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# BOUNDING APPROACH TO FAULT DETECTION OF UNCERTAIN DYNAMIC SYSTEMS

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**Abstract:** This paper deals with a fault detection method taking model uncertainties into account. Only sensor imperfections are considered; moreover every uncertain parameter is represented by a bounded variable. By this membership approach, a model represents a set of feasible behaviours; therefore, interval analysis is used to build consistency tests based on the parity space approach. A fault is detected when the residual vector is outside its domain of feasible values. Unfortunately, the dependencies between bounded variables make the previous domain too complex to be exactly evaluated. The proposed solution leads to an overestimation obtained by using a paving algorithm. *Copyright © 2000 IFAC*

**Keywords:** Fault detection, Structured uncertainties, Bounding method, Residue.

## 1. INTRODUCTION

Fault detection methods often use the concept of analytical redundancy based on a mathematical model of the studied system. A problem met in the field of Fault Detection (F.D.) schemes lies in the fact that a model does not define a complete and accurate description of the physical system behaviour (Patton, *et al.*, 1989). Nevertheless, a modelling error has not to be confused with a fault. Even if the structure of the model is known, the values of some parameters are inaccessible. Instead of representing these uncertainties by Gaussian stochastic variables, they may be defined by a bounded set of feasible values. This approach, known as bounding approach, allows to describe straight parameter uncertainties in every matrix of the state representation (Adrot, *et al.*, 1999), instead of putting them together in an unique additive term in each equation of the model.

This paper focuses on a fault detection method taking the structure and the magnitude of every uncertain parameter described by a bounded variable into account. Thus, using interval analysis, the set of feasible residual values is built according to uncertainty ranges. This work allows to define the normal operation field of the studied system. A parity space approach is chosen to avoid the wrapping effect due to the use of interval arithmetic for recursive systems. In this way, a static representation of a dynamic model can be obtained by stacking sensory observations on a finite time horizon. Thus, this paper has the following structure. At first, the notion of abstract space is presented to tackle models containing bounded variables. After defining the

chosen model, two different static forms are given. The section 3 establishes the equivalence between these ones. The principle of consistency tests is developed in section 4, where the problems caused by dependencies between bounded variables are detailed. An algorithm is proposed to analyse the obtained residuals, which are non-linear in bounded variables. At last, an example is presented.

## 2. STATIC FORMS

### 2.1. Presentation of the model

Structured models take the lack of knowledge on a physical system into account by indicating which parameters are uncertain. These uncertainties are known as multiplicative ones since they affect straight model parameters. Moreover, they are described by bounded variables, which only bounds are known. To tackle this kind of models, the concept of abstract space, denoted by  $\mathcal{A}()$ , is introduced. If  $\mathbf{z}$  is a vector of bounded variables, in other words, if it is only known by the space to which it may belong, then this space is given as  $\mathcal{A}(\mathbf{z})$  (Ploix, *et al.*, 1999).

Only dynamic systems given by linear discrete state equations are considered.  $\mathbf{A}$  and  $\mathbf{B}$  respectively define the certain state and actuator matrices:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \quad \mathbf{x}_k \in \mathbb{R}^n, \mathbf{u}_k \in \mathbb{R}^l. \quad (1)$$

$\mathbf{u}_k$  and  $\mathbf{x}_k$  respectively define the state and actuator input vectors at time  $k$ . Structured uncertainties only affect the measurement system and allow to take the tolerance and the noise of each sensor into account.

Therefore, the measurement system can be written as:

$$\mathbf{y}_k = \mathbf{C}(\boldsymbol{\theta}_k) \mathbf{x}_k + \mathbf{T} \boldsymbol{\delta}_k, \text{ with } \mathbf{C}(\boldsymbol{\theta}_k) = (\mathbf{I}_m + \boldsymbol{\theta}_k) \mathbf{C}_0, (2)$$

where  $\mathbf{C}$  and  $\mathbf{I}_m$  are respectively the observation matrix and the  $m$ -order identity matrix. The vector  $\mathbf{y}_k$  in  $\mathbb{R}^m$  defines the sensor outputs and the term  $\mathbf{C}_0 \mathbf{x}_k$  represents the true values of measured variables, since  $\mathbf{C}_0$  is the nominal part of  $\mathbf{C}$ . The diagonal matrix  $\boldsymbol{\theta}_k$ , which non-zero components are noted  $\theta_{k,i}$  describes the uncertainties due to sensor tolerances. If the relative precision on the  $i^{\text{th}}$  sensor is  $\sigma_i \in [0, 1[$ , then the abstract space of the bounded variable  $\theta_{k,i}$  is an interval given as:  $\mathcal{A}(\theta_{k,i}) = [-\sigma_i, \sigma_i]$ . The vector  $\boldsymbol{\delta}$  represents bounded sensory noises and is composed of  $m$  bounded variables  $\delta_{k,i}$ . It is always possible to boil down to the case all variables have the same bounds (Adrot, *et al.*, 1999); therefore,  $\boldsymbol{\delta}$  may be normalised:  $\mathcal{A}(\delta_{k,i}) = [-1, 1]$ . Moreover, since noises are centred and independent,  $\mathbf{T}$  is a diagonal matrix too indicating the magnitude of sensor noises.

This approach is attractive because the model is close to the physical system. Moreover bounded variables are interesting since they describe straight parameter uncertainties to build thresholds more naturally.

