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SEMI-LINEAR SUB-ELLIPTIC EQUATIONS ON THE HEISENBERG GROUP WITH A SINGULAR POTENTIAL

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ABSTRACT. In this work, we study the Dirichlet problem for a class of semi-linear sub-elliptic equations on the Heisenberg group with a singular potential. The singularity is controlled by Hardy's inequality, and the nonlinearity is controlled by Sobolev's inequality. We prove the existence of a nontrivial solution for a homogenous Dirichlet problem.

Key words: Heisenberg group, Hardy's inequality, Sobolev's inequality, Singular potential

A.M.S. Classification

1. INTRODUCTION

In this work, we study the partial differential equations on the Heisenberg group \mathbb{H}^d . Let us recall that the Heisenberg group is the space \mathbb{R}^{2d+1} of the (non commutative) law of product

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + 2((y|x') - (y'|x))).$$

The left invariant vector fields are

$$X_j = \partial_{x_j} + 2y_j \partial_s, \quad Y_j = \partial_{y_j} - 2x_j \partial_s, \quad j = 1, \dots, d \quad \text{and} \quad S = \partial_s = \frac{1}{4}[Y_j, X_j].$$

In all that follows, we shall denote by $Z_j = X_j$ and $Z_{j+d} = Y_j$ for $j \in \{1, \dots, d\}$. We fix here some notations :

$$z = (x, y) \in \mathbb{R}^{2d}, \quad w = (z, s) \in \mathbb{H}^d, \quad \rho(z, s) = (|z|^4 + |s|^2)^{1/4}$$

where ρ is the Heisenberg distance. Moreover, the Laplacian-Kohn operator on \mathbb{H}^d and Heisenberg gradient is given by

$$\Delta_{\mathbb{H}^d} = \sum_{j=1}^n X_j^2 + Y_j^2; \quad \nabla_{\mathbb{H}^d} = (Z_1, \dots, Z_{2d}).$$

Let Ω be an open and bounded domain of \mathbb{H}^d , we define thus the associated Sobolev space as following

$$H^1(\Omega, \mathbb{H}^d) = \left\{ f \in L^2(\Omega); \nabla_{\mathbb{H}^d} f \in L^2(\Omega) \right\}$$

and $H_0^1(\Omega, \mathbb{H}^d)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega, \mathbb{H}^d)$.

We consider the following semi-linear Dirichlet problem

$$(1.1) \quad \begin{cases} -\Delta_{\mathbb{H}^d} u - \mu V u = \lambda u + |u|^{p-2} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

where V is a positive potential function which admits the singularity on Ω , λ is a real constant and $2 < p < 2 + \frac{2}{d}$; the index $2^* = 2 + \frac{2}{d}$ is the critical index of Sobolev's inequality on the Heisenberg group [9, 15, 16, 17, 22]

$$(1.2) \quad \|u\|_{L^{2^*}(\Omega)} \leq C_{\Omega} \|u\|_{H^1(\Omega, \mathbb{H}^d)},$$

for all $u \in H_0^1(\Omega, \mathbb{H}^d)$.

The potential function V is controlled by the following Hardy's inequality,

$$(1.3) \quad \int_{\Omega} V(w) |u(w)|^2 dw \leq \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2,$$

for all $u \in H_0^1(\Omega, \mathbb{H}^d)$.

We will prove in the next section the following results

- If $0 \in \Omega$, Hardy's inequality (1.3) holds for

$$V(z, s) = \frac{d^4}{(d+1)^2} \rho(z, s)^{-2}.$$

- If $0 \in \Omega$, Hardy's inequality (1.3) also holds for a softer potential

$$V(z, s) = d^2 \frac{|z|^2}{\rho(z, s)^4},$$

see also [12, 18].

- In the Lemma 2.7, we prove that Hardy's inequality (1.3) holds for

$$V(z, s) = \bar{\mu} \rho_c(z, s)^{-2},$$

where ρ_c defined in (2.8) is the distance to a sub-manifold Σ_c of codimension ≥ 2 and $\bar{\mu}$ is a constant.

Theorem 1.1. *Assume that the potential function V satisfies (1.3), then for any $\lambda > 0$ and any $0 \leq \mu < 1$, the Dirichlet problem (1.1) admits a nontrivial solution in $H_0^1(\Omega, \mathbb{H}^d)$.*

The Dirichlet problem (1.1) on the Heisenberg group is a natural generalization of the classical problem on \mathbb{R}^d , see [5, 7, 8, 10, 11, 14, 20] and their references. The subellipticity of the operator $\Delta_{\mathbb{H}^d}$ implies that any weak solution of the Dirichlet problem (1.1) belongs to $C^\infty(\Omega \setminus \Sigma_c)$ (see [25]). The issue of regularity of a weak solution near $\partial\Omega \cup \Sigma_c$ is a very delicate problem.

This paper is organized as follows. In section 2, we recall Poincaré's inequality and prove Hardy's inequality on the Heisenberg group; Section 3 deals with the study of the eigenvalue problem; finally, we prove the existence of a weak solution in Sections 4 and 5 by using Rabinowitz's Theorem and the Palais-Smale Theorem.

2. HARDY'S INEQUALITIES ON \mathbb{H}^d

The following density theorem is very useful. Let us give here an other simple proof than [2] (see also [3]),

Theorem 2.1. *We have that $C_0^\infty(\Omega \setminus \{(0, 0)\})$ is dense in $H_0^1(\Omega, \mathbb{H}^d)$ for $d \geq 1$.*

Proof : By definition of $H_0^1(\Omega, \mathbb{H}^d)$, it suffices to show that

$$\mathcal{C}_0^\infty(\Omega) \subset \overline{\mathcal{C}_0^\infty(\Omega \setminus \{(0,0)\}, \mathbb{H}^d)}^{\|\cdot\|_{H^1}}.$$

Let φ be a cut-off function for which

$$(2.1) \quad \varphi(\eta) = \begin{cases} 0 & \text{if } 0 < \eta \leq 1, \\ 1 & \text{if } \eta \geq 2. \end{cases}$$

For $u \in \mathcal{C}_0^\infty(\Omega, \mathbb{H}^d)$, let $\varepsilon > 0$ small enough, and we set $u_\varepsilon(z, s) = \varphi(\frac{1}{\varepsilon}\rho(z, s)) u(z, s)$. So $u_\varepsilon \in \mathcal{C}_0^\infty(\Omega \setminus \{(0,0)\}, \mathbb{H}^d)$ and we have

$$\|u_\varepsilon - u\|_{H^1(\Omega)}^2 = \|\nabla_{\mathbb{H}^d}(u_\varepsilon - u)\|_{L^2(\Omega)}^2 + \|u_\varepsilon - u\|_{L^2(\Omega)}^2.$$

Dominated convergence theorem implies that

$$\|u_\varepsilon - u\|_{L^2(\Omega)}^2 \rightarrow 0,$$

and

$$\int_{\Omega} |\varphi(\frac{1}{\varepsilon}\rho(z, s)) - 1|^2 |\nabla_{\mathbb{H}^d} u(z, s)|^2 dz ds \rightarrow 0, \text{ when } \varepsilon \rightarrow 0.$$

On the other hand, we have

$$\begin{aligned} & \int_{\Omega} |\nabla_{\mathbb{H}^d}(\frac{1}{\varepsilon}\rho(z, s))|^2 |\varphi'(\frac{1}{\varepsilon}\rho(z, s))|^2 |u(z, s)|^2 dz ds \\ &= \frac{1}{\varepsilon^2} \int_{\Omega} \frac{|z|^2}{\rho(z, s)^2} |\varphi'(\frac{1}{\varepsilon}\rho(z, s))|^2 |u(z, s)|^2 dz ds \\ &\leq \frac{1}{\varepsilon^2} \|u\|_{L^\infty(\Omega)}^2 \|\varphi'\|_{L^\infty(\Omega)}^2 \int_{\{(z,s); \varepsilon \leq \rho(z,s) \leq 2\varepsilon\}} dz ds \\ &\leq C \frac{1}{\varepsilon^2} \varepsilon^{2d+2} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Remark that this proof also show that the density theorem is not true for classical Sobolev space in a 2 dimensional case.

