Minimal hypersurfaces in ${}^{n}\times$, total curvature and index

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Abstract

In this paper, we consider minimal hypersurfaces in the product space $\mathbb{H}^n \times \mathbb{R}$. We begin by studying examples of rotation hypersurfaces and hypersurfaces invariant under hyperbolic translations. We then consider minimal hypersurfaces with finite total curvature. This assumption implies that the corresponding curvature goes to zero uniformly at infinity. We show that surfaces with finite total intrinsic curvature have finite index. The converse statement is not true as shown by our examples which also serve as useful barriers.

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1 Introduction

In this paper, we focus on complete oriented minimal hypersurfaces M immersed in $\mathbb{H}^n \times \mathbb{R}$ equiped with the product metric.

In Section 3, we study the family $\{\mathcal{C}_a, a > 0\}$ of hypersurfaces invariant under rotations about the vertical geodesic $\{0\} \times \mathbb{R} \subset \mathbb{H}^n \times \mathbb{R}$ ("catenoids") and the family $\{\mathcal{M}_d, d > 0\}$ of hypersurfaces invariant under hyperbolic translations. These examples generalize to higher dimensions some of the minimal surfaces constructed in [21, 19, 20].

In particular, we prove that the n-dimensional catenoids $\mathcal{C}_a$ have vertical heights bounded from above by $\pi/(n-1)$ (Proposition 3.2). In Section 3.3, we describe the maximal stable rotationally invariant domains on $\mathcal{C}_a$ and we prove that the catenoids have index 1 (Theorem 3.5). We also give an interpretation in terms of the envelope of the family $\mathcal{C}_a$ (Corollary 3.7). Finally, we observe that the half-catenoid $\mathcal{C}_a \cap \left(\mathbb{H}^n \times \mathbb{R}_+\right)$ is not maximally stable.

We describe the minimal hypersurfaces invariant under hyperbolic translations in Theorem 3.8. In particular, we find a hypersurface $\mathcal{M}_1$ which is a complete non-entire vertical graph over a half-space bounded by some hyperplane Π in $\mathbb{H}^n \times \{0\}$. It takes infinite value data on Π and zero asymptotic boundary value data. When $d < 1$, the hypersurface $\mathcal{M}_d$ is an entire vertical graph. When $d > 1$, it is a bi-graph over the exterior of an equidistant hypersurface of $\mathbb{H}^n \times \{0\}$.

In Section 4, we consider the relationships between finiteness of the total curvature and finiteness of the index. In dimension 2, we consider the curvature integrals $\int_M |A_M|^2$ and $\int_M |K_M|$, where A_M is the second fundamental form of the immersion and K_M the Gauss curvature. Finiteness of these integrals implies that the corresponding curvatures tend to zero uniformly at infinity; finiteness of the latter implies finiteness of the index of the Jacobi (stability) operator (Theorem 4.1). The converse statements do not hold. On the one hand, the catenoids $\mathcal{C}_a$ have finite index although they have infinite total intrinsic curvature. This is in contrast with the case of minimal surfaces in Euclidean 3-space ([12]) and with the case of surfaces with constant mean curvature 1 in hyperbolic 3-space ([11, 10]). Note that catenoids have finite total extrinsic curvature. On the other hand, the surfaces invariant under hyperbolic translations are stable graphs, their curvature goes to zero at infinity although they have infinite total curvature. The proof we give of Theorem 4.1 relies mainly on Simons' equation and the de Giorgi-Moser-Nash method which shows that finite total curvature implies that the curvature tends to zero uniformly at infinity. We point out that the finiteness of the intrinsic total curvature has deep consequences. Under this assumption on M, L. Hauswirth and H. Rosenberg ([14], Theorem 3.1) have indeed shown that the total intrinsic curvature is quantified, that the ends of M are asymptotic to Scherk type surfaces and obtained a C^2-control on the curvature at infinity. In dimension $n \geq 3$, we give an upper bound of the index in terms of the total extrinsic curvature (Theorem 4.3).

In Section 5, using the catenoids $\mathcal{C}_a$ as barriers, we prove some symmetry and characterization results for minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ whose boundary consists of two congruent convex hypersurfaces in parallel slices (Theorem 5.1). We point out that the hypersurfaces $\mathcal{M}_d$ ($d < 1$ and $d = 1$) have been used in [22, 23] as barriers for the Dirichlet problem and that they play a crucial role for some existence theorem for the vertical minimal surface equation.

Finally, we point out that most of our results may be established if the ambient space is one of the product spaces $\mathbb{H}^n \times \mathbb{R}^k$ or $\mathbb{H}^n \times \mathbb{H}^k$.

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2 General framework

2.1 Notations

We consider hypersurfaces M immersed in the space $\widehat{M} := \mathbb{H}^n \times \mathbb{R}$ equiped with the product metric $\widehat{g} = g_{\mathbb{B}} + dt^2$, where $g_{\mathbb{B}}$ is the hyperbolic metric,

$$g_{\mathbb{B}} := \big(\frac{2}{1-|x|^2}\big)^2 \big(dx_1^2 + \cdots + dx_n^2\big). \tag{2.1}$$

We have chosen the ball model $\mathbb{B}$ for the n-dimensional hyperbolic space $\mathbb{H}^n$.

2.2 Jacobi operator, Index, Jacobi fields

Let $M^n \looparrowright \widehat{M}^{n+1}$ be an orientable minimal hypersurface in an oriented Riemannian manifold $\widehat{M}$ with metric $\widehat{g}$. Let N_M be a unit normal field along M and let A_M be the second fundamental form of the immersion with respect to N_M. Let $\widehat{\mathrm{Ric}}$ be the normalized Ricci curvature of $\widehat{M}$. The second variation of the volume functional gives rise to the *Jacobi operator* (or stability operator) J_M of M (see [25, 17, 8]),

$$J_M := -\Delta_M - \big(|A_M|^2 + \widehat{\mathrm{Ric}}(N_M)\big), \tag{2.2}$$

where Δ_M is the (non-positive) Laplacian on M (for the induced metric).

Given a relatively compact regular domain Ω on the hypersurface M, we let $\mathrm{Ind}(\Omega)$ denote the number of negative eigenvalues of J_M for the Dirichlet problem on Ω (this is well defined because Ω is compact). The *index* of M is defined to be the supremum ($\leq +\infty$)

$$\mathrm{Ind}(M) := \sup\{\mathrm{Ind}(\Omega) \mid \Omega \Subset M\}, \tag{2.3}$$

taken over all relatively compact regular domains.

Let $\lambda_1(\Omega)$ be the least eigenvalue of the operator J_M with Dirichlet boundary conditions in Ω. Recall that a relatively compact regular domain Ω is said to be *stable*, if $\lambda_1(\Omega) > 0$; *unstable*, if $\lambda_1(\Omega) < 0$; *stable-unstable*, if $\lambda_1(\Omega) = 0$. More generally, we say that a domain Ω is stable if any relatively compact subdomain is stable.

Properties 2.1 *Recall the following properties.*

1. *Let Ω be a stable-unstable relatively compact domain. Then, any smaller domain is stable while any larger domain is unstable (monotonicity of Dirichlet eigenvalues).*
2. *Of particular interest are the solutions of the equation $J_M(u) = 0$. We call such functions* Jacobi fields *on M. Let $X_a : M^n \looparrowright (\widehat{M}^{n+1}, \widehat{g})$ be a one-parameter family of oriented minimal immersions, with variation field $V_a = \frac{\partial X_a}{\partial a}$ and unit normal N_a. Then, the function $\widehat{g}(V_a, N_a)$ is a Jacobi field on M ([1], Theorem 2.7).*
3. *Let Ω be a relatively compact domain on a minimal manifold M. If there exists a positive function u on Ω such that $J_M(u) \geq 0$, then Ω is stable ([13], Theorem 1).*

3 Examples of minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

In this section we give examples of minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$. We use these examples as guidelines and counter-examples to study the relationships between index properties of the Jacobi operator and the finiteness of some total curvature of M, see Theorems 4.1 and 4.3. We also use them as barriers for a symmetry and characterization result in Section 5.

3.1 Rotation hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

We first consider rotation hypersurfaces about a vertical geodesic axis in $\mathbb{H}^n \times \mathbb{R}$. Up to isometry, we can assume the rotation axis to be $\{0\} \times \mathbb{R}$. Recall that we take the ball model for $\mathbb{H}^n$.

Take the vertical plane $\mathbb{V} := \{(x_1, \dots, x_n, t) \in \widehat{M} \mid x_1 = \cdots = x_{n-1} = 0\}$ and consider a generating curve $\big(\tanh(f(t)/2), t\big)$ for some positive function f which represents the hyperbolic distance to the axis $\mathbb{R}$, at height t.

We define a *rotation hypersurface* $M \looparrowright \widehat{M}$ by the "parametrization"

$$X : \begin{cases} \mathbb{R}_+ \times S^{n-1} \to \widehat{M}, \\ (t, \xi) \mapsto \big(\tanh(f(t)/2)\xi, t\big), \end{cases} \tag{3.4}$$

where $\xi = (\xi_1, \dots, \xi_n)$ is a point in the unit sphere S^{n-1} and $\tanh(\rho/2)\xi$ stands for the point $(\tanh(\rho/2)\xi_1, \dots, \tanh(\rho/2)\xi_n)$ in the ball $\mathbb{B}$.

