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► **To cite this version:**

Philippe Carmona. Directed polymer in random environment and last passage percolation. ESAIM: Probability and Statistics, 2010, 14, pp.263–270. hal-00315297

**HAL Id: hal-00315297**

**<https://hal.science/hal-00315297>**

Submitted on 27 Aug 2008

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# Directed polymer in random environment and last passage percolation\*

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August 28, 2008

## Abstract

The sequence of random probability measures  $\nu_n$  that gives a path of length  $n$ ,  $\frac{1}{n}$  times the sum of the random weights collected along the paths, is shown to satisfy a large deviations principle with good rate function the Legendre transform of the free energy of the associated directed polymer in a random environment.

Consequences on the asymptotics of the typical number of paths whose collected weight is above a fixed proportion are then drawn.

**Keywords:** directed polymer, random environment, partition function, last passage percolation.

**Mathematic Classification :** 60K37

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\*the author acknowledges the support of the French Ministry of Education through the ANR BLAN07-2184264 grant

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# 1 Introduction

## Last passage percolation

To each site  $(k, x)$  of  $\mathbb{N} \times \mathbb{Z}^d$  is assigned a random weight  $\eta(k, x)$ . The  $(\eta(k, x))_{k \geq 1, x \in \mathbb{Z}^d}$  are taken IID under the probability measure  $\mathbf{Q}$ . The set of oriented paths of length  $n$  starting from the origin is

$$\Omega_n = \left\{ \omega = (\omega_0, \dots, \omega_n) : \omega_i \in \mathbb{Z}^d, \omega_0 = 0, |\omega_i - \omega_{i-1}| = 1 \right\}.$$

The weight (energy, reward) of a path is the sum of weights of visited sites:

$$H_n = H_n(\omega, \eta) = \sum_{k=1}^n \eta(k, \omega_k) \quad (n \geq 1, \omega \in \Omega_n).$$

Observe that when  $\eta(k, x)$  are Bernoulli( $p$ ) distributed

$$\mathbf{Q}(\eta(k, x) = 1) = 1 - \mathbf{Q}(\eta(k, x) = 0) = p \in (0, 1),$$

the quantity  $\frac{H_n}{n}(\omega, \eta)$  is the proportion of *open* sites visited by  $\omega$ , and it is natural to consider for  $0 < \rho < 1$ ,

$$N_n(\rho) = \text{number of paths of length } n \text{ such that } H_n(\omega, \eta) \geq n\rho.$$

The problem of  $\rho$ -percolation, as we learnt it from Comets, Popov and Vachkovskaia [8] and Kesten and Sidoravicius [12], is to study the behaviour of  $N_n(\rho)$  for large  $n$  and different values of  $\rho$ .

## Directed polymer in a random environment

We are going to consider fairly general environment distributions, by requiring first that they have exponential moments of any order:

$$\lambda(\beta) = \log \mathbf{Q}\left(e^{\beta \eta(k, x)}\right) < +\infty \quad (\beta \in \mathbb{R}),$$

and second that they satisfy a logarithmic Sobolev inequality (see e.g. [2]): in particular we can apply our result to bounded support and Gaussian environments.

The polymer measure is the random probability measure defined on the set of oriented paths of length  $n$  by:

$$\mu_n(\omega) = (2d)^{-n} \frac{e^{\beta H_n(\omega, \eta)}}{Z_n(\beta)} \quad (\omega \in \Omega_n),$$

with  $Z_n(\beta)$  the partition function

$$Z_n(\beta) = Z_n(\beta, \eta) = (2d)^{-n} \sum_{\omega \in \Omega_n} e^{\beta H_n(\omega, \eta)} = \mathbf{P} \left( e^{\beta H_n(\omega, \eta)} \right),$$

where  $\mathbf{P}$  is the law of simple random walk on  $\mathbb{Z}^d$  starting from the origin. Bolthausen [3] proved the existence of a deterministic limiting free energy

$$p(\beta) = \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbf{Q}(\log Z_n(\beta)) = \mathbf{Q} \text{ a.s. } \lim_{n \rightarrow +\infty} \frac{1}{n} \log Z_n(\beta).$$

Thanks to Jensen's inequality, we have the upper bound  $p(\beta) \leq \lambda(\beta)$  and it is conjectured (and partially proved, see [7, 6]) that the behaviour of a typical path under the polymer measure is diffusive iff  $\beta \in \mathcal{C}_\eta$  the critical region

$$\mathcal{C}_\eta = \{\beta \in \mathbb{R} : p(\beta) = \lambda(\beta)\}.$$

In dimension  $d = 1$ ,  $\mathcal{C}_\eta = \{0\}$  and in dimensions  $d \geq 3$ ,  $\mathcal{C}_\eta$  contains a neighborhood of the origin (see [3, 9]).