Now, we state the following precise Poincaré inequality,

Theorem 2.2. *Let Ω be a sub-domain of \mathbb{H}^d bounded in some direction of (z_1, \dots, z_{2d}) , that is, there exist $R > 0$ and $1 \leq j_0 \leq 2d$ such that $0 < r = |z_{j_0}| \leq R$ for all $(z, s) \in \Omega$. Then for any $u \in H_0^1(\Omega, \mathbb{H}^d)$,*

$$(2.2) \quad \int_{\Omega} |u|^2 dz ds \leq 4R^2 \int_{\Omega} |\nabla_{\mathbb{H}^d} u|^2 dz ds.$$

Remark : By using the inequality (2.2), we can use $\|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}$ as a norm on $H_0^1(\Omega, \mathbb{H}^d)$. The Poincaré inequality (2.2) holds for any $\Omega \subset \{(z, s) \in \mathbb{H}^d; \rho(z, s) \leq R\}$. We can also obtain the Poincaré inequality from Bony's maximum principle for general Hörmander's vector fields but with a non-precise constant (see [6, 21]). The proof given here is a modification of L. D'Ambrosio [13].

Proof : Using the density results of Theorem 2.1, take $u \in \mathcal{C}_0^\infty(\Omega \setminus \{(0,0)\})$ and let $T(z, s) = (T_1(z, s), \dots, T_{2d}(z, s))$ be a C^1 vector function on Ω . Denote by

$$\operatorname{div}_{\mathbb{H}^d} T = \sum_{j=1}^{2d} Z_j T_j,$$

we have

$$\begin{aligned}
\int_{\Omega} (\operatorname{div}_{\mathbb{H}^d} T) |u|^2 \, dzds &= -2 \int_{\Omega} \langle T, \nabla_{\mathbb{H}^d} u \rangle u \, dzds \\
&\leq 2 \int_{\Omega} |\langle T, \nabla_{\mathbb{H}^d} u \rangle u| \, dzds \\
&\leq 2 \left[\int_{\Omega} |\nabla_{\mathbb{H}^d} u|^2 \, dzds \right]^{\frac{1}{2}} \left[\int_{\Omega} |T|^2 |u|^2 \, dzds \right]^{\frac{1}{2}} \\
&\leq \int_{\Omega} |\nabla_{\mathbb{H}^d} u|^2 \, dzds + \int_{\Omega} |T|^2 |u|^2 \, dzds.
\end{aligned}$$

Thus

$$\int_{\Omega} \left(\operatorname{div}_{\mathbb{H}^d} T - |T|^2 \right) |u|^2 \, dzds \leq \int_{\Omega} |\nabla_{\mathbb{H}^d} u|^2 \, dzds.$$

For $\varepsilon > 0$, let us choose $T := T_{\varepsilon} = -\frac{1}{2} \frac{\nabla_{\mathbb{H}^d} r_{\varepsilon}}{r_{\varepsilon}}$ where $r_{\varepsilon} = (r^2 + \varepsilon^2)^{\frac{1}{2}}$ and $r = |z_{j_0}|$. Then

$$\begin{aligned}
\operatorname{div}_{\mathbb{H}^d} T_{\varepsilon} &= -\frac{1}{2} \frac{1}{r_{\varepsilon}^2} [r_{\varepsilon} \Delta_{\mathbb{H}^d} r_{\varepsilon} - |\nabla_{\mathbb{H}^d} r_{\varepsilon}|^2] \\
\operatorname{div}_{\mathbb{H}^d} T_{\varepsilon} - |T_{\varepsilon}|^2 &= -\frac{1}{2} \frac{\Delta_{\mathbb{H}^d} r_{\varepsilon}}{r_{\varepsilon}} + \frac{1}{4} \frac{|\nabla_{\mathbb{H}^d} r_{\varepsilon}|^2}{r_{\varepsilon}^2}
\end{aligned}$$

since

$$\begin{aligned}
\nabla_{\mathbb{H}^d} r_{\varepsilon} &= \frac{\nabla_{\mathbb{H}^d} r^2}{2r_{\varepsilon}} = \frac{r}{r_{\varepsilon}}, \quad |\nabla_{\mathbb{H}^d} r_{\varepsilon}|^2 = \frac{r^2}{r_{\varepsilon}^2} \\
\Delta_{\mathbb{H}^d} r_{\varepsilon} &= \frac{\Delta_{\mathbb{H}^d} r^2}{2r_{\varepsilon}} - \frac{\nabla_{\mathbb{H}^d} r^2}{2r_{\varepsilon}^2} \nabla_{\mathbb{H}^d} r_{\varepsilon} = \frac{1}{r_{\varepsilon}} - \frac{r^2}{r_{\varepsilon}^3}.
\end{aligned}$$

So,

$$\operatorname{div}_{\mathbb{H}^d} T_{\varepsilon} - |T_{\varepsilon}|^2 = -\frac{1}{2r_{\varepsilon}^2} + \frac{3r^2}{4r_{\varepsilon}^4},$$

and for any $(z, s) \in \Omega$,

$$\lim_{\varepsilon \rightarrow 0} \operatorname{div}_{\mathbb{H}^d} T_{\varepsilon} - |T_{\varepsilon}|^2 = \frac{1}{4} \frac{1}{r^2} \geq \frac{1}{4} \frac{1}{R^2}.$$

From the dominated convergence theorem, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} [\operatorname{div}_{\mathbb{H}^d} T_{\varepsilon} - |T_{\varepsilon}|^2] |u|^2 \, dzds \geq \frac{1}{4} \frac{1}{R^2} \int_{\Omega} |u|^2 \, dzds,$$

thus

$$\frac{1}{4} \frac{1}{R^2} \int_{\Omega} |u|^2 \, dzds \leq \int_{\Omega} |\nabla_{\mathbb{H}^d} u|^2 \, dzds.$$

Let us give a very easy proof of the classical Hardy inequality by using a radial vector field. Let U be a bounded domain of \mathbb{R}^d , $d > 2$, $H_0^1(U)$ is the usual Sobolev space.

Lemma 2.3. *We have, for any $u \in H_0^1(U)$,*

$$(2.3) \quad \left(\frac{d}{2} - 1 \right)^2 \int_U \frac{u^2}{|x|^2} dx \leq \|\nabla_x u\|_{L^2(U)}^2.$$

Proof : As $C_0^\infty(U \setminus \{0\})$ is dense $H_0^1(U)$, we have restricted ourselves to a function u in $C_0^\infty(U \setminus \{0\})$. The proof mainly consists of an integration by parts with respect to the radial vector field R ,

$$R = \sum_{j=1}^d x_j \partial_{x_j}.$$

We notice that $R(|x|^{-2}) = -2|x|^{-2}$ and $\operatorname{div}R = d$, so

$$\begin{aligned} \int_U \frac{u^2}{|x|^2} dx &= -\frac{1}{2} \int_U R(|x|^{-2}) dx \\ &= \frac{1}{2} \int_U \operatorname{div}(R) \frac{u^2}{|x|^2} dx + \frac{1}{2} \int_U \frac{1}{|x|^2} R(u^2) dx \\ \left(1 - \frac{d}{2}\right) \int_U \frac{u^2}{|x|^2} dx &= \frac{1}{2} \int_U \frac{1}{|x|^2} R(u^2) dx = \int_U \sum_{j=1}^d \frac{u}{|x|} \frac{x_j}{|x|} \partial_{x_j} u dx. \end{aligned}$$

By Cauchy-Schwarz,

$$\left(\frac{d}{2} - 1\right) \int_U \frac{|u|^2}{|x|^2} dx \leq \left(\int_U \sum_{j=1}^d |\partial_{x_j} u|^2 dx\right)^{\frac{1}{2}} \left(\int_U \frac{|u|^2}{|x|^2} dx\right)^{\frac{1}{2}}.$$

On the Heisenberg group, if we introduce the radial vector field

$$R = \sum_{j=1}^d (x_j \partial_{x_j} + y_j \partial_{y_j}) = \sum_{j=1}^d (x_j X_j + y_j Y_j),$$

then we immediately obtain for $d > 1$,

$$(d-1)^2 \int_\Omega \frac{u^2}{\rho(z,s)^2} dz ds \leq (d-1)^2 \int_\Omega \frac{u^2}{|z|^2} dz ds \leq \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2$$

for any $u \in H_0^1(\Omega, \mathbb{H}^d)$.

By using the idea inspired from the radial vector field, we now prove the following Hardy inequality.

Lemma 2.4. *For $d \geq 1$, we have that*

$$(2.4) \quad \left(\frac{d^2}{d+1}\right)^2 \int_\Omega \frac{u^2}{\rho^2} dz ds \leq \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2,$$

for any $u \in H_0^1(\Omega, \mathbb{H}^d)$.

