows that for any  $m > 0$

$$\mathbb{P}[z^{(N)}(X_1, \tau_1) > m] \leq \frac{|\theta_2| \mathbb{E} \|P(\phi_{\Delta_N}(\tau_1 t))\|_{L^2(\pi)}}{m N^{1/\alpha}} = C N^{-1} \rightarrow 0,$$

as  $N \rightarrow +\infty$ . Likewise one can show that for any  $\lambda' \in [0, 1]$

$$\mathbb{P}[z_0^{(N)}(\lambda') > m] \rightarrow 0, \quad \text{as } N \rightarrow +\infty. \quad (7.20)$$

We have therefore

$$\begin{aligned} \mathbb{E}|III_N| &\leq N(t+1) \|g''\|_\infty \int_0^1 \mathbb{E} \left[ R^2(X_1, X_0, \tau_1), |z_0^{(N)}(\lambda')| \geq a \mid \mathcal{G}_0 \right] d\lambda' \\ &\leq C N(t+1) \|g''\|_\infty \left\{ N^{-1} \|\chi\|_{L^2(\pi)}^2 \int_0^1 \mathbb{P} \left[ |z_0^{(N)}(\lambda')| \geq a \right] d\lambda' \right. \\ &\quad \left. + N^{-2/\alpha} \mathbb{E} \|P(\phi_{\Delta_N}(\tau_1 t))\|_{L^2(\pi)}^2 \right\} \rightarrow 0, \quad \text{as } N \rightarrow +\infty \end{aligned}$$

by virtue of (7.19) and (7.20).

Suppose now that  $a > 0$  and  $\text{supp } \varphi \subset (-a, a)$ . Note that

$$\begin{aligned} &\sum_{n=1}^{[Nt]} \mathbb{E} \left[ (Z_{n,N})^2 1[-a < Z_{n,N} < a] \mid \mathcal{G}_{n-1} \right] \\ &= \theta_1^2 \sum_{n=1}^{[Nt]} \mathbb{E} \left[ (Z_{n,N}^{(1)})^2 1[-a < Z_{n,N} < a] \mid \mathcal{G}_{n-1} \right] \\ &\quad + \theta_2^2 \sum_{n=1}^{[Nt]} \mathbb{E} \left[ (Z_{n,N}^{(2)})^2 1[-a < Z_{n,N} < a] \mid \mathcal{G}_{n-1} \right] \\ &\quad + 2\theta_1 \theta_2 \sum_{n=1}^{[Nt]} \mathbb{E} \left[ Z_{n,N}^{(1)} Z_{n,N}^{(2)} 1[-a < Z_{n,N} < a] \mid \mathcal{G}_{n-1} \right]. \end{aligned}$$

Denote the terms on the right hand side by  $U_N$ ,  $V_N$  and  $W_N$ . For appropriate constants  $C, C' > 0$  we have

$$\begin{aligned} \mathbb{E}|W_N| &\leq C \left\{ N \left\{ \mathbb{E} [|\theta_1 Z_{1,N}^{(1)}|^2] \right\}^{1/2} \left\{ \mathbb{E} [|\theta_2 Z_{1,N}^{(2)}|^2], |\theta_2 Z_{1,N}^{(2)}| < 10a \right\}^{1/2} \right. \\ &\quad \left. + N |\theta_1 \theta_2| \mathbb{E} \left[ |Z_{1,N}^{(1)} Z_{1,N}^{(2)}|, |\theta_1 Z_{1,N}^{(1)}| > 9a \right] \right\} \end{aligned}$$

Denote the first and second term appearing in the braces on the right hand side of the above estimate by  $W_N^{(1)}$  and  $W_N^{(2)}$  respectively. We have

$$\begin{aligned} \mathbb{P}[|\theta_2 Z_{n,N}^{(2)}| > \lambda] &\leq \mathbb{P}[|\theta_2|t(X_n)\tau_n > N^{1/\alpha}\lambda/2] \\ &+ \int_0^{+\infty} e^{-\tau}\pi[|\theta_2|P(\phi_{\Delta_N}(\tau t)) > N^{1/\alpha}\lambda/2]d\tau \leq \frac{C}{N\lambda^\alpha} \end{aligned} \quad (7.21)$$

for all  $\lambda > 0$  and  $N \geq 1$  and a certain constant  $C > 0$ , independent of  $n \geq 0$ . Using (7.21) and an elementary estimate

$$\left\{ \mathbb{E}[[\theta_1 Z_{1,N}^{(1)}]^2] \right\}^{1/2} \leq CN^{-1/2}\|\chi\|_{L^2(\pi)}$$

we obtain

$$W_N^{(1)} \leq C\|\chi\|_{L^2(\pi)} \int_0^{10a} \lambda^{1-\alpha} d\lambda \leq C'\|\chi\|_{L^2(\pi)} a^{2-\alpha} \quad (7.22)$$

for some constants  $C, C' > 0$ .