## 2.2. First static form

From (1), the relation linking up the state vector  $\mathbf{x}_{k+j}$  to  $\mathbf{x}_k$  and inputs can be easily expressed. After multiplying this one by  $\mathbf{C}_0$  and piling  $\mathbf{x}_{k+j}$  on the time horizon  $[k, k+s]$ , the following relation is deduced:

$$\begin{aligned} \mathbf{L}_0 \mathbf{X}_{k,s} &= \mathbf{C}_{s,0} \mathbf{x}_k + \mathbf{H}_{s,0} \mathbf{U}_{k,s-1}, & (3) \\ \text{with: } \underbrace{\mathbf{Z}_{k,s}}_{z \in \{u, x, y\}} &= \begin{bmatrix} \mathbf{z}_k \\ \vdots \\ \mathbf{z}_{k+s} \end{bmatrix}, \mathbf{L}_0 = \text{diag}_{s+1}(\mathbf{C}_0), \\ \mathbf{C}_{s,0} &= \begin{bmatrix} \mathbf{C}_0 \\ \mathbf{C}_0 \mathbf{A} \\ \vdots \\ \mathbf{C}_0 \mathbf{A}^s \end{bmatrix} \text{ and } \mathbf{H}_{s,0} = \begin{bmatrix} 0 & \cdots & 0 \\ \mathbf{C}_0 \mathbf{B} & & 0 \\ \vdots & \ddots & \vdots \\ \mathbf{C}_0 \mathbf{A}^{s-1} \mathbf{B} & \cdots & \mathbf{C}_0 \mathbf{B} \end{bmatrix}. \end{aligned}$$

The matrix  $\mathbf{L}_0$  is bloc diagonal containing  $s+1$  times the matrix  $\mathbf{C}_0$ . Let us note  $s_m = (s+1)m$ . After piling measurements (2) on the same time horizon, the following relation is deduced:

$$\begin{aligned} \mathbf{Y}_{k,s} &= (\mathbf{I}_{s_m} + \boldsymbol{\Theta}_{k,s}) \mathbf{L}_0 \mathbf{X}_{k,s} + \text{diag}_{s+1}(\mathbf{T}) \Delta_{k,s}, & (4) \\ \text{with: } \boldsymbol{\Theta}_{k,s} &= \begin{bmatrix} \boldsymbol{\theta}_k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \boldsymbol{\theta}_{k+s} \end{bmatrix} \text{ and } \Delta_{k,s} = \begin{bmatrix} \boldsymbol{\delta}_k \\ \vdots \\ \boldsymbol{\delta}_{k+s} \end{bmatrix}. \end{aligned}$$

Unlike (Adrot, *et al.*, 1999), the uncertain parameters are assumed to depend on time in this paper. Thus, a parameter  $z$  expressed at different times  $k$  and  $t$  is defined by two independent bounded variables  $z_k$  and  $z_t$  with the same abstract space  $\mathcal{A}(z_k) = \mathcal{A}(z_t)$ .

From both relations (3) and (4), the *first static form* is deduced. Let us notice that generalised parity space

methods are based on a similar static form obtained for a certain model (Chow, *et al.*, 1984). By analogy, external and internal forms are defined. They allow to respectively calculate residuals and assess their sets of feasible values. Since the former one only contains known variables, uncertain parameters and state variables are put together in the latter one. Therefore, the first static form is cut into two pieces, given as:

$$\begin{aligned} \mathbf{Y}_{k,s} - \mathbf{H}_{s,0} \mathbf{U}_{k,s-1} &= (\mathbf{I}_{s_m} + \boldsymbol{\Theta}_{k,s}) \mathbf{C}_{s,0} \mathbf{x}_k + \cdots \\ &\cdots + \boldsymbol{\Theta}_{k,s} \mathbf{H}_{s,0} \mathbf{U}_{k,s-1} + \text{diag}_{s+1}(\mathbf{T}) \Delta_{k,s}. \end{aligned} \quad (5)$$

The structure (5) can always lead to the elimination of the unknown state vector. Let us consider a row vector  $\boldsymbol{\Omega}^T$  imposing a de-coupling from the nominal part of the internal form depending on  $\mathbf{x}_k$ :

$$\boldsymbol{\Omega}^T \mathbf{C}_{s,0} = 0. \quad (6)$$

A minimal time horizon always exists for which the matrix  $\mathbf{C}_{s,0}$  defined in (3) is not full row rank, which is the existence condition for  $\boldsymbol{\Omega}$  (as shown in (Adrot, *et al.*, 1999) by applying the Cayley-Hamilton theorem on the state matrix). To obtain a parity relation, (5) is multiplied by  $\boldsymbol{\Omega}^T$ . Thus, the external form  $p^{\text{ext}}$  and the external form  $p^{\text{int}}$  are respectively defined by:

$$p_{k+s}^{\text{ext}} = \boldsymbol{\Omega}^T (\mathbf{Y}_{k,s} - \mathbf{H}_{s,0} \mathbf{U}_{k,s-1}), \quad (7)$$

$$p_{k+s}^{\text{int}} = \boldsymbol{\Omega}^T \boldsymbol{\Theta}_{k,s} \{ \mathbf{C}_{s,0} \mathbf{x}_k + \mathbf{H}_{s,0} \mathbf{U}_{k,s-1} \} + \boldsymbol{\Omega}^T \text{diag}_{s+1}(\mathbf{T}) \Delta_{k,s}$$

At first, the term in brackets in the previous internal form is replaced by the vector  $\mathbf{L}_0 \mathbf{X}_{k,s}$  (3), which expression can be easily deduced from (4). Then, by using the matrix inversion lemma:

$$(\mathbf{I}_{s_m} + \boldsymbol{\Theta}_{k,s})^{-1} = \mathbf{I}_{s_m} - \boldsymbol{\Theta}_{k,s} (\mathbf{I}_{s_m} + \boldsymbol{\Theta}_{k,s})^{-1}, \quad (8)$$

$p^{\text{int}}$ , de-coupled from state variables, becomes:

$$\begin{aligned} p_{k+s}^{\text{int}} &= \boldsymbol{\Omega}^T \boldsymbol{\Theta}_{k,s} (\mathbf{I}_{s_m} + \boldsymbol{\Theta}_{k,s})^{-1} \mathbf{Y}_{k,s} + \cdots \\ &\cdots + \boldsymbol{\Omega}^T (\mathbf{I}_{s_m} + \boldsymbol{\Theta}_{k,s})^{-1} \text{diag}_{s+1}(\mathbf{T}) \Delta_{k,s}. \end{aligned} \quad (9)$$

The matrix  $\mathbf{I}_{s_m} + \boldsymbol{\Theta}_{k,s}$  being regular and the calculus of  $\boldsymbol{\Omega}$  being classic, the previous relation can be evaluated with the help of interval arithmetic because unknown state variables are eliminated. Under ideal circumstances (no noise and no fault), residuals are non-zero since they depend on uncertain parameters.