The basic tangent vectors to the immersion X are

$$T(t, \xi) := T_{t,\xi} X(\partial_t) = \big(\frac{f_t(t)}{2\cosh^2(f(t)/2)}\xi, 1\big),$$

where f_t is the derivative of f with respect to t, and

$$U(t, \xi, u) := T_{t,\xi} X(u) = \big(\tanh(f(t)/2)u, 0\big),$$

where $u \in T_\xi S^{n-1}$ is a unit vector.

We collect basic formulas in the next proposition whose proof is straightforward.

Proposition 3.1 *Let $(M, g_M) \looparrowright (\widehat{M}, \widehat{g})$ be an isometric immersion. We have the following formulas in the parametrization X on $\mathbb{R} \times S^{n-1}$.*

1. *The induced metric g_M is given by*

$$g_M = \big(1 + f_t^2(t)\big)dt^2 + \sinh^2(f(t))g_S\,, \tag{3.5}$$

   where g_S is the canonical metric on S^{n-1}.

2. *The Riemannian measure $d\mu_M$ for the metric g_M is given by*

$$d\mu_M = \big(1 + f_t^2(t)\big)^{1/2} \sinh^{n-1}(f(t))\, dt\, d\mu_S, \tag{3.6}$$

   where $d\mu_S$ is the canonical measure on the sphere.

3. The unit normal field to the immersion can be chosen to be

$$N_M(t,\xi) = (1+f_t^2(t))^{-1/2}\Big(\frac{-1}{2\cosh^2(f(t)/2)}\xi, f_t(t)\Big)\,. \tag{3.7}$$

In particular, the vertical component of the unit normal field is given by

$$v_M(t) := f_t(t)\big(1+f_t^2(t)\big)^{-1/2}. \tag{3.8}$$

At the point $X(t,\xi)$, the principal directions of curvature of M are

- the tangent to the meridian curve in the vertical 2-plane
$$\mathbb{V}_\xi = \{\big(\tanh(\rho/2)\xi, t\big) \mid (\rho,t)\in\mathbb{R}^2\},$$
- the vectors tangent to the distance sphere $X(t,S^{n-1})$ at ξ in the hyperbolic slice $\mathbb{H}^n\times\{t\}$, where the restriction of the second fundamental form A_M is a scalar multiple of the identity.

The principal curvatures with respect to N_M are

- $k_n(t)$, the principal curvature in the direction tangent to the meridian curve, given by
$$k_n(t) = -f_{tt}(t)(1+f_t^2(t))^{-3/2}, \tag{3.9}$$
- the principal curvatures in the directions tangent to $X(t,S^{n-1})$ at $X(t,\xi)$,
$$k_1(t) = \cdots = k_{n-1}(t) = \coth(f(t))(1+f_t^2(t))^{-1/2}. \tag{3.10}$$

We conclude that the mean curvature $H(t)$ of the rotation hypersurface $M \looparrowright \widehat{M}$ with respect to the unit normal N_M is given by

$$nH(t) = -f_{tt}(t)(1+f_t^2(t))^{-3/2} + (n-1)\coth(f(t))(1+f_t^2(t))^{-1/2}, \tag{3.11}$$

or

$$nf_t(t)\,\sinh^{n-1}(f(t))\,H(t) = \partial_t\Big(\sinh^{n-1}(f(t))(1+f_t^2(t))^{-1/2}\Big). \tag{3.12}$$

3.2 Catenoids in $\mathbb{H}^n\times\mathbb{R}$

In this Section, we describe the minimal rotation hypersurfaces about $\{0\}\times\mathbb{R}$, in $\mathbb{H}^n\times\mathbb{R}$. By analogy with the Euclidean case, we call them *catenoids*. They are the higher dimensional counterparts of the catenoids constructed in [21].
Given some $a>0$, let $\big(I_a, f(a,\cdot)\big)$ denote the maximal solution of the Cauchy problem

$$\left\{\begin{array}{lcl} f_{tt} &=& (n-1)\coth(f)(1+f_t^2),\\ f(0) &=& a>0,\\ f_t(0) &=& 0,\end{array}\right. \tag{3.13}$$

where f_t and f_{tt} are the first and second derivatives of f with respect to t.

Proposition 3.2 *For $a>0$, the maximal solution $\big(I_a, f(a,\cdot)\big)$ gives rise to the generating curve C_a, $t\mapsto\big(\tanh(f(a,t)),t\big)$ (*catenary*), of a complete minimal rotation hypersurface $\mathcal{C}_a$ (*catenoid*) in $\mathbb{H}^n\times\mathbb{R}$, with the following properties.*

1. *The interval I_a is of the form $I_a =]-T(a), T(a)[$ for some finite positive number $T(a)$ and $f(a,\cdot)$ is an even function of the second variable.*

2. *For all $t \in I_a$, $f(a,t) \geq a$.*

3. *The derivative $f_t(a,\cdot)$ is positive on $]0,T(a)[$, negative on $]-T(a),0[$.*

4. *The function $f(a,\cdot)$ is a bijection from $[0,T(a)[$ onto $[a,\infty[$, with inverse function $\lambda(a,\cdot)$ given by*

$$\lambda(a,\rho) = \sinh^{n-1}(a) \int_a^\rho \big(\sinh^{2n-2}(u) - \sinh^{2n-2}(a)\big)^{-1/2}\, du. \tag{3.14}$$

5. *The catenoid $\mathcal{C}_a$ has finite vertical height $h_R(a)$,*

$$h_R(a) = 2\sinh^{n-1}(a) \int_a^\infty \big(\sinh^{2n-2}(u) - \sinh^{2n-2}(a)\big)^{-1/2}\, du. \tag{3.15}$$

6. *The function $a \mapsto h_R(a)$ increases from 0 to $\frac{\pi}{(n-1)}$ when a increases from 0 to infinity. Furthermore, given $a \neq b$, the generating catenaries C_a and C_b intersect at exactly two symmetric points.*

Proof. *Assertion 1* follows from the Cauchy-Lipschitz theorem for some positive $T(a)$ which is finite as we will see below.

Assertion 2 follows from the fact that $\sinh^{n-1}(f(a,t))\big(1+f_t^2(a,t)\big)^{-1/2} = \sinh^{n-1}(a)$ for all $t \in]-T(a),T(a)[$ (see (3.12)).

Assertion 3 is clear.

Assertion 4. According to Assertion 3, $t \mapsto f(a,t)$ is increasing so that it has a limit when t tends to $T(a)$ and this limit must be infinite because we took a maximal solution. It follows that the inverse function $\lambda(a,\cdot)$ maps $[a,\infty[$ onto $[0,T(a)[$ and that $\lambda_\rho(a,f(a,t))f_t(a,t) \equiv 1$. Finally, we find that $\lambda_\rho(a,\rho) = \sinh^{n-1}(a)\big(\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a)\big)^{-1/2}$ on $]a,\infty[$ and the formula for $\lambda(a,\rho)$ follows because $f(a,0) = a$. Note that the integral (3.14) converges at $u = a$.

Assertion 5. We have that $h_R(a) = 2T(a)$, where

$$T(a) = \lim_{\rho\to\infty} \lambda(a,\rho) = \sinh^{n-1}(a) \int_a^\infty \big(\sinh^{2n-2}(u) - \sinh^{2n-2}(a)\big)^{-1/2}\, du,$$

where the integral converges at both a and ∞.

Assertion 6. By a change of variables, we can write

$$T(a) = \sinh(a) \int_1^\infty \big(v^{2n-2}-1\big)^{-1/2}\big(\sinh^2(a)v^2+1\big)^{-1/2}\, dv$$

and compute the derivative

$$T'(a) = \cosh(a) \int_1^\infty \big(v^{2n-2}-1\big)^{-1/2}\big(\sinh^2(a)v^2+1\big)^{-3/2}\, dv > 0.$$

Note that

$$\sinh(a)\big(v^{2n-2}-1\big)^{-1/2}\big(\sinh^2(a)v^2+1\big)^{-1/2} \leq v^{-1}\big(v^{2n-2}-1\big)^{-1/2}$$

and that the right-hand side is in $L^1([1,\infty[)$ for $n \geq 2$, so that we can take the limits under the integral and obtain that $\lim_{a\to 0} T(a) = 0$ and $\lim_{a\to\infty} T(a) = \int_1^\infty v^{-1}\big(v^{2n-2}-1\big)^{-1/2}\,dv$. The last integral can be calculated explicitly because $\big(\arctan\sqrt{v^N-1}\big)' = \frac{N}{2\sqrt{v^N-1}}$. The last assertion follows by considering the function $\lambda(a,\rho)-\lambda(b,\rho)$ and by using the monotonicity of $T(a)$. □

Remark. The above proposition shows that the catenoids in $\mathbb{H}^n \times \mathbb{R}$ have uniformly bounded finite vertical height. This is in contrast with the Euclidean catenoids ($n \geq 3$) which have finite, yet unbounded, vertical heights.

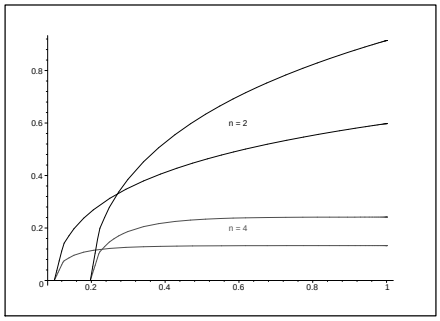

Figure 1: Catenaries $n = 2, 4$

Notations. Let N_a denote the unit normal to the catenoid $\mathcal{C}_a$, let A_a denote its second fundamental form relative to the normal N_a and let $d\mu_a$ denote its Riemannian measure. When $n = 2$, let K_a denote the Gauss curvature of $\mathcal{C}_a$. We state the following proposition for later purposes.