On the other hand, using Chebyshev's inequality we get

$$\mathbb{P}[|\theta_1 Z_{n,N}^{(1)}| > \lambda] \leq \frac{C\|\chi\|_{L^2(\pi)}^2}{N\lambda^2} \quad (7.23)$$

for all  $\lambda > 0$ . The constant  $C > 0$  appearing here does not depend on  $N, n$  and  $\lambda$ . Thus, for some constants  $C, C' > 0$ , we have

$$\begin{aligned} W_N^{(2)} &\leq CN\Delta \mathbb{E} \left[ |\theta_1 Z_{1,N}^{(1)}|, |\theta_1 Z_{1,N}^{(1)}| > 9a \right] \\ &\leq CN\Delta \left\{ \mathbb{E} \left[ [\theta_1 Z_{1,N}^{(1)}]^2, |\theta_1 Z_{1,N}^{(1)}| > 9a \right] \right\}^{1/2} \mathbb{P}^{1/2} \left[ |\theta_1 Z_{1,N}^{(1)}| > 9a \right] \\ &\stackrel{(7.23)}{\leq} C' \left\{ \mathbb{E} \left[ R(X_1, X_0, \tau_0)^2, |\theta_1 Z_{1,N}^{(1)}| > 9a \right] \right\}^{1/2} \rightarrow 0, \end{aligned} \quad (7.24)$$

as  $N \rightarrow +\infty$ . We have proved therefore that

$$\limsup_{n \rightarrow +\infty} \mathbb{E}|W_N| \leq Ca^{2-\alpha},$$

where  $a > 0$  can be chosen to be as small as we wish. We have shown therefore that  $\lim_{n \rightarrow +\infty} \mathbb{E}|W_N| = 0$ .

Note that

$$\begin{aligned} \mathbb{E}|V_N| &\leq CNE[[\theta_2 Z_{n,N}^{(2)}]^2, |\theta_2 Z_{n,N}^{(2)}| < 10a] + \\ &+ CN^{1-2/\alpha} \mathbb{E}[\phi_{\Delta_N}^2(\tau_0 t(X_0)), |\theta_1 Z_{n,N}^{(1)}| > 9a] \end{aligned} \quad (7.25)$$

for some constant  $C > 0$ . Denote the first and the second terms on the right hand side of (7.25) by  $V_N^{(1)}, V_N^{(2)}$  respectively. Using (7.21) we obtain

$$V_N^{(1)} \leq C \int_0^{10a} \lambda^{1-\alpha} d\lambda \leq C'a^{2-\alpha} \quad (7.26)$$

for some constant  $C' > 0$ . This term can be made arbitrarily small by choosing a sufficiently small  $a > 0$ . On the other hand, from Chebyshev's inequality

$$\begin{aligned} V_N^{(2)} &\leq CN\Delta^2\mathbb{P}[|\theta_1 R_0(X_1, X_0, \tau_1)| \geq 9aN^{1/2}] \\ &\leq C'\Delta^2\mathbb{E}[|\theta_1 R_0(X_1, X_0, \tau_1)|^2, |\theta_1 R_0(X_1, X_0, \tau_1)| \geq 9aN^{1/2}] \rightarrow 0, \end{aligned}$$

both a.s. and in the  $L^1$  sense, as  $N \rightarrow +\infty$ . Finally, we can write that

$$U_N = \tilde{U}_N - \bar{U}_N,$$

where

$$\tilde{U}_N := \theta_1^2 \sum_{n=1}^{[Nt]} \mathbb{E} \left[ (Z_{n,N}^{(1)})^2 \mid \mathcal{G}_{n-1} \right]$$

and

$$\bar{U}_N := \theta_1^2 \sum_{n=1}^{[Nt]} \mathbb{E} \left[ (Z_{n,N}^{(1)})^2 1_{[|Z_{n,N}| > a \mid \mathcal{G}_{n-1}]} \right].$$

We have, by the ergodic theorem,

$$\tilde{U}_N = \frac{\theta_1^2}{N} \sum_{n=1}^{[Nt]} [P\Psi^2(X_{n-1}) + P\chi^2(X_{n-1}) - (P\chi)^2(X_{n-1})] \rightarrow \frac{1}{2}\sigma^2\theta_1^2 t, \quad \text{as } N \rightarrow +\infty,$$

both a.s. and in the  $L^1$  sense. Here

$$\sigma^2 := 2 \left\{ \|\Psi\|_{L^2(\pi)}^2 + \|\chi\|_{L^2(\pi)}^2 - \|P\chi\|_{L^2(\pi)}^2 \right\}.$$

We can also estimate, using stationarity of  $\{(\tau_n, X_n), n \geq 1\}$ , that

$$\mathbb{E}|\bar{U}_N| \leq C\theta_1^2\mathbb{E}[R_0^2(X_1, X_0, \tau_1), A_N],$$

where  $A_N$  is the event that either  $|\theta_1 R_0(X_1, X_0, \tau_1)| > aN^{1/2}/2$ , or  $|\theta_2 Z_{1,N}^{(2)}| > a$  and  $C > 0$  is a certain constant. The conclusion of Proposition 7.4 therefore follows from the  $L^2$ -integrability of  $R_0(X_1, X_0, \tau_1)$  and (7.21).  $\square$

7.2.4. *The proof of Theorem 2.10.* We have

$$\sum_{n=0}^{n(Nt)} t(X_n)\tau_n \leq Nt < \sum_{n=0}^{n(Nt)+1} t(X_n)\tau_n, \quad t > 0.$$

On the other hand, with  $K_N := N^\alpha$  and  $s := T_{K_N}^{-1}(t)$  we have

$$\sum_{n=0}^{[K_N s]} t(X_n)\tau_n = K_N^{1/\alpha} T_{K_N}(s) \geq t \geq T_{K_N}(s-) \geq \sum_{n=0}^{[K_N s]-1} t(X_n)\tau_n,$$



hence,  $n(Nt) = K_N T_{K_N}^{-1}(t) - 1$ . From the definitions of processes  $Y_N(\cdot)$ ,  $T_{K_N}(\cdot)$  and  $B_{K_N}(\cdot)$ , see (2.22) and (2.20), we conclude that

$$Y_N(t) = \frac{1}{K_N^{1/2}} \sum_{n=0}^{K_N T_{K_N}^{-1}(t)} \Psi(X_n) \tau_n + o(1) B_{K_N}(T_{K_N}^{-1}(t)) + o(1).$$

By Skorochod's embedding theorem there exists a family of processes  $(\hat{B}_N(\cdot), \hat{T}_N(\cdot))$  such that

- 1) the law of  $(\hat{B}_N(\cdot), \hat{T}_N(\cdot))$  is identical with that of  $(B_{K_N}(\cdot), T_{K_N}(\cdot))$  for each  $N \geq 1$ ,
- 2)  $(\hat{B}_N(\cdot), \hat{T}_N(\cdot))$  converges a.s., in  $D[0, +\infty)$  topology, to  $(\hat{B}(\cdot), \hat{T}(\cdot))$  and  $\hat{B}(\cdot)$  is a Brownian motion,  $\hat{T}(\cdot)$  is a subordinator process and they are independent.