## 2.3. Second static form

Instead of expressing  $\mathbf{x}_{k+j}$  according to the initial state vector  $\mathbf{x}_k$  as previously,  $\mathbf{x}_{k+j}$  is deduced from  $\mathbf{x}_{k+j-1}$ . Then, state and input vectors are stacked on a time horizon  $[k, k+s]$  leading to the next equation:

$$\begin{aligned} \mathbf{M} \mathbf{X}_{k,s} &= \mathbf{N} \mathbf{U}_{k,s-1}, & (10) \\ \text{where: } \mathbf{M} &= \begin{bmatrix} \mathbf{A} & -\mathbf{I}_n & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \mathbf{A} & -\mathbf{I}_n \end{bmatrix} \text{ and } \mathbf{N} = -\text{diag}_s(\mathbf{B}). \end{aligned}$$



Using the relations (4) and (10), a new static form, called *second static form*, is finally deduced:

$$\begin{bmatrix} (\mathbf{I}_{s_m} + \Theta_{k,s}) \mathbf{L}_0 \\ \mathbf{M} \end{bmatrix} \mathbf{X}_{k,s} = \dots \\ \dots = \begin{bmatrix} 0 & \mathbf{I}_{s_m} \\ \mathbf{N} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{k,s-1} \\ \mathbf{Y}_{k,s} \end{bmatrix} - \begin{bmatrix} \text{diag}_{s+1}(\mathbf{T}) \Delta_{k,s} \\ 0 \end{bmatrix}. \quad (11)$$

By analogy with the first static form, let us note  $\tilde{\mathbf{C}}_{s,0}$  the nominal part of the left-hand side of this equality and  $\tilde{\mathbf{H}}_{s,0}$  the matrix depending on observations:

$$\tilde{\mathbf{C}}_{s,0} = \begin{bmatrix} \mathbf{L}_0 \\ \mathbf{M} \end{bmatrix}, \quad \tilde{\mathbf{H}}_{s,0} = \begin{bmatrix} 0 & \mathbf{I}_{s_m} \\ \mathbf{N} & 0 \end{bmatrix}. \quad (12)$$

A parity relation is obtained by multiplying (11) by a suited row vector  $\tilde{\mathbf{\Omega}}^T$  satisfying:

$$\tilde{\mathbf{\Omega}}^T \tilde{\mathbf{C}}_{s,0} = \begin{bmatrix} \tilde{\mathbf{\Omega}}_L^T & \tilde{\mathbf{\Omega}}_M^T \end{bmatrix} \begin{bmatrix} \mathbf{L}_0 \\ \mathbf{M} \end{bmatrix} = 0. \quad (13)$$

The section 3 will show that a minimal time horizon always exists for which the matrix  $\tilde{\mathbf{C}}_{s,0}$  defined in (12) is full row rank. In this way, the external and internal forms are respectively defined by:

$$\tilde{\mathbf{p}}_{k+s}^{\text{ext}} = \tilde{\mathbf{\Omega}}^T \tilde{\mathbf{H}}_{s,0} \begin{bmatrix} \mathbf{U}_{k,s-1} \\ \mathbf{Y}_{k,s} \end{bmatrix}, \quad (14)$$

$$\tilde{\mathbf{p}}_{k+s}^{\text{int}} = \tilde{\mathbf{\Omega}}_L^T (\Theta_{k,s} \mathbf{L}_0 \mathbf{X}_{k,s} + \text{diag}_{s+1}(\mathbf{T}) \Delta_{k,s}).$$

By using (4) to express  $\mathbf{L}_0 \mathbf{X}_{k,s}$  and the lemma (8) on the previous relation, an internal form is deduced, where the unknown state vector is eliminated:

$$\tilde{\mathbf{p}}_{k+s}^{\text{int}} = \tilde{\mathbf{\Omega}}_L^T \Theta_{k,s} (\mathbf{I}_{s_m} + \Theta_{k,s})^{-1} \mathbf{Y}_{k,s} + \dots \\ \dots + \tilde{\mathbf{\Omega}}_L^T (\mathbf{I}_{s_m} + \Theta_{k,s})^{-1} \text{diag}_{s+1}(\mathbf{T}) \Delta_{k,s}. \quad (15)$$

Like the first static form, the internal form is de-coupled from state variables, which is an advantage in comparison with the model proposed in (Adrot, et al., 1999).

### 3. RELATION BETWEEN BOTH STATIC FORMS

Both matrices  $\mathbf{C}_{s,0}$  (3) and  $\tilde{\mathbf{C}}_{s,0}$  (12) are certain. The existence condition of  $\mathbf{\Omega}$  depends on the time horizon and for a sufficient value of  $s$ , an infinity of vectors  $\mathbf{\Omega}$  satisfying (6) exists. Now, the aim is the justification of the equivalence between both static forms for a given time horizon, leading to the same existence condition for  $\tilde{\mathbf{\Omega}}$  having to verify (13). Let us note:

$$\mathbf{A}_s = \begin{bmatrix} \mathbf{I}_n \\ \vdots \\ \mathbf{A}^s \end{bmatrix} \text{ and } \mathbf{B}_s = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \mathbf{A}^{s-1} \mathbf{B} & \dots & \mathbf{B} \end{bmatrix}. \quad (16)$$

For generalising, the symbol  $\mathbf{\Omega}^T$  is replaced by  $\mathbf{W}$  defining a matrix instead of a row vector. In fact, all independent parity relations are interesting in

diagnosis because they conserve all the attainable information embodied in the model.