## The main theorem

The connection between Last passage percolation and Directed polymer in random environment is made by the family  $(\nu_n)_{n \in \mathbb{N}}$  of random probability measures on the real line:

$$\nu_n(A) = \frac{1}{|\Omega_n|} \sum_{\omega \in \Omega_n} \mathbf{1}_{(\frac{H_n}{n}(\omega, \eta) \in A)} = \mathbf{P} \left( \frac{H_n}{n}(\omega, \eta) \in A \right).$$

Indeed,

$$N_n(\rho) = \sum_{\omega \in \Omega_n} \mathbf{1}_{(H_n(\omega, \eta) \geq n\rho)} = (2d)^n \nu_n([\rho, +\infty)).$$

The main result of the paper is

**Theorem 1.**  *$\mathbf{Q}$  almost surely, the family  $(\nu_n)_{n \in \mathbb{N}}$  satisfies a large deviations principle with good rate function  $I = p^*$  the Legendre transform of the free energy of the directed polymer.*

Let  $m = \mathbf{Q}(\eta(k, x))$  be the average weight of a path  $m = \mathbf{Q}(\frac{H_n}{n}(\omega, \eta))$ . It is natural to consider the quantities:

$$N_n(\rho) = \begin{cases} \sum_{\omega \in \Omega_n} \mathbf{1}_{(H_n(\omega, \eta) \geq n\rho)} & \text{if } \rho \geq m, \\ \sum_{\omega \in \Omega_n} \mathbf{1}_{(H_n(\omega, \eta) \leq n\rho)} & \text{if } \rho < m. \end{cases}$$

A simple exchange of limits  $\beta \rightarrow \pm\infty$ , and  $n \rightarrow +\infty$ , yields the following

$$\rho^\pm = \mathbf{Q} \text{ a.s. } \lim_{n \rightarrow +\infty} \max_{\omega \in \Omega_n} \pm \frac{H_n}{n}(\omega, \eta) = \lim_{\beta \rightarrow +\infty} \frac{p(\pm\beta)}{\beta} \in [0, +\infty].$$

Repeating the proof of Theorem 1.1 of [8] gives

**Corollary 2.** *For  $-\rho^- < \rho < \rho^+$ , we have  $\mathbf{Q}$  almost surely,*

$$\lim_{n \rightarrow +\infty} (N_n(\rho))^{\frac{1}{n}} = (2d)e^{-I(\rho)}.$$

We can then translate our knowledge of the critical region  $\mathcal{C}_\eta$ , into the following remark. Let

$$\mathcal{V}_\eta = \{\rho \in \mathbb{R} : I(\rho) = \lambda^*(\rho)\}.$$

In dimension  $d = 1$ ,  $\mathcal{V}_\eta = \{m\}$  and in dimensions  $d \geq 3$ ,  $\mathcal{V}_\eta$  contains a neighbourhood of  $m$ .

This means that in dimensions  $d \geq 3$ , the typical large deviation of  $\frac{H_n}{n}(\omega, \eta)$  close to its mean is the same as the large deviation of  $\frac{1}{n}(\eta_1 + \dots + \eta_m)$  close to its mean, with  $\eta_i$  IID. There is no influence of the path  $\omega$  : this gives another justification to the name weak-disorder region given to the critical set  $\mathcal{C}_\eta$ .

## 2 Proof of the main theorem

Observe that for any  $\beta \in \mathbb{R}$  we have:

$$\int e^{\beta n x} d\nu_n(x) = \mathbf{P} \left( e^{\beta H_n(\omega, \eta)} \right) = Z_n(\beta) \quad \mathbf{Q} \text{ a.s.} \quad (1)$$

Consequently, since  $e^u + e^{-u} \geq e^{|u|}$ , we obtain for any  $\beta > 0$ ,

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \left( \int e^{\beta n |x|} d\nu_n(x) \right) \leq p(\beta) + p(-\beta) < +\infty,$$

and the family  $(\nu_n)_{n \geq 0}$  is exponentially tight (see Dembo and Zeitouni[10], or Feng and Kurtz[11]). We only need to show now that for a lower semi-continuous function  $I$ , and for  $x \in \mathbb{R}$

$$\lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \nu_n((x - \delta, x + \delta)) = I(x), \quad (2)$$

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \nu_n([x - \delta, x + \delta]) = I(x). \quad (3)$$