As a result of the above the law of  $\hat{Y}_N(t) := \hat{Z}_N(\hat{T}_N^{-1}(t))$ ,  $t \geq 0$  is identical with that of  $Y_N(\cdot)$ . This implies that  $\hat{Y}_N(\cdot) \rightarrow \hat{Y}(\cdot)$  a.s. in  $D[0, +\infty)$  topology, as  $N \rightarrow +\infty$ , where  $\hat{Y}(\cdot)$  is a Mittag-Leffler process, see [11]. In consequence also the processes  $Y_N(\cdot)$ ,  $N \geq 1$  converge weakly over  $D[0, +\infty)$  to a Mittag-Leffler process.

#### APPENDIX A. THE PROOF OF THEOREM 5.1

Throughout the appendix we use the notation introduced in Section 5. Recall that  $\{Z_n : n \geq 1\}$  is a stationary sequence as introduced in that section.

**A.1. The proof of part i).** Below, we formulate a certain estimate of the total variation distance for counting random variable and a suitable Poisson random variable. It is taken from [10], see Theorem 5, p. 258 and also Proposition 4. 3, p. 268 there. Before formulating the result let us introduce an auxiliary notation

$$K_1(q) := -\frac{1}{q} \log(1 - q) = 1 + \frac{q}{2} + \frac{q^2}{3} + \dots,$$

$$K_2(q) = q^{-2}[-\log(1 - q) - q] = \frac{1}{2} + \frac{q}{3} + \frac{q^2}{4} + \dots,$$

for  $|q| < 1$ . For any random variables  $X, Y$  denote by  $d(X, Y)$  the total variation distance between their laws, given by  $d(X, Y) = \sup_A |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|$ .

**Theorem A.1.** *Suppose that:*

- 1)  $\{\mathcal{G}_i, i \geq 0\}$  is a filtration of  $\sigma$ -algebras, with  $\mathcal{G}_0 := \{\emptyset, \Omega\}$ ,
- 2)  $\tau$  is a stopping time,
- 3) we have a family of events  $\{A_i, i \geq 1\}$  such that  $A_i \in \mathcal{G}_i$ ,  $i \geq 1$ ,
- 4)  $N = \sum_{i=1}^{\tau} 1_{A_i}$ ,
- 5) Let  $p_i := \mathbb{P}(A_i | \mathcal{G}_{i-1})$ ,  $i \geq 1$ ,  $0 < a < b$  and  $\varepsilon, \delta \in (0, 1)$ . We shall assume that

$$\mathbb{P}(a \leq \sum_{i=1}^{\tau} p_i \leq b, \sum_{i=1}^{\tau} p_i^2 \leq \varepsilon) \geq 1 - \delta.$$

Then,

$$d(N, N_a) \leq \alpha\varepsilon + b - a + 2\delta, \quad (\text{A.1})$$

where  $\alpha := (1/2)[K_1(\sqrt{\varepsilon})]^2 + K_2(\sqrt{\varepsilon})$  and  $N_a$  is a Poisson random variable with the parameter  $a$ .

Let  $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$  and let  $\mathcal{M}_{loc}(\mathbb{R}_{+,0}^2)$  be the Polish space of all locally finite measures on  $\mathbb{R}_{+,0}^2$ , equipped with the metric

$$\rho(\mu, \nu) := \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{\|\mu - \nu\|_{TV,n}}{1 + \|\mu - \nu\|_{TV,n}},$$

where  $\|\cdot\|_{TV,n}$  is the total variation norm of the measure restricted to  $[0, n] \times (\mathbb{R} \setminus (-1/n, 1/n))$ ,  $n \geq 1$  and  $\mu, \nu \in \mathcal{M}_{loc}(\mathbb{R}_{+,0}^2)$ . On the space  $\mathbb{R}_{+,0}^2 := [0, +\infty) \times \mathbb{R}_0$  we consider a random measure defined on sets of the form  $[0, t] \times A$ , where  $t > 0$  and  $A$  is Borel, by

$$\mathcal{N}_N[[0, t] \times A] := \sum_{n=1}^{[Nt]} 1_A \left( \frac{Z_n}{N^{1/\alpha}} \right). \quad (\text{A.2})$$

Our first result is given by the following.