Proof : By using the density theorem, we prove the inequality (2.4) for the function $u \in C_0^\infty(\Omega \setminus \{(0,0)\})$. Then the proof mainly consists of an integration by parts with respect to the radial vector field $R_{\mathbb{H}^d}$ adapted to the structure of \mathbb{H}^d , namely

$$R_{\mathbb{H}^d} = 2s\partial_s + \sum_{j=1}^d (x_j \partial_{x_j} + y_j \partial_{y_j}) = \frac{s}{2d} \sum_{j=1}^d [Y_j, X_j] + \sum_{j=1}^d (x_j X_j + y_j Y_j).$$

We notice that $R_{\mathbb{H}^d}(\rho^{-2}) = -2\rho^{-2}$ and $\operatorname{div} R_{\mathbb{H}^d} = 2d + 2$. We have

$$\begin{aligned} \int_{\Omega} \frac{u^2}{\rho(z, s)^2} dz ds &= -\frac{1}{2} \int_{\Omega} R_{\mathbb{H}^d}(\rho(z, s)^{-2}) u^2 dz ds \\ &= \frac{1}{2} \int_{\Omega} \rho^{-2} R_{\mathbb{H}^d}(u^2) dz ds + \frac{1}{2} \int_{\Omega} \rho^{-2} u^2 \operatorname{div} R_{\mathbb{H}^d} dz ds. \end{aligned}$$

This gives

$$\begin{aligned} -d \int_{\Omega} \frac{u^2}{\rho^2} dz ds &= \int_{\Omega} \sum_{j=1}^d \frac{u}{\rho} \left(\frac{x_j}{\rho} X_j + \frac{y_j}{\rho} Y_j \right) u dz ds - \frac{1}{2d} \int_{\Omega} Y_j \left(\frac{s}{\rho^2} \right) u (X_j u) dz ds \\ &\quad + \frac{1}{2d} \int_{\Omega} X_j \left(\frac{s}{\rho^2} \right) u (Y_j u) dz ds \\ &= \left(1 + \frac{1}{d} \right) \int_{\Omega} \sum_{j=1}^d \left(\frac{x_j u}{\rho^2} X_j u + \frac{y_j u}{\rho^2} Y_j u \right) dz ds \\ &\quad + \frac{1}{d} \int_{\Omega} \sum_{j=1}^d \frac{s}{\rho^6} [|z|^2 y_j - s x_j] u X_j u dz ds \\ &\quad - \frac{1}{d} \int_{\Omega} \sum_{j=1}^d \frac{s}{\rho^6} [|z|^2 x_j + s y_j] u Y_j u dz ds, \end{aligned}$$

then

$$\begin{aligned} -d^2 \int_{\Omega} \frac{u^2}{\rho^2} dz ds &= \int_{\Omega} \sum_{j=1}^d \left[(d+1) - \frac{s^2}{\rho^4} \right] \left(\frac{x_j u}{\rho^2} X_j u + \frac{y_j u}{\rho^2} Y_j u \right) dz ds \\ &\quad + \int_{\Omega} \sum_{j=1}^d \frac{s |z|^2}{\rho^4} \left[\frac{y_j u}{\rho^2} X_j u - \frac{x_j u}{\rho^2} Y_j u \right] dz ds \\ &= \int_{\Omega} \sum_{j=1}^d \left[\left((d+1) - \frac{s^2}{\rho^4} \right) \frac{x_j}{\rho} + \frac{s |z|^2 y_j}{\rho^4} \frac{y_j}{\rho} \right] \frac{u}{\rho} X_j u dz ds \\ &\quad + \int_{\Omega} \sum_{j=1}^d \left[\left((d+1) - \frac{s^2}{\rho^4} \right) \frac{y_j}{\rho} - \frac{s |z|^2 x_j}{\rho^4} \frac{x_j}{\rho} \right] \frac{u}{\rho} Y_j u dz ds. \end{aligned}$$

Setting

$$A(z, s) = \sum_{j=1}^d \left\{ \left[\left((d+1) - \frac{s^2}{\rho^4} \right) \frac{x_j}{\rho} + \frac{s |z|^2 y_j}{\rho^4} \frac{y_j}{\rho} \right]^2 + \left[\left((d+1) - \frac{s^2}{\rho^4} \right) \frac{y_j}{\rho} - \frac{s |z|^2 x_j}{\rho^4} \frac{x_j}{\rho} \right]^2 \right\},$$

Cauchy-Schwarz inequality implies

$$d^2 \int_{\Omega} \frac{|u|^2}{\rho^2} dz ds \leq \left(\int_{\Omega} A(z, s) \frac{|u|^2}{\rho^2} dz ds \right)^{\frac{1}{2}} \left(\int_{\Omega} \sum_{j=1}^d (|X_j u|^2 + |Y_j u|^2) dz ds \right)^{\frac{1}{2}}.$$

For $A(z, s)$, we have

$$\begin{aligned}
A(z, s) &= \left((d+1) - \frac{s^2}{\rho^4} \right)^2 \frac{|z|^2}{\rho^2} + \frac{s^2 |z|^4 |z|^2}{\rho^8 \rho^2} \\
&= \frac{|z|^2}{\rho^2} \left[(d+1)^2 - (2d+1) \frac{s^2}{\rho^4} \right] \\
&= \frac{|z|^2}{\rho^6} \left[(d+1)^2 |z|^4 + d^2 s^2 \right] \\
&= \frac{|z|^2}{\rho^6} \left[(2d+1) |z|^4 + d^2 (|z|^4 + s^2) \right] \\
&\leq (d+1)^2 \frac{|z|^2}{\rho^2} \leq (d+1)^2.
\end{aligned}$$

So, we deduce the inequality (2.4).

The Hardy inequality on the Heisenberg group \mathbb{H}^d is first proven in [18, 12] for a softer potential.

Lemma 2.5. *We have, for any $u \in H_0^1(\Omega, \mathbb{H}^d)$,*

$$(2.5) \quad d^2 \int_{\Omega} \frac{|z|^2}{|z|^4 + s^2} |u|^2 \, dz ds \leq \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2.$$

The singularity of potential in the Hardy inequalities (2.3), (2.4) and (2.5) is a isolate point of domain. We consider now the general case when the singularity is on a sub-manifold. We have first the following density result:

Lemma 2.6. *Let Ω be a bounded domain of \mathbb{R}^{2d+1} and Σ_c a sub-manifold of Ω such that $\dim \Sigma_c \leq 2d-1$. Then $C_0^\infty(\Omega \setminus \Sigma_c)$ is dense in the space $H_0^1(\Omega, \mathbb{H}^d)$.*

Proof : As $H_0^1(\Omega, \mathbb{H}^d)$ is a Hilbert space, it is enough to prove that the orthogonal of $C_0^\infty(\Omega \setminus \Sigma_c)$ in $H_0^1(\Omega, \mathbb{H}^d)$ is $\{0\}$. Let u be in this space. For any v in $C_0^\infty(\Omega \setminus \Sigma_c)$, we have

$$(u, v)_{L^2} + (\nabla_{\mathbb{H}^d} u, \nabla_{\mathbb{H}^d} v)_{L^2} = 0.$$

By integration by part,

$$\forall v \in C_0^\infty(\Omega \setminus \Sigma_c), \quad \langle u - \Delta_{\mathbb{H}^d} u, v \rangle = 0,$$

this implies that, as a distribution,

$$\text{Supp}(u - \Delta_{\mathbb{H}^d} u) \subset \Sigma_c.$$

Since $u - \Delta_{\mathbb{H}^d} u$ belong to the classical Sobolev space $H^{-1}(\Omega)$ and except 0, no distribution of $H^{-1}(\Omega)$ can be supported in a submanifold of dimension $\leq (2d+1)-2$. Thus $u - \Delta_{\mathbb{H}^d} u = 0$ on Ω . Taking the L^2 scalar product with $u \in H_0^1(\Omega, \mathbb{H}^d)$ implies that $u \equiv 0$. This completes the proof of Lemma 2.6.

We consider now the hyper-surface $\Sigma = \{(x, y, s) \in \Omega : g(x, y, s) = s + f(x, y) = 0\}$ where Ω is a neighborhood of 0 in \mathbb{H}^d . Assume that

$$(2.6) \quad \Sigma_c = \{w \in \Omega : g(w) = 0, \nabla_{\mathbb{H}^d} g(w) = 0\},$$

is a sub-manifold of dimension $(2d+1) - r - 1$, $r \geq 1$.

Lemma 2.7. *Assume that Σ_c is a sub-manifold of dimension $2d - r$ and $r \geq 1$. Then, there exists $\bar{\mu} > 0$ such that for any $u \in H_0^1(\Omega, \mathbb{H}^d)$,*

$$(2.7) \quad \bar{\mu} \int_{\Omega} \frac{u^2}{\rho_c^2} dw \leq \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2$$

with

$$(2.8) \quad \rho_c(w) = (g^2(w) + |\nabla_{\mathbb{H}^d} g(w)|^4)^{1/4}.$$

We refer to the proof of this lemma to [4] and also [2]. The constant $\bar{\mu}$ depends, of course on Σ_c , but in many interesting cases, it depends only on the dimension of Σ_c .