From these, we shall infer that  $(\nu_n)_{n \in \mathbb{N}}$  follows a large deviations principle with good rate function  $I$ . Eventually, equation (1) and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\beta) = p(\beta)$$

will imply, by Varadhan's lemma that  $I$  and  $p$  are Legendre conjugate:

$$I(x) = p^*(x) = \sup_{\beta \in \mathbb{R}} (x\beta - p(\beta)).$$

The strategy of proof finds its origin in Varadhan's seminal paper [13], and has already successfully been applied in [5]. Let us define for  $\lambda > 0, x \in \mathbb{Z}, a \in \mathbb{R}$

$$V_n^{(\lambda)}(x, a; \eta) = \log \mathbf{P}^x \left( e^{-\lambda |H_n(\omega, \eta) - a|} \right) = V^{(\lambda)}(0, a; \tau_{0,x} \circ \eta),$$

with  $\tau_{k,x}$  the translation operator on the environment defined by :

$$\tau_{k,x} \circ \eta(i, y) = \eta(k + i, x + y),$$

and  $\mathbf{P}^x$  the law of simple random walk starting from  $x$ .

*Step 1* The functions  $v_n^{(\lambda)}(a) = \mathbf{Q}(V^{(\lambda)}(0, a; \eta))$  satisfy the inequality

$$v_{n+m}^{(\lambda)}(a+b) \geq v_n^{(\lambda)}(a) + v_m^{(\lambda)}(b) \quad (n, m \in \mathbb{N}; a, b \in \mathbb{R}). \quad (4)$$

*Proof.* Since  $|H_{n+m} - (a+b)| \leq |H_n - b| + |(H_{n+m} - H_n) - a|$  we have

$$\begin{aligned} V_{n+m}^{(\lambda)}(x, a; \eta) &\geq \log \mathbf{P}^x \left( e^{-\lambda |H_n - b|} e^{-\lambda |(H_{n+m} - H_n) - a|} \right) \\ &= \log \mathbf{P}^x \left( e^{-\lambda |H_n - b|} e^{V_m^{(\lambda)}(0, a; \tau_{n, S_n} \circ \eta)} \right) \\ &= \log \sum_y \mathbf{P}^x \left( e^{-\lambda |H_n - b|} \mathbf{1}_{(S_n = y)} \right) e^{V_m^{(\lambda)}(0, a; \tau_{n, y} \circ \eta)} \\ &= V_n^{(\lambda)}(x, b; \eta) + \log \left( \sum_y \sigma_n(y) e^{V_m^{(\lambda)}(0, a; \tau_{n, y} \circ \eta)} \right) \\ &\geq V_n^{(\lambda)}(x, b; \eta) + \sum_y \sigma_n(y) V_m^{(\lambda)}(0, a; \tau_{n, y} \circ \eta) \quad (\text{Jensen's inequality}), \end{aligned}$$

with  $\sigma_n$  the probability measure on  $\mathbb{Z}^d$ :

$$\sigma_n(y) = \frac{1}{V_n^{(\lambda)}(x, b; \eta)} \mathbf{P}^x \left( e^{-\lambda |H_n - b|} \mathbf{1}_{(S_n = y)} \right) \quad (y \in \mathbb{Z}^d).$$

Observe that the random variables  $\sigma_n(y)$  are measurable with respect to the sigma field  $\mathcal{G}_n = \sigma(\eta(i, x) : i \leq n, x \in \mathbb{Z}^d)$ , whereas the random variables  $V_m^{(\lambda)}(0, a; \tau_{n,y} \circ \eta)$  are independent from  $\mathcal{G}_n$ . Hence, by stationarity,

$$\begin{aligned}
v_{n+m}^{(\lambda)}(x, a; \eta) &= \mathbf{Q}\left(V_{n+m}^{(\lambda)}(x, a; \eta)\right) \\
&\geq v_n^{(\lambda)}(b) + \sum_y \mathbf{Q}(\sigma_n(y)) \mathbf{Q}\left(V_m^{(\lambda)}(0, a; \tau_{n,y} \circ \eta)\right) \\
&= v_n^{(\lambda)}(b) + \sum_y \mathbf{Q}(\sigma_n(y)) v_m^{(\lambda)}(a) \\
&= v_n^{(\lambda)}(b) + v_m^{(\lambda)}(a) \mathbf{Q}\left(\sum_y \sigma_n(y)\right) \\
&= v_n^{(\lambda)}(b) + v_m^{(\lambda)}(a).
\end{aligned}$$

□

*Step 2* There exists a function  $I^{(\lambda)} : \mathbb{R} \rightarrow \mathbb{R}^+$  convex, non negative, Lipschitz with constant  $\lambda$ , such that

$$-\lim_{n \rightarrow \infty} \frac{1}{n} v_n^{(\lambda)}(a_n) = I^{(\lambda)}(\xi) \quad (\text{if } \frac{a_n}{n} \rightarrow \xi \in \mathbb{R}). \quad (5)$$