**Lemma A.2.**  $\{\mathcal{N}_N, N \geq 1\}$  when considered as  $\mathcal{M}_{loc}(\mathbb{R}_{+,0}^2)$ -valued random elements are weakly convergent to the law of a Poisson measure  $\mathcal{N}$  on  $\mathbb{R}_{+,0}^2$  with intensity  $\nu_*(d\lambda) := C_*(\lambda)|\lambda|^{-1-\alpha} dtd\lambda$ , where

$$C_*(\lambda) := \begin{cases} \alpha c_*^-, & \text{when } \lambda < 0, \\ \alpha c_*^+, & \text{when } \lambda > 0. \end{cases} \quad (\text{A.3})$$

*Proof.* To abbreviate let us write  $k_N(t) := [Nt] - 1$ . Suppose that  $A$  is of the form  $A = \bigcup_{i=1}^m Z_i$ , where  $Z_i := (a_i, b_i] \times I_i$ . Here  $(a_i, b_i]$ ,  $i = 1, \dots, m$  are pairwise disjoint while  $I_i$  are finite unions of disjoint intervals. We shall show that for such a set

$$\mathcal{N}_N[A] \Rightarrow \mathcal{N}[A], \quad \text{as } N \rightarrow +\infty. \quad (\text{A.4})$$

According to [15], p. 209, this implies the weak convergence in question. We have

$$\mathcal{N}_N(A) = \sum_{i=1}^m \sum_{k_N(a_i) < n \leq k_N(b_i)} 1_{I_i} \left( \frac{Z_n}{N^{1/\alpha}} \right).$$

As a consequence of assumption (5.3) we have

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left| \sum_{i=1}^m \sum_{k_N(a_i) < n \leq k_N(b_i)} \mathbb{E} \left[ 1_{I_i} \left( \frac{Z_n}{N^{1/\alpha}} \right) \mid \mathcal{G}_{n-1} \right] - \sum_{i=1}^m (b_i - a_i) \nu_*(I_i) \right| = 0.$$

We also have

$$\begin{aligned} & \mathbb{E} \left[ \sum_{i=1}^m \sum_{k_N(a_i) < n \leq k_N(b_i)} \left\{ \mathbb{E} \left[ 1_{I_i} \left( \frac{Z_n}{N^{1/\alpha}} \right) \mid \mathcal{G}_{n-1} \right] \right\}^2 \right] \\ & \leq \sum_{i=1}^m N(b_i - a_i + 1) \mathbb{E} \left\{ \mathbb{E} \left[ 1_{I_i} \left( \frac{Z_1}{N^{1/\alpha}} \right) \mid \mathcal{G}_0 \right] \right\}^2 \rightarrow 0, \end{aligned} \quad (\text{A.5})$$

as  $N \rightarrow +\infty$ , by virtue of (5.4). Theorem A.1 implies then (A.4).  $\square$

**Lemma A.3.** *For any  $\varepsilon > 0$  and  $T \geq 1$  there exists  $\Delta \geq 1$  such that*

$$\mathbb{E} \left| \sup_{t \in [0, T]} \frac{1}{N^\alpha} \sum_{n=0}^{[Nt]} Z_n 1_{[\Delta, +\infty)} \left( \frac{Z_n}{N^{1/\alpha}} \right) \right| < \varepsilon. \quad (\text{A.6})$$

*Proof.* Using stationarity of  $\{Z_n, n \geq 1\}$  we can estimate the expression on the left hand side of (A.6) by

$$\begin{aligned} & N^{1-1/\alpha} T \mathbb{E}[|Z_0|, |Z_0| \geq \Delta N^{1/\alpha}] = N^{1-1/\alpha} T \int_{\Delta N^{1/\alpha}}^{+\infty} \mathbb{P}(|Z_0| > \lambda) d\lambda \\ & \leq C N^{1-1/\alpha} T \int_{\Delta N^{1/\alpha}}^{+\infty} \frac{d\lambda}{\lambda^\alpha} = \frac{C}{\alpha - 1} N^{1-1/\alpha} T (\Delta N^{1/\alpha})^{1-\alpha} \leq C' \Delta^{1-\alpha} < \varepsilon, \end{aligned} \quad (\text{A.7})$$

provided that  $\Delta$  is sufficiently large (recall  $\alpha \in (1, 2)$  in this case).  $\square$

Define a random measure

$$\mathcal{N}'_N[[0, t] \times A] := \sum_{n=1}^{[Nt]} \mathbb{E} \left[ 1_A \left( \frac{Z_n}{N^{1/\alpha}} \right) \mid \mathcal{G}_{n-1} \right]. \quad (\text{A.8})$$

As a consequence of (5.3) we conclude that  $\mathcal{N}'_N \Rightarrow \nu_*(d\lambda)dt$ , as  $N \rightarrow +\infty$ . The mapping  $\mathcal{H} : \mathcal{M}_{loc}(\mathbb{R}_{+,0}^2) \rightarrow D[0, +\infty)$  given by  $\mathcal{H} : \mu \mapsto \mathcal{H}(\mu)$ , where

$$\mathcal{H}(\mu)(t) := \int_{|\lambda| \in (\delta, \Delta)} \lambda \mu([0, t] \times d\lambda), \quad t \geq 0 \quad (\text{A.9})$$

is continuous at any  $\mu \in \mathcal{M}_{loc}(\mathbb{R}_{+,0}^2)$ , for which  $\Delta > \delta > 0$  and  $0 > -\delta > -\Delta$  are not atoms. Applying the continuous mapping theorem, see Theorem 2.7, p. 21 of [2], for  $\mathcal{H}$  we obtain that

$$\frac{1}{N^{1/\alpha}} \sum_{n=1}^{[Nt]} \mathbb{E} \left[ Z_n 1_{[\delta, \Delta)} \left( \frac{|Z_n|}{N^{1/\alpha}} \right) \mid \mathcal{G}_{n-1} \right] \Rightarrow t \int_{|\lambda| \in [\delta, \Delta)} \lambda \nu_*(d\lambda), \quad \text{as } N \rightarrow +\infty. \quad (\text{A.10})$$

Since the limit is deterministic the convergence holds also in probability.