Here we present a proof for a model case in \mathbb{H}^1 to precise the constant $\bar{\mu}$. We take $g(x, y, s) = s + 2xy$, then

$$(2.9) \quad \begin{aligned} \Sigma &= \{(x, y, s) \in \mathbb{H}^1 : s + 2xy = 0\}, \\ \Sigma_c &= \{(x, 0, 0), x \in \mathbb{R}\}. \end{aligned}$$

Lemma 2.8. *Let Σ_c as in (2.9), then, we have for any $u \in H^1(\mathbb{H}^1)$,*

$$(2.10) \quad \frac{2^2}{5 + 2^8} \int_{\mathbb{H}^1} \frac{u^2}{\rho_c^2} dw \leq \|\nabla_{\mathbb{H}^1} u\|_{L^2(\mathbb{H}^1)}^2$$

Proof : We rectify Σ by setting $x' = x$, $y' = y$, $s' = s + 2xy$, so the vector fields X and Y change to $X' = \partial_{x'} + 4y'\partial_{s'}$, $Y' = \partial_{y'}$ and

$$\rho_c(x', y', s') = ((4y')^4 + s'^2)^{1/4}.$$

Let R be a radial vector field

$$\begin{aligned} R &= X'(s')Y' + 2^3 s' \partial_{s'} \\ &= 4y'Y' + 2^3 s' \partial_{s'} \\ &= 4y'Y' + 2s'[Y', X'], \end{aligned}$$

where $R(\rho_c^{-2}) = -8\rho_c^{-2}$ and $\text{div}R = 12$. Using the density Lemma 2.6, we have for $u \in C_0^\infty(\mathbb{H}^1 \setminus \{(0, 0)\})$,

$$\begin{aligned} \int_{\mathbb{H}^1} \frac{u^2}{\rho_c^2} dz' ds' &= -\frac{1}{8} \int_{\mathbb{H}^1} u^2 R(\rho_c^{-2}) dz' ds' \\ -\frac{1}{2} \int_{\mathbb{H}^1} \frac{u^2}{\rho_c^2} dz' ds' &= \int_{\mathbb{H}^1} \frac{y'}{\rho_c^2} u Y' u dz' ds' - \frac{1}{2} \int_{\mathbb{H}^1} Y' \left(\frac{s'}{\rho_c^2} \right) u X' u dz' ds' \\ &\quad + \frac{1}{2} \int_{\mathbb{H}^1} X' \left(\frac{s'}{\rho_c^2} \right) u Y' u dz' ds' \\ &= \int_{\mathbb{H}^1} \frac{y'}{\rho_c^2} u Y' u dz' ds' + \int_{\mathbb{H}^1} 2^8 \frac{y'^3 s'}{\rho_c^6} u X' u dz' ds' \\ &\quad + \int_{\mathbb{H}^1} \left[\frac{2y'}{\rho_c^2} - \frac{2y' s'^2}{\rho_c^6} \right] u Y' u dz' ds' \\ &= \int_{\mathbb{H}^1} \left[\frac{3y'}{\rho_c} - \frac{2y' s'^2}{\rho_c^5} \right] \frac{u}{\rho_c} Y' u dz' ds' + \int_{\mathbb{H}^1} 2^8 \frac{y'^3 s'}{\rho_c^5} \frac{u}{\rho_c} X' u dz' ds' \end{aligned}$$

We then obtain

$$\frac{1}{2} \int_{\mathbb{H}^1} \frac{|u|^2}{\rho_c^2} dz' ds' \leq \left(\int_{\mathbb{H}^1} A(z', s') \frac{|u|^2}{\rho^2} dz' ds' \right)^{\frac{1}{2}} \|\nabla_{\mathbb{H}^1} u\|_{L^2(\mathbb{H}^1)},$$

with

$$\begin{aligned} A(z', s') &= \left[\frac{3y'}{\rho_c} - 2 \frac{y' s'^2}{\rho_c^5} \right]^2 + 2^{16} \frac{y'^6 s'^2}{\rho_c^{10}} \\ &= 3^2 \frac{y'^2}{\rho_c^2} - 12 \frac{y'^2 (\rho_c^4 - 2^8 y'^4)}{\rho_c^6} + 4 \frac{y'^2 (\rho_c^4 - 2^8 y'^4)^2}{\rho_c^{10}} + 2^{16} \frac{y'^6 (\rho_c^4 - 2^8 y'^4)}{\rho_c^{10}} \\ &= \frac{y'^2}{\rho_c^2} \left[1 + 2^8 (4 + 2^8) \frac{y'^4}{\rho_c^4} + 2^8 (2^{10} - 2^{16}) \frac{y'^8}{\rho_c^8} \right] \\ &\leq \frac{1}{2^4} \left[1 + \frac{1}{2^8} 2^8 (4 + 2^8) \right] \\ &\leq \frac{1}{2^4} (5 + 2^8). \end{aligned}$$

3. VARIATIONAL FORMULATION AND EIGENVALUE PROBLEM

Thanks to Hardy's inequality (1.3) and Poincaré's inequality (2.2),

$$(3.1) \quad \|u\|_{\mu} = \left(\int_{\Omega} [|\nabla_{\mathbb{H}^d} u(z, s)|^2 - \mu V(z, s) |u(z, s)|^2] dz ds \right)^{\frac{1}{2}}$$

is equivalent to the norm on $H_0^1(\Omega, \mathbb{H}^d)$ for all $0 \leq \mu < 1$, so that we will use $\|\cdot\|_{\mu}$ as the norm of $H_0^1(\Omega, \mathbb{H}^d)$.

We will use the variational method to study the Dirichlet problem (1.1). We define the following energy functional on $H_0^1(\Omega, \mathbb{H}^d)$:

$$(3.2) \quad I_{\mu, \lambda}(u) = \frac{1}{2} \int_{\Omega} [|\nabla_{\mathbb{H}^d} u|^2 - \mu V |u|^2] dz ds - \frac{1}{p} \int_{\Omega} |u|^p dz ds - \frac{\lambda}{2} \int_{\Omega} |u|^2 dz ds.$$

Similar to the classical case, $I_{\mu, \lambda}(\cdot)$ is well-defined on $H_0^1(\Omega, \mathbb{H}^d)$ and belongs to $C^1(H_0^1(\Omega, \mathbb{H}^d); \mathbb{R})$.

We say that $u \in H_0^1(\Omega, \mathbb{H}^d)$ is a weak solution of the Dirichlet problem (1.1), if for any $v \in C_0^\infty(\Omega)$, there holds

$$\int_{\Omega} [\nabla_{\mathbb{H}^d} u \overline{\nabla_{\mathbb{H}^d} v} - \mu V u \bar{v}] dz ds - \int_{\Omega} |u|^{p-2} u \bar{v} dz ds - \lambda \int_{\Omega} u \bar{v} dz ds = 0$$

So a weak solution $u \in H_0^1(\Omega, \mathbb{H}^d)$ of the Dirichlet problem (1.1) is a critical point of $I_{\mu, \lambda}$. The Euler-Lagrange equation of the variational problem (3.2) is exactly the semilinear equation in (1.1), and we have

$$\langle I'_{\mu, \lambda}(u), v \rangle = \int_{\Omega} [\nabla_{\mathbb{H}^d} u \overline{\nabla_{\mathbb{H}^d} v} - \mu V u \bar{v} - |u|^{p-2} u \bar{v} - \lambda u \bar{v}] dz ds = 0$$

for any $v \in H_0^1(\Omega, \mathbb{H}^d)$.

Since we consider the Dirichlet problem (1.1) for any $\lambda > 0$, we cannot use the direct method to prove the existence of the critical point for $I_{\mu, \lambda}$. We need to use the Mountain Pass Theorem and the Linking Theorem of Rabinowitz (see [23, 24, 26]).

Thus that we study firstly the spectral decomposition of $H_0^1(\Omega, \mathbb{H}^d)$ with respect to the operator $-\Delta_{\mathbb{H}^d} - \mu V$ where the singular potential V satisfies Hardy's inequality (1.3). This eigenvalue problem has also its independent interest. We have the following proposition.

Proposition 3.1. *Let $0 \leq \mu < 1$. Then there exist $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \leq \dots \rightarrow +\infty$, such that for each $k \geq 1$, the following Dirichlet problem*

$$(3.3) \quad \begin{cases} -\Delta_{\mathbb{H}^d} \phi_k - \mu V \phi_k = \lambda_k \phi_k, & \text{in } \Omega \\ \phi_k|_{\partial\Omega} = 0 \end{cases}$$

admits a nontrivial solution in $H_0^1(\Omega, \mathbb{H}^d)$. Moreover, $\{\phi_k\}_{k \geq 1}$ constitutes an orthonormal basis of Hilbert space $H_0^1(\Omega, \mathbb{H}^d)$.

