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Minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, total curvature and index

Pierre Bérard and Ricardo Sa Earp

27th August 2008

Abstract

In this paper, we consider minimal hypersurfaces in the product space $\mathbb{H}^n \times \mathbb{R}$. We study the relation between the notions of finite total curvature and index of the stability operator. We study examples of rotation hypersurfaces and hypersurfaces invariant under hyperbolic translations; they serve as counter-examples and are useful barriers for many geometric problems.¹

1 Introduction

In this paper, we study complete minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, introducing suitable notions of finite total curvature and their relationship with the index of the Jacobi operator (stability operator). Roughly speaking, “finite total curvature implies finite index”, but the converse is not true (compare with [11, 10]). With this scope in mind, we construct and study the model minimal hypersurfaces that serve as examples and counter-examples for our results, *i.e.* minimal hypersurfaces of revolution and minimal hypersurfaces invariant by hyperbolic translations.

We have at first focused the case $n = 2$. However, we very fast came to realize that *all* the examples, geometric properties and results hold for hypersurfaces in the $(n + 1)$ -dimensional product space $\mathbb{H}^n \times \mathbb{R}$. For instance, we prove that the 1-parameter family of n -dimensional catenoids, say $\{\mathcal{C}_a, a > 0\}$, has the same behavior for all $n \geq 2$. For example, the vertical distance between the two asymptotic boundary

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components of any catenoid \mathcal{C}_a , *i.e.* the *height* of the family, is less than or equal to $\frac{\pi}{n-1}$. If $a \rightarrow \infty$, the height of \mathcal{C}_a increases with a and converges to $\frac{\pi}{n-1}$. For $n = 2$ this fact was established in [22, 20], with applications given in [25].

We also remark that the unit normal to the catenoids converges to a vertical unit field $\pm \frac{\partial}{\partial t}$ at infinity and that the two ends converge geometrically to two copies of the slice $\mathbb{H}^n \times \{0\}$, with finite total curvature.

In Section 5, we use the geometric classification of the catenoids to give symmetry and characterization results for compact minimal hypersurfaces whose boundary consists of two congruent convex hypersurfaces in parallel slices.

In Sections 2 and 4, we study the domains of stability of the catenoids and we deduce that their index is 1. More precisely, we show that the two (symmetric) spheres where the catenoid \mathcal{C}_a touches the *envelope* of the family determines a maximal connected weakly stable compact domain.

Let us now state our main results about total curvature and index in dimension 2.

Let $M \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be a complete minimal immersion of a surface M into $\mathbb{H}^2 \times \mathbb{R}$. We prove that if the extrinsic total curvature $\int_M |E_M| d\mu_M$ is finite then E_M , the extrinsic curvature of M , tends uniformly to zero at infinity. We remark that the catenoids have finite extrinsic total curvature but infinite intrinsic total curvature. As shown in [14], an interesting class is the class of complete minimal surfaces with finite total intrinsic curvature in $\mathbb{H}^2 \times \mathbb{R}$. Such surfaces behave like Scherk surfaces, *i.e.* each end converges geometrically to a vertical plane (the unit normal converge to a horizontal field). In Section 2, using a different method from [14], we prove that if the intrinsic total curvature $\int_M |K_M| d\mu_M$ is finite then K_M , the Gauss (intrinsic) curvature of M , tends uniformly to zero at infinity. We also prove that the index of the Jacobi operator of M is finite. On the other hand, there exists a complete (non-entire) minimal graph, say M_1 , over a half-plane $P_+ \subset \mathbb{H}^2 \times \{0\}$, with $\partial P_+ = \gamma$, such that M_1 takes infinite boundary value over γ and zero asymptotic boundary value over an arc C of the asymptotic boundary of $\mathbb{H}^2 \times \{0\}$. This graph is given by an explicit formula (see [21]) and belongs to a one parameter family of minimal surfaces invariant by hyperbolic translations. The geometry of this family has been studied in [25]. The asymptotic behavior of M_1 is as follows. Near γ the graph approaches the vertical plane $\gamma \times \mathbb{R}$ asymptotically and geometrically so that both its intrinsic and extrinsic curvatures go to 0. Near C the graph approaches the slice $\mathbb{H}^2 \times \{0\}$ asymptotically and geometrically so that its intrinsic curvature goes to -1 and its extrinsic curvature goes to 0. Of course, M_1 has infinite total curvature. In Section 3, we find a one parameter family of minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ with a similar geometric behavior.

Notice that by Gauss equation the assumption that K_M vanishes is equivalent to the assumption that both the third component v_M of the unit normal and the extrinsic

curvature E_M vanish. If the Gauss curvature K_M of M vanishes identically on an open subset then M is part of a vertical plane. More generally, we establish that if $v_M^2 \leq 1/5$, there exists a constant C such that if the extrinsic total curvature $\int_M |E_M| d\mu_M$ is less than C then M is a vertical plane.

In Section 4, we generalize part of the previous results to dimension n , $n \geq 3$. Let $M \looparrowright \mathbb{H}^n \times \mathbb{R}$ be a complete minimal immersion of a hypersurface M into $\mathbb{H}^n \times \mathbb{R}$. We prove that if the total curvature $\int_M |A_M|^n d\mu_M$ is finite then A_M , the second fundamental form of M , tends to zero uniformly at infinity and the index of the Jacobi operator of M is finite. More precisely, we give an upper estimate of the index in term of the total curvature. On the other hand, for $n \geq 2$, we have examples of complete minimal hypersurfaces invariant by hyperbolic translations which are stable and have with infinite total curvature, see Section 3.

Finally, we would like to mention that most of our results may be established if the ambient space is one of the product spaces $\mathbb{H}^n \times \mathbb{R}^k$ or $\mathbb{H}^n \times \mathbb{H}^k$. In [6], we study families of examples of rotation and translation H -hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ with some applications to symmetry and uniqueness results in the spirit of Section 3 and 5.

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2 Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, total curvature and index

In this section we investigate minimal surfaces in $\widehat{M} := \mathbb{H}^2 \times \mathbb{R}$ with the product metric \widehat{g} . For \mathbb{H}^2 , we consider the ball model $\mathbb{B} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ with the hyperbolic metric $g_{\mathbb{B}} := 4(1 - (x^2 + y^2))(dx^2 + dy^2)$. We begin with a description of the index of the family of catenoids using the explicit formulas given in [22]. We provide a summary of the main geometric data of the catenoids for completeness. Motivated by these examples we then state a general theorem.

2.1 Jacobi operator and the index

In this section we recall some definitions to be used later on.

Let $M^n \looparrowright \widehat{M}^{n+1}$ be an orientable minimal hypersurface in a Riemannian manifold \widehat{M} with metric \widehat{g} . Let N_M be a unit normal field along M and let A_M be the second

fundamental form of the immersion with respect to N_M . Let $\widehat{\text{Ric}}$ be the normalized Ricci curvature of \widehat{M} . The second variation of the volume functional gives rise to the *Jacobi operator* (or *stability operator*) J_M of M (see [26, 17, 9]),

$$(2.1) \quad J_M := -\Delta_M - (|A_M|^2 + \widehat{\text{Ric}}(N_M, N_M))$$

where Δ_M is the (non-positive) Laplacian on M (for the induced metric).

Given a compact regular domain Ω on the hypersurface M , we let $\text{Ind}(\Omega)$ denote the number of negative eigenvalues of J_M for the Dirichlet problem on Ω (this is well defined because Ω is compact). The *index* of M is defined to be

$$(2.2) \quad \text{Ind}(M) := \sup\{\text{Ind}(\Omega) \mid \Omega \Subset M\}.$$

2.2 Catenoids in $\mathbb{H}^2 \times \mathbb{R}$ and their index

Let M be a rotation surface about the \mathbb{R} axis in \widehat{M} . It can be parametrized by the hyperbolic distance ρ to the axis \mathbb{R} and by the angle φ . Such a parametrization can be written

$$(2.3) \quad X(\rho, \varphi) := (\tanh(\rho/2) \cos \varphi, \tanh(\rho/2) \sin \varphi, \lambda(\rho)).$$

We denote $\frac{d\lambda}{d\rho}$ by $\dot{\lambda}$ and $\frac{d}{d\rho}$ by ∂_ρ .

The meridian curves of minimal rotation surfaces about \mathbb{R} in \widehat{M} are given by the differential equation

$$(2.4) \quad \sinh(\rho) \dot{\lambda}(\rho) (1 + \dot{\lambda}^2(\rho))^{-1/2} = \sinh(a)$$

for some constant a (see [22]).

The case $a = 0$ corresponds to horizontal (totally geodesic) hyperbolic planes. Changing λ to $-\lambda$ if necessary, we may assume that $a > 0$. The equation shows that $\rho \geq a$ and the function λ is given, up to an additive constant (which corresponds to a vertical translation in \widehat{M}), by

$$(2.5) \quad \lambda(a, \rho) = \sinh(a) \int_a^\rho (\sinh^2(u) - \sinh^2(a))^{-1/2} du.$$

This gives a family of catenary-like curves C_a and corresponding catenoids \mathcal{C}_a in \widehat{M} . The above parametrization actually gives half-catenoids. It follows easily, using the results of [18, 19], that we have a whole catenoid after making a symmetry across the $\mathbb{H}^2 \times \{0\}$ plane.