Denote by  $\mathcal{L}_N$  the law of the random element  $\mathcal{N} : \Omega \rightarrow \mathcal{M}_{loc}(\mathbb{R}_{+,0}^2)$ . Observe that  $\mathbb{E}\mathcal{N}([0, t] \times \{-\Delta, -\delta, \delta, \Delta\}) = 0$  therefore the set  $\mathcal{D}$  that consists of possible discontinuity points of  $\mathcal{H}$  satisfies  $\mathcal{L}_N(\mathcal{D}) = 0$ . Using the continuous mapping theorem we obtain

$$\mathcal{N}_N \circ \mathcal{H}^{-1} \Rightarrow \mathcal{N} \circ \mathcal{H}^{-1}, \quad \text{as } N \rightarrow +\infty, \quad (\text{A.11})$$

hence

$$\frac{1}{N^{1/\alpha}} \sum_{n=0}^{[Nt]} Z_n 1[\delta < |Z_n|/N^{1/\alpha} < \Delta] \Rightarrow \int_{[\delta < |\lambda| < \Delta]} \lambda \mathcal{N}([0, t] \times d\lambda), \quad \text{as } N \rightarrow +\infty \quad (\text{A.12})$$

as stochastic processes in  $D[0, +\infty)$ . Let

$$\begin{aligned} S_N^{\delta, \Delta}(t) &:= \frac{1}{N^{1/\alpha}} \sum_{n=0}^{[Nt]} Z_n 1[\delta < |Z_n|/N^{1/\alpha} < \Delta] \\ &\quad - \frac{1}{N^{1/\alpha}} \sum_{n=1}^{[Nt]} \mathbb{E} [Z_n 1[\delta < |Z_n|/N^{1/\alpha} < \Delta] | \mathcal{G}_{n-1}], \\ S_N^{\Delta}(t) &:= \frac{1}{N^{1/\alpha}} \sum_{n=0}^{[Nt]} Z_n 1[|Z_n|/N^{1/\alpha} < \Delta] \\ &\quad - \frac{1}{N^{1/\alpha}} \sum_{n=1}^{[Nt]} \mathbb{E} [Z_n 1[|Z_n|/N^{1/\alpha} < \Delta] | \mathcal{G}_{n-1}], \end{aligned} \quad (\text{A.13})$$

and  $S_N(t) := N^{-1/\alpha} M_{[Nt]}$ .

Combining (A.10), (A.12) with Lemma A.3 we conclude that for any  $0 < \delta < \Delta < +\infty$

$$S_N^{\delta, \Delta}(\cdot) \Rightarrow Z_{\delta, \Delta}(\cdot), \quad \text{as } N \rightarrow +\infty \quad (\text{A.14})$$

as stochastic processes in  $D[0, +\infty)$ , where

$$Z_{\delta, \Delta}(t) := \int_{[\delta < |\lambda| < \Delta]} \lambda \mathcal{N}([0, t] \times d\lambda) - t \int_{[\delta < |\lambda| < \Delta]} \lambda \nu_*(\lambda) d\lambda, \quad t \geq 0.$$

Using Theorem 14.27, p. 312 of [3] we conclude that for any  $T > 0$

$$\lim_{\delta \rightarrow 0^+} \sup_{t \in [0, T]} |Z_{\delta, \Delta}(t) - Z_{\Delta}(t)| = 0, \quad \text{a.s.}, \quad (\text{A.15})$$

where  $Z_{\Delta}(\cdot) := Z_{0, \Delta}(\cdot)$ . We show that

**Lemma A.4.** *For any  $\Delta, \varepsilon, \eta > 0$  and  $T \geq 1$  there exists  $\delta \in (0, \Delta)$  sufficiently small so that*

$$\limsup_{N \rightarrow +\infty} \mathbb{P} \left[ \rho_{D[0, T]}(S_N^{\delta, \Delta}, S_N^{\Delta}) \geq \varepsilon \right] < \eta. \quad (\text{A.16})$$

Here  $\rho_{D[0, T]}$  is the Skorochod metric on  $D[0, T]$ .

*Proof.* Since  $S_N^{\Delta}(t) = S_N^{\delta, \Delta}(t) + s_N^{\delta}(t)$ , to prove (A.16) it suffices only to show that for any  $\eta > 0$  there exists  $\delta \in (0, 1)$  such that

$$\limsup_{N \rightarrow +\infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |s_N^{\delta}(t)| \right]^2 < \eta. \quad (\text{A.17})$$

Let  $F(\lambda) = \mathbb{P}[|Z_0| > \lambda]$ . By Doob's inequality we can estimate

$$\begin{aligned}
 \mathbb{E} \left[ \sup_{t \in [0, T]} |s_{N, \delta}(t)| \right]^2 &\leq \frac{C}{N^{2/\alpha}} \sum_{n=1}^{[NT]} \mathbb{E} \left[ Z_n^2 1_{[0, \delta]} \left( \frac{|Z_n|}{N^{1/\alpha}} \right) \right] \\
 &= C(T+1)N^{1-2/\alpha} \mathbb{E} \left[ Z_0^2 1_{[0, \delta]} \left( \frac{|Z_0|}{N^{1/\alpha}} \right) \right] \\
 &= -C(T+1)N^{1-2/\alpha} \int_0^{\delta N^{1/\alpha}} \lambda^2 F(d\lambda) \\
 &= 2C(T+1)N^{1-2/\alpha} \int_0^{\delta N^{1/\alpha}} \lambda F(\lambda) d\lambda - C(T+1)N^{1-2/\alpha} \lambda^2 F(\lambda) \Big|_{\lambda=0}^{\delta N^{1/\alpha}} \\
 &\leq 2C'(T+1)N^{1-2/\alpha} \int_0^{\delta N^{1/\alpha}} \frac{\lambda d\lambda}{\lambda^\alpha + 1} + C''\delta^{2-\alpha} \sim C'''\delta^{2-\alpha},
 \end{aligned} \tag{A.18}$$

as  $N \rightarrow +\infty$ , for some constants  $C, C', C'' > 0$ . Estimate (A.17) then follows, upon a suitable choice of  $\delta$ .  $\square$

As a corollary of the above result, (A.15) and (A.14) we conclude that

$$S_N^\Delta \Rightarrow Z_\Delta, \quad \text{as } N \rightarrow +\infty. \tag{A.19}$$

To finish the proof of part i) of the theorem we shall need the following two results.