Proposition 3.3 *For $a > 0$, the n-dimensional catenoid $\mathcal{C}_a$ in $\mathbb{H}^n \times \mathbb{R}$ has infinite volume and finite* total extrinsic curvature $\int_{\mathcal{C}_a} |A_a|^n\,d\mu_a$. *When $n = 2$, the catenoid $\mathcal{C}_a$ has infinite* total intrinsic curvature $\int_{\mathcal{C}_a} |K_a|\,d\mu_a$.

Proof. We can restrict to the upper half-catenoid, $\mathcal{C}_{a,+} = \mathcal{C}_a \cap (\mathbb{H}^n \times \mathbb{R}_+)$, which admits the parametrization

$$Y(a,\rho,\xi) := \big(\tanh(\rho/2)\xi, \lambda(a,\rho)\big), \quad \rho \geq a.$$

The geometric data of $\mathcal{C}_{a,+}$ are readily calculated. In particular,

$$|A_a|^2(\rho) = n(n-1)\big(\frac{\sinh^{n-1}(a)\cosh(\rho)}{\sinh^n(\rho)}\big)^2,$$

and

$$d\mu_a = \sinh^{2n-2}(\rho)\big(\sinh^{2n-2}(\rho)-\sinh^{2n-2}(a)\big)^{-1/2}\,d\rho\,d\mu_S.$$

The first assertion follows ($|A_a|^n d\mu_a$ tends to zero exponentially at infinity). For the second assertion, we use Gauss equation and minimality to get that

$$K_a = \widehat{K}_a - \frac{1}{2}|A_a|^2 = -v_a^2 - \frac{1}{2}|A_a|^2,$$

where $\widehat{K}_a$ is the sectional curvature of the 2-plane tangent to $\mathcal{C}_a$ in the ambient space $\mathbb{H}^2 \times \mathbb{R}$ and where v_a is the vertical component of the unit normal to $\mathcal{C}_a$,

$$v_a(\rho) = \widehat{g}(N_a, \partial_t) = \sinh^{1-n}(\rho)\big(\sinh^{2n-2}(\rho)-\sinh^{2n-2}(a)\big)^{1/2}.$$

Assertion 2 follows because v_a tends to 1 at infinity on $\mathcal{C}_{a,+}$. □

3.3 Catenoids in $\mathbb{H}^n \times \mathbb{R}$, stability properties

Recall that the catenoid $\mathcal{C}_a$ is generated by the curve $t \mapsto \big(\tanh(f(a,t)/2), t\big)$ in the vertical plane $\mathbb{V}$, where $f(a,\cdot)$ is the maximal solution of the Cauchy problem (3.13). This yields the parametrization

$$X(a,t,\xi) = \big(\tanh(f(a,t)/2)\xi, t\big) \tag{3.16}$$

for $\mathcal{C}_a$, with $t \in \mathbb{R}$ and $\xi \in S^{n-1}$. According to Property 2.1 (2), we have two Jacobi fields on the catenoid $\mathcal{C}_a$.

• The *vertical Jacobi field* $v(a,t)$ comes from the vertical translations $(x,t) \mapsto (x, t+\tau)$ in $\mathbb{H}^n \times \mathbb{R}$. It is given by $v(a,t) = \widehat{g}(N_a, \partial_t)$, where N_a is the unit normal to $\mathcal{C}_a$. According to (3.7), it is given by the formula

$$v(a,t) = f_t(a,t)\big(1 + f_t^2(a,t)\big)^{-1/2}, \tag{3.17}$$

where f_t stands for the derivative with respect to the variable t. Because $t \mapsto f(a,t)$ is even, the function $t \mapsto v(a,t)$ is odd.

• The *variation Jacobi field* $e(a,t)$ comes from the variations with respect to the parameter a. It is given by $e(a,t) = \widehat{g}(N_a, \frac{\partial X}{\partial a})$. According to (3.7) and (3.16), the function $e(a,t)$ is given

$$e(a,t) = -f_a(a,t)\big(1 + f_t^2(a,t)\big)^{-1/2}, \tag{3.18}$$

where f_a stands for the derivative with respect to the variable a. Because $t \mapsto f(a,t)$ is even, the function $t \mapsto e(a,t)$ is even.

• The Jacobi fields $v(a,t)$ and $e(a,t)$ have nice expressions when restricted to the upper-half $\mathcal{C}_{a,+} = \mathcal{C}_a \cap (\mathbb{H}^n \times \mathbb{R}_+)$ of the catenoid $\mathcal{C}_a$. Indeed, recall that the function $f(a,\cdot) : [0, T(a)[\to [0, \infty[$ has an inverse function $\lambda(a,\rho)$ given by (3.14). Using the relationships

$$\lambda\big(a, f(a,t)\big) \equiv t \quad \text{for} \quad t \geq 0$$

and

$$\lambda_\rho(a,f) f_t \equiv 1 \quad \text{and} \quad \lambda_a(a,f) + \lambda_\rho(a,f) f_a \equiv 0,$$

we get the following expressions for $v(a,t)$ and $e(a,t)$ for $t \geq 0$,

$$\left\{ \begin{array}{rcl} v(a,t) & = & \big(1 + \lambda_\rho^2(a, f(a,t))\big)^{-1/2}, \\ \\ e(a,t) & = & v(a,t)\lambda_a(a, f(a,t)). \end{array} \right. \tag{3.19}$$

For $\rho \geq a$, define the functions $v_1(a,\rho)$, $A_1(a,\rho)$ and $B_1(a,\rho)$, by the following formulas

$$\left\{ \begin{array}{rcl} v_1(a,\rho) & = & \big(1 + \lambda_\rho^2(a,\rho)\big)^{-\frac{1}{2}} = \big(\dfrac{\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a)}{\sinh^{2n-2}(\rho)}\big)^{\frac{1}{2}} \\ \\ A_1(a,\rho) & = & \dfrac{\cosh(a)}{\cosh(\rho)} \Big(\dfrac{\sinh(a)}{\sinh(\rho)}\Big)^{n-2}, \\ \\ B_1(a,\rho) & = & \cosh(a) \int_1^{\frac{\sinh(\rho)}{\sinh(a)}} (v^{2n-2} - 1)^{-\frac{1}{2}} (\sinh^2(a) v^2 + 1)^{-\frac{3}{2}}\, dv. \end{array} \right. \tag{3.20}$$

From (3.14), we can write

$$\lambda(a,\rho) = \sinh(a) \int_1^{\frac{\sinh(\rho)}{\sinh(a)}} \big(v^{2n-2} - 1\big)^{-\frac{1}{2}} \big(\sinh^2(a) v^2 + 1\big)^{-\frac{1}{2}}\, dv \tag{3.21}$$

and compute λ_a,

$$\lambda_a(a,\rho) = -\cosh(a)\sinh^{n-2}(a)\tanh(\rho)\big(\sinh^{2n-2}(\rho)-\sinh^{2n-2}(a)\big)^{-\frac{1}{2}} + \\ +\cosh(a)\int_1^{\frac{\sinh(\rho)}{\sinh(a)}}(v^{2n-2}-1)^{-\frac{1}{2}}(\sinh^2(a)v^2+1)^{-\frac{3}{2}}\,dv.$$

We obtain,

$$\lambda_a(a,\rho)v_1(a,\rho) = -A_1(a,\rho)+B_1(a,\rho)v_1(a,\rho). \tag{3.22}$$

We summarize the relevant properties in the following lemma whose proof is straightforward.

Lemma 3.4 *Define the functions $A(a,t)$ and $B(a,t)$ for $t\geq 0$ by*

$$A(a,t)=A_1(a,f(a,t)),\quad B(a,t)=B_1(a,f(a,t)), \tag{3.23}$$

see Formulas(3.20). Then,

$$e(a,t)=-A(a,t)+B(a,t)v(a,t),\quad \textit{for}\ \ t\geq 0. \tag{3.24}$$

Furthermore, for $t\geq 0$,

1. *$A(a,t)>0,\ A(a,0)=1$ and $\lim_{t\to T(a)}A(a,t)=0$,*
2. *$B(a,t)>0,\ B(a,0)=0$ and $\lim_{t\to T(a)}B(a,t)=C(a)$, where $C(a)=\cosh(a)\int_1^{\infty}(v^{2n-2}-1)^{-\frac{1}{2}}(\sinh^2(a)v^2+1)^{-\frac{3}{2}}\,dv$.*
3. *$v(a,t)=v_1(a,f(a,t))$ for $t>0$, so that $v(a,t)>0$ for $t>0$, $v(a,0)=0$ and $\lim_{t\to T(a)}v(a,t)=1$.*

Notation. For $\alpha<\beta\in[0,T(a)]$, let $\mathcal{D}(\alpha,\beta)$ denote the rotationally symmetric domain

$$\mathcal{D}_a(\alpha,\beta)=X(a,]\alpha,\beta[,S^{n-1}). \tag{3.25}$$

In particular, $\mathcal{D}_a(0,T(a))$ is the half-vertical catenoid $\mathcal{C}_{a,+}=\mathcal{C}_a\cap(\mathbb{H}^n\times\mathbb{R}_+)$.