Remark that the first eigenvalue λ_1 is characterized by the following Poincaré inequality

$$(3.4) \quad \|u\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda_1} \int_{\Omega} (|\nabla_{\mathbb{H}^d} u|^2 - \mu V |u|^2) dz ds$$

for all $u \in H_0^1(\Omega, \mathbb{H}^d)$.

The first step of the proof is the following compact embedding result

Lemma 3.2. *Let $\Omega \in \mathbb{H}^d$ be a bounded open domain. Then $H_0^1(\Omega, \mathbb{H}^d)$ is compactly embedded to $L^2(\Omega)$.*

We can prove this result by the continuous embedding of $H_0^1(\Omega, \mathbb{H}^d)$ into usual the Sobolev space $H_0^{1/2}(\Omega)$, then the compact embedding of $H^{1/2}(\Omega)$ into $L^2(\Omega)$. But the first embedding requires some careful extension results. We refer to [19] for a complete and elegant proof of this compact embedding result.

Proof of Proposition 3.1

Denote by $L_\mu = -\Delta_{\mathbb{H}^d} - \mu V$ the operator defined on the Hilbert space $H_0^1(\Omega, \mathbb{H}^d)$ with the norm $\|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}$, then Hardy's inequality (1.3) implies

$$(L_\mu u, u)_{L^2(\Omega)} \geq (1 - \mu) \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2 > 0, \quad \forall u \in H_0^1(\Omega, \mathbb{H}^d),$$

and

$$(L_\mu u, v) = (u, L_\mu v), \quad \forall u, v \in H_0^1(\Omega, \mathbb{H}^d).$$

Hence it is positive, definite and self-adjoint on $H_0^1(\Omega, \mathbb{H}^d)$. The Lax-Milgram Theorem implies that for any $g \in H^{-1}(\Omega; \mathbb{H}^d)$, the following Dirichlet problem

$$\begin{cases} L_\mu u = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a unique solution u belonging to $H_0^1(\Omega, \mathbb{H}^d)$, where $H^{-1}(\Omega; \mathbb{H}^d)$ is the dual space of $H_0^1(\Omega, \mathbb{H}^d)$, $g \in H^{-1}(\Omega; \mathbb{H}^d)$ if $g \in \mathcal{D}'(\Omega)$ and there exists $C > 0$ such that

$$|\langle g, \varphi \rangle| \leq C \|\varphi\|_{H_0^1(\Omega, \mathbb{H}^d)}$$

for all $\varphi \in C_0^\infty(\Omega)$ with the norm

$$\|g\|_{H^{-1}(\Omega, \mathbb{H}^d)} = \sup_{\varphi \in C_0^\infty(\Omega)} \frac{|\langle g, \varphi \rangle|}{\|\varphi\|_{H_0^1(\Omega, \mathbb{H}^d)}}.$$

Then

$$\nabla_{\mathbb{H}^d} : L^2(\Omega) \rightarrow H^{-1}(\Omega; \mathbb{H}^d) \quad \text{and} \quad \Delta_{\mathbb{H}^d} : H_0^1(\Omega, \mathbb{H}^d) \rightarrow H^{-1}(\Omega; \mathbb{H}^d)$$

are continuous. The inverse operator L_μ^{-1} of L_μ is well defined and it is a continuous map from $H^{-1}(\Omega, \mathbb{H}^d)$ into $H_0^1(\Omega, \mathbb{H}^d)$.

The compact embedding $i : H_0^1(\Omega, \mathbb{H}^d) \rightarrow L^2(\Omega)$ and the continuous embedding $i^* : L^2(\Omega) \rightarrow H^{-1}(\Omega, \mathbb{H}^d)$ imply that $K_\mu = L_\mu^{-1} \circ i^* \circ i : H_0^1(\Omega, \mathbb{H}^d) \rightarrow H_0^1(\Omega, \mathbb{H}^d)$ is a compact

and self adjoint operator. So the spectrum of the compact operator K_μ is $\{\eta_k\}$ such that $\eta_k > 0, k \geq 1$ and $\eta_k \rightarrow 0$. If $\{\phi_k\}$ are the associated normal eigenvectors, we have that

$$K_\mu \phi_k = \eta_k \phi_k, \quad \forall k \geq 1,$$

and $\{\phi_k\}$ form a complete basis of Hilbert space $H_0^1(\Omega, \mathbb{H}^d)$, which completes the proof of Proposition 3.1.

4. EXISTENCE OF CRITICAL POINTS

We prove now the following existence result of critical points for the variational functional $I_{\mu,\lambda}$ which gives the weak solution for the Dirichlet problem (1.1).

Theorem 4.1. *Let $0 \leq \mu < 1, \lambda > 0$, then $I_{\mu,\lambda}$ admits at least one nontrivial critical point on $H_0^1(\Omega, \mathbb{H}^d)$.*

We recall now the well-known Palais-Smale condition.

Definition 4.2. *Let E be a Banach space, $I \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. We say that I satisfies the $(PS)_c$ condition, if for any sequence $\{u_n\} \subset E$ with the properties :*

$$I(u_n) \rightarrow c \quad \text{and} \quad \|I'(u_n)\|_{E'(\Omega)} \rightarrow 0,$$

there exists a subsequence which is convergent, where $I'(\cdot)$ is the Frechet differentiation of I and E' is the dual space of E . If this holds for any $c \in \mathbb{R}$, we say that I satisfies the (PS) condition.

We will prove in the next section the following result

Theorem 4.3. *Let $0 \leq \mu < 1, \lambda > 0$, then $I_{\mu,\lambda}$ satisfies the (PS) condition on $H_0^1(\Omega, \mathbb{H}^d)$.*

Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \leq \dots \rightarrow +\infty$ be the eigenvalues of $-\Delta_{\mathbb{H}^d} - \mu V$ in Proposition 3.1. We consider firstly the case $0 < \lambda < \lambda_1$ and we use the following Mountain Pass Theorem to prove the existence of a critical point for $I_{\mu,\lambda}$:

Theorem 4.4. *(see [1, 23])*

Let E be a Banach space and $I \in C^1(E, \mathbb{R})$. We suppose that $I(0) = 0$ and satisfies that

- (i) there exist $R > 0, a > 0$ such that if $\|u\|_E = R$, then $I(u) \geq a$;*
- (ii) there exists $e \in E$ such that $\|e\| > R$ and $I(e) < a$. If I satisfies the $(PS)_c$ condition with*

$$c = \inf_{h \in \Gamma} \max_{t \in [0,1]} I(h(t)), \quad \text{where } \Gamma = \{h \in C([0,1]; E); h(0) = 0 \text{ and } h(1) = e\},$$

then c is a critical value of I and $c \geq a$.

We check the above conditions for $I = I_{\mu,\lambda}$ on $E = H_0^1(\Omega, \mathbb{H}^d)$. We have $I_{\mu,\lambda}(0) = 0$. For $u \in H_0^1(\Omega, \mathbb{H}^d)$, Sobolev's inequality (1.2) and Hardy's inequality imply that

$$\|u\|_{L^p(\Omega)} \leq C_\Omega \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)} \leq \frac{C_\Omega}{(1-\mu)^{1/2}} \|u\|_\mu.$$

Then, for $0 < \lambda < \lambda_1$

(4.1)

$$I_{\mu,\lambda}(u) \geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|_\mu^2 - \frac{C_1}{p} \|u\|_\mu^p \geq C_1 \|u\|_\mu^2 \left(\frac{1}{2C_1} \left(1 - \frac{\lambda}{\lambda_1}\right) - \frac{1}{p} \|u\|_\mu^{p-2} \right)$$

where

$$C_1 = \left(\frac{C_\Omega}{(1-\mu)^{1/2}} \right)^p > 1.$$

Let

$$R_0 = \left(\frac{p}{2C_1} \left(1 - \frac{\lambda}{\lambda_1}\right) \right)^{\frac{1}{p-2}} > 0.$$

Then for any $0 < R < R_0$,

$$(4.2) \quad \inf_{\|u\|_\mu=R} I_{\mu,\lambda}(u) = a(R) > 0.$$

So $I_{\mu,\lambda}$ satisfies condition (i) of Theorem 4.4.