**Lemma A.5.** *For any  $T > 0$*

$$\lim_{\Delta \rightarrow +\infty} \sup_{t \in [0, T]} |Z(t) - Z_\Delta(t)| = 0, \quad a.s.$$

where  $Z(\cdot)$  is a Levy process with the characteristic functional

$$\mathbb{E} e^{i\xi Z(t)} = \exp \left\{ t \int_{\mathbb{R}} (e^{i\xi\lambda} - 1 - i\xi\lambda) |\lambda|^{-1-\alpha} C_*(\lambda) d\lambda \right\}. \tag{A.20}$$

*Proof.* Note that for  $\Delta' > \Delta$  we can write by Doob's inequality

$$\begin{aligned}
 \mathbb{E} \left[ \sup_{t \in [0, T]} |Z_{\Delta'}(t) - Z_\Delta(t)| \right] &\leq C \mathbb{E} [|Z_{\Delta'}(T) - Z_\Delta(T)|] \\
 &= C \mathbb{E} \left| \int_{|\lambda| \in (\Delta, \Delta')} \lambda \mathcal{N}([0, T], d\lambda) \right| \leq CT \int_{|\lambda| \in (\Delta, \Delta')} |\lambda| \nu_*(\lambda) d\lambda < \varepsilon,
 \end{aligned}$$

provided  $\Delta > 1$  is sufficiently large. Process  $Z(\cdot)$  is then an a.s. uniform limit on compact sets of  $Z_{\Delta_n}(\cdot)$  for some  $\Delta_n \uparrow +\infty$ .  $\square$

**Lemma A.6.** *For any  $T \geq 1, \eta > 0$  there exists  $\Delta > 1$  sufficiently large so that*

$$\limsup_{N \rightarrow +\infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |S_N^\Delta(t) - S_N(t)| \right] < \eta. \tag{A.21}$$

*Proof.* Note that

$$S_N(t) \frac{1}{N^{1/\alpha}} \sum_{n=1}^{\lfloor Nt \rfloor} [Z_n - \mathbb{E}[Z_n | \mathcal{G}_{n-1}]] = S_N^\Delta(t) + R_N^\Delta(t), \quad (\text{A.22})$$

where

$$\begin{aligned} R_N^\Delta(t) &: \frac{Z_0}{N^{1/\alpha}} 1_{[\Delta, +\infty)} \left( \frac{|Z_0|}{N^{1/\alpha}} \right) \\ &+ \frac{1}{N^{1/\alpha}} \sum_{n=1}^{\lfloor Nt \rfloor} \left[ Z_n 1_{[\Delta, +\infty)} \left( \frac{|Z_n|}{N^{1/\alpha}} \right) - \mathbb{E}[Z_n 1_{[\Delta, +\infty)} \left( \frac{|Z_n|}{N^{1/\alpha}} \right) | \mathcal{G}_{n-1}] \right]. \end{aligned} \quad (\text{A.23})$$

Since  $R_N^\Delta(\cdot)$  is a martingale we can write by Doob's inequality

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |R_N^\Delta(t)| \right] \leq C \mathbb{E} R_N^\Delta(T) \rightarrow 0, \quad \text{as } N \rightarrow +\infty,$$

by the argument as in the proof of Lemma A.3.  $\square$

**A.2. The proof of part ii).** Observe that the proof of (A.12) goes through also for  $\alpha \in (0, 1)$  without assumption (5.1). Let  $\hat{Z}_{\delta, \Delta}(\cdot)$  be the compound Poisson process appearing on the right hand side of (A.12). For any  $\delta > 0$  denote also

$$\hat{S}_N^\delta(t) := \frac{1}{N^{1/\alpha}} \sum_{n=0}^{\lfloor Nt \rfloor} Z_n 1_{[|Z_n|/N^{1/\alpha} < \delta]} \quad (\text{A.24})$$

and

$$\bar{S}_N^\delta(t) := \frac{1}{N^{1/\alpha}} \sum_{n=0}^{\lfloor Nt \rfloor} Z_n 1_{[|Z_n|/N^{1/\alpha} > \delta]}. \quad (\text{A.25})$$

**Lemma A.7.** *For any  $T \geq 1$  and  $\varepsilon > 0$  there exists  $\delta \in (0, 1)$  such that*

$$\limsup_{N \rightarrow +\infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |\hat{S}_N^\delta(t)| \right] < \varepsilon. \quad (\text{A.26})$$

*Proof.* Observe that

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} |\hat{S}_N^\delta(t)| \right] &\leq C(T+1) N^{1-1/\alpha} \int |Z_0| 1_{[0, \delta)} \left( \frac{|Z_0|}{N^{1/\alpha}} \right) d\mathbb{P} \\ &\leq C(T+1) N^{1-1/\alpha} \int_0^{\delta N^{1/\alpha}} \mathbb{P}(|Z_0| > \lambda) d\lambda \leq C(T+1) \delta^{1-\alpha} \end{aligned}$$

and the conclusion of the lemma follows upon choosing a suitable small  $\delta > 0$ .  $\square$