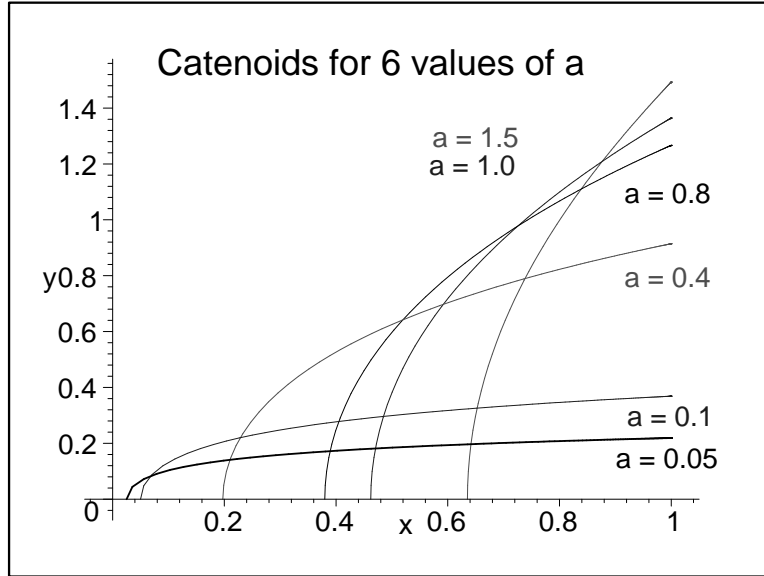


Figure 1: Generating curves for the catenoids

Consider the Jacobi operator $J_{\mathcal{C}_a} := -\Delta_{\mathcal{C}_a} - (|A_{\mathcal{C}_a}|^2 + \widehat{\text{Ric}}(N_{\mathcal{C}_a}, N_{\mathcal{C}_a}))$ (see Section 2.1). We define the *Jacobi potential* by $S_{\mathcal{C}_a} := |A_{\mathcal{C}_a}|^2 + \widehat{\text{Ric}}(N_{\mathcal{C}_a}, N_{\mathcal{C}_a})$. We investigate the compact domains of \mathcal{C}_a for which the operator $J_{\mathcal{C}_a}$ (with Dirichlet boundary conditions) is non-negative. More precisely, we prove the following theorem.

Theorem 2.1 *The bounded domain determined by the circles where the catenoid \mathcal{C}_a touches the envelope of the family is a maximal weakly stable domain. The index of the catenoid \mathcal{C}_a (i.e. of the Jacobi operator $J_{\mathcal{C}_a}$) is one.*

In order to prove this theorem, we need to relate the operator $J_{\mathcal{C}_a}$ with geometric properties of the catenoids. For this purpose, we recall some formulas established in [22].

It turns out that the expression of the Laplacien $\Delta_{\mathcal{C}_a}$ in the (ρ, φ) parameters is rather complicated. We will use another parametrization of \mathcal{C}_a , with parameters (s, φ) , where s is the arc length along the meridian curves, see [22] where these parameters were

introduced. More precisely, we introduce the functions $R(a, s)$, $U(a, s)$ and $\Lambda(a, s)$ defined as follows.

$$(2.6) \quad \begin{cases} \cosh(R(a, s)) & := \cosh(a) \cosh(s), \text{ and } R(a, s) > 0, \\ U(a, s) & := (\cosh^2(a) \cosh^2(s) - 1)^{1/2}, \\ \Lambda(a, s) & := \lambda(a, R(a, s)) = \sinh(a) \int_0^s U^{-1}(a, t) dt. \end{cases}$$

We introduce the parametrization Y of \mathcal{C}_a given by,

$$(2.7) \quad Y(a, s, \varphi) := (\tanh(R(a, s)/2) \cos \varphi, \tanh(R(a, s)/2) \sin \varphi, \Lambda(a, s)).$$

This function is at first defined for $s \geq 0$ as a parametrization of the upper part of the catenoid. It actually extends to a parametrization of the whole catenoid.

Properties 2.2 *The geometric data of the catenoid \mathcal{C}_a in the parameters (s, φ) are given as follows.*

1. *The Riemannian metric is given by*

$$G_{\mathcal{C}_a}(s, \varphi) = \begin{pmatrix} 1 & 0 \\ 0 & U^2(a, s) \end{pmatrix}.$$

2. *The Riemannian measure is given by*

$$d\mu_{\mathcal{C}_a}(s, \varphi) = U(a, s) ds d\varphi.$$

3. *The unit normal field can be chosen to be*

$$(2.8) \quad N_a(s, \varphi) = \left(-\frac{\sinh(a) \cos \varphi}{2 \cosh^2(R(a, s)/2) U(a, s)}, \dots \right. \\ \left. \dots -\frac{\sinh(a) \sin \varphi}{2 \cosh^2(R(a, s)/2) U(a, s)}, \frac{\cosh(a) \sinh(s)}{U(a, s)} \right).$$

4. *The intrinsic (sectional) curvature of \mathcal{C}_a for the induced metric does not depend on φ and is given by*

$$K_{\mathcal{C}_a}(s) = -\frac{\ddot{U}(a, s)}{U(a, s)} = -1 - \frac{\sinh^2(a)}{U^4(a, s)},$$

where $\dot{}$ denotes the derivative with respect to s .

5. The extrinsic curvature $E_{\mathcal{C}_a}$ of \mathcal{C}_a does not depend on φ and is given by

$$(2.9) \quad E_{\mathcal{C}_a}(s) = -\frac{1}{2}|A_{\mathcal{C}_a}|^2(s) = -\sinh^2(a)\frac{U^2(a, s) + 1}{U^4(a, s)}.$$

6. The Jacobi potential $S_{\mathcal{C}_a} := |A_{\mathcal{C}_a}|^2 + \widehat{\text{Ric}}(N_{\mathcal{C}_a}, N_{\mathcal{C}_a})$ does not depend on φ and is given by

$$S_{\mathcal{C}_a}(s) = \sinh^2(a)\frac{U^2(a, s) + 2}{U^4(a, s)}.$$

7. The Laplacian is given by

$$-\Delta_{\mathcal{C}_a} = -\left\{\frac{\partial^2}{\partial s^2} + \frac{\dot{U}}{U}\frac{\partial}{\partial s} + \frac{1}{U^2}\frac{\partial^2}{\partial \varphi^2}\right\}.$$

8. The Jacobi operator is given by

$$(2.10) \quad J_{\mathcal{C}_a} = -\left\{\frac{\partial^2}{\partial s^2} + \frac{\dot{U}}{U}\frac{\partial}{\partial s} + \frac{1}{U^2}\frac{\partial^2}{\partial \varphi^2} + \sinh^2(a)\frac{U^2 + 2}{U^4}\right\}.$$

9. The quadratic form associated with the Jacobi operator is given by

$$\mathcal{J}_{\mathcal{C}_a}(f) = \int_{\mathcal{C}_a} \left(\left(\frac{\partial f}{\partial s}\right)^2 + U^{-2}\left(\frac{\partial f}{\partial \varphi}\right)^2 - S_{\mathcal{C}_a}f^2 \right) U \, ds \, d\varphi$$

Properties 2.3 We have the following global properties.

1. The catenoids have infinite area and infinite total intrinsic curvature.
2. The total extrinsic curvature of the catenoids

$$T(E_{\mathcal{C}_a}) := \int_{\mathcal{C}_a} |E_{\mathcal{C}_a}| d\mu_{\mathcal{C}_a}$$

is finite and the extrinsic curvature goes to zero at infinity.

As a function of a , $T(E_{\mathcal{C}_a})$ increases from 4π ($a = 0$) to infinity, with $T(E_{\mathcal{C}_a})$ equivalent to $2\pi^2 \sinh(a)$ when a tends to infinity.

3. The integral of the Jacobi potential on the catenoids

$$T(S_{\mathcal{C}_a}) := \int_{\mathcal{C}_a} S_{\mathcal{C}_a} d\mu_{\mathcal{C}_a}$$

is finite.

As a function of a , $T(S_{\mathcal{C}_a})$ increases from 8π ($a = 0$) to infinity, with $T(S_{\mathcal{C}_a})$ equivalent to $\pi^2 \sinh(a)$ when a tends to infinity.

4. The unit normal to the catenoids tends to the vertical vector $(0, 0, 1)$ at infinity. Furthermore, the intrinsic curvature $K_{\mathcal{C}_a}$ tends to -1 at infinity.

We can now begin to study the Jacobi operator which we denote by J_a for simplicity. Recall that a *Jacobi field* on \mathcal{C}_a is a function u (corresponding to the normal field uN_M) which satisfies $J_a(u) = 0$.

The geometric context provides interesting Jacobi fields. More precisely, if N_a is the unit normal along the catenoid \mathcal{C}_a , and if \mathcal{K} is a Killing field on $(\widehat{M}, \widehat{g})$, then $u_{\mathcal{K}} := \widehat{g}(N_a, \mathcal{K})$ is a Jacobi field along \mathcal{C}_a , see [26].