Theorem 3.5 *The stability properties of the rotationally symmetric domains $\mathcal{D}_a(\alpha,\beta)$ on the catenoid $\mathcal{C}_a$ are as follows.*

1. *There exists some $\sigma(a)\in]0,T(a)[$ such that the relatively compact domain $\mathcal{D}_a(-\sigma(a),\sigma(a))$ is stable-unstable. Hence, for any $\alpha\in]0,\sigma(a)[$, the domain $\mathcal{D}_a(-\alpha,\alpha)$ is stable; for any $\alpha\in]\sigma(a),T(a)[$, the domain $\mathcal{D}_a(-\alpha,\alpha)$ is unstable.*
2. *There exists some $\tau(a)\in]0,T(a)[$ such that*
   (a) *the (non relatively compact) domain $\mathcal{D}_a(-\tau(a),T(a))$ is stable,*
   (b) *for any $\alpha\in]\tau(a),T(a)[$, there exists some $\beta(\alpha)\in]\tau(a),T(a)[$ such that the domain $\mathcal{D}_a(-\alpha,\beta(\alpha))$ is stable-unstable.*
3. *The catenoid $\mathcal{C}_a$ has index 1.*

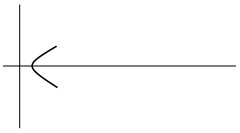

Figure 2: Case 1

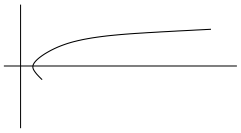

Figure 3: Case 2a

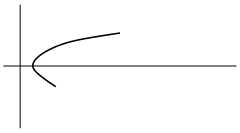

Figure 4: Case 2b

The above domains are generated by the portions of curves illustrated in the following figures.
Proof. *Assertion 1.* Consider the function $e(a,t)$. According to Lemma 3.4, $e(a,0) = -1$ and $\lim_{t\to T(a)} e(a,t) = C(a) > 0$, so that it must vanish at least once on $]0,T(a)[$. It turns out (compare with Lemma 3.6 below) that $e(a,\cdot)$ has a unique positive zero $\sigma(a)$. Because $e(a,t)$ is even in t, it does not vanish in the open set $\mathcal{D}_a(-\sigma(a),\sigma(a))$ and satisfies $J_a(e) = 0$ in $\mathcal{D}_a(-\sigma(a),\sigma(a))$, and $e|\partial\mathcal{D}_a(-\sigma(a),\sigma(a)) = 0$. This means that $\mathcal{D}_a(-\sigma(a),\sigma(a))$ is a stable-unstable domain. The second assertion follows from Property 2.1 (1).

Assertion 2. Take any $\alpha \in]0,T(a)[$ and define the function $w(a,\alpha,t)$ by

$$w(a,\alpha,t) = e(a,\alpha)v(a,t) + v(a,\alpha)e(a,t), \quad \text{for } t \in]-T(a),T(a)[. \tag{3.26}$$

This is a Jacobi field on $\mathcal{C}_a$ and furthermore $w(a,\alpha,-\alpha) = 0$, because v is odd and e is even with respect to t. Note also that $w(a,\alpha,0) = -v(a,\alpha) < 0$.

Lemma 3.6 *The function $w(a,\alpha,\cdot)$ vanishes only once on $]-T(a),0[$ and vanishes at most once on $]0,T(a)[$.*

Let us prove the first assertion of the Lemma, the proof of the second assertion is similar. Assume that $w(a,\alpha,\cdot)$ has at least two consecutive zeroes $\alpha_1 < \alpha_2$ in the interval $]-T(a),0[$. The domain $\mathcal{D}_a(\alpha_1,\alpha_2)$ would then be stable-unstable because $J_a(w) = 0$ on $\mathcal{D}_a(\alpha_1,\alpha_2)$ and because w vanishes on $\partial\mathcal{D}_a(\alpha_1,\alpha_2)$. On the other-hand, the Jacobi field v satisfies $J_a(v) = 0$ and $v < 0$ in $\mathcal{D}_a(\alpha_1,\alpha_2)$. By Property 2.1 (3), we have that $\lambda_1(\mathcal{D}_a(\alpha_1,\alpha_2)) > 0$ which contradicts the fact that this domain is stable-unstable. This proves the lemma.

In order to determine whether the function $w(a,\alpha,\cdot)$ vanishes on $]0,T(a)[$ or not, it is sufficient to look at the behaviour of $w(a,\alpha,t)$ when t tends to $T(a)$ from below. For this purpose, we use the expression (3.24) for $e(a,t)$ and we write

$$w(a,\alpha,t) = -A(a,t)v(a,\alpha) + v(a,t)\big(e(a,\alpha) + B(a,t)v(a,\alpha)\big).$$

Using Lemma 3.4, we can write

$$W(a,\alpha) := \lim_{t\to T(a)} w(a,\alpha,t) = e(a,\alpha) + C(a)v(a,\alpha).$$

If $W(a,\alpha) \le 0$, then $w(a,\alpha,t)$ does not vanish on $]0,T(a)[$ and in fact on $]-\alpha,T(a)[$; if $W(a,\alpha) > 0$, then $w(a,\alpha,t)$ has one and only one zero $\beta(\alpha)$on $]0,T(a)[$.

We now observe that $W(a,t) := e(a,t) + C(a)v(a,t)$ is a Jacobi field on $]0,T(a)[$ which take the value -1 at 0 and the value $C(a)v(a,\sigma(a)) > 0$ at $\sigma(a)$. It follows from Lemma 3.6 that $W(a,\cdot)$ has one and only one positive zero $\tau(a) \in]0,\sigma(a)[$. We have that $W(a,t) \le 0$ on $]0,\tau(a)[$, so that for any $\alpha \in]0,\tau(a)]$, the function $w(a,\alpha,t)$ has only one zero $-\alpha$ on $]-T(a),T(a)[$. This proves the Assertion 2(a). On the other-hand, $W(a,t) > 0$ on $]\tau(a),T(a)[$, so that for any $\alpha \in]\tau(a),T(a)[$, the function $w(a,\alpha,t)$ has a unique positive zero $\beta(\alpha) \in]0,T(a)[$. This proves the Assertion 2(b).

Assertion 3. Assertion 1 shows that $\mathcal{C}_a$ has index at least 1. In order to show that the index is at most one, we use Fourier decomposition with respect to the variable ξ and an extra stability argument.

Recall that we work in the ball model for $\mathbb{H}^n$. Let γ be a geodesic through 0 in $\mathbb{H}^n$. Up to a rotation, we may assume that $\gamma(s) = \big(\tanh(s/2), 0, \cdots, 0\big)$. Let $\mathbb{H}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{B} \mid x_1 > 0\}$ and let $\mathcal{C}_{a,\gamma+} = \mathcal{C}_a \cap (\mathbb{H}^n_+ \times \mathbb{R})$. We call this set a *half-horizontal catenoid*.

Claim 1. A half-horizontal catenoid $\mathcal{C}_{a,\gamma+}$ is stable.

To prove the claim, we shall find a positive Jacobi field on $\mathcal{C}_{a,\gamma+}$.
Let $z = x + iy$ denote the complex coordinate in $\mathbb{H}^2$ (ball model). We consider the group of hyperbolic isometries along the geodesic γ and we extend these isometries slice-wise as isometries in $\mathbb{H}^2 \times \mathbb{R}$. We then have the one-parameter group of isometries

$$(z; t) \mapsto \big(\frac{e^\tau(1+z) - (1-z)}{e^\tau(1+z) + (1-z)}; t\big) \text{ in } \mathbb{H}^2 \times \mathbb{R}.$$

The associated Killing vector-field in $\mathbb{H}^2 \times \mathbb{R}$ is given by $\mathcal{K}_\gamma(z; t) = \big(\frac{1}{2}(1 - z^2); 0\big)$ or, in the (x, y) coordinates, $\mathcal{K}_\gamma(x, y; t) = \big(\frac{1}{2}(1 - x^2 + y^2), -xy; 0\big)$ which can be written as

$$\mathcal{K}_\gamma(x, y; t) = \frac{1}{2}\big(1 + x^2 + y^2)\big)(1, 0; 0) - x(x, y; 0)$$

where $(1, 0; 0)$ and $(x, y; 0)$ are seen as vectors in $\mathbb{R}^2 \times \mathbb{R} = T_{(x,y;t)}\mathbb{H}^2 \times \mathbb{R}$.

This formula can easily be generalized to higher dimensions as

$$\mathcal{K}_\gamma(x; t) = \frac{1}{2}(1 + |x|^2(e_1; 0) - x_1(x; 0),$$

where $x = (x_1, \cdots, x_n), e_1 = (1, 0, \cdots, 0), |x|^2 = x_1^2 + \cdots + x_n^2$, and where $(e_1; 0)$ and $(x; 0)$ are seen as vectors in $\mathbb{R}^n \times \mathbb{R} = T_{(x,t)}\mathbb{H}^n \times \mathbb{R}$. Writing the point x in the parametrization X as $x = \tanh(f(a, t)/2)\xi$, we obtain that

$$\mathcal{K}_\gamma(\tanh(f(a, t)/2)\xi; t) = \frac{1}{2}\big(1 + \tanh^2(f/2)\big)(e_1; 0) - \tanh^2(f/2)\xi_1(\xi; 0).$$

Using the fact that $(1 + f_t^2)^{-1/2} = \big(\frac{\sinh(a)}{\sinh(f)}\big)^{n-1}$ on $\mathcal{C}_a$, we find that the Killing field $\mathcal{K}_\gamma$ gives rise to the horizontal Jacobi field

$$h_\gamma(a, t, \xi) = \big(\frac{\sinh(a)}{\sinh(f(a, t))}\big)^{n-1}\xi_1$$

which is positive on $\mathcal{C}_{a,\gamma+}$.