*Proof.* This is a standard subadditivity argument (see e.g. Varadhan [13] or Alexander [1]) combined with the Lipschitz property of  $V^{(\lambda)}$ : from  $|H_n - a| \leq |H_n - b| + |a - b|$  we infer that

$$V_n^{(\lambda)}(0, a; \eta) \geq V_n^{(\lambda)}(0, a; \eta) + \lambda|a - b|.$$

□

*Step 3*  $\mathbf{Q}$  almost surely, for any  $\xi \in \mathbb{R}$ ,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbf{P}\left(e^{-\lambda|H_n - a_n|}\right) = I^{(\lambda)}(\xi). \quad (6)$$

*Proof.* Since the functions are Lipschitz, it is enough to prove that for any fixed  $\xi \in \mathbb{Q}$ , (6) holds a.s. This is where we use the restrictive assumptions made on the distribution of the environment. If the distribution of  $\eta$  is with bounded support, or Gaussian, or more generally satisfies a logarithmic Sobolev inequality, then it has the gaussian concentration of measure

property (see[2]): for any 1-Lipschitz function  $F$  of independent random variables distributed as  $\eta$ ,

$$\mathbf{P}(|F - \mathbf{P}(F)| \geq r) \leq 2e^{-r^2/2} \quad (r > 0).$$

It is easy to prove, as in Proposition 1.4 of [4], that the function

$$(\eta(k, x), k \leq n, |x| \leq n) \rightarrow \log \mathbf{P} \left( e^{-\lambda|H_n(\omega, \eta) - a|} \right)$$

is Lipschitz, with respect to the euclidean norm, with Lipschitz constant at most  $\lambda\sqrt{n}$ . Therefore, the Gaussian concentration of measure yields

$$\mathbf{Q} \left( \left| V_n^{(\lambda)}(0, a; \eta) - v_n^{(\lambda)}(a) \right| \geq u \right) \leq 2e^{-\frac{\lambda^2 u^2}{2n}}.$$

We conclude by a Borel Cantelli argument combined with (5)

□

Observe that for fixed  $\xi \in \mathbb{R}$ , the function  $\lambda \rightarrow I^{(\lambda)}(\xi)$  is increasing ; we shall consider the limit:

$$I(\xi) = \lim_{\lambda \uparrow +\infty} \uparrow I^{(\lambda)}(\xi)$$

which is by construction non negative, convex and lower semi continuous.

*Step 4* The function  $I$  satisfy (2) and (3).

*Proof.* Given,  $\xi \in \mathbb{R}$  and  $\lambda > 0, \delta > 0$ , we have

$$\mathbf{P} \left( \left| \frac{H_n}{n}(\omega, \eta) - \xi \right| \leq \delta \right) = \mathbf{P} \left( e^{-\lambda n \left| \frac{H_n}{n}(\omega, \eta) - \xi \right|} \geq e^{-\lambda n \delta} \right) \leq e^{\lambda n \delta} \mathbf{P} \left( e^{-\lambda |H_n - n\xi|} \right).$$

Therefore,

$$\begin{aligned} \limsup \frac{1}{n} \log \nu_n([\xi - \delta, \xi + \delta]) &\leq \lambda \delta - I^{(\lambda)}(\xi) \\ \limsup_{\delta \rightarrow 0} \limsup \frac{1}{n} \log \nu_n([\xi - \delta, \xi + \delta]) &\leq -I^{(\lambda)}(\xi) \end{aligned}$$

and we obtain by letting  $\lambda \rightarrow +\infty$ ,

$$\limsup_{\delta \rightarrow 0} \limsup \frac{1}{n} \log \nu_n([\xi - \delta, \xi + \delta]) \leq -I(\xi).$$



Given  $\xi \in \mathbb{R}$  such that  $I(\xi) < +\infty$ , and  $\delta > 0$ , we have for  $\lambda > 0$ ,

$$\mathbf{P} \left( \left| \frac{H_n}{n} - \xi \right| < \delta \right) \geq \mathbf{P} \left( e^{-\lambda|H_n - n\xi|} - e^{-\lambda\delta n} \right).$$

Hence, if we choose  $\lambda > 0$  large enough such that  $\lambda\delta > I(\xi) \geq I^{(\lambda)}(\xi)$ , we obtain

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \nu_n((\xi - \delta, \xi + \delta)) \geq -I^{(\lambda)}(\xi) \geq -I(\xi)$$

and therefore

$$\liminf_{\delta \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{n} \log \nu_n((\xi - \delta, \xi + \delta)) \geq -I(\xi).$$

□

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