Natural Jacobi fields on the catenoids \mathcal{C}_a

- Since vertical translations are isometries in \widehat{M} , the vertical field ∂_t is a Killing field and hence $v_a := \widehat{g}(N_a, \partial_t)$ is a Jacobi field along \mathcal{C}_a . In the (s, φ) parameters, the function v_a only depends on s and is given by

$$(2.11) \quad v_a(s) = \cosh(a) \frac{\sinh(s)}{U(a, s)}.$$

- Translations along a geodesic generate isometries in the hyperbolic plane (see [23]) and hence on \widehat{M} by taking isometries in each slice, with associated Killing (horizontal) field \mathcal{H} in \widehat{M} . We choose the ball model

$$\mathbb{B} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}, \quad g_{\mathbb{B}} := 4(1 - (x^2 + y^2))^{-2}(dx^2 + dy^2)$$

for the hyperbolic plane \mathbb{H}^2 and consider the one-parameter subgroup of isometries associated with the geodesic whose image is $] -1, 1[\times \{0, 0\}$ in $\mathbb{H}^2 \times \mathbb{R}$. This subgroup is given by

$$(2.12) \quad K_{\tau} : (z, t) \rightarrow \left(\frac{e^{\tau}(1+z) - (1-z)}{e^{\tau}(1+z) + (1-z)}, t \right),$$

where we have used the complex coordinate $z := x + iy$ in \mathbb{B} . The corresponding Killing field \mathcal{H} in $\mathbb{H}^2 \times \mathbb{R}$ is

$$(2.13) \quad \mathcal{H}(z, t) = \left(\frac{1}{2}(1 - z^2), 0 \right) \quad \text{or} \quad \mathcal{H}(x, y, t) = \left(\frac{1}{2}(1 - x^2 + y^2), -xy, 0 \right).$$

We denote the associated Jacobi field by $h_a := \widehat{g}(N_a, \mathcal{H})$. It is given in the (s, φ) parameters by

$$(2.14) \quad h_a(s, \varphi) = -\sinh(a) \frac{\cos \varphi}{U(a, s)}.$$

Observations. A *vertical graph* in \widehat{M} is a subset of the form $\{(x, u(x)) \mid x \in \Omega\}$ for some function $u : \Omega \rightarrow \mathbb{R}$, where Ω is a domain in \mathbb{H}^2 . The Killing field ∂_t is transversal to vertical graphs. A *horizontal graph* with respect to the geodesic $] - 1, 1[\times \{0, 0\}$ is a subset G_h of \widehat{M} with the property that in each slice $\mathbb{H}^2 \times \{a\}$ the equidistant curves to the geodesic $] - 1, 1[\times \{0, a\}$ intersect G_h transversally and at most at one point. The Killing field \mathcal{H} is transverse to these horizontal graphs. Of course, there is a natural notion of horizontal graph with respect to any given horizontal geodesic γ contained in the slice $\mathbb{H}^2 \times \{0\}$.

• The family \mathcal{C}_a of minimal catenoids gives rise to a variation vector field $\mathcal{E}(a, s, \varphi) := \frac{\partial}{\partial a} Y(a, s, \varphi)$ and the function $\widehat{g}(\mathcal{E}(a, s, \varphi), N_a)$ is yet another Jacobi field (this is because all the surfaces in the family are minimal, compare with the classical case of variations of geodesics; it can be shown that this function is precisely the derivative of the mean curvature when a varies, see [17]). This function actually only depends on s and we have

$$(2.15) \quad e_a(s) = \sinh^2(a) \frac{\cosh(s)}{U^2(a, s)} - \cosh^2(a) \frac{\sinh(s)}{U(a, s)} \int_0^s \frac{\sinh^2(t)}{U^3(a, t)} dt.$$

Remark. One can also check that the above three functions are Jacobi fields by a direct (easy but lengthy) computation.

Lemma 2.4 *The function e_a is smooth and even. It satisfies $e_a(0) = 1$ and $\dot{e}_a(0) = 0$ and it has exactly one positive zero which we denote by $z(a)$.*

Proof. The first two assertions are clear. For the last assertion, take the expression (2.15) into account and notice that, for a fixed value of a , the function $\sinh^2(a) \frac{\cosh(s)}{U(a, s)}$ decreases from $\sinh(a)$ to $\tanh^2(a)$ whereas the function $\cosh^2(a) \sinh(s) \int_0^s \frac{\sinh^2(t)}{U^3(a, t)} dt$ increases from 0 to infinity when s increases from 0 to infinity. \square

Geometric interpretation of $e_a(s)$

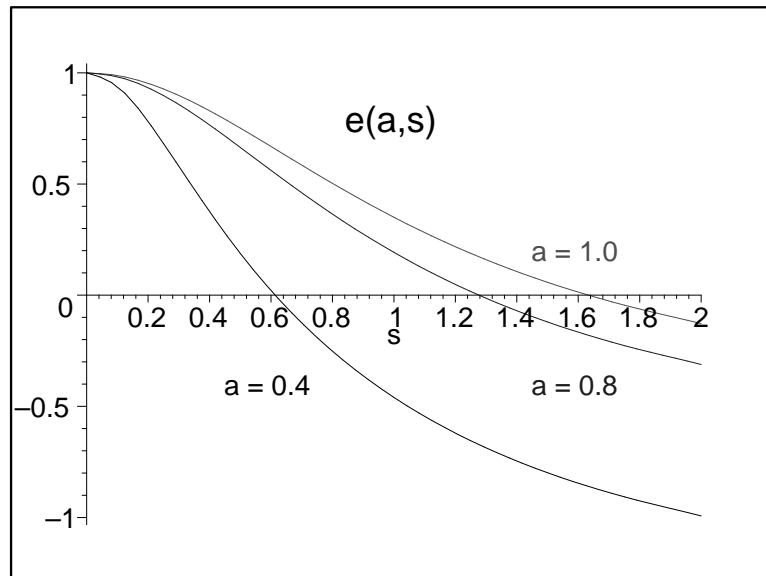
We consider the catenary-like curves

$$(2.16) \quad Y_1(a, s) := \left(\tanh(R(a, s)/2), \Lambda(a, s) \right),$$

where $R(a, s)$ and $\Lambda(a, s)$ are given by (2.6).

The *envelope* of this family of curves is given by the equation

$$\text{Det} \left(\partial_a Y_1(a, s), \partial_s Y_1(a, s) \right) = 0.$$

Figure 2: The Jacobi field e_a

Using (2.5) and (2.6), one finds that

$$\Lambda(a, s) = \sinh(a) \int_1^s (v^2 - 1)^{-1/2} (\cosh^2(a)v^2 - 1)^{-1/2} dv$$

and the previous envelope condition is seen to be equivalent to $e_a(s) = 0$. This means that the envelope of the family C_a is given by $Y_1(a, z(a))$.

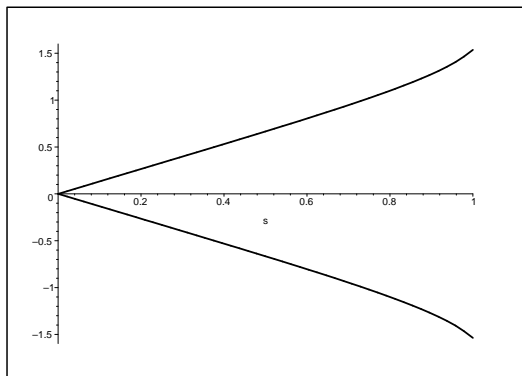


Figure 3: Envelope of the catenaries

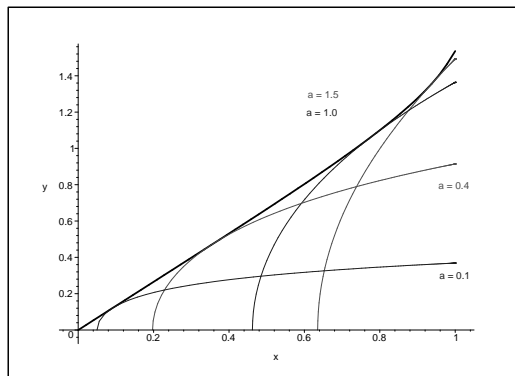


Figure 4: Catenaries and their envelope

We now consider the following parts of the catenoids C_a .

$$(2.17) \quad \mathcal{C}_a((s_1, s_2), (\varphi_1, \varphi_2)) := \{Y(a, s, \varphi) \mid s \in (s_1, s_2), \varphi \in (\varphi_1, \varphi_2)\}.$$

We have the following proposition.

Proposition 2.5 *With the above notations.*

1. The domain $\mathcal{C}_a([0, \infty[, [0, 2\pi])$ is weakly stable.
2. The domains $\mathcal{C}_a([s_1, \infty), [0, 2\pi])$ are stable for all $s_1 > 0$.
3. The domain $\mathcal{C}_a([-z(a), z(a)], [0, 2\pi])$ is weakly stable.
4. The domains $\mathcal{C}_a([-S, S], [0, 2\pi])$ are stable for $0 < S < z(a)$ and unstable for $S > z(a)$.
5. The domains $\mathcal{C}_a(]-\infty, \infty[, [\phi - \pi/2, \phi + \pi/2,])$ are weakly stable for any ϕ .