Claim 2. On S^{n-1} equiped with the standard Riemannian metric, there exists an orthonormal basis of spherical harmonics $Y_k, k \geq 0$ with the property that the nodal domains of the $Y_k, k \geq 1$ are contained in hemispheres.

The property is clearly true on S^1 and can be proved by induction on the dimension, using polar coordinates centered at a given point on the sphere.

Claim 3. The Jacobi operator on $\mathcal{C}_a$ can be written as

$$J_a = L_{a,t} - q(a,t)\Delta_{S,\xi},$$

where $L_{a,t}$ is a Sturm-Liouville operator on the t variable, with coefficients depending only on a and t, where $q(a,t)$ is a positive function and where $\Delta_{S,\xi}$ is the Laplacian of the sphere S^{n-1} acting on the ξ-variable. This claim follows immediately from Formulas (3.5) and (3.6) for the metric and the Riemannian measure on a rotation hypersurface and from the expression for the quadratic form associated with J_a.

Assume that the index of $\mathcal{C}_a$ is at least 2. Then, there exists some $S \in]0, T(a)[$ such that J_a has at least two negative eigenvalues $\lambda_1(S) < \lambda_2(S) < 0$ in $\mathcal{C}_a(-S, S)$ (we only consider Dirichlet boundary conditions). Because the least eigenvalue $\lambda_1(S)$ is simple, a corresponding eigenfunction u must be invariant under rotations (*i.e.* only depends on the variable t) and say positive. Consider an eigenfunction v associated with $\lambda_2(S)$. We claim that v cannot be invariant under rotations. Indeed, v would otherwise depend only on the variable t, it would be orthogonal to u and hence it would have to vanish on $]-S, S[$. This would contradict the fact that the domains $\mathcal{C}_a(-S, 0)$ and $\mathcal{C}_a(0, S)$ are stable. Since v is not rotationally invariant, there exists some $p \geq 1$ and some $v_p \neq 0$ in the decomposition into spherical harmonics with respect to the second variable, $v(t, \xi) = \sum_{k=0}^{\infty} v_k(t) Y_k(\xi)$. We would have $J_a(v_p Y_p) = \lambda_2(S) v_p Y_p$. Using Claim 2 and the fact that $\lambda_2(S) < 0$, this would mean that any nodal domain of $v_p Y_p$ is unstable, in contradiction with Claim 1.

Assuming that the index is at least 2 therefore yields a contradiction and hence the index of $\mathcal{C}_a$ is exactly one. □

Remark. It follows from the positivity of the Jacobi field $v(a,t)$ for $t \in]0, T(a)[$ that the upper half-catenoid $\mathcal{C}_{a,+}$ is stable (in the sense that any relatively compact domain Ω contained in $\mathcal{C}_{a,+}$ is stable, see Section 2.2). The second assertion in the preceding theorem says more. Indeed, there exists some $\tau(a) \in]0, T(a)[$ such that the non-compact domain $\mathcal{D}_a(-\tau(a), T(a))$ is stable and stricly contains $\mathcal{C}_{a,+}$. This is different from what happens for Euclidean catenoids. Indeed, the half-catenoid $\mathcal{C}_{a,+}$ in $\mathbb{R}^3$ is a *maximal* stable domain ([18]). We study this phenomenon with more details in [5].

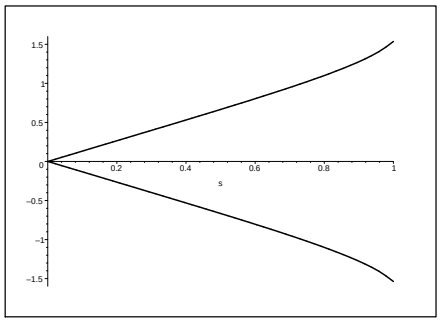

Figure 5: Envelope, $n = 2$

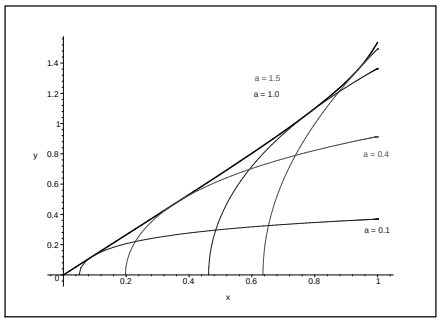

Figure 6: Catenaries, envelope

Geometric interpretation. According to Proposition 3.2, Assertion (6), two distinct catenaries C_a and C_b meet at exactly two symmetric points, $m_\pm(a,b)$. Fixing a and letting b tend to a, the points $m_\pm(a,b)$ tend to limit points $m_\pm(a)$ which correspond to the points where the catenary C_a

touches the envelope of the family of catenaries $\{C_a\}_{a>0}$. According to [26], §58, page 127 ff, the condition defining the envelope a family Γ_a, given by the parametrization $\big(x(a,t), y(a,t)\big)$, is the condition

$$\begin{vmatrix} x_a(a,t) & x_t(a,t) \\ y_a(a,t) & y_t(a,t) \end{vmatrix} = 0.$$

Specializing to catenaries, we find that the envelope condition is precisely the condition that $e(a,t) = 0$. Therefore, the value $\sigma(a)$ is precisely the value of t at which the catenary C_a touches the envelope of the family, see Figure 6.

Corollary 3.7 *The stable-unstable domain $\mathcal{D}_a(-\sigma(a), \sigma(a))$ is precisely the symmetric, rotation invariant compact domain bounded by the two spheres where the catenoid $\mathcal{C}_a$ touches the envelope of the family.*

3.4 Translation invariant hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

Let γ be a complete geodesic through 0 in the ball model $\mathbb{B}$ of the hyperbolic space $\mathbb{H}^n$, parametrized by the signed distance ρ to 0. Let $\mathbb{P}$ be the hyperbolic hyperplane orthogonal to γ at 0. We consider the hyperbolic translations along the geodesics passing through 0 in $\mathbb{P}$. The image of a point of γ under these translations is an equidistant hypersurface to $\mathbb{P}$ in $\mathbb{H}^n$. We can extend these translations "slice-wise" to give positive isometries of $\mathbb{H}^n \times \mathbb{R}$ which we call *hyperbolic translations*.

A generating curve $(\tanh(\rho/2), \mu(\rho))$ in the vertical Euclidean plane $\gamma \times \mathbb{R}$ gives rise, under the previous isometries, to a translation invariant hypersurface $M \hookrightarrow \mathbb{H}^n \times \mathbb{R}$, whose intersection with the slice $\mathbb{H}^n \times \{\mu(\rho)\}$ is the equidistant hypersurface to $\mathbb{P} \times \{\mu(\rho)\}$ in the slice, at distance ρ.

The principal directions of curvature of M are the tangent vector to the generating curve and the directions tangent to the equidistant hypersurface. The corresponding principal curvatures are given respectively by

$$k_G(\rho) = \ddot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-3/2}, \tag{3.27}$$

and

$$k_E(\rho) = \dot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-1/2} \tanh(\rho). \tag{3.28}$$

It follows that the mean curvature of M is given by

$$nH(\rho) = \ddot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-3/2} + (n-1)\dot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-1/2} \tanh(\rho)$$

or, equivalently, by

$$nH(\rho)\cosh^{n-1}(\rho) = \partial_\rho\Big(\dot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-1/2} \cosh^{n-1}(\rho)\Big). \tag{3.29}$$

This formula allows us to study constant mean curvature hypersurfaces invariant by hyperbolic translations. In this paper we only consider the case $H = 0$ and we refer to [6] for the case $H \neq 0$.

3.5 Translation invariant minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

In this section, we establish the following theorem which generalizes the 2-dimensional result of [22].