**Lemma A.8.** *For any  $T \geq 1$ ,  $\varepsilon > 0$  and  $\eta > 0$  there exists  $\Delta > 1$  such that*

$$\limsup_{N \rightarrow +\infty} \mathbb{P} \left[ \sup_{t \in [0, T]} |\bar{S}_N^\Delta(t)| \geq \varepsilon \right] < \eta. \quad (\text{A.27})$$

*Proof.* Observe that for any  $\beta \in (0, \alpha)$  we have

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |\bar{S}_N^\Delta(t)|^\beta \right] \leq \frac{1}{N^{\beta/\alpha}} \mathbb{E} \left[ \sum_{n=0}^{\lfloor Nt \rfloor} |Z_n| 1_{[|Z_n|/N^{1/\alpha} > \Delta]} \right]^\beta. \quad (\text{A.28})$$

We use an elementary inequality  $(\sum_i a_i)^\beta \leq \sum_i a_i^\beta$  that holds for arbitrary  $a_i \geq 0$  and  $\beta \in (0, 1)$ . By stationarity of  $\{Z_n, n \geq 0\}$  we obtain that the left hand side of (A.28) can be estimated by

$$\begin{aligned} & C(T+1)N^{1-\beta/\alpha} \int |Z_0|^\beta 1_{[\Delta, +\infty)} \left( \frac{|Z_0|}{N^{1/\alpha}} \right) d\mathbb{P} \\ & \leq C(T+1)N^{1-\beta/\alpha} \int_{\Delta N^{1/\alpha}}^{\infty} \lambda^{\beta-1} \mathbb{P}(|Z_0| > \lambda) d\lambda \leq C(T+1)\Delta^{\beta-\alpha} \end{aligned}$$

and the conclusion of the lemma follows upon choosing suitably large  $\Delta > 1$ .  $\square$

Let  $Z(t) := \int \lambda \mathcal{N}([0, t] \times d\lambda)$ . It is an  $\alpha$ -stable process with the characteristic functional

$$\mathbb{E} e^{i\xi Z(t)} = \exp \left\{ t \int_{-\infty}^{+\infty} (e^{i\xi\lambda} - 1) |\lambda|^{-1-\alpha} C_*(\lambda) d\lambda \right\}. \quad (\text{A.29})$$

The conclusion of part ii) of the theorem then follows from the above lemmas and the following result.

**Lemma A.9.** *For any  $T \geq 1$ ,  $\varepsilon > 0$  and  $\eta > 0$  there exist  $\Delta > 1 > \delta > 0$  such that*

$$\mathbb{P} \left[ \sup_{t \in [0, T]} |Z(t) - \hat{Z}_{\delta, \Delta}(t)| \geq \varepsilon \right] < \eta. \quad (\text{A.30})$$

*Proof.* We have

$$Z(t) = \hat{Z}_{\delta, \Delta}(t) + z_\delta(t) + Z^\Delta(t),$$

where  $z_\delta(t) := \int_{|\lambda| \leq \delta} \lambda \mathcal{N}([0, t] \times d\lambda)$  and  $Z^\Delta(t) := \int_{|\lambda| \geq \Delta} \lambda \mathcal{N}([0, t] \times d\lambda)$ . We estimate  $z_\delta(\cdot)$  using

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |z_\delta(t)| \right] \leq \mathbb{E} \left[ \int |\lambda| \mathcal{N}([0, T] \times d\lambda) \right] \leq C \int_0^\delta \lambda^{-\alpha} d\lambda \leq C\delta^{1-\alpha}.$$

As for  $Z^\Delta(t)$  we have  $Z^\Delta(t) = \sum_{t_i \leq t} \xi_i 1_{[|\xi_i| \geq \Delta]}$ , where  $(t_i, \xi_i)$  are the atoms of the random measure  $\mathcal{N}(\cdot)$ . Hence, for any  $\beta \in (0, \alpha)$

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} |Z^\Delta(t)|^\beta \right] \leq \mathbb{E} \left[ \sum_{t_i \leq T} |\xi_i| 1_{[|\xi_i| \geq \Delta]} \right]^\beta \\ & \leq \mathbb{E} \left[ \sum_{t_i \leq T} |\xi_i|^\beta 1_{[|\xi_i| \geq \Delta]} \right] \int_{|\lambda| \geq \Delta} |\lambda|^\beta \mathcal{N}([0, T] \times d\lambda) \leq \frac{C}{\Delta^{\alpha-\beta}} \end{aligned}$$

is as small as we wish for a sufficiently large  $\Delta > 1$ .  $\square$



## APPENDIX B. THE PROOF OF LEMMA 7.3

Let  $S_0 := Z_0$  and  $S_N := \sum_{n=0}^N Z_n$ . By virtue of the ergodic theorem we know that  $N^{-1}S_N \rightarrow Y$ , as  $N \uparrow \infty$ , both a.s. and in  $L^1(\mathbb{P})$ . Choosing an arbitrary  $\varepsilon > 0$ , for  $\mathbb{P}$ -a.s.  $\omega$  we can find  $N_0(\omega)$  so that  $|N^{-1}S_N(\omega) - Y(\omega)| < \varepsilon$  for all  $N \geq N_0(\omega)$ . We can estimate therefore

$$\begin{aligned} \frac{1}{N} \max\{Z_0(\omega), \dots, Z_N(\omega)\} &\leq \frac{1}{N} \max\{Z_0(\omega), \dots, Z_{N_0(\omega)-1}(\omega)\} \\ &+ \frac{1}{N} \max\{Z_{N_0(\omega)}(\omega), \dots, N^{-1}Z_N(\omega)\}. \end{aligned}$$