For the proof of this proposition, recall the following classical result which can be found in Glazman [13] or Fischer-Colbrie – Schoen [12].

Theorem 2.6 *Let M be a complete non compact Riemannian manifold and let q be a continuous, real valued function on M . For any bounded domain Ω in M , the first Dirichlet eigenvalue of the operator $-\Delta + q$ (where Δ is the non-positive Laplacian on M) is non-negative if and only if there exists a positive function u on Ω which satisfies $(-\Delta + q)u \geq 0$ on Ω .*

The proof of Proposition 2.5 follows easily from the previous theorem making use of the Jacobi fields v_a, h_a and e_a we defined above. \square

Remarks. The domain $\mathcal{C}_a([0, \infty[, [0, 2\pi])$ is a vertical graph and, for all ϕ , the domains $\mathcal{C}_a(]-\infty, \infty[, [\phi - \pi/2, \phi + \pi/2,])$ are horizontal graphs so that Assertions 1 and 5 also follow from classical results on the stability of graphs (see the observations made after equation (2.14)).

Proof of Theorem 2.1. In order to prove this theorem, we only need consider an exhaustion of the catenoid \mathcal{C}_a by compact domains. For $S > 0$, we consider the family of domains $\mathcal{C}_{a,S} := \mathcal{C}_a([-S, S], [0, 2\pi])$. Using the function e_a and the fact that $\mathcal{C}_{a,z(a)}$ is weakly stable, we may conclude that $\mathcal{C}_{a,S}$ is stable for $S < z(a)$ and that $\mathcal{C}_{a,S}$ has index at least one if $S > z(a)$. This proves the first statement of the theorem.

Let f be an eigenfunction of J_a with Dirichlet boundary conditions in $\mathcal{C}_{a,S}$, associated with an eigenvalue λ . Decompose $f(s, \varphi)$ into a Fourier series in φ , with Fourier coefficients $f_p(s)$. These coefficients (when different from zero) are eigenfunctions of a family (indexed by $p \in \mathbb{Z}$) of Sturm-Liouville operators on the interval $[-S, S]$, namely

$$\ddot{f}_p(s) + \frac{p^2}{U^2(s)} f_p(s) - \lambda f_p(s) = 0,$$

with Dirichlet boundary conditions $f_p(-S) = f_p(S) = 0$.

It follows from the fifth assertion of Proposition 2.5 that eigenfunctions of J_{C_a} corresponding to negative eigenvalues have vanishing coefficients f_p for $p \neq 0$ i.e. they only depend on the s variable.

If the index of $\mathcal{C}_{a,S}$ were at least 2 for some $S > 0$, a radial eigenfunction corresponding to the second negative eigenvalue would have two nodal domains and hence one zero in $] - S, S[$. We would find an unstable compact domain in $\mathcal{C}_a([0, \infty[, [0, 2\pi])$, a contradiction with the first assertion in Proposition 2.5. \square

Note. See also Theorem 4.1 for a higher dimensional result.

2.3 Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature

For oriented minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ we have the following general theorem in which we consider two possible notions of total curvature.

Theorem 2.7 *Let $M \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be an oriented complete minimal immersion with unit normal field N_M . Let $v_M := \hat{g}(N_M, \partial_t)$ be the vertical component of N_M , let A_M be the second fundamental form of M and let K_M be the intrinsic curvature of M .*

1. *If the integral $\int_M |A_M|^2 d\mu_M$ is finite then A_M tends to zero uniformly at infinity.*
2. *If the integral $\int_M |K_M| d\mu_M$ is finite then A_M, v_M and K_M tend to zero uniformly at infinity.*
3. *If the integral $\int_M |K_M| d\mu_M$ is finite then the Jacobi operator of M has finite index.*
4. *Assume that $5v_M^2 \leq 1$. Then there exists a universal constant C such that if the integral $\int_M |A_M|^2 d\mu_M$ is less than C then M is a vertical plane.*
5. *Assume that the integral $\int_M |A_M|^2 d\mu_M$ is finite and that there exists a compact set $\Omega \subset M$ and a positive constant c such that $v_M^2 \leq 1 - c < 1$ on $M \setminus \Omega$. Then the Jacobi operator of the immersion has finite index.*

Remarks.

1. There are examples of surfaces which are invariant under hyperbolic translations and such that A_M tends to zero at infinity, but with $\int_M |A_M|^2 d\mu_M$ infinite (see [22] and Section 3.4).

2. For complete orientable minimal surfaces in \mathbb{R}^3 , finiteness of the index is equivalent to finiteness of the intrinsic curvature (see [11]). No such statement can hold in $\mathbb{H}^2 \times \mathbb{R}$. Indeed, there are examples of complete minimal surfaces which are stable and invariant under a group of hyperbolic translations (see [22] and Section 3.4) and hence whose total curvatures are infinite, so that the converse of Assertion 3 is false.
3. The assumption $\int_M |K_M| d\mu_M$ finite has also been considered by L. Hauswirth and H. Rosenberg ([14]) who deduce from this hypothesis (using different methods) that v_M tends to zero at infinity and that M is conformally a compact surface with a finite number of punctures.
4. Assertion 4 generalizes the following facts : (i) A minimal surface whose intrinsic curvature K_M is zero is part of a vertical plane $\gamma \times \mathbb{R}$ (where γ is a geodesic in \mathbb{H}^2). Indeed, we have $K_M = -\frac{1}{2}|A_M|^2 - v_M^2$ and hence M is totally geodesic with horizontal normal vector. (ii) A complete minimal surface whose total intrinsic curvature is less than 2π is a vertical plane (see [14]).
5. We do not know whether the sole assumption $\int_M |A_M|^2 d\mu_M$ finite is sufficient to insure the finiteness of the index of the Jacobi operator of M .
6. We shall partly generalize this theorem for higher dimensions in Section 4.

Proof of Theorem 2.7.

Fact 1. The function $|A_M|$ satisfies a non-linear elliptic inequality.

Indeed, the second fundamental form A_M of the minimal immersion $M \looparrowright \widehat{M}$ satisfies Simons' equation whose general form (see [26]) is

$$D^2 A = -(A \circ \widetilde{A} + \widetilde{A} \circ A) + \widehat{R}(A) + \widehat{R}'.$$

Under our assumption, the manifold \widehat{M} is locally symmetric so that the term \widehat{R}' which involves the covariant derivative of \widehat{R} is zero. Since M has codimension 1, the right-hand side simplifies. Indeed, one has $(A \circ \widetilde{A} + \widetilde{A} \circ A) = |A|^2 A$ (see [26]) when the codimension is one.

It follows easily from the explicit formula for $\widehat{R}(A)$ (see [26]) that we have (in dimension 2)

$$|\langle \widehat{R}(A), A \rangle| \leq 4|A|^2.$$

It then follows as in [26] that the norm of the second fundamental form satisfies the following differential inequality (in the sense of distributions)

$$(2.18) \quad -\Delta|A| \leq |A|^3 + 4|A|.$$

In the case at hand a careful computation shows that one can give a precise expression for the term $\langle \widehat{R}(A), A \rangle$, namely,

$$(2.19) \quad \widehat{R}(A) = (5\widehat{K}_M + 1)A$$

where \widehat{K}_M is the sectional curvature of \widehat{M} for the 2-plane tangent to M .

Fact 2. The surface M satisfies a Sobolev inequality.

Indeed, the ambient space $\widehat{M} := \mathbb{H}^2 \times \mathbb{R}$ has non-positive sectional curvature and is simply connected. It follows that its injectivity radius is infinite. It follows from [15, 16] that M satisfies the following Sobolev inequality, there exists a positive constant $S(M)$ such that

$$(2.20) \quad \|f\|_{n/(n-1)} \leq S(M)\|df\|_1 \quad \text{for all } f \in C_0^1(M).$$

Fact 3. General analytic inequalities.

We now state general analytic results which are adapted from [4, 5], see also [7]. We state them for all dimensions since we will use them again in Section 4. We only sketch the proofs.

Notations. Let us first fix some notations. Let M be a complete non-compact Riemannian manifold with dimension $n \geq 2$. Fix a point $x_0 \in M$ and let $d(x)$ denote the Riemannian distance from the point $x \in M$ to the point x_0 .