Theorem 3.8 *There exists a 1-parameter family $\{\mathcal{M}_d,\, d>0\}$ of complete embedded minimal hypersurfaces in $\mathbb{H}^n\times\mathbb{R}$ invariant under hyperbolic translations. The hypersurfaces are $\mathcal{M}_d$ stable (in the sense of the Jacobi operator), their principal curvatures go uniformly to zero at infinity, but they have infinite total curvatures.*
More precisely,

1. *If $d>1$, the hypersurface $\mathcal{M}_d$ consists of the union of two symmetric vertical graphs over the exterior of an equidistant hypersurface in the slice $\mathbb{H}^n\times\{0\}$. It is also a horizontal graph, and hence stable.*
   The family $\mathcal{M}_d$ has finite vertical height $h_T(d)$, a function which decreases from infinity to $\pi/(n-1)$. In particular, it is bounded from below by $\pi/(n-1)$, the upper bound of the heights of the family of catenoids. Furthermore, the asymptotic boundary of $\mathcal{M}_d$ consists of the union of two copies of an hemisphere $S^{n-1}_+\times\{0\}$ of $\partial_\infty\mathbb{H}^n\times\{0\}$ in parallel slices $t=\pm S(d)$, glued with the finite cylinder $\partial S^{n-1}_+\times[-S(d),S(d)]$.
2. *If $d=1$, the hypersurface $\mathcal{M}_1$ is a complete stable vertical graph over a half-space in $\mathbb{H}^n\times\{0\}$, bounded by a totally geodesic hyperplane P. It takes infinite boundary value data on P and constant asymptotic boundary value data. Furthermore, the asymptotic boundary of $\mathcal{M}_1$ is the union of a spherical cap S in $\partial_\infty\mathbb{H}^n\times\{c\}$ with a half-vertical cylinder over ∂S.*
3. *If $d<1$, the hypersurface $\mathcal{M}_d$ is an entire stable vertical graph with finite vertical height. Furthermore, its asymptotic boundary consists of a homologically non-trivial $(n-1)$-sphere in $\partial_\infty\mathbb{H}^n\times\mathbb{R}$.*

Proof. In the minimal case, Equation (3.29) can be written

$$\dot\mu(\rho)\big(1+\dot\mu^2(\rho)\big)^{-1/2}\cosh^{n-1}(\rho)=d, \tag{3.30}$$

for some constant d which satisfies $d\le\cosh^{n-1}(\rho)$ for all ρ for which the solution exists. Changing μ to $-\mu$ if necessary, we may assume that d is non-negative and hence, $\dot\mu(\rho)\ge 0$ and $\dot\mu(\rho)=d\big(\cosh^{2n-2}(\rho)-d^2\big)^{-1/2}$ whenever the square root exists. We have to consider three cases, $d>1$, $d=1$ and $d<1$.

$\boxed{d>1}$ Let $d=:\cosh^{n-1}(a)$, with $a>0$. It follows from Equation (3.30) that

$$\dot\mu(\rho)=\cosh^{n-1}(a)\big(\cosh^{2n-2}(\rho)-\cosh^{2n-2}(a)\big)^{-1/2}.$$

Up to a vertical translation, the solution $\mu_+(a,\rho)$ of Equation (3.30) is given by

$$\mu_+(a,\rho)=\cosh^{n-1}(a)\int_a^\rho\big(\cosh^{2n-2}(r)-\cosh^{2n-2}(a)\big)^{-1/2}\,dr \tag{3.31}$$

or, making $\cosh(r)=\cosh(a)\,t$,

$$\mu_+(a,\rho)=\cosh(a)\int_1^{\frac{\cosh(\rho)}{\cosh(a)}}(t^{2n-2}-1)^{-\frac12}\,(\cosh^2(a)t^2-1)^{-\frac12}\,dt. \tag{3.32}$$

These integrals converge at $\rho=a$ (*resp.* at $t=1$) and at infinity and

$$h_T(d):=2\cosh(a)\int_1^\infty(t^{2n-2}-1)^{-1/2}\,(\cosh^2(a)t^2-1)^{-1/2}\,dt \tag{3.33}$$

is the height of the hypersurface $\mathcal{M}_{\cosh^{n-1}(a)}$. The function $h_T(d)$ is decreasing in d, tends to infinity when d tends to 1_+ and to $\pi/(n-1)$ when d tends to infinity. (Hints. When a tends to zero, use the fact that (3.33) is bigger than some constant times the integral $\int_1^2 \big((t-1)(\cosh(a)t-1)\big)^{-1/2}\,dt$ which can be computed explicitly. When a tends to infinity, use the fact that $\int (t^N-1)^{-1/2}t^{-1}\,dt = \frac{2}{N}\arctan\sqrt{t^N-1}$.) The assertions on the asymptotic boundary are clear.

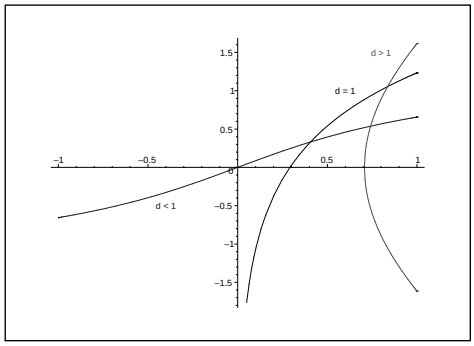

Figure 7: Generatrices translation invariant hypersurfaces, $n = 2$

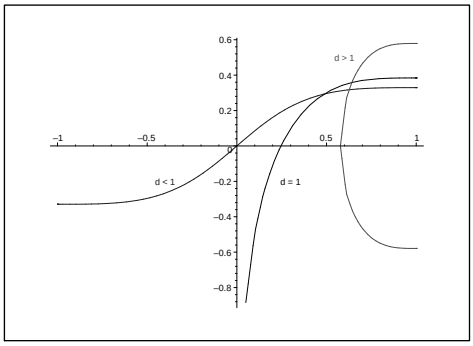

Figure 8: Generatrices translation invariant hypersurfaces, $n = 4$

$\boxed{d=1}$ It follows from Equation (3.30) that

$$\dot\mu(\rho) = \big(\cosh^{2n-2}(\rho) - 1\big)^{-1/2},$$

so that, when $d = 1$, the solution is given by

$$\mu_0(\rho) = \int_b^\rho \big(\cosh^{2n-2}(r) - 1\big)^{-1/2}\,dr, \tag{3.34}$$

for some constant $b > 0$, and $\mu_0(\rho)$ tends to $-\infty$ when ρ tends to zero and to a finite value when ρ tends to infinity. The corresponding hypersurface is complete. It is a vertical graph so that it is stable. The assertion on the asymptotic boundary is clear.

$\boxed{0<d<1}$ In this case, Equation (3.30) gives the following solution (up to a vertical translation),

$$\mu_-(d,\rho) = d\int_0^\rho \big(\cosh^{2n-2}(r) - d^2\big)^{-1/2}\,dr. \tag{3.35}$$

The corresponding curve can be extended by symmetry and we get a complete hypersurface in a vertical slab with finite height. This surface is an entire vertical graph (hence stable). The assertion on the asymptotic boundary is clear. □

The generating curves for translation invariant minimal hypersurfaces are given in Fig. 7 and 8. Note that they cannot meet tangentially at finite distance.

Remark. Using the catenoids and the minimal translations hypersurfaces, the second author and E. Toubiana have extended the 2-dimensional results of their paper [22] to higher dimensions, see [23].

4 Index and total curvature for minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

4.1 Dimension two, $M^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$

For oriented minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ we have the following general theorem in which we consider two possible notions of total curvature.

Theorem 4.1 *Let $M \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be a complete oriented minimal immersion with unit normal field N_M. Let $v_M := \hat{g}(N_M, \partial_t)$ be the vertical component of N_M, let A_M be the second fundamental form of M and let K_M be the intrinsic curvature of M.*

1. *If the integral $\int_M |A_M|^2 \, d\mu_M$ is finite, then A_M tends to zero uniformly at infinity.*
2. *If the integral $\int_M |K_M| \, d\mu_M$ is finite, then A_M, v_M and K_M tend to zero uniformly at infinity.*
3. *If the integral $\int_M |K_M| \, d\mu_M$ is finite, then the Jacobi operator of M has finite index.*

Remarks.

1. For complete orientable minimal surfaces in $\mathbb{R}^3$, finiteness of the index is equivalent to finiteness of the intrinsic curvature (see [12, 11, 10]). No such statement can hold in $\mathbb{H}^2 \times \mathbb{R}$. Indeed, the surfaces $\mathcal{M}_d$ ([22] and Section 3.5) are stable complete minimal surfaces, invariant under a group of hyperbolic translations. Their total curvatures are infinite, so that the converse to Assertion 3 is false.
2. The assumption $\int_M |K_M| \, d\mu_M$ finite is natural in view of Huber's theorem. In [14], L. Hauswirth and H. Rosenberg show that this assumption actually implies that the total intrinsic curvature is an integer multiple of 2π. There are actually many examples of such surfaces ([9, 14]).
3. As pointed out in the introduction, Assertion 2 is contained in [14], Theorem 3.1 whose proof actually gives a C^2-control on the curvature at infinity. We provide a simple proof of Assertion 2 for completeness.

One can slightly improve the above theorem with the following proposition.

Proposition 4.2 *The notations are the same as in Theorem 4.1.*

1. *Assume that $5v_M^2 \leq 1$. Then there exists a universal constant C such that if the integral $\int_M |A_M|^2 \, d\mu_M$ is less than C then M is a vertical plane.*
2. *Assume that the integral $\int_M |A_M|^2 \, d\mu_M$ is finite and that there exists a compact set $\Omega \subset M$ and a positive constant c such that $v_M^2 \leq 1 - c < 1$ on $M \setminus \Omega$. Then the Jacobi operator of the immersion has finite index.*

Remarks.

1. Assertion 1 generalizes the following facts : (i) A minimal surface whose intrinsic curvature K_M is zero is part of a vertical plane $\gamma \times \mathbb{R}$ (where γ is a geodesic in $\mathbb{H}^2$). Indeed, we have $K_M = -\frac{1}{2}|A_M|^2 - v_M^2$ and hence M is totally geodesic with horizontal normal vector. (ii) A complete minimal surface whose total intrinsic curvature is less than 2π is a vertical plane (see [14]).

2. We do not know whether the sole assumption $\int_M |A_M|^2\, d\mu_M$ finite is sufficient to insure the finiteness of the index of the Jacobi operator of M.

Proof of Theorem 4.1 and Proposition 4.2.