①. Let  $\tilde{\mathbf{W}}$  be a matrix orthogonal to  $\tilde{\mathbf{C}}_{s,0}$  (13).  $\tilde{\mathbf{W}}$  is full row rank in order to obtain independent residuals; therefore, its rank is equal to:  $s_{\tilde{\mathbf{W}}} = s_m + s_n - \text{rank}(\tilde{\mathbf{C}}_{s,0})$ .  $\tilde{\mathbf{W}}$  is partitioned into two blocs as follows:

$$\tilde{\mathbf{W}} \begin{bmatrix} \mathbf{L}_0 \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{W}}_L & \tilde{\mathbf{W}}_M \end{bmatrix} \begin{bmatrix} \mathbf{L}_0 \\ \mathbf{M} \end{bmatrix} = 0, \quad \begin{matrix} \tilde{\mathbf{W}}_L \in \mathbb{R}^{s_{\tilde{\mathbf{W}}} \times s_m} \\ \tilde{\mathbf{W}}_M \in \mathbb{R}^{s_{\tilde{\mathbf{W}}} \times s_n} \end{matrix} \quad (17)$$

By multiplying the previous relation (17) on the right hand side by the matrix  $\mathbf{A}_s$  (16) and after noticing that  $\mathbf{M} \mathbf{A}_s$  equals 0, the following equality is obtained:

$$\tilde{\mathbf{W}}_L \mathbf{C}_{s,0} = 0. \quad (18)$$

At first,  $\tilde{\mathbf{W}}_L$  is necessary full row rank; otherwise a non-zero vector  $\omega$  exists so that  $\omega^T \tilde{\mathbf{W}}_L$  is null. Thus, by multiplying (17) by  $\omega^T$ , the equality  $\omega^T \tilde{\mathbf{W}}_M \mathbf{M} = 0$  is deduced. Because of its structure, the matrix  $\mathbf{M}$  (10) is necessary full row rank, implying  $\omega^T \tilde{\mathbf{W}}_M = 0$ . Therefore, a contradiction exists since a non-zero vector  $\omega$  exists so that  $\omega^T [\tilde{\mathbf{W}}_L \ \tilde{\mathbf{W}}_M] = 0$ , whereas the matrix  $\tilde{\mathbf{W}}$  is full row rank by assumption.

Moreover, the Singular Value Decomposition of the matrix  $\mathbf{C}_{s,0}$  allows to obtain its kernel. Let us note  $s_c$  the rank of the matrix  $\mathbf{C}_{s,0}$ ;  $\mathbf{U}$  and  $\mathbf{V}^T$  two orthogonal matrices which columns are respectively noted  $\mathbf{u}_i$  and  $\mathbf{v}_i$ . The non-zero singular values  $\varepsilon_i$ ,  $i \in \{1, \dots, s_c\}$ , of  $\mathbf{C}_{s,0}$  are put together in the diagonal matrix  $\mathbf{\Sigma}$ :

$$\mathbf{C}_{s,0} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \text{ with } \begin{matrix} \mathbf{U} \in \mathbb{R}^{s_m \times s_m} \\ \mathbf{V} \in \mathbb{R}^{n \times n} \end{matrix}, \text{ or } \mathbf{C}_{s,0} = \sum_{i=1}^{s_c} \varepsilon_i \mathbf{u}_i \mathbf{v}_i^T.$$

From this, the following equivalencies are deduced:

$$\omega^T \mathbf{C}_{s,0} = 0 \Leftrightarrow \sum_{i=1}^{s_c} \varepsilon_i (\omega^T \mathbf{u}_i) \mathbf{v}_i^T = 0 \Leftrightarrow \omega^T \mathbf{u}_i = 0, \quad \forall i$$

Thus, if  $\omega^T \mathbf{u}_i$  is null for all  $i$ , then  $\omega^T \mathbf{C}_{s,0}$  equals 0. The opposite implication is trivial too since  $\omega^T \mathbf{u}_i$  is a scalar and  $\mathbf{v}_i$  is a non-zero vector (otherwise,  $\mathbf{V}$  can not be an orthogonal matrix). So the solutions of  $(\omega^T \mathbf{u}_i) \mathbf{v}_i^T = 0$  are the ones of  $\omega^T \mathbf{u}_i = 0$ . Thus, since the matrix  $\mathbf{U}$  is orthogonal,  $\omega^T$  belongs to the space generated by the  $s_w = s_m - s_c$  independent row vectors  $\mathbf{u}_{s_c+1}^T, \dots, \mathbf{u}_{s_m}^T$ . In conclusion, from the previous result and the relation (18), since the matrix  $\tilde{\mathbf{W}}_L$  is full row rank, a regular matrix  $\tilde{\mathbf{K}}$  of order  $s_w$  exists so that:

$$\tilde{\mathbf{W}}_L = \tilde{\mathbf{K}} [0 \ \mathbf{I}_{s_w}] \mathbf{U}^T. \quad (19)$$

②. Let  $\mathbf{W}$  be a matrix orthogonal to  $\mathbf{C}_{s,0}$  (6).  $\mathbf{W}$  is full row rank in order to obtain independent residuals; so



its rank is equal to  $s_w$  and a regular matrix  $\mathbf{K}$  of order  $s_w$  exists so that:

$$\mathbf{W} = \mathbf{K} \begin{bmatrix} 0 & \mathbf{I}_{s_w} \end{bmatrix} \mathbf{U}^T. \quad (20)$$

Therefore, the existence of  $\tilde{\mathbf{W}}$  ① (that is to say the fact that  $\tilde{\mathbf{C}}_{s,0}$  is not full row rank) necessary implies the one of  $\mathbf{W}$  ② since  $\tilde{\mathbf{W}}_L$  satisfies (6). The converse is true. Since  $\mathbf{M}$  is full row rank, if  $\mathbf{W}$  ② exists, then:

$$\tilde{\mathbf{W}}_L = \mathbf{W} \text{ and } \tilde{\mathbf{W}}_M = -\mathbf{W} \mathbf{L}_0 \mathbf{M}^T (\mathbf{M} \mathbf{M}^T)^{-1}.$$

Thus, a matrix  $\tilde{\mathbf{W}} = [\tilde{\mathbf{W}}_L \ \tilde{\mathbf{W}}_M]$  verifying ① is perfectly defined. In conclusion, the existence conditions of  $\mathbf{W}$  and  $\tilde{\mathbf{W}}$  are equivalent. If  $\mathbf{C}_{s,0}$  is not full row rank, so does  $\tilde{\mathbf{C}}_{s,0}$ , and the reciprocal is true.