For $S > R \geq 0$, define the sets

$$(2.21) \quad \begin{cases} B(R) & := \{x \in M \mid d(x) < R\} \\ E(R) & := \{x \in M \mid d(x) > R\} \\ A(R, S) & := \{x \in M \mid R < d(x) < S\} \end{cases}$$

Theorem 2.8 *Let M be a complete non-compact Riemannian manifold with dimension n . Assume that there exists a positive constant $S(M)$ such that M satisfies the Sobolev inequality*

$$(2.22) \quad \|f\|_{n/(n-1)} \leq S(M)\|df\|_1 \quad \text{for all } f \in C_0^1(M).$$

Let u be a non-negative function on M . Assume that $u \in C^0(M)$, $u \in H_{\text{loc}}^1(M)$ and

$$(2.23) \quad u \in L^n(M) \quad \text{with respect to the Riemannian measure } \mu_M.$$

Assume furthermore that there exist non-negative constants a, b such that

$$(2.24) \quad -\Delta u \leq au^3 + bu.$$

Then there exist positive constants $A_i = A_i(S(M), a, b, n)$ for $i \in \{1, 2, 3, 4\}$ and a radius R_1 defined by the condition $A_1 \int_{E(R_1)} u^n d\mu_M \leq 1$ such that, for any $R \geq R_1$,

$$\|u\|_{\infty, E(2R)} := \sup\{u(x) \mid x \in E(2R)\} \leq A_2 \left(\int_{E(R)} u^n d\mu_M \right)^{1/n}.$$

Furthermore, if $(\int_M u^n d\mu_M)^{1/n} \leq A_3$, then

$$\|u\|_{\infty} := \sup\{u(x) \mid x \in M\} \leq A_4 \left(\int_M u^n d\mu_M \right)^{1/n}.$$

In particular, under the assumption that $\int_M u^n d\mu_M$ finite, u tends to zero uniformly at infinity on M .

Sketch the proof of Theorem 2.8

- The general idea of the proof is to use the non-linear inequality (2.24) and the Sobolev inequality (2.22) to obtain general inequalities on L^p norms of u that will allow us to apply the de Giorgi-Moser-Nash iteration method to get an estimate on the L^∞ norm.

- Take any $\xi \in C_0^\infty(M, \mathbb{R}_+)$, multiply (2.24) by $\xi^2 u^{2k-1}$ (for any $k \geq 1$) and integrate on M (wrt the Riemannian measure $d\mu_M$) to get

$$\frac{2k-1}{k^2} \int \xi^2 |du^k|^2 \leq a \int \xi^2 u^{2k+2} + b \int \xi^2 u^{2k} + \frac{2}{k} \int u^k |d\xi| \xi |du^k|,$$

and hence, for all $k \geq 1$ and all $\xi \in C_0^\infty(M, \mathbb{R}_+)$,

$$(2.25) \quad \frac{2k-1.5}{k^2} \int \xi^2 |du^k|^2 \leq a \int \xi^2 u^{2k+2} + b \int \xi^2 u^{2k} + 2 \int |d\xi|^2 u^{2k}.$$

Applying the Sobolev inequality (2.22) to $f = \xi^2 u^{2k}$ we find

$$(2.26) \quad \|\xi u^k\|_{2n/(n-1)}^2 \leq S(M) (2\|\xi u^k\|_2^2 + \|\xi |du^k|\|_2^2 + \|d\xi|u^k\|_2^2)$$

and, using (2.25),

$$\begin{aligned} \|\xi u^k\|_{2n/(n-1)}^2 &\leq S(M) \{2\|\xi u^k\|_2^2 + \dots \\ &\quad \dots \|\xi u^k\|_2^2 + \frac{k^2}{2k-1.5} [a\|\xi u^{k+1}\|_2^2 + b\|\xi u^k\|_2^2 + 2\|d\xi|u^k\|_2^2]\}. \end{aligned}$$

We finally obtain

$$(2.27) \quad \|\xi u^k\|_{2n/(n-1)}^2 \leq C_1 k \{ \|\xi u^{k+1}\|_2^2 + \|\xi u^k\|_2^2 + \|d\xi|u^k\|_2^2 \}.$$

for some constant $C_1 = C_1(S(M), a, b, n)$.

The general idea is that from some control on $\|u\|_{2k}$ we will get a control on $\|u\|_{2k \cdot \frac{n}{n-1}}$ and then, by iteration starting from $k = n/2$, control on $\|u\|_\infty$.

• In order to proceed with the above idea, we need some further control on u and in particular on u^{n+2} . For this purpose, we use the following lemma whose proof can be found in [5] (Proposition 3.1 and Corollary 3.2 page 278).

Lemma 2.9 *Under the assumptions of Theorem 2.8, there exist positive constants $A_i, i \in \{5, 6, 7, 8\}$, and a positive radius R_2 such that*

1. *The constants A_i only depend on $S(M), a, b$ and n ,*
2. *The radius R_2 is defined by $A_5 \int_{E(R_2)} u^n d\mu_M \leq 1$ and, for all $R \geq R_2$,*

$$(a) \quad \left(\int_{E(R+2)} u^{n+2} d\mu_M \right)^{1/(n+2)} \leq A_6 \left(\int_{E(R)} u^n d\mu_M \right)^{1/n} \text{ for } n \geq 2,$$

$$(b) \quad \int_{E(R+2)} |du^{n/2}| d\mu_M \leq A_6 \int_{E(R)} u^n d\mu_M \text{ for } n \geq 2,$$

$$(c) \quad \left(\int_{E(R+2)} u^{n^2/(n-2)} d\mu_M \right)^{(n-2)/n^2} \leq A_6 \left(\int_{E(R)} u^n d\mu_M \right)^{1/n} \text{ for } n \geq 3.$$

In particular, $u \in L^{n+2}(M)$ and $|du^{n/2}| \in L^2(M)$ for $n \geq 2$ and $u \in L^{n^2/(n-2)}$ for $n \geq 3$. Furthermore, if $\left(\int_M u^n d\mu_M \right)^{1/n} \leq A_7$, then the above inequalities hold with $E(R)$ and $E(R+2)$ replaced by M and A_6 replaced by A_8 .

In order to apply the de Giorgi-Moser-Nash iteration method, we need the following lemma (see [5] Lemma 4.2 page 282). Note that the factor involving k in the right-hand side of (2.28) comes from an interpolation argument to get rid of the term involving $\|\xi u^{k+1}\|_2$ in (2.27).

Lemma 2.10 *Define $\alpha = 2, \beta = 1$ when $n = 2$ and $\alpha = n/(n-2), \beta = (n-2)^2$ when $n \geq 3$. Under the assumptions of Theorem 2.8, there exist positive constants $A_i, i \in \{9, 10\}$ and a positive radius $R_3 \geq R_2$ such that for all $R \geq R_3$, for all $k \geq 1$ and all $\xi \in C_0^\infty(E(R), \mathbb{R}_+)$,*

$$(2.28) \quad \|\xi u^k\|_{2\alpha}^2 \leq A_8 k \left\{ (1 + k^{\alpha\beta}) \|\xi u^k\|_2^2 + \|d\xi|u^k\|_2^2 \right\}.$$

Furthermore, if $\int_M u^n d\mu_M < A_{10}$ then the preceding inequality holds for all $\xi \in C_0^\infty(M, \mathbb{R}_+)$.

In order to finish the proof of Theorem 2.8, we apply the de Giorgi-Moser-Nash method. Take R_3 as in Lemma 2.10 and choose $S > 2R > R > R_3$. Define the sequences

$$R_0 := R, R_i := R_{i-1} + 2^{-i}R = 2R(1 - 2^{-(i+1)})$$

and

$$S_0 := 2S, S_i := S_{i-1} - 2^{-i}S = 2S - S(1 - 2^{-(i+1)}).$$

Define $A_i := A(R_i, S_i)$ and choose $\xi_i \in C_0^\infty(M)$ such that $0 \leq \xi_i \leq 1, |d\xi_i| \leq 5^i$ and $\text{Supp}(\xi_i) \subset A_i, \xi_i|_{A_{i+1}} = 1$. Let $\chi_i = \mathbf{1}_{\text{Supp}(\xi_i)}$.

Lemma 2.10 gives

$$\|\xi_i u^k\|_{2\alpha}^2 \leq A_8 k \left\{ (1 + k^{\alpha\beta}) \|\chi_i u^k\|_2^2 + 25^i \|\xi_i u^k\|_2^2 \right\}$$

and

$$\|\xi_i u^k\|_{2\alpha}^2 \leq A_8 k (26^i + k^{\alpha\beta}) \|\chi_i u^k\|_2^2.$$

Take $k = k_i := \gamma\alpha^i$ for some $\gamma > 0$ and $i \in \mathbb{N}$. We obtain

$$\|u\|_{2k_{i+1}, A_{i+1}} \leq \left\{ A_8 \gamma \alpha^i [26^i + (\gamma\alpha^i)^{\alpha\beta}] \right\}^{1/(2\gamma\alpha^i)} \|u\|_{2k_i, A_i}$$

and finally

$$\|u\|_{2k_{i+1}, A_{i+1}} \leq B_1 e^{B_2 i \alpha^{-i}} \|u\|_{2k_i, A_i}.$$

Since the series $\sum_{i=0}^\infty i \alpha^{-i}$ converges, we conclude that

$$\|u\|_{\infty, A_{2R, S}} \leq B_4 \|u\|_{2\gamma, A_{R, 2S}}.$$

Taking $\gamma = n/2$, we have the estimate of Theorem 2.8. □

Proof of Theorem 2.7 continued.