Fact 1. The function $u := |A_M|$ satisfies the non-linear elliptic inequality

$$-u\,\Delta_M u \leq u^4 + (5\widehat{K}_M + 1)u^2 \leq u^4 + 4u^2, \tag{4.36}$$

where $\widehat{K}_M$ is the sectional curvature of the 2-plane TM in $\mathbb{H}^2 \times \mathbb{R}$.
This formula follows from J. Simons' equation for minimal submanifolds ([25]), from our context ($\mathbb{H}^2 \times \mathbb{R}$ is locally symmetric and we work in codimension 1), and from explicit curvature computations in $\mathbb{H}^2 \times \mathbb{R}$.

Fact 2. The surface M satisfies the Sobolev inequality

$$\|f\|_2 \leq S(M)\|df\|_1 \tag{4.37}$$

for some positive constant $S(M)$ and all C^1 functions f with compact support. This follows from [15, 16] and the fact that $\mathbb{H}^2 \times \mathbb{R}$ is simply-connected and non-positively curved.

Fact 3. Curvature computations in $\mathbb{H}^2 \times \mathbb{R}$ give

$$\widehat{K}_M = -v_M^2 \quad \text{and} \quad \widehat{\mathrm{Ric}}(N_M, N_M) = -(1 - v_M^2). \tag{4.38}$$

The fact that v_M is a Jacobi field implies that

$$-\Delta_M v_M = v_M^3 + (|A_M|^2 - 1)v_M \tag{4.39}$$

and a similar inequality for $|v_M|$.

Theorem, Assertion 1. Following the general ideas of [24], we use (4.36) and (4.37), to estimate the L^p-norms of u and the classical de Giorgi-Moser-Nash method to estimate $\|u\|_\infty$ outside big balls. The details appear in the proof of Theorem 4.1, p. 282 of [4], where it is observed that the proof only uses the facts that u satisfies Simons' inequality and M a Sobolev inequality.
Theorem, Assertion 2. By Gauss equation, the Gauss curvature K_M of M satisfies

$$K_M = -\frac{1}{2}|A_M|^2 - v_M^2. \tag{4.40}$$

The assumption implies that both integrals $\int_M |A_M|^2\, d\mu_M$ and $\int_M v_M^2\, d\mu_M$ are finite. By Assertion 1, we already know that $|A_M|$ tends to zero at infinity, and hence that it is bounded. Equation (4.39) then tells us that $|v_M|$ satisfies an elliptic inequality similar to (4.36) and we can again apply the de Giorgi-Moser-Nash method to conclude.

Theorem, Assertion 3 and Proposition, Assertion 2. According to Section 2.2 and to the above curvature calculations, the Jacobi operator can be written as $J_M = -\Delta_M + 1 - |A_M|^2 - v_M^2$. We now follow [3], Section 2. It follows from Assertion 2 in the Theorem that J_M is bounded from below, essentially self-adjoint and that its essential spectrum lies above 1. Because the eigenvalues below the essential spectrum can only accumulate at $-\infty$ or at the bottom of the essential spectrum, it follows that J_M has finite index.

Proposition, Assertion 1. It is a classical fact ([7] and [24]) that Simons' inequality (4.36) can be improved to

$$|du|^2 - u\,\Delta_M u \leq u^4 + (5\widehat{K}_M + 1)u^2.$$

We then use the expression of $\widehat{K}_M$ and the assumption on v_M to obtain

$$(a) \quad |du|^2 - u\,\Delta_M u \leq u^4.$$

Multiply equation (a) by some function with compact support ξ (to be chosen later on) and integrate by parts to obtain,

$$2\int_M \xi^2|du|^2 + 2\int_M \xi u\langle du, d\xi\rangle \leq \int_M \xi^2 u^4.$$

Using Cauchy-Schwarz inequality, we obtain

$$(b) \quad \int_M \xi^2|du|^2 \leq \int_M \xi^2 u^4 + \int_M u^2|d\xi|^2.$$

Plug the function $f = \xi u^2$ into Sobolev inequality (4.37) to obtain

$$\int_M \xi^2 u^4 \leq S\big(\int_M |d(\xi u^2)|\big) \leq 2S\big(\int_M u^2|d\xi|\big)^2 + 8S\big(\int_M u^2\big)\big(\int_M u\xi|du|\big)^2,$$

where we have noted S for $S(M)$ and used the fact that $\int_M u^2$ is finite. Using Cauchy-Schwarz again, we find

$$(c) \quad \int_M \xi^2 u^4 \leq 2S\big(\int_M u^2|d\xi|\big)^2 + 8S\big(\int_M u^2\big)\int_M \xi^2|du|^2.$$

Plug (c) into (a) to get

$$\big(1 - 8S\int_M u^2\big)\int_M \xi^2|du|^2 \leq 2S\big(\int_M u^2|d\xi|\big)^2 + \int_M u^2|d\xi|^2.$$

We now assume that $8S\int_M u^2 < 1$ and we choose a family of functions ξ_R such that ξ_R is equal to 1 in $B(x_0, R)$ (the ball with radius R centered at some $x_0 \in M$), ξ_R is equal to 0 outside $B(x_0, 2R)$ and $|d\xi_R| \leq 2/R$. Letting R tend to infinity and using the fact that $\int_M u^2$ is finite, we obtain that $du = 0$. Since M has infinite volume, it follows that $u = 0$. □

4.2 Higher dimension, $M^n \looparrowright \mathbb{H}^n \times \mathbb{R}, n \geq 3$

Recall the formula for the Jacobi operator,

$$J_M := -\Delta_M - \big(|A_M|^2 + \widehat{\mathrm{Ric}}(N_M)\big)$$

where N_M is a unit normal field along M and A_M the second fundamental form of M with respect to N_M (Section 2.2).

Let $v_M := \hat{g}(N_M, \partial_t)$ be the vertical component of the unit normal vector N_M. A simple computation gives that $\widehat{\mathrm{Ric}}(N_M) = -(n-1)(1 - v_M^2)$. It follows that the Jacobi operator of M is given by

$$J_M := -\Delta_M + (n-1)(1 - v_M^2) - |A_M|^2. \tag{4.41}$$

We have the following theorem.

Theorem 4.3 *Let $M^n \looparrowright \mathbb{H}^n \times \mathbb{R}$ a complete oriented minimal immersion. Assume that M has* finite total curvature, i.e. $\int_M |A_M|^n\, d\mu_M < \infty$.

1. *For $n \geq 2$, the second fundamental form A_M tends to zero uniformly at infinity.*

2. *For $n \geq 3$, the Jacobi operator of the immersion has finite index and, more precisely, there exists a universal constant $C(n)$ such that*

$$\mathrm{Ind}(J_M) \leq C(n) \int_M |A_M|^n \, d\mu_M. \tag{4.42}$$

Remarks.
(i) The examples $\mathcal{M}_d$ prove that the converse statements in the previous theorems are not true in general, see Section 3.5.
(ii) Note that we state the second assertion of Theorem 4.3 only for $\dim(M) \geq 3$ (our proof does not apply in dimension 2, see [2]).

Proof. As in the proof of Theorem 4.1, the manifold M satisfies a Sobolev inequality of the form (4.37), namely

$$\|f\|_{n/(n-1)} \leq S(M)\|df\|_1 \quad \text{for all} \quad f \in C_0^1(M)$$

for some constant $S(M)$. Furthermore, the second fundamental form A_M satisfies the following Simons' equation (compare with (4.36)),

$$-\Delta|A_M| \leq |A_M|^3 + C(n)|A_M|,$$

for some constant $C(n)$ which only depends on the dimension (this follows from the expression of the term $\widehat{R}(A)$ as given in [25].

The de Giorgi-Moser-Nash technique applies (see [4], Theorem 4.1) and it follows that $|A_M|$ tends to zero uniformly at infinity.

Since $|v_M| \leq 1$, the operator J_M is bounded from below and essentially self-adjoint. Furthermore, its index is less than or equal to the index of the operator $-\Delta - |A_M|^2$ which is also bounded from below and essentially self-adjoint. The estimate (4.42) then follows by applying Theorem 39 in [2]. □

Remark. The preceding results can be generalized to minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}^k$ or $\mathbb{H}^n \times \mathbb{H}^k$.

5 Applications

In this section, we use the examples constructed in Section 3 as barriers to prove some general results on minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$. These results generalize results obtained in [19] for dimension 2. Similar results hold for H-hypersurfaces as well, see [19, 6].