By definition, the matrix  $\tilde{\mathbf{W}}_L$  belongs to  $\mathbb{R}^{s_{\tilde{\mathbf{W}}} \times s_m}$ . Thus, from (19), the following equality is deduced:

$$s_{\tilde{\mathbf{W}}} = s_m + s_n - \text{rank}(\tilde{\mathbf{C}}_{s,0}) = s_m - \text{rank}(\mathbf{C}_{s,0}) = s_w.$$

Therefore, both static forms lead to the same number of independent parity relations for an identical time horizon since matrices  $\mathbf{C}_{s,0}$  and  $\tilde{\mathbf{C}}_{s,0}$  have exactly the same number of singular values equal to zero. Thus, the choice of  $s$  for the second static form can be deduced from the study of the first static form and the relations (19) and (20) leads to:

$$\tilde{\mathbf{W}}_L = \mathbf{Q} \mathbf{W}, \text{ where the matrix } \mathbf{Q} \text{ is a regular.} \quad (21)$$

Now, the aim is the justification of the equivalence between parity relations obtained from both static forms. The letter  $p$  is replaced by the bold letter  $\mathbf{p}$  representing a vector of several independent parity relations. At first, a relationship between external forms  $\mathbf{p}_{k+s}^{\text{ext}}$  (7) and  $\tilde{\mathbf{p}}_{k+s}^{\text{ext}}$  (14) is searched. From the expressions of  $\tilde{\mathbf{p}}_{k+s}^{\text{ext}}$  and  $\tilde{\mathbf{W}}$ , after using the equality  $\mathbf{N} = \mathbf{M} \mathbf{B}_s$ , the following relation is deduced:

$$\tilde{\mathbf{p}}_{k+s}^{\text{ext}} = \tilde{\mathbf{W}}_M \mathbf{M} \mathbf{B}_s \mathbf{U}_{k,s-1} + \tilde{\mathbf{W}}_L \mathbf{Y}_{k,s}.$$

From the equality (17) and the relation:  $\mathbf{L}_0 \mathbf{B}_s = \mathbf{H}_{s,0}$ , after using the equality (21) and by the expression of  $\mathbf{p}_{k+s}^{\text{ext}}$  (7), the previous expression becomes:

$$\tilde{\mathbf{p}}_{k+s}^{\text{ext}} = \tilde{\mathbf{W}}_L (-\mathbf{H}_{s,0} \mathbf{U}_{k,s-1} + \mathbf{Y}_{k,s}) = \mathbf{Q} \mathbf{p}_{k+s}^{\text{ext}}.$$

The equivalence between both internal forms is deduced straight from expressions (9) and (15) by using the property (21):  $\tilde{\mathbf{p}}_{k+s}^{\text{int}} = \mathbf{Q} \mathbf{p}_{k+s}^{\text{int}}$ . Since the matrix  $\mathbf{Q}$  is regular, when a de-coupling of parity relations according to space variables is obtained, residuals are structurally equivalent for both static forms and the same time horizon. Contrary to an uncertain state matrix  $\mathbf{A}(\theta_k)$  (Adrot, *et al.*, 1999); there is no advantage to use the second static form in comparison with the first one in the case of our paper.

## 4. FAULT DETECTION METHOD

### 4.1. Principle

To simplify notations in this section, references to the fact that residuals depend on time are omitted. At first, the parity vector  $\mathbf{p}^{\text{ext}}$  is calculated from its external form, which only depends on measurements (7). Then, interval arithmetic (Moore, 1979) is used to evaluate the associated internal form containing bounded variables. The obtained abstract space  $\mathcal{A}(\mathbf{p}^{\text{int}})$  defines all the feasible values of  $\mathbf{p}^{\text{ext}}$  which are consistent with the model. Therefore, it represents the normal operation field of the physical system; thus, a fault is detected if  $\mathbf{p}^{\text{ext}}$  does not belong to  $\mathcal{A}(\mathbf{p}^{\text{int}})$ .

In a F.D. scheme, more than one residual is needed. If a bounded variable, called *common variable*, occurs in several parity relations, to check if  $\mathbf{p}^{\text{ext}}$  is inside  $\mathcal{A}(\mathbf{p}^{\text{int}})$  is not equivalent to test if each component  $p^{\text{ext},i}$ ,  $i \in \{1, \dots, s_w\}$ , of the previous vector belongs to the associated abstract space  $\mathcal{A}(\mathbf{p}^{\text{int},i})$ . Moreover, internal forms are non-linear in bounded variables, what leads to complicated domains. Now, the aim of the next section is the presentation of the problems due to dependencies between parity relations and the proposition of a method to realise consistency tests.

### 4.2. Dependencies between bounded variables

If  $z$  is a bounded variable,  $\mathcal{A}(z)$  is an interval  $[\underline{z}, \bar{z}]$  (in case this space is connected),  $\underline{z}$  and  $\bar{z}$  denoting its lower and upper bounds. A bounded vector  $\mathbf{z}$  is an ordered  $n$ -tuple consisting of interval components  $z_i$ . If these ones are independent, then  $\mathcal{A}(\mathbf{z})$  is an orthotope which edges are aligned with the coordinate axes associated with every  $z_i$ .