Assertion 1. Under the assumption $\int_M |A_M|^2 d\mu_M < \infty$ i.e. $\int_M u^2 d\mu_M < \infty$, we can apply Theorem 2.8 and conclude that A_M tends to zero at infinity on M .

Assertion 2. As in Equation (2.11), the function v_M is a Jacobi field on M . The following formula for the Ricci curvature follows from a simple computation,

$$\widehat{\text{Ric}}(N_M, N_M) = -(1 - \hat{g}(N_M, \partial_t)^2) = -(1 - v_M^2).$$

It follows that v_M satisfies the equation

$$-\Delta v_M = v_M^3 + (|A_M|^2 - 1)v_M,$$

and the same equation holds for the function $|v_M|$ as well.

Under the assumption of Assertion 2, $\int_M |K_M| d\mu_M$ finite, we have that both integrals $\int_M |A_M|^2 d\mu_M$ and $\int_M v_M^2 d\mu_M$ are finite. We can again apply Theorem 2.8 and conclude that both $|A_M|$ and v_M (and hence K_M) tend to zero at infinity.

Assertion 3. Assertion 2 implies that the operator $J_M := -\Delta_M + 1 - |A_M|^2 - v_M^2$ is bounded from below, essentially self-adjoint and that its essential spectrum lies above 1 (indeed the essential spectrum only depends on the behaviour of the operator at infinity, see [4], Section 2). Because the eigenvalues below the essential spectrum can only accumulate at $-\infty$ or at the bottom of the essential spectrum, it follows that J_M has finite index.

Assertion 4. Let $u := |A|$. Recall that $\widehat{K} = -v_M^2$. According to Equation (2.19) and under the assumption that $5v_M^2 \leq 1$, Simons' inequality can be written as

$$|du|^2 - u\Delta u \leq u^4 - (5\widehat{K}_M + 1)u^2 \leq u^4$$

(see [1] for an explanation why the term $|du|^2$ appears). Using the Sobolev inequality $\|f_1\|_2^2 \leq C_S \|df_1\|_1^2$ and taking $f_1 = f^2$ for some smooth f with compact support, we have that

$$\int_M f^4 \leq 4C_S \int_M f^2 \int_M |df|^2.$$

We now choose $\lambda \in C_0^1(M)$, multiply Simons' equation by $\lambda^2 u^{2k-2}$ and we integrate to obtain the following inequalities,

$$\int_M \lambda^2 u^{2k-2} |du|^2 + \int_M \lambda^2 u^{2k-1} (-\Delta u) \leq \int_M \lambda^2 u^{2k+2}$$

and then

$$2k^{-1} \int_M \lambda^2 |du^k|^2 + 2 \int_M \lambda u^{2k-1} \langle d\lambda, du \rangle \leq \int_M \lambda^2 u^{2k+2}$$

and, using Cauchy-Schwarz inequality with some parameter α small,

$$(2k - \alpha)k^{-2} \int_M \lambda^2 |du^k|^2 \leq \int_M \lambda^2 u^{2k+2} + \alpha^{-1} \int_M |d\lambda|^2 u^{2k}$$

for all $\lambda \in C_0^1(M)$ and $k \geq 1$.

We now apply the above Sobolev inequality to $f = \lambda u$ and choose $k = 1$ to get

$$\int_M \lambda^4 u^4 \leq 8C_S \int_M \lambda^2 u^2 \left(\int_M \lambda^2 |du|^2 + \int_M |d\lambda|^2 u^2 \right)$$

and

$$\int_M \lambda^2 |du|^2 \leq (2 - \alpha)^{-1} \left[\int_M \lambda^2 u^4 + \alpha^{-1} \int_M |d\lambda|^2 u^2 \right]$$

and then

$$(1 - 8C_S(2 - \alpha)^{-1} \int_M \lambda^2 u^2) \int_M \lambda^2 u^4 \leq 8C_S(1 + 1/(\alpha(2 - \alpha))) \int_M \lambda^2 u^2 \int_M |d\lambda|^2 u^2.$$

We can now choose $\lambda(x) = \zeta(d(x_0, x)/R)$ where $d(x_0, x)$ is the distance from x to x_0 on M , $R > 0$ a real parameter and $\zeta \in C_0^1(\mathbb{R}, [0, 1])$, $\zeta \equiv 1$ on $[0, 1]$, 0 outside $[0, 2]$ and $|\zeta'| \leq 2$ and we let R tend to infinity. The right-hand side of the above inequality tends to 0 when R tends to infinity. If $1 - 8C_S(2 - \alpha)^{-1} \int_M u^2 > 0$ then $1 - 8C_S(2 - \alpha)^{-1} \int_M \lambda^2 u^2 > 0$ also and it follows that $\int_M u^4 = 0$. Hence M is totally geodesic and since $5v_M^2 \leq 1$, it must be a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in \mathbb{H}^2 .

Assertion 5. The assumptions imply that $|A_M|$ tends to zero uniformly at infinity (*cf.* Assertion 1) and hence that the essential spectrum of the Jacobi operator is above the given number c . As in the proof of Assertion 3, it follows that it has finite index. \square

3 Examples of minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

In this section we give examples of minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$. Such examples are useful for instance because they provide barriers for existence theorems, see Section 5 and [25, 6].

3.1 Rotation hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

We first describe rotation hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$. We consider the ball model \mathbb{B} (the unit ball in \mathbb{R}^n) for \mathbb{H}^n , with metric

$$(3.29) \quad g_{\mathbb{B}} := \left(\frac{2}{1 - |x|^2} \right)^2 (dx_1^2 + \cdots + dx_n^2).$$

We consider $\widehat{M} := \mathbb{H}^n \times \mathbb{R}$ with the product metric

$$(3.30) \quad \widehat{g} := g_{\mathbb{B}} + dt^2.$$

In the vertical plane $P := \{(x_1, \dots, x_n, t) \in \widehat{M} \mid x_1 = \cdots = x_{n-1} = 0\}$ we consider the generating curve $(\tanh(\rho/2), \lambda(\rho))$ for positive values of ρ , the hyperbolic distance to the axis \mathbb{R} .

We define a *rotation hypersurface* $M \looparrowright \widehat{M}$ by the “parametrization”

$$(3.31) \quad \begin{cases} X & : \mathbb{R}_+ \times S^{n-1} \rightarrow \widehat{M} \\ X & : (\rho, \xi) \mapsto (\tanh(\rho/2)\xi, \lambda(\rho)) \end{cases}$$

where $\tanh(\rho/2)\xi = (\tanh(\rho/2)\xi_1, \dots, \tanh(\rho/2)\xi_n) \in \mathbb{B}$, for $\xi = (\xi_1, \dots, \xi_n) \in S^{n-1}$ a point in the unit sphere.

Let

$$T(\rho, \xi) := T_{\rho, \xi} X(\partial_\rho) = \left(\frac{\xi}{2 \cosh^2(\rho/2)}, \dot{\lambda}(\rho) \right)$$

and

$$U(\rho, \xi, u) := T_{\rho, \xi} X(u) = (\tanh(\rho/2)u, 0)$$

where $u \in T_\xi S^{n-1}$.

The metric \widehat{g} is given in the parametrization X by the formula (see (3.30))

$$(3.32) \quad \widehat{g}_{X(\rho, \xi)} = 4 \cosh^4(\rho/2) g_E + dt^2$$

where g_E is the Euclidean metric on \mathbb{B} and we have the relations

$$\begin{aligned}
X^*\widehat{g}(\partial_\rho, \partial_\rho) &= \widehat{g}(T, T) &= 1 + \dot{\lambda}^2(\rho) \\
X^*\widehat{g}(\partial_\rho, u) &= \widehat{g}(T, U) &= 0 \\
X^*\widehat{g}(u, u) &= \widehat{g}(U, U) &= \sinh^2(\rho)
\end{aligned}$$

where $u \in T_\xi S^{n-1}$. This means that

$$(3.33) \quad X^*\widehat{g} = (1 + \dot{\lambda}^2(\rho))d\rho^2 + \sinh^2(\rho)g_S$$

where g_S is the canonical metric on S^{n-1} .

Let g_M be the metric induced by \widehat{g} on M from the immersion $M \looparrowright \widehat{M}$.

Properties 3.1 *On $\mathbb{R} \times S^{n-1}$, and in the parametrization X ,*

1. *the metric g_M is given by*

$$(3.34) \quad g_M = (1 + \dot{\lambda}^2(\rho))d\rho^2 + \sinh^2(\rho)g_S;$$

2. *the Riemannian measure $d\mu_M$ is given by*

$$(3.35) \quad d\mu_M = (1 + \dot{\lambda}^2(\rho))^{1/2} \sinh^{n-1}(\rho) d\rho d\mu_S$$

where $d\mu_S$ is the canonical measure on the sphere ;

3. *the unit normal field to the immersion can be chosen to be*

$$(3.36) \quad N_M(\rho, \xi) = \left(-\frac{\dot{\lambda}(\rho)(1 + \dot{\lambda}^2(\rho))^{-1/2}}{2 \cosh^2(\rho/2)} \xi, (1 + \dot{\lambda}^2(\rho))^{-1/2} \right)$$

Using (3.32), one can check immediately that

$$\widehat{g}(N, N) = 1 \quad \text{and} \quad \widehat{g}(N, T) = \widehat{g}(N, U) = 0.$$

In order to compute the curvatures of M we need to compute the second fundamental form A_M (or A for short) or the principal curvatures. At the point $X(\rho, \xi)$, the principal directions are

1. the tangent to the meridian curve in the vertical 2-plane

$$P_\xi = \{(\tanh(\rho/2)\xi, t) \mid (\rho, t) \in \mathbb{R}^2\},$$

2. the vectors tangent to $X(\rho, S^{n-1})$ at ξ i.e. $T_{(\rho, \xi)}X(T_\xi S^{n-1})$ where the restriction of A is a scalar multiple of the identity.