The first term on the right hand side clearly tends to 0 as  $N \uparrow \infty$  while any of the terms under the second maximum can be written as

$$\begin{aligned} \frac{S_{n+1}(\omega)}{n+1} \frac{n+1}{N} - \frac{S_n(\omega)}{n} \frac{n}{N} &\leq (Y(\omega) + \varepsilon) \frac{n+1}{N} - (Y(\omega) - \varepsilon) \frac{n}{N} \\ &< 2\varepsilon + \frac{Y(\omega)}{N} \end{aligned}$$

for all  $N_0(\omega) \leq n \leq N$ . From here we obtain the a.s. statement of the lemma. To conclude the convergence in  $L^1(\mathbb{P})$ , it suffices to observe that  $N^{-1} \max\{Z_0, \dots, Z_N\}$  is bounded by the  $L^1(\mathbb{P})$  convergent sequence  $N^{-1}S_N$ .  $\square$

## REFERENCES

- [1] Basile, G., Olla, S., Spohn, H., *Energy transport in stochastically perturbed lattice dynamics*, <http://arxiv.org/abs/0805.3012v1>.
- [2] Billingsley, P., *Convergence of probability measures*, 2-nd edition, Wiley and Sons, New York, 1999,
- [3] Breiman, L., *Probability*, Addison-Wesley series in Statistics, (1968).
- [4] Brown, B. M., Eagleson, G. K., *Martingale convergence to infinite divisible laws with finite variance*, TAMS, **162**, 449-453, (1971).
- [5] Csáki, E. and Csörgö, M.: *On additive functionals of Markov chains*, Journal of Theoretical Probability, Vol. 8, No. 4, 1995.
- [6] De Masi, A. and Ferrari, P. A. and Goldstein, S. and Wick, W. D., *An invariance principle for reversible Markov processes. Applications to random motions in random environments*, J. Statist. Phys., **55**, (1989), 787–855,
- [7] Derriennic, Y., Lin, M., *The central limit theorem for Markov chains started at a point*, Probab. Theory Related Fields, **125**, (2003), 73–76,
- [8] Durrett, R., Resnick, S., *Functional limit theorems for dependent variables*, Ann. of Prob. **6**, 829-846, (1978).
- [9] Ethier, S. N., Kurtz, T. G., *Markov processes. Characterization and convergence* Wiley & Sons, (1986).
- [10] Freedman, D., *The Poisson approximation for dependent events*, Ann. of Prob., **2**, pp 256-269, 1974.
- [11] Fujita, Y., *A probabilistic approach to Volterra equations in Banach spaces*, Differential and Int. Equations, **5** (1992), 769-776
- [12] Goldstein, S., *Antisymmetric functionals of reversible Markov processes*, Ann. Inst. H. Poincaré Probab. Statist., **31**, (1995), 177–190,
- [13] Gordin, M. I., *The central limit theorem for stationary processes*, Dokl. Akad. Nauk SSSR, **188**, pp 739–741, 1969.
- [14] Ibragimov I.A., Linnik Yu. V., *Independent and stationary sequences of random variables.*, Wolters-Noordhoff Publishing, Groningen (1971)

- [15] Jagers, P., *Aspects of random measures and point processes*, Advances in Prob., **3**, Marcel-Dekker, 1974.
- [16] Kipnis, C., Landim, C., *Scaling limit of Interacting Particle Systems*, Springer-Verlag, (1999)
- [17] Kipnis, C. and Varadhan, S. R. S., *Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions*, Comm. Math. Phys., **104**, 1986, pp 1–19.
- [18] Lax, P., *Functional analysis*, Wiley & Sons, (2002)
- [19] S. Lepri, R. Livi, A. Politi, Thermal Conduction in classical low-dimensional lattices, Phys. Rep. **377**, 1-80 (2003).
- [20] Maxwell, M. and Woodroffe, M., *Central limit theorems for additive functionals of Markov chains*, Ann. Probab., **28**, (2000), pp 713–724.
- [21] A. Mellet, S. Mischler, C. Mouhot, Fractional diffusion limit for collisional kinetic equations, arXiv:0809.2455v1.
- [22] S. Olla, Central limit theorems for tagged particles and diffusions in random environment, Notes of the course given at Etats de la recherche: Milieux Aleatoires, 23-25 November 2000, Panorama et Syntheses 12, SMF (2001).
- [23] Sato, K., (1999), *Levy Processes and Infinite Divisible Distributions*. Cambridge Univ. Press.

CEREMADE, UMR CNRS 7534  
 UNIVERSITÉ DE PARIS DAUPHINE,  
 PLACE DU MARÉCHAL DE LATTRE DE TASSIGNY  
 75775 PARIS CEDEX 16 - FRANCE.

AND  
 INSTITUTE OF MATHEMATICS  
 UNIVERSITÉ CATHOLIQUE DE LOUVAIN  
 PLACE DE L'UNIVERSITÉ 1 B-1348 LEUVEN (LOUVAIN), BELGIUM

INSTITUTE OF MATHEMATICS  
 UMCS  
 PL. MARIJ CURIE-SKŁODOWSKIEJ 1  
 LUBLIN 20-031, POLAND  
 komorow@hektor.umcs.lublin.pl  
<http://hektor.umcs.lublin.pl/~komorow>

CEREMADE, UMR CNRS 7534  
 UNIVERSITÉ DE PARIS DAUPHINE,  
 PLACE DU MARÉCHAL DE LATTRE DE TASSIGNY  
 75775 PARIS CEDEX 16 - FRANCE.  
 olla@ceremade.dauphine.fr  
<http://www.ceremade.dauphine.fr/~olla>