Now, let us consider a vector field  $\mathbf{f}(z_1, \dots, z_n)$  composed of two functions  $f_1$  and  $f_2$  depending on  $n$  independent bounded variables  $z_i$ . If both functions have no common variable, then they are independent and  $\mathcal{A}(\mathbf{f})$  leads to a rectangle in the space  $(f_1, f_2)$ . In this case, interval formalism is therefore well suited. Otherwise,  $\mathcal{A}(\mathbf{f})$  becomes more complicated. Common variables imply dependencies modifying the shape of  $\mathcal{A}(\mathbf{f})$  against its circumscribed axis-aligned orthotope, noted  $\square \mathcal{A}(\mathbf{f})$ , built by evaluating independently  $f_1$  and  $f_2$  and assuming no variable is common. If both functions  $f_1$  and  $f_2$  are linear in common variables,  $\mathcal{A}(\mathbf{f})$  is a parallelotope (which can be connected if  $\mathbf{f}$  is linear in all bounded variables). Otherwise, the shape of  $\mathcal{A}(\mathbf{f})$  may be complex and non-connected. Thus, the notion of abstract space allows to replace interval formalism, which cannot represent such domains.

Now, let us show that  $\square \mathcal{A}(\mathbf{f})$  always contains  $\mathcal{A}(\mathbf{f})$ . Let us consider the following interval vector field in  $\mathbb{R}^p$ , where the bounded vectors  $\mathbf{z}_C$  and  $\mathbf{z}_{NC,i}$  contain respectively all common variables to at least two



components of  $\mathbf{f}$  and all non-common variables present in the  $i^{\text{th}}$  component  $f_i$ :

$$\mathbf{f}(\mathbf{z}_C, \mathbf{z}_{NC,1 \dots p}) = [f_1(\mathbf{z}_C, \mathbf{z}_{NC,1}) \cdots f_p(\mathbf{z}_C, \mathbf{z}_{NC,p})]^T.$$

All bounded variables are independent. The orthotope  $\square \mathcal{A}(\mathbf{f})$  is built by replacing the vector  $\mathbf{z}_C$  of each component  $f_i$  by a new independent bounded vector  $\mathbf{z}_{C,i}$  with the same abstract space  $\mathcal{A}(\mathbf{z}_{C,i}) = \mathcal{A}(\mathbf{z}_C)$ :

$$\mathbf{f}_{\square}(\mathbf{z}_{C,1 \dots p}, \mathbf{z}_{NC,1 \dots p}) = [f_1(\mathbf{z}_{C,1}, \mathbf{z}_{NC,1}) \cdots f_p(\mathbf{z}_{C,p}, \mathbf{z}_{NC,p})]^T$$

Thus,  $\square \mathcal{A}(\mathbf{f})$  is given by the abstract space  $\mathcal{A}(\mathbf{f}_{\square})$ . Inversely, to obtain  $\mathbf{f}$  from  $\mathbf{f}_{\square}$ , the vectors  $\mathbf{z}_{C,i}$  have to be identical. In these conditions, all feasible values of  $\mathbf{f}$  are necessary feasible values of  $\mathbf{f}_{\square}$ ; the reciprocal being false because of previous constraints. Therefore,  $\mathcal{A}(\mathbf{f})$  is necessary included in  $\square \mathcal{A}(\mathbf{f})$ .

Now, let us show that  $\square \mathcal{A}(\mathbf{f})$  is circumscribed to  $\mathcal{A}(\mathbf{f})$ : at least one point of each boundary hyperplane of  $\square \mathcal{A}(\mathbf{f})$  belongs to  $\mathcal{A}(\mathbf{f})$ . Let us consider the component  $f_i$  of  $\mathbf{f}$  and let us assume its upper bound  $\bar{f}_i$  is obtained for the arguments  $\mathbf{z}_C^*$  and  $\mathbf{z}_{NC,i}^*$ . Therefore, for all feasible values of  $\mathbf{z}_C$  and  $\mathbf{z}_{NC,i}$ , the inequality  $f_i(\mathbf{z}_C, \mathbf{z}_{NC,i}) \leq \bar{f}_i$  is satisfied. The upper bound of the  $i^{\text{th}}$  component  $f_{\square,i}$  of  $\mathbf{f}_{\square}$  is:

$$\sup(f_i(\mathbf{z}_{C,i}, \mathbf{z}_{NC,i}), \mathbf{z}_{C,i} \in \mathcal{A}(\mathbf{z}_{C,i}), \mathbf{z}_{NC,i} \in \mathcal{A}(\mathbf{z}_{NC,i})).$$

Since both abstract spaces  $\mathcal{A}(\mathbf{z}_C)$  and  $\mathcal{A}(\mathbf{z}_{C,i})$  are identical, this maximum is necessary equal to  $\bar{f}_i$ . Since  $\mathcal{A}(\mathbf{f})$  is included in  $\square \mathcal{A}(\mathbf{f})$ , then a boundary hyperplane of  $\square \mathcal{A}(\mathbf{f})$ , which Cartesian equation is  $f_{\square,i}(\mathbf{z}_{C,i}, \mathbf{z}_{NC,i}) = \bar{f}_i$ , contains the point  $f_i(\mathbf{z}_C^*, \mathbf{z}_{NC,i}^*)$ . This property holds for both bounds of every  $f_i$ , thus  $\square \mathcal{A}(\mathbf{f})$  is circumscribed to  $\mathcal{A}(\mathbf{f})$ .

Therefore,  $\square \mathcal{A}(\mathbf{f})$  is an overestimation of  $\mathcal{A}(\mathbf{f})$ . To compare individually each residual with its abstract space boils down to use  $\square \mathcal{A}(\mathbf{f})$  since the assumption of independence is implicit. Even if it is easier to build  $\square \mathcal{A}(\mathbf{f})$  than  $\mathcal{A}(\mathbf{f})$ , a part of the information is thus lost and the number of no-detections may become important in a F.D. procedure.

#### 4.3. Consistency tests

For both static forms presented in section 2, internal forms are non-linear in bounded variables. Because of dependencies between parity relations, the abstract space  $\mathcal{A}(\mathbf{p}^{\text{int}})$  is often too complicated to be evaluated. Therefore, the aim is the determination of an overestimation  $\mathcal{A}_{\text{ov}}(\mathbf{p}^{\text{int}})$  of  $\mathcal{A}(\mathbf{p}^{\text{int}})$ , which is less prejudicial to a F.D. procedure than  $\square \mathcal{A}(\mathbf{p}^{\text{int}})$ . To solve this problem, the notion of paving presented in (Jaulin and Walter, 1993) is used. The paving  $\mathcal{A}_{\text{ov}}(\mathbf{p}^{\text{int}})$  is a subset of the residual space based upon axis-aligned orthotopes (called boxes too) of  $\mathbb{R}^{\text{sw}}$ .