In order to compute the principal curvatures, it suffices to consider $M \cap \mathbb{H}_{\xi,u}^2 \times \mathbb{R}$ where $\xi \in S^{n-1}$, $u \in T_\xi S^{n-1}$ and $\mathbb{H}_{\xi,u}^2$ is the totally geodesic hyperbolic 2-plane in \mathbb{H}^n through (ξ, u) . From these consideration and the computations in [22], it follows that

1. $k_n(\rho)$, the principal curvature in the direction tangent to the meridian curve, is given by

$$(3.37) \quad k_n(\rho) = \ddot{\lambda}(\rho)(1 + \dot{\lambda}^2(\rho))^{-3/2},$$

2. the principal curvatures in the directions tangent to S^{n-1} through $X(\rho, \xi)$ are all equal to

$$(3.38) \quad k_1(\rho) = \dots = k_{n-1}(\rho) = \coth(\rho)\dot{\lambda}(\rho)(1 + \dot{\lambda}^2(\rho))^{-1/2}.$$

We conclude that the mean curvature equation of a rotation hypersurface $M \looparrowright \widehat{M}$ with respect to the unit normal pointing upwards is given by

$$nH(\rho) \sinh(\rho) = \sinh(\rho)\ddot{\lambda}(\rho)(1 + \dot{\lambda}^2(\rho))^{-3/2} + (n-1) \cosh(\rho)\dot{\lambda}(\rho)(1 + \dot{\lambda}^2(\rho))^{-1/2}$$

or, equivalently

$$(3.39) \quad nH(\rho) \sinh^{n-1}(\rho) = \partial_\rho \left(\sinh^{n-1}(\rho)\dot{\lambda}(\rho)(1 + \dot{\lambda}^2(\rho))^{-1/2} \right).$$

Remark. One can also use an adaptation of the flux formula to derive this equation (see [6] where we consider H -rotation hypersurfaces and applications).

3.2 Catenoids in $\mathbb{H}^n \times \mathbb{R}$

In this Section, we describe catenoids, *i.e.* minimal rotation hypersurfaces about \mathbb{R} , in $\mathbb{H}^n \times \mathbb{R}$. They are the higher dimensional counterparts of the catenoids constructed in [22].

According to the mean curvature equation (3.39), minimal rotation hypersurfaces about \mathbb{R} in $\mathbb{H}^n \times \mathbb{R}$ are given by the differential equation

$$(3.40) \quad \sinh^{n-1}(\rho)\dot{\lambda}(\rho)(1 + \dot{\lambda}^2(\rho))^{-1/2} = d := \sinh^{n-1}(a)$$

for some constant d which we write as $\sinh^{n-1}(a)$. Putting aside the trivial case of horizontal (totally geodesic) hyperplanes, we may assume (changing if necessary λ

to $-\lambda$) that $a > 0$ and hence that $\dot{\lambda}(\rho) > 0$. It follows from this equation that λ is defined for $\rho \geq a$ and that we have (up to vertical translations)

$$(3.41) \quad \begin{cases} \dot{\lambda}(\rho) &= \sinh^{n-1}(a) (\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a))^{-1/2}, \\ \lambda(\rho) &= \sinh^{n-1}(a) \int_a^\rho (\sinh^{2n-2}(u) - \sinh^{2n-2}(a))^{-1/2} du, \end{cases}$$

where the integral converges at $\rho = a$ and $\rho = \infty$. Note that the corresponding $X(\rho, \xi)$ describes the upper-half of a catenoid \mathcal{C}_a which is obtained after a symmetry with respect to the slice $\mathbb{H}^n \times \{0\}$.

We collect the following formulas for convenience.

$$(3.42) \quad (1 + \dot{\lambda}^2(\rho))^{1/2} = \sinh^{n-1}(\rho) (\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a))^{-1/2},$$

$$(3.43) \quad \begin{cases} \ddot{\lambda}(\rho) &= -(n-1) \sinh^{n-1}(a) \sinh^{2n-3}(\rho) \cosh(\rho) \\ &\quad (\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a))^{-3/2}, \\ \ddot{\lambda}(\rho) &= -(n-1) \sinh^{n-1}(a) \frac{\cosh(\rho)}{\sinh^n(\rho)} (1 + \dot{\lambda}^2(\rho))^{3/2}. \end{cases}$$

In particular, the curve generating the catenoids has strictly negative curvature κ tending to zero exponentially at infinity in the vertical 2-plane $\{\rho, t\}$ and it is concave. As in dimension 2, the upper parts of the different catenary curves intersect exactly once.

The Riemannian measure on the catenoids is given by

$$(3.44) \quad d\mu_{\mathcal{C}_a} = \sinh^{2n-2}(\rho) (\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a))^{-1/2} d\rho d\mu_S$$

We can rewrite the function λ for the catenoids \mathcal{C}_a as

$$\lambda(a, \rho) = \sinh(a) \int_1^{\sinh(\rho)/\sinh(a)} (\sinh^2(a)v^2 + 1)^{-1/2} (v^{2n-2} - 1)^{-1/2} dv.$$

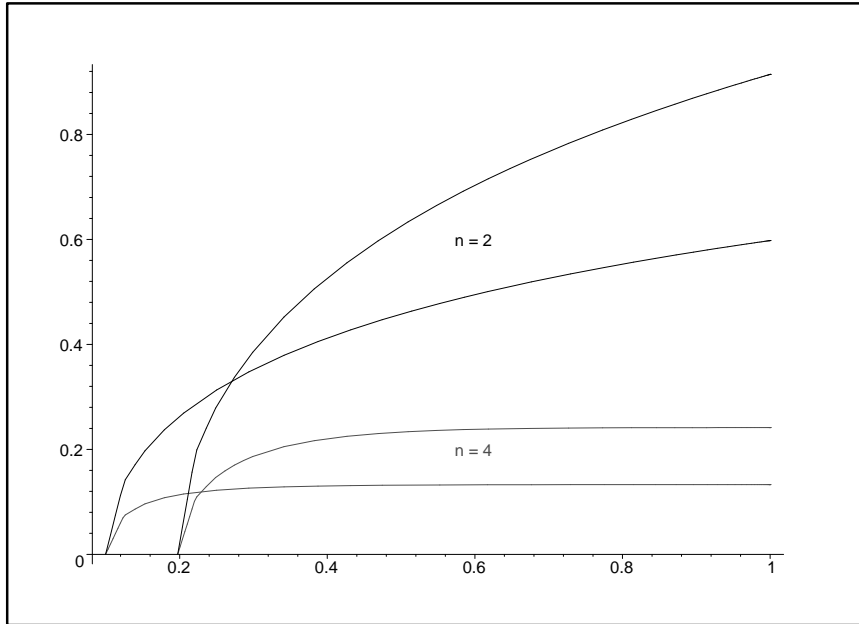


Figure 5: Catenaries in dimension 2 and 4

Proposition 3.2 *When ρ tends to infinity, $\lambda(a, \rho)$ tends to the finite value*

$$(3.45) \quad R(a) := \sinh(a) \int_1^\infty (\sinh^2(a)v^2 + 1)^{-1/2} (v^{2n-2} - 1)^{-1/2} dv.$$

Furthermore, the function R increases from 0 to $\pi/(2n-2)$ when a increases from 0 to ∞ . This means that the catenoids in the family have finite height bounded from above by $\pi/(n-1)$.

Proof. Hint $\int \frac{dv}{v(v^N - 1)^{1/2}} = \frac{2}{N} \arctan \sqrt{v^N - 1}$. □

One can check directly that $\ddot{\lambda}(\rho) < 0$ and that, for $a_2 > a_1$, $\lambda_{a_2}(\rho) - \lambda_{a_1}(\rho)$ increases from a negative value for $\rho \geq a_2$ to infinity.

We collect the following properties for later use.