For condition (ii) of Theorem 4.4, take $u \in H_0^1(\Omega, \mathbb{H}^d)$ such that $\|u\|_\mu = R > 0$, then for $\theta \geq 0$,

$$(4.3) \quad I_{\mu,\lambda}(\theta u) = \frac{\theta^2}{2} \int_\Omega [|\nabla_{\mathbb{H}^d} u|^2 - \mu V(z, s) |u|^2] dz ds$$

$$(4.4) \quad - \frac{\theta^p}{p} \int_\Omega |u|^p dz ds - \frac{\lambda \theta^2}{2} \int_\Omega |u|^2 dz ds.$$

Since $p > 2$, thus

$$\lim_{\theta \rightarrow +\infty} I_{\mu,\lambda}(\theta u) = -\infty.$$

Then, there exists $\theta_1 > 0$ large enough such that for $e = \theta_1 u$, we have $\|e\|_\mu > R$ and $I_\mu(\theta_1 u) < 0 < a(R)$. Set now

$$\Gamma = \{ h \in C([0, 1]; H_0^1(\Omega, \mathbb{H}^d)); h(0) = 0 \text{ and } h(1) = e \},$$

then by continuity, we have

$$c = \inf_{h \in \Gamma} \max_{t \in [0, 1]} I_{\mu,\lambda}(h(t)) \geq a(R) > 0,$$

and c is a local minimum. Theorem 4.3 implies that the $(PS)_c$ condition is satisfied. So $c > 0$ is a critical value by using Theorem 4.4 and the critical point is $u \in H_0^1(\Omega, \mathbb{H}^d)$, which is nontrivial. We have proved Theorem 4.4 for $0 < \lambda < \lambda_1$.

We need now the following Linking theorem from Rabinowitz [23].

Theorem 4.5. *Let E be a Banach space with $E = Y \oplus X$, where $\dim Y < \infty$. Suppose that $I \in C^1(E, \mathbb{R})$ and satisfies*

(i) *there exist $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$;*

(ii) *there exist $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $A \equiv (\bar{B}_R \cap Y) \oplus \{r e, 0 < r < R\}$, then $I|_{\partial A} \leq 0$.*

If I satisfies the $(PS)_c$ condition with

$$c = \inf_{h \in \Gamma} \max_{u \in A} I(h(u)), \text{ where } \Gamma = \{h \in C(\bar{A}, E); h|_{\partial A} = id\},$$

then c is a critical value of I and $c \geq \alpha$.

Remark 4.6. *Suppose $I|_Y \leq 0$ and there are an $e \in \partial B_1 \cap X$ and $\bar{R} > \rho$ such that $I(u) \leq 0$ for $u \in Y \oplus \text{span}\{e\}$ and $\|u\| \geq \bar{R}$, then for any large R , we have $I|_{\partial A} \leq 0$ where $A = (\bar{B}_R \cap Y) \oplus \{r e, 0 < r < R\}$.*

We assume now that there is k such that $\lambda_k \leq \lambda < \lambda_{k+1}$, where λ_k is the k -th eigenvalue of the operator $-\Delta_{\mathbb{H}^d} - \mu V$. Let $Y = \text{span} \{\phi_1, \dots, \phi_k\}$, where ϕ_k is the eigenfunction corresponding to λ_k . Then Proposition 3.1 implies that $H_0^1(\Omega, \mathbb{H}^d) = Y \oplus X$ where $X = \text{span} \{\phi_l; l > k\}$. Thus we have

$$(4.5) \quad \int_{\Omega} |\nabla_{\mathbb{H}^d} y|^2 - \mu |y|^2 \, dzds \leq \lambda_k \int_{\Omega} |y|^2 \, dzds, \quad \forall y \in Y$$

and

$$(4.6) \quad \int_{\Omega} |\nabla_{\mathbb{H}^d} u|^2 - \mu V |u|^2 \, dzds \geq \lambda_{k+1} \int_{\Omega} |u|^2 \, dzds, \quad \forall u \in X.$$

We will show that $I_{\mu, \lambda}$ satisfies the conditions (i), (ii) of Theorem 4.5 on $H_0^1(\Omega, \mathbb{H}^d)$.

Proposition 4.7. *Assume that $0 \leq \mu < 1$ and $\lambda_k \leq \lambda < \lambda_{k+1}$. There exist $\rho, \alpha > 0$ such that $I_{\mu, \lambda}|_{\partial B_\rho \cap X} \geq \alpha$ where $X = \text{span} \{\phi_l; l > k\}$.*

Proof : For any $u \in X$, $\lambda_k \leq \lambda < \lambda_{k+1}$, we obtain from equation (4.6), Hardy's inequality and Poincaré's inequality that

$$\begin{aligned} I_{\mu, \lambda}(u) &= \frac{1}{2} \int_{\Omega} [|\nabla_{\mathbb{H}^d} u|^2 - \mu V |u|^2] \, dzds \\ &\quad - \frac{1}{p} \int_{\Omega} |u|^p \, dzds - \frac{\lambda}{2} \int_{\Omega} |u|^2 \, dzds \\ &\geq \frac{1}{2} \frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}} \|u\|_{\mu}^2 - \frac{C_1}{p} \|u\|_{\mu}^p \\ &\geq C_1 \|u\|_{\mu}^2 \left(\frac{1}{2C_1} \frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}} - \frac{1}{p} \|u\|_{\mu}^{p-2} \right). \end{aligned}$$

Let

$$\rho_0 = \left(\frac{p}{2C_1} \left(1 - \frac{\lambda}{\lambda_{k+1}} \right) \right)^{\frac{1}{p-2}} > 0.$$

Then for any $0 < \rho < \rho_0$,

$$(4.7) \quad \inf_{u \in X; \|u\|_{\mu} = \rho} I_{\mu, \lambda}(u) = \alpha(\rho) > 0.$$

Thus $I_{\mu, \lambda}$ satisfies the condition (i) of Theorem 4.5.

Proposition 4.8. *Assume that $0 \leq \mu < 1$ and $\lambda_k \leq \lambda < \lambda_{k+1}$. Then $I_{\mu, \lambda}$ verifies (ii) of Theorem 4.5 with $e = \phi_{k+1}$ and $Y = \text{span} \{\phi_1, \dots, \phi_k\}$.*

Proof : We prove Proposition 4.8 using Remark 4.6. For any $y \in Y$, we have from (4.5) that

$$\begin{aligned} I_{\mu, \lambda}(y) &= \frac{1}{2} \int_{\Omega} [|\nabla_{\mathbb{H}^d} y|^2 - \mu V |y|^2] \, dzds \\ &\quad - \frac{1}{p} \int_{\Omega} |y|^p \, dzds - \frac{\lambda}{2} \int_{\Omega} |y|^2 \, dzds \\ &\leq \frac{1}{2} \frac{\lambda_k - \lambda}{\lambda_k} \|y\|_{\mu}^2 - \frac{1}{p} \|y\|_{L^p(\Omega)}^p \\ &\leq 0 \end{aligned}$$

Let $e = \phi_{k+1}$ the $(k+1)$ -th eigenfunction of L_{μ} and $y \in Y$, let us the following claim

$$(4.8) \quad I_{\mu, \lambda}(y + \theta \phi_{k+1}) \rightarrow -\infty \text{ as } \theta \rightarrow +\infty,$$

which will prove Remark 4.6.

Since $\{\phi_j\}$ is an orthonormal basis of $H_0^1(\Omega, \mathbb{H}^d)$, we have for all $i, j \in \mathbb{N}, i \neq j$

$$(4.9) \quad \int_{\Omega} [\nabla_{\mathbb{H}^d} \phi_i \overline{\nabla_{\mathbb{H}^d} \phi_j} - \mu V \phi_i \overline{\phi_j}] dz ds = \lambda_i \int_{\Omega} \phi_i \overline{\phi_j} dz ds = 0.$$

Let $y = \sum_{i=1}^k \alpha_i \phi_i \in Y$, then for $\theta \geq 1$,

$$\begin{aligned} I_{\mu, \lambda}(y + \theta \phi_{k+1}) &= I_{\mu, \lambda}(y) + I_{\mu, \lambda}(\theta \phi_{k+1}) + \int_{\Omega} [\nabla_{\mathbb{H}^d} y \overline{\nabla_{\mathbb{H}^d}(\theta \phi_{k+1})} - \mu V y \overline{(\theta \phi_{k+1})}] dz ds \\ &\quad - \lambda \int_{\Omega} y \overline{(\theta \phi_{k+1})} dz ds - \frac{1}{p} \int_{\Omega} |y + \theta \phi_{k+1}|^p dz ds \\ &\quad + \frac{1}{p} \int_{\Omega} |y|^p dz ds + \frac{1}{p} \int_{\Omega} |\theta \phi_{k+1}|^p dz ds \\ &= I_{\mu, \lambda}(y) + I_{\mu, \lambda}(\theta \phi_{k+1}) - \frac{1}{p} \int_{\Omega} |y + \theta \phi_{k+1}|^p dz ds \\ &\quad + \frac{1}{p} \int_{\Omega} |y|^p dz ds + \frac{1}{p} \int_{\Omega} |\theta \phi_{k+1}|^p dz ds. \end{aligned}$$