Contrary to Jaulin, *et al.*, (1993), the proposed subset is not necessary compact and is composed of overlapping boxes. In fact,  $\mathcal{A}_{\text{ov}}(\mathbf{p}^{\text{int}})$  is defined by the union of boxes obtained by the uniform subdivision of order  $q \in \mathbb{N}^*$  (Moore, 1979) of the parameter space  $\mathcal{A}(\Theta, \Delta)$ . The box  $\mathcal{A}(\Theta, \Delta)$  is subdivided into  $q^{2s_m}$  nonoverlapping boxes  $\mathcal{E}_{q,i}$  of  $\mathbb{R}^{2s_m}$  obtained by cutting the abstract space of each bounded variable into  $q$  equal pieces. Thus, all  $\mathcal{E}_{q,i}$  have the same size and their union exactly describes  $\mathcal{A}(\Theta, \Delta)$ . For every box  $\mathcal{E}_{q,i}$ , the domain  $\square \mathcal{A}(\mathbf{p}^{\text{int}}(\Theta, \Delta)) / (\Theta, \Delta) \in \mathcal{E}_{q,i}$  is determined, which union defines  $\mathcal{A}_{\text{ov}}(\mathbf{p}^{\text{int}})$ . The greater is  $q$ , the smaller is the overestimation. Thus, to obtain an accurate abstract space  $\mathcal{A}_{\text{ov}}(\mathbf{p}^{\text{int}})$ , time consuming is high. In the following, because of its simplicity, this approach is called *basic paving method*.

However, for diagnosis, there is no need to build entirely  $\mathcal{A}_{\text{ov}}(\mathbf{p}^{\text{int}})$ . Checking if the vector  $\mathbf{p}^{\text{ext}}$  belongs to  $\mathcal{A}_{\text{ov}}(\mathbf{p}^{\text{int}})$  all the time is sufficient. Therefore, only the area close to  $\mathbf{p}^{\text{ext}}$  is worth studying. The recursive algorithm presented in the following is used to find boxes  $\mathcal{E}_{q,i}$  such that each  $\square \mathcal{A}(\mathbf{p}^{\text{int}}(\Theta, \Delta)) / (\Theta, \Delta) \in \mathcal{E}_{q,i}$  contain  $\mathbf{p}^{\text{ext}}$ . It works on a F.I.F.O list  $L$ , in which each element is associated with a kept box  $\mathcal{E}_i$  of  $\mathcal{A}(\Theta, \Delta)$ . Since a box can be directly described by an interval vector,  $L$  memorises their upper and lower bounds. The idea is to subdivide these boxes and then keep their children in  $L$  only if they contain  $\mathbf{p}^{\text{ext}}$ . If  $L$  becomes empty, then a fault is detected:  $\mathbf{p}^{\text{ext}}$  does not belong to  $\mathcal{A}(\mathbf{p}^{\text{int}})$  since it is not inside  $\mathcal{A}_{\text{ov}}(\mathbf{p}^{\text{int}})$ . At first,  $\mathcal{E}_1 = \mathcal{A}(\Theta, \Delta)$  and  $r=1$  are imposed, the list  $L$  is initialised with the bounds of uncertain parameters. In the following, the necessary steps are listed:

- (1) Take the first element  $L_1$  from the list  $L$  and remove it from  $L$ .
- (2) By assuming that  $L_1$  corresponds to the box  $\mathcal{E}_1$ , evaluate  $\square \mathcal{A}(\mathbf{p}^{\text{int}}(\Theta, \Delta)) / (\Theta, \Delta) \in \mathcal{E}_1$ .
- (3) Check if  $\mathbf{p}^{\text{ext}}$  belongs to  $\square \mathcal{A}(\mathbf{p}^{\text{int}}(\Theta, \Delta)) / (\Theta, \Delta) \in \mathcal{E}_1$ . If this test is false, go to step (1)
- (4) Cut  $\mathcal{E}_1$  along one of its edges into two boxes  $\mathcal{E}_{1,1}$  and  $\mathcal{E}_{1,2}$  such that  $\mathcal{E}_1 = \mathcal{E}_{1,1} \cup \mathcal{E}_{1,2}$  and memorise the associated bounds at the end of  $L$ .
- (5) Stop the algorithm if  $L$  is empty because a fault has been detected, else go to step (6).
- (6) If all boxes of the list  $L$  correspond to an uniform subdivision of  $\mathcal{A}(\Theta, \Delta)$  of order  $q=2^r$ , do  $r=r+1$ .
- (7) Stop the algorithm if  $r$  is greater than a threshold specified by the user, else go to step (1).

If this test is true, a normal functioning is assumed. The precision of the F.D. procedure depends on the threshold. The higher it is, the smaller is the overestimation, but the more important is time consuming. To illustrate that, let us take this example:

$$\mathbf{p}(\mathbf{z}) = \begin{bmatrix} 2z_1^2 + z_2 \\ z_1 - z_2 \end{bmatrix} \text{ with } \mathcal{A}(z_1) = [-1, 2] \\ \mathcal{A}(z_2) = [1, 3]$$