Theorem 5.1 *Let $\Gamma \subset \mathbb{H}^n$ be a compact embedded hypersurface and consider two copies of Γ in different slices, $\Gamma_- = \Gamma \times \{-a\}$ and $\Gamma_+ = \Gamma \times \{a\} \subset \mathbb{H}^n \times \mathbb{R}$, for some $a > 0$. Assume that Γ is convex.*

Let $M \subset \mathbb{H}^n \times \mathbb{R}$ be a compact immersed minimal hypersurface such that $\partial M = \Gamma_- \cup \Gamma_+$. Then,

$$2a < \frac{\pi}{n-1} \quad \textit{(the height of the family of catenoids).}$$

Furthermore, if M is embedded,

1. *M is symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$.*

2. *The parts of M above and below the slice of symmetry are vertical graphs.*

3. *If Γ is a horizontal graph and symmetric with respect to a hyperbolic hyperplane P, then M is a horizontal graph and symmetric with respect to the vertical hyperplane $P \times \mathbb{R}$. In particular, if Γ is an $(n-1)$-sphere, then M is part of a catenoid.*

Proof. We reason ab absurdo. Suppose that the height of M is greater than or equal to $\frac{\pi}{n-1}$, that is $2a \geq \frac{\pi}{n-1}$. We recall that the height of the family of n-dimensional catenoids $\{\mathcal{C}_\rho,\ \rho \in (0,\infty)\}$ is bounded from above by $\frac{\pi}{n-1}$, but each catenoid $\mathcal{C}_\rho$ has height strictly less than $\frac{\pi}{n-1}$. Now as M is compact, there is a (hyperbolic) radius ρ_0 big enough such that M is strictly contained inside the vertical cylinder $\mathbb{M}_{\rho_0}$ of radius ρ_0 (where $\mathbb{M}_{\rho_0}$ is a cylinder over a $n-1$ sphere $\mathbb{S}_{\rho_0} \subset \mathbb{H}^2 \times \{0\}$ of radius ρ_0) containing M in its mean convex side. Recall that, by the geometry of the catenoids, the catenoid $\mathcal{C}_{\rho_0}$ whose distance to the t-axis is ρ_0 is contained in the closure of the non mean convex side of $\mathbb{M}_{\rho_0}$ touching $\mathbb{M}_{\rho_0}$ just along the $n-1$ sphere $\mathbb{S}_{\rho_0}$. Hence, M is strictly contained in the connected component of $\mathbb{H}^2 \times \mathbb{R} \setminus \mathcal{C}_{\rho_0}$ that contains the t axis of $\mathcal{C}_{\rho_0}$. Notice that the whole family of catenoids $\mathcal{C}_\rho$ is strictly contained in the slab of $\mathbb{H}^2 \times \mathbb{R}$ with boundary $\Gamma_- \cup \Gamma_+$. Starting from $\rho = \rho_0$, making $\rho \to 0$, that is moving the family of catenoids $\{\mathcal{C}_\rho,\ \rho \leq \rho_0\}$ towards M, we will find a first interior point of contact with some $\mathcal{C}_\rho$ and M, since the family of catenoid cannot touch the boundary of M. We arrive at a contradiction, by the the maximum principle. The proof of the first part of the statement is completed.

Now using the family of slices $\mathbb{H} \times \{t\} \cup \mathbb{H} \times \{-t\}$ coming from the infinity towards M we get, by maximum principle, that M is entirely contained in the closed slab whose boundary is the slices $\mathbb{H} \times \{a\} \cup \mathbb{H} \times \{-a\}$ and $(\mathbb{H} \times \{a\} \cup \mathbb{H} \times \{-a\}) \cap M = \partial M$.

In the same way, considering the family of vertical hyperplanes, we get that M is contained in the mean convex side of the vertical cylinder $\mathbb{M}_\Gamma$ over Γ and $\mathbb{M}_\Gamma \cap M = \partial M$. Now using Alexandrov Reflection Principle on the slices, moving the slices from $t = a$ towards $t = 0$, by vertical reflections, we get that the reflection of $M^+ = M \cap \{t \geqslant 0\}$, the part of M above $t = 0$, on the horizontal slice $t = 0$, is above $M^- = M \cap \{t \leqslant 0\}$. Moreover, we find that M^+ is a vertical graph. In the same way, moving the slices from $t = -a$ towards $t = 0$, doing vertical reflections, we get that the reflection of $M^- = M \cap \{t \leqslant 0\}$, the part of M above $t = 0$, on the horizontal slice $t = 0$, is below $M^+ = M \cap \{t \geq 0\}$. We conclude that $M^- = M^+$, hence both M^+ and M^- are vertical graphs and M is symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$. Therefore, the proof of the second part of the Statement is completed.

Let us assume now that $P \subset \mathbb{H} \times \{0\}$ is a hyperplane of symmetry of Γ. Consider the vertical hyperplane $\mathbb{P} = P \times \mathbb{R}$ and the family of hyperplanes $\mathbb{P}_t$ at signed distance t from $\mathbb{P}$ obtained from $\mathbb{P}$ by horizontal translations along an oriented geodesic γ orthogonal to $\mathbb{P}$ at the origin. Choosing $|t|$ big enough, we move the family $\mathbb{P}_t$ towards $\mathbb{P}$ (in the two sides of $\mathbb{H}^2 \times \mathbb{R} \setminus \mathbb{P}$), doing Alexandrov Reflection Principle on $\mathbb{P}_t$, taking into account that Γ is a horizontal graph and that the symmetric of ∂M on $\mathbb{P}_t$ stays on the slices $t = \pm a$, so that it does not touch the interior of M. We can argue as before to conclude that $\mathbb{P}$ is a hyperplane of symmetry of M. Of course, is Γ is rotationally symmetry then M is a minimal hypersurface of revolution. Henceforth, by the classification theorem, M is part of a catenoid. This completes the proof of the theorem. □

References

[1] Lucas Barbosa, Jonas Gomes, and Alexandre Silveira. Foliation of 3-dimensional space forms by surfaces with constant mean curvature. *Bol. Soc. Bras. Mat.*, 18:1–12, 1987.

[2] Pierre Bérard and Gérard Besson. Number of bound states and estimates on some geometric invariants. *J. Funct. Anal.*, 94:375–396, 1990.

[3] Pierre Bérard, Manfredo do Carmo, and Walcy Santos. The index of constant mean curvature surfaces in hyperbolic 3-space. *Math. Z.*, 224:313–326, 1997.

[4] Pierre Bérard, Manfredo do Carmo, and Walcy Santos. Complete hypersurfaces with constant mean curvature and finite total curvature. *Ann. Global Anal. Geom.*, 16:273–290, 1998.

[5] Pierre Bérard and Ricardo Sa Earp. Lindeloef's theorem for catenoids, revisited. *arXiv:0907.4294*, 2009.

[6] Pierre Bérard and Ricardo Sa Earp. H-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ and applications. *Mat. Contemp.*, 34, 2008.

[7] Manfredo do Carmo and C.K. Peng. Stable complete minimal hypersurfaces. In *Proceedings Beijing Symposium on Diff. Geom. and Diff. Eq., ed. by S.S. Chern and W.W. Tsun*, pages 1349–1358, 1980.

[8] Tobias H. Colding and William P. Minicozzi. Estimates for parametric elliptic integrands. *IMRN International Mathematics Research Notices*, 6:291–297, 2002.

[9] Pascal Collin and Harold Rosenberg. Construction of harmonic diffeomorphisms and minimal graphs. *arXiv:0701.547v1*, 2007.

[10] Levi Lopes de Lima and Wayne Rossman. On the index of constant mean curvature 1 surfaces in hyperbolic space. *Indiana Univ. Math. J.*, 47:685–723, 1998.

[11] Manfredo P. do Carmo and Alexandre M. Da Silveira. Index and total curvature of surfaces with constant mean curvature. *Proc. Amer. Math. Soc.*, 110:1009–1015, 1990.

[12] Doris Fischer-Colbrie. On complete minimal surfaces with finite Morse index in three-manifolds. *Invent. Math.*, 82:121–132, 1985.

[13] Doris Fischer-Colbrie and Richard Schoen. The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.*, 33:199–211, 1980.

[14] Laurent Hauswirth and Harold Rosenberg. Minimal surfaces of finite total curvature in $\mathbb{H} \times \mathbb{R}$. *Mat. Contemp.*, 31:65–80, 2006.

[15] David Hoffman and Joel Spruck. Sobolev and isoperimetric inequalities for riemannian submanifolds. *Comm. Pure Appl. Math.*, 27:715–727, 1974.

[16] David Hoffman and Joel Spruck. A correction to : Sobolev and isoperimetric inequalities for riemannian submanifolds. *Comm. Pure Appl. Math.*, 28:765–766, 1975.

[17] H. Blaine Lawson, Jr. *Lectures on minimal submanifolds. Vol. I*, volume 9 of *Mathematics Lecture Series*. Publish or Perish Inc., Wilmington, Del., second edition, 1980.

[18] Lorenz Lindeloef. Sur les limites entre lesquelles le caténoïde est une surface minimale. *Math. Annalen*, 2:160–166, 1870.

[19] Barbara Nelli, Ricardo Sa Earp, Walcy Santos, and Eric Toubiana. Uniqueness of H-surfaces in $\mathbb{H}^2 \times \mathbb{R}, |H| \leq 1/2$, with boundary one or two parallel horizontal circles. *Ann. Global Anal. Geom.*, 33:307–321, 2008.

[20] Ricardo Sa Earp. Parabolic and hyperbolic screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$. *Journal of the Australian Math. Soc.*, 85:113–143, 2008.

[21] Ricardo Sa Earp and Eric Toubiana. Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. *Illinois J. Math.*, 49:1323–1362, 2005.

[22] Ricardo Sa Earp and Eric Toubiana. An asymptotic theorem for minimal surfaces and existence results for minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$. *Math. Annalen*, 342:309–331, 2008.

[23] Ricardo Sa Earp and Eric Toubiana. Minimal graphs in $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{R}^{n+1}$. *arXiv:0908.4170*, 2009.

[24] Richard Schoen, Leon Simon, and Shing-Tung Yau. Curvature estimates for minimal hypersurfaces. *Acta Math.*, 134:275–288, 1975.

[25] James Simons. Minimal varieties in riemannian manifolds. *Ann. of Math.*, 88:62–105, 1968.

[26] Georges Valiron. *Équations fonctionnelles. Applications.* Masson, 1950.

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