By using the following inequality

$$(4.10) \quad |a + b|^p \geq |a|^p + |b|^p - c_p (|a|^{p-1}|b| + |a||b|^{p-1}), \quad \forall a, b \in \mathbb{R}, p > 1,$$

and from the fact $I_{\mu, \lambda}(y) \leq 0$, we have

$$\begin{aligned} I_{\mu, \lambda}(y + \theta \phi_{k+1}) &\leq I_{\mu, \lambda}(\theta \phi_{k+1}) + c_p \int_{\Omega} (|y|^{p-1} |\theta \phi_{k+1}| + |y| |\theta \phi_{k+1}|^{p-1}) dz ds \\ &\leq \frac{\theta^2}{2} \left(\|\phi_{k+1}\|_{\mu}^2 - \lambda \|\phi_{k+1}\|_{L^2(\Omega)}^2 \right) - \frac{\theta^p}{p} \|\phi_{k+1}\|_{L^p(\Omega)}^p \\ &\quad + c_p \int_{\Omega} \left(|y|^{p-1} |\theta \phi_{k+1}| + |y| |\theta \phi_{k+1}|^{p-1} \right) dz ds \\ &\leq \frac{\theta^2}{2} \left\{ (\lambda_{k+1} - \lambda) \|\phi_{k+1}\|_{L^2(\Omega)}^2 - \frac{2\theta^{p-2}}{p} \|\phi_{k+1}\|_{L^p(\Omega)}^p \right. \\ &\quad \left. + 2c_p \int_{\Omega} (\theta^{-1} |y|^{p-1} |\phi_{k+1}| + \theta^{p-3} |y| |\phi_{k+1}|^{p-1}) dz ds \right\}. \end{aligned}$$

Now $p > 2$ and $\|\phi_{k+1}\|_{L^p(\Omega)}^p > 0$ imply

$$I_{\mu, \lambda}(y + \theta \phi_{k+1}) \rightarrow -\infty \text{ as } \theta \rightarrow +\infty.$$

We have proved Proposition 4.8.

Now Proposition 4.7 and Proposition 4.8 imply that, if $0 \leq \mu < 1$ and $\lambda_k \leq \lambda < \lambda_{k+1}$, $I_{\mu, \lambda}$ satisfies the assumptions of Theorem 4.5 and the Palais-Smale condition with

$$c = \inf_{h \in \Gamma} \max_{u \in A} I_{\mu, \lambda}(h(u)),$$

$$\text{and } A \equiv (\bar{B}_T \cap Y) \oplus \{\theta e, 0 < \theta < T\} > 0.$$

Then $I_{\mu, \lambda}$ has a critical value c and a nontrivial critical point $u \in H_0^1(\Omega, \mathbb{H}^d)$, since $I_{\mu, \lambda}(u) = c \geq \alpha > 0$. We have proved Theorem 4.1.

5. THE PALAIS-SMALE CONDITION

We prove now Theorem 4.3, i.e. we verify the (PS) condition for $I_{\mu,\lambda}$ on $H_0^1(\Omega, \mathbb{H}^d)$.

Lemma 5.1. *Let $0 \leq \mu < 1, \lambda > 0, c \in \mathbb{R}$, then any sequence $\{u_n\} \in H_0^1(\Omega, \mathbb{H}^d)$ satisfying*

$$(5.1) \quad I_{\mu,\lambda}(u_n) \rightarrow c \quad \text{and} \quad \|I'_{\mu,\lambda}(u_n)\|_{H^{-1}(\Omega, \mathbb{H}^d)} \rightarrow 0,$$

is bounded in $H_0^1(\Omega, \mathbb{H}^d)$.

Proof : We use

$$\|u\|_{\mu} = \left(\int_{\Omega} \left[|\nabla_{\mathbb{H}^d} u|^2 - \mu V(z, s) |u|^2 \right] dz ds \right)^{1/2} \approx \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}$$

as norm of $H_0^1(\Omega, \mathbb{H}^d)$. Let $\{u_n\} \in H_0^1(\Omega, \mathbb{H}^d)$ be the sequence of Lemma 5.1, then

$$(5.2) \quad \frac{1}{2} \|u\|_{\mu}^2 - \frac{1}{p} \|u_n\|_{L^p(\Omega)}^p - \frac{\lambda}{2} \|u_n\|_{L^2(\Omega)}^2 \rightarrow c,$$

and

$$(5.3) \quad \begin{aligned} \langle I'_{\mu,\lambda}(u_n), \varphi \rangle &= \int_{\Omega} \left[\nabla_{\mathbb{H}^d} u_n \nabla_{\mathbb{H}^d} \bar{\varphi} - (\mu V(z, s) u_n + |u_n|^{p-2} u_n + \lambda u_n) \bar{\varphi} \right] dz ds \\ &= o(1) \|\varphi\|_{H_0^1(\Omega, \mathbb{H}^d)}. \end{aligned}$$

We suppose that $\{u_n\}$ is not bounded in $H_0^1(\Omega, \mathbb{H}^d)$, i. e.

$$\|\nabla_{\mathbb{H}^d} u_n\|_{L^2(\Omega)} \rightarrow +\infty,$$

and let $w_n = \frac{u_n}{\|\nabla_{\mathbb{H}^d} u_n\|_{L^2(\Omega)}}$, then $\|\nabla_{\mathbb{H}^d} w_n\|_{L^2(\Omega)} = 1$. So, there exist $w \in H_0^1(\Omega, \mathbb{H}^d)$ and a subsequence still denoted by $\{w_n\}$ such that

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega, \mathbb{H}^d).$$

Now, for any $q = \nu 2 + (1 - \nu) 2^*$ with $0 < \nu < 1$, we have

$$\|w_n - w\|_{L^q(\Omega)}^q \leq \|w_n - w\|_{L^2(\Omega)}^{\nu 2} \|w_n - w\|_{L^{2^*}(\Omega)}^{(1-\nu) 2^*}.$$

Then, Sobolev's inequality (1.2) and the compact embedding result of Lemma 3.2 imply the following strong limit

$$w_n \rightarrow w \text{ in } L^q(\Omega) \text{ for } 1 \leq q < 2^*,$$

which implies also the weak limit, for $2 < p < 2^*$,

$$\int_{\Omega} |w_n|^{p-2} w_n \bar{\varphi} dz ds \rightarrow \int_{\Omega} |w|^{p-2} w \bar{\varphi} dz ds,$$

for any $\varphi \in C_0^\infty(\Omega)$. Divide (5.3) by $\|\nabla_{\mathbb{H}^d} u_n\|_{L^2(\Omega)}$, we get for any $\varphi \in C_0^\infty(\Omega)$,

$$(5.4) \quad \begin{aligned} &\int_{\Omega} \left[\nabla_{\mathbb{H}^d} w_n \nabla_{\mathbb{H}^d} \bar{\varphi} - \mu V(z, s) w_n \bar{\varphi} \right] dz ds \\ &- \int_{\Omega} |u_n|^{p-2} w_n \bar{\varphi} dz ds - \lambda \int_{\Omega} w_n \bar{\varphi} dz ds \\ &= \frac{o(1)}{\|\nabla_{\mathbb{H}^d} u_n\|_{L^2(\Omega)}} \|\varphi\|_{H_0^1(\Omega, \mathbb{H}^d)}. \end{aligned}$$

Passing to the limit, we obtain for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} & \int_{\Omega} \left[\nabla_{\mathbb{H}^d} w \nabla_{\mathbb{H}^d} \bar{\varphi} - \mu V(z, s) w \bar{\varphi} \right] dz ds - \lambda \int_{\Omega} w \bar{\varphi} dz ds \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{p-2} w_n \bar{\varphi} dz ds \\ &= \lim_{n \rightarrow \infty} \|\nabla_{\mathbb{H}^d} u_n\|_{L^2(\Omega)}^{p-2} \int_{\Omega} |w_n|^{p-2} w_n \bar{\varphi} dz ds, \end{aligned}$$

which implies that

$$\int_{\Omega} |w|^{p-2} w \bar{\varphi} dz ds = 0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

Then $w = 0$ a.e. in Ω and so

$$\lim_{n \rightarrow \infty} \int_{\Omega} |w_n|^2 dz ds = \int_{\Omega} |w|^2 dz ds = 0.$$