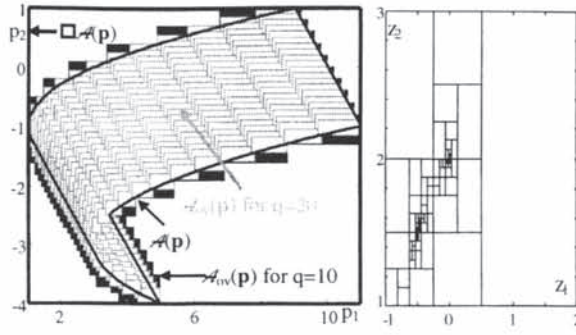


Fig. 1. Example of paving

Both abstract spaces  $\mathcal{A}(\mathbf{p})$  and  $\square\mathcal{A}(\mathbf{p})$  are plotted on the left in the figure 1. Then, two overestimations  $\mathcal{A}_{ov}(\mathbf{p})$  obtained by a basic paving method with respectively an uniform subdivision of order  $q=10$  and  $q=20$  are drawn. They imply to build 100 and 400 boxes. The second picture, on the right, presents the subdivision of the parameter space  $\mathcal{A}(\mathbf{z})$  obtained with the proposed algorithm by testing if the point (2,2) belongs to  $\mathcal{A}(\mathbf{p})$ . It shows two explored areas around the exact solutions (-0.5,1.5) and (0,2); therefore, the answer is affirmative. The chosen threshold is  $q=10$ , which corresponds to a uniform subdivision of order  $2^9$ . With the basic method,  $2^{18}$  boxes must be calculated whereas our algorithm only needs 261 ones; thus time consuming is smaller.

## 5. EXAMPLE

Let us consider the following system:

$$\mathbf{A} = \begin{bmatrix} 0.8 & 1 \\ 0 & 0.2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C}_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \mathbf{T} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.01 \end{bmatrix},$$

$$\text{with } |\theta_{k,i}| \leq \sigma_i, \sigma_1 = 0.02, \sigma_2 = 0.1, |\delta_{k,i}| \leq 1, i \in \{1,2\}$$

Since the rank of the matrix  $\mathbf{C}_{1,0}$  is 2, two parity relations are obtained for a time horizon equal to 1. The input is piece wise constant with a magnitude comprised between -1 and 1. At time 0.2s, a fault occurs on the second sensor during 0.4s. The relative precision  $\sigma_2$  becomes 0.16. From  $t=0.8s$  to  $t=1.2s$ , a noise with an uniform distribution bounded by  $\pm 0.35$  is added on the first sensor. At last, a bias equal to 0.2 is added on this one at time  $t=1.4s$  during 0.4s. Then internal and external forms are determined, where  $y_{k,i}$  denotes the  $i^{\text{th}}$  measurement,  $i \in \{1,2\}$ , at time  $k$ . In order to obtain a compact expression, a function  $\mu$  is introduced, where  $\rho$  represents a real argument. Notice that  $\mu_k(2,.)$  and  $\mu_{k+1}(1,.)$  appear in both components of the following internal form:

$$\mathbf{p}_{k+1}^{\text{ext}} = \begin{bmatrix} y_{k,1} - 5y_{k,2} + 5y_{k+1,1} \\ 4y_{k,2} - 10y_{k+1,1} + 5y_{k+1,2} - 5u_k \end{bmatrix},$$

$$\mathbf{p}_{k+1}^{\text{int}} = \begin{bmatrix} \mu_k(1,0.1) - 5\mu_k(2,0.01) + 5\mu_{k+1}(1,0.1) \\ 4\mu_k(2,0.01) - 10\mu_{k+1}(1,0.1) + 5\mu_{k+1}(2,0.01) \end{bmatrix},$$

where  $\mu_k(i,\rho) = (\theta_{k,i}y_{k,i} + \rho\delta_{k,i}) / (1 + \theta_{k,i})$ .

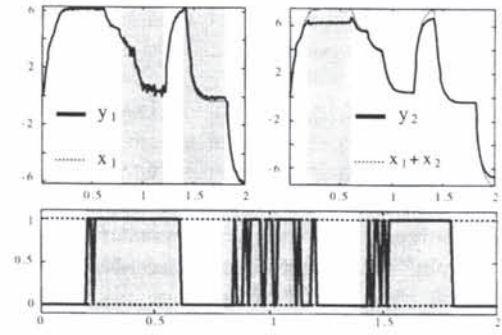


Fig. 2. Fault detection procedure

Since several common variables exist and the internal form is non-linear, a F.D. procedure based on the algorithm presented in the section 4 is applied. Measurements and associated combinations of state variables are plotted in figure 2. Grey areas indicate when a fault is present. The last curve gives the result of consistency tests: a value equal to 1 indicates when a fault is detected. During a normal operation, the parity vector  $\mathbf{p}^{\text{ext}}$  always belongs to the abstract space  $\mathcal{A}(\mathbf{p}^{\text{int}})$ . When a fault is present, the detection depends on the operating point and uncertainty values. Thus, the detection is successful in spite of important uncertainty ranges.

## 6. CONCLUSION

An alternative to traditional F.D. methods for uncertain models is proposed. Both static forms, presented in a previous article, are adapted according to the modeling of sensor imperfections. In fact, the residuals obtained from these ones are equivalent for the studied model. Because of dependencies between residuals due to common variables, the normal operating field is often too complicated to be exactly evaluated. Therefore, a paving method is proposed to detect faults while limiting time consuming, which increases quickly with the number of uncertainties and the accuracy of the detection.

## 7. REFERENCES

- Adrot O., D. Maquin, J. Ragot, (1999). Fault detection with model parameter structured uncertainties, *ECC'99*.
- Chow E.Y., A.S. Willsky, (1984). Analytical Redundancy and the Design of Robust Failure Detection System. *IEEE Trans. Aut. Control*, **AC-29** (7), 603-614.
- Jaulin L., E. Walter, (1993), Guaranteed nonlinear parameter estimation from bounded-error data via interval analysis, *Math. & Comp. in Sim.*, 123-137.
- Moore R.E., (1979). *Methods and applications of interval analysis*, SIAM, Philadelphia.
- Patton R.J., P.M. Frank, R.N. Clark, (1989). *Fault diagnosis in dynamic systems*, Prentice Hall, Englewood Cliffs.
- Ploix S., O. Adrot, J. Ragot, (1999). Parameter uncertainty computation in static linear models, *38<sup>th</sup> IEEE CDC'99*, 1916-1920.