Divide (5.2) by $\|\nabla_{\mathbb{H}^d} u_n\|_{L^2(\Omega)}^2$, we obtain

$$(5.5) \quad \begin{aligned} & \frac{1}{2} \int_{\Omega} [|\nabla_{\mathbb{H}^d} w_n|^2 - \mu V(z, s) |w_n|^2] dz ds \\ & - \frac{1}{p} \int_{\Omega} |u_n|^{p-2} |w_n|^2 dz ds - \frac{\lambda}{2} \int_{\Omega} |w_n|^2 dz ds = o(1), \end{aligned}$$

where

$$o(1) = \frac{c}{\|\nabla_{\mathbb{H}^d} u_n\|_{L^2(\Omega)}^2}.$$

Passing to the limit in (5.5),

$$(5.6) \quad \frac{1}{2} - \frac{1}{2} \lim_{n \rightarrow +\infty} \int \mu V(z, s) |w_n|^2 dz ds = \frac{1}{p} \lim_{n \rightarrow +\infty} \int_{\Omega} |u_n|^{p-2} |w_n|^2 dz ds,$$

and taking $\varphi = w_n$ in (5.4), we have

$$(5.7) \quad 1 - \lim_{n \rightarrow +\infty} \int \mu V(z, s) |w_n|^2 dz ds = \lim_{n \rightarrow +\infty} \int_{\Omega} |u_n|^{p-2} |w_n|^2 dz ds.$$

We multiply (5.6) by p , we obtain from (5.7) and Hardy's inequality that

$$\begin{aligned} \frac{p}{2} - 1 &= \left(\frac{p}{2} - 1\right) \lim_{n \rightarrow +\infty} \int \mu V(z, s) |w_n|^2 dz ds \\ &\leq \left(\frac{p}{2} - 1\right) \lim_{n \rightarrow +\infty} \mu \int |\nabla_{\mathbb{H}^d} w_n|^2 dz ds \\ &\leq \left(\frac{p}{2} - 1\right) \mu. \end{aligned}$$

That is impossible since $\mu < 1, p > 2$ and this implies that $\{u_n\}$ is bounded in $H_0^1(\Omega, \mathbb{H}^d)$.

Lemma 5.2. *Under the assumption of Lemma 5.1, $\{u_n\}$ possesses a convergent subsequence in $H_0^1(\Omega, \mathbb{H}^d)$.*

Proof : Since $\{u_n\}$ is a bounded sequence in $H_0^1(\Omega, \mathbb{H}^d)$, there exists $u \in H_0^1(\Omega, \mathbb{H}^d)$ such that for any $2 < p < 2^*$,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H_0^1(\Omega, \mathbb{H}^d), \\ u_n &\rightarrow u \quad \text{in } L^p(\Omega), \\ |u_n|^{p-2} u_n &\rightarrow |u|^{p-2} u \quad \text{in } L^{\frac{p}{p-1}}(\Omega). \end{aligned}$$

By passing to the limit in (5.3), we have

$$\langle I'_{\mu,\lambda}(u), \varphi \rangle = 0$$

for any $\varphi \in C_0^\infty(\Omega)$ which is dense in $H_0^1(\Omega, \mathbb{H}^d)$. Then

$$I'_{\mu,\lambda}(u_n) \rightarrow 0 \quad \text{in } H^{-1}(\Omega, \mathbb{H}^d)$$

implies that

$$\lim_{n \rightarrow \infty} \langle I'_{\mu,\lambda}(u_n) - I'_{\mu,\lambda}(u), u_n - u \rangle = 0.$$

We rewrite the norm of $H_0^1(\Omega, \mathbb{H}^d)$

$$\begin{aligned} \|u_n - u\|_\mu^2 &= \langle I'_{\mu,\lambda}(u_n) - I'_{\mu,\lambda}(u), u_n - u \rangle \\ &\quad + \int_\Omega [|u_n|^{p-2} u_n - |u|^{p-2} u] \overline{(u_n - u)} \, dzds + \lambda \int_\Omega |u_n - u|^2 \, dzds. \end{aligned}$$

From Hölder's inequality, we have

$$\begin{aligned} &\int_\Omega [|u_n|^{p-2} u_n - |u|^{p-2} u] \overline{(u_n - u)} \, dzds \\ &\leq \| |u_n|^{p-2} u_n - |u|^{p-2} u \|_{L^{\frac{p}{p-1}}(\Omega)} \|u_n - u\|_{L^p(\Omega)}, \end{aligned}$$

so we can deduce that $\|u_n - u\|_\mu^2 \rightarrow 0$, $n \rightarrow \infty$.

REFERENCES

- [1] Ambrosetti A., Rabinowitz P.H., Dual variational methods in critical point theory and applications., *J.Functional Analysis.*, **14**, 349-381 (1973).
- [2] Bahouri H., Chemin J-Y., Gallagher I., Precised Hardy inequalities on \mathbb{R}^d and on the Heisenberg group \mathbb{H}^d ., *Séminaire Equations aux dérivées partielles. Ecole Polytech, Palaiseau 2004-2005.*
- [3] Bahouri H., Xu C-J., Chemin J-Y., Trace and trace lifting theorems in weighted Sobolev spaces., *J. Inst. Math. Jussieu.*, **4**, 509-552 (2005).
- [4] Bahouri H., Xu C-J., Chemin J-Y., Trace theorem on the Heisenberg group., to appear at "Ann. Inst. Fourier".
- [5] Benci V., Cerami G., Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{\frac{N+2}{N-2}}$ in \mathbb{R}^n ., *J.Funct.Anal.*, **88**(1), 90-117 (1990).
- [6] Bony J.M., Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés., *Ann. Inst. Fourier.*, **19**, 227-304 (1969).
- [7] Cerami G., Solimini S., Struwe M., Some existence results for superlinear elliptic boundary value problems involving critical exponents., *J.Funct.Anal.*, **69**, 289-306 (1986).
- [8] Chaudhuri N., Ramaswamy M., Existence of positive solutions of some semilinear elliptic equations with singular coefficients., *Proc. Royal Soc. Ed. Sec. A.*, **131**, 1275-1295 (2001).
- [9] Chemin J-Y., Xu C-J., Inclusions de Sobolev en calcul de Weyl-Hörmander et champs de vecteurs sous-elliptiques., *Ann. Sci. école Norm. Sup.*, **4**, 719-751 (1997).
- [10] Chen J., Existence of solutions for a nonlinear PDE with an inverse square potential., *J. Differential Equations.*, **195**, 497-519 (2003).
- [11] Chen J., On a semilinear elliptic equation with singular term and Hardy-Sobolev critical growth., *Math.Nach.*, **208**(8), 838-850 (2007).

- [12] D'Ambrosio L, Some Hardy Inequalities on the Heisenberg Group., Translation in Differ. Equ., **40**, 552-564 (2004).
- [13] D'Ambrosio L, Hardy-type inequalities related to degenerate elliptic differential operators., Annali della Scuola Normale Superiore di Pisa, 451-486 (2005).
- [14] Ferrero A., Gazzola F., Existence of solutions for singular critical growth semilinear elliptic equations., J.Differential Equations, **177**, 494-522 (2001).
- [15] Folland G. B., Stein E. M., Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group., Comm. Pure Appl. Math., **27**, 429-522 (1974).
- [16] Folland G. B., Subelliptic estimates and function spaces on nilpotent Lie groups., Ark. Math., **13**, 161-207 (1975).
- [17] Loiudice A., Improved inequalities on the Heisenberg group., Nonlinear Analysis., **62**, 953-962 (2005).
- [18] Garofalo N., Lanconelli E., Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation., Annales de l'institut de Fourier, 331-356 (1990).
- [19] Garofalo N., Lanconelli E., Existence and non existence results for semilinear equations on the Heisenberg group., Indiana Univ.Math.J., **41**, 71-97 (1992).
- [20] Ghoussoub N., Yuan C., Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents., Trans. Amer. Math. Soc., **352**(12), 5703-5743 (2000).
- [21] Jerison D., The Poincaré inequality for vector fields satisfying Hörmander's condition., Duke Math.J., **53**(2), 503-523 (1986).
- [22] Jerison D., Lee J., Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem., J. Amer. Math. Soc. **1**, 1-13 (1988)
- [23] Rabinowitz P.H., Minimax methods in critical points theory with applications to differential equation., CBMS Reg. Conf. Ser. Math. Vol. 65 (Amer. Math. Soc., Providence, RI, 1986).
- [24] Struwe S., Variational methods. Application to nonlinear partial differential equations and Hamiltonian systems., Second Edition, Springer-Verlag (1996).
- [25] Xu C-J., The Dirichlet problems for a class of semilinear sub-elliptic equations., Nonlinear Anal., **37**, 1039-1049 (1999).
- [26] Willem M., Minimax Theorems., Birkäuser, Boston, 1996.