Properties 3.3 *We have the following geometric data for the catenoids \mathcal{C}_a .*

1. *The unit normal field is given by*

$$(3.46) \quad N_{\mathcal{C}_a}(\rho, \xi) = \left(-\frac{\sinh^{n-1}(a)\xi}{2 \sinh^{n-1}(\rho) \cosh^2(\rho/2)}, \dots \right. \\ \left. \dots \frac{(\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a))^{1/2}}{\sinh^{n-1}(\rho)} \right).$$

2. The principal curvatures are given by

$$(3.47) \quad \begin{cases} k_n(\rho) = -(n-1) \sinh^{n-1}(a) \frac{\cosh(\rho)}{\sinh^n(\rho)}, \\ k_i(\rho) = \sinh^{n-1}(a) \frac{\cosh(\rho)}{\sinh^n(\rho)}, 1 \leq i \leq n-1. \end{cases}$$

3. The second symmetric function of curvature S_{2, \mathcal{C}_a} is given by

$$(3.48) \quad S_{2, \mathcal{C}_a} := \sum_{1 \leq i < j \leq n} k_i k_j = -\frac{n(n-1)}{2} \left(\frac{\sinh^{n-1}(a) \cosh(\rho)}{\sinh^n(\rho)} \right)^2.$$

Note that by minimality, $|A_{\mathcal{C}_a}|^2 = -2S_{2, \mathcal{C}_a}$.

4. The vertical component of the normal vector $N_{\mathcal{C}_a}$ is given by

$$(3.49) \quad v_{\mathcal{C}_a}(\rho) := \widehat{g}(N_{\mathcal{C}_a}, \partial_t) = \frac{(\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a))^{1/2}}{\sinh^{n-1}(\rho)}.$$

5. The Ricci curvature of \widehat{M} in the direction $N_{\mathcal{C}_a}$ is given by $\widehat{\text{Ric}}(N_{\mathcal{C}_a}, N_{\mathcal{C}_a}) = -(n-1)(1 - v_{\mathcal{C}_a}^2)$ and hence,

$$(3.50) \quad \widehat{\text{Ric}}(N_{\mathcal{C}_a}, N_{\mathcal{C}_a}) = -(n-1) \frac{\sinh^{2n-2}(a)}{\sinh^{2n-2}(\rho)}.$$

For $n = 2$, we recover the results of [22]. As for global invariants, we can summarize the properties of the catenoids as follows, see Equations (3.44) and (3.48).

Properties 3.4 *The catenoids have the following global behavior.*

1. The catenoids have infinite volume (as a matter of fact, exponential volume growth).
2. The catenoids have finite total curvature, $\int_{\mathcal{C}_a} |A_{\mathcal{C}_a}|^n d\mu_{\mathcal{C}_a} < \infty$.
3. The catenoids have finite height bounded from above by $\pi/(n-1)$.

Remark. Note that it follows from Properties (3.3) that $|A_{\mathcal{C}_a}|^n d\mu_{\mathcal{C}_a}$ decreases exponentially as $e^{-(n-1)^2 \rho} d\rho d\mu_S$. By Theorem 4.2 in the next section, this implies that the index of \mathcal{C}_a is finite for $n \geq 3$. See Theorem 4.1 for a more precise result.

3.3 Translation invariant hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

Let γ be a complete geodesic through 0 in the ball model \mathbb{B} of the hyperbolic space \mathbb{H}^n , parametrized by the signed distance ρ to 0. Let \mathbb{P} be the hyperbolic hyperplane orthogonal to γ at 0. We consider the hyperbolic translations along the geodesics passing through 0 in \mathbb{P} . The image of a point of γ under these translations is an equidistant hypersurface to \mathbb{P} in \mathbb{H}^n . We can extend these translations “slice-wise” to give positive isometries of $\mathbb{H}^n \times \mathbb{R}$ which we will call hyperbolic translations.

A generating curve $(\tanh(\rho/2), \mu(\rho))$ in the vertical Euclidean plane $\gamma \times \mathbb{R}$ gives rise, under the previous isometries, to a translation invariant hypersurface $M \hookrightarrow \mathbb{H}^n \times \mathbb{R}$ whose intersection with the slice $\mathbb{H}^n \times \{\mu(\rho)\}$ is the equidistant hypersurface to $\mathbb{P} \times \{\mu(\rho)\}$ in the slice, at distance ρ .

The principal directions of M are the tangent vector to the generating curve and the directions tangent to the equidistant hypersurface. The corresponding principal curvatures, with respect to the normal pointing upwards, are given respectively by

$$(3.51) \quad k_G(\rho) = \ddot{\mu}(\rho)(1 + \dot{\mu}^2(\rho))^{-3/2},$$

and

$$(3.52) \quad k_E(\rho) = \dot{\mu}(\rho)(1 + \dot{\mu}^2(\rho))^{-1/2} \tanh(\rho).$$

It follows that the mean curvature of M , with respect to the normal pointing upwards, is given by

$$nH = \ddot{\mu}(\rho)(1 + \dot{\mu}^2(\rho))^{-3/2} + (n - 1)\dot{\mu}(\rho)(1 + \dot{\mu}^2(\rho))^{-1/2} \tanh(\rho)$$

or, equivalently, by

$$(3.53) \quad nH \cosh^{n-1}(\rho) = \partial_\rho \left(\dot{\mu}(\rho)(1 + \dot{\mu}^2(\rho))^{-1/2} \cosh^{n-1}(\rho) \right).$$

This formula allows us to study constant mean curvature hypersurfaces invariant by hyperbolic translations. In this paper we only consider the case $H = 0$ and we refer to [6] for the case $H \neq 0$.

3.4 Translation invariant minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

In this section, we establish the following result.

Theorem 3.5 *There exists a 1-parameter family $\{M_d, d > 0\}$ of complete embedded minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ invariant under hyperbolic translations. The hypersurfaces are stable (in the sense of the Jacobi operator). More precisely,*

1. *If $d > 1$ then M_d consists of the union of two symmetric vertical graphs over the exterior of an equidistant hypersurface in the slice $\mathbb{H}^n \times \{0\}$. Furthermore M_d is a horizontal graph, and hence it is stable.*

The asymptotic boundary of M_d is topologically an $(n-1)$ -sphere that is homologically trivial in $\partial_\infty \mathbb{H}^n \times \mathbb{R}$. More precisely, the asymptotic boundary consists of the union of two copies of a hemisphere $S_+^{n-1} \times \{0\}$ of $\partial_\infty \mathbb{H}^n \times \{0\}$ in parallel slices $t = \pm T(a)$, glued with the finite cylinder $\partial S_+^{n-1} \times [-T(a), T(a)]$, where $d =: \cosh^{n-1}(a)$. The vertical height of M_d is $2T(a)$. The height of the family M_d is a decreasing function in d and varies from infinity (when $d \rightarrow 1$) to $\pi/(n-1)$ (when $d \rightarrow \infty$). It is therefore bounded from below by $\pi/(n-1)$, the upper bound of the heights of the family of catenoids.

2. *If $d = 1$ then M_1 consists of a complete (stable, non-entire) vertical graph over a half-space in $\mathbb{H}^n \times \{0\}$, bounded by a totally geodesic hyperplane P . It takes infinite boundary value data on P and constant asymptotic boundary value data. The asymptotic boundary of M_1 is the union of a spherical cap S of $\partial_\infty \mathbb{H}^n \times \{c\}$ with a half-vertical cylinder over ∂S .*
3. *If $d < 1$ then M_d is an entire (stable) vertical graph with finite vertical height. Its asymptotic boundary consists of a homologically non-trivial $(n-1)$ -sphere in $\partial_\infty \mathbb{H}^n \times \mathbb{R}$.*

Proof. In the minimal case, Equation (3.53) can be written

$$(3.54) \quad \dot{\mu}(\rho)(1 + \dot{\mu}^2(\rho))^{-1/2} \cosh^{n-1}(\rho) = d,$$

for some constant d which satisfies $d \leq \cosh^{n-1}(\rho)$ for all ρ for which the solution exists. Changing μ to $-\mu$ if necessary, we may assume that d is non-negative and hence, $\dot{\mu}(\rho) \geq 0$ and $\dot{\mu}(\rho) = d(\cosh^{2n-2}(\rho) - d^2)^{-1/2}$ whenever the square root exists. We have to consider three cases, $d > 1$, $d = 1$ and $d < 1$.

$d > 1$ Let $d =: \cosh^{n-1}(a)$, with $a > 0$. It follows from Equation (3.54) that

$$\dot{\mu}(\rho) = \cosh^{n-1}(a) (\cosh^{2n-2}(\rho) - \cosh^{2n-2}(a))^{-1/2}.$$

Up to a vertical translation, the solution $\mu_+(a, \rho)$ of Equation (3.54) is given by

$$(3.55) \quad \mu_+(a, \rho) = \cosh^{n-1}(a) \int_a^\rho (\cosh^{2n-2}(r) - \cosh^{2n-2}(a))^{-1/2} dr$$

or, making $\cosh(r) = \cosh(a)t$,

$$(3.56) \quad \mu_+(a, \rho) = \cosh(a) \int_1^{\cosh(\rho)/\cosh(a)} (t^{2n-2} - 1)^{-1/2} (\cosh^2(a)t^2 - 1)^{-1/2} dt.$$

These integrals converge at $\rho = a$ (*resp.* at $t = 1$) and at infinity with limit value

$$(3.57) \quad T(a) := \cosh(a) \int_1^\infty (t^{2n-2} - 1)^{-1/2} (\cosh^2(a)t^2 - 1)^{-1/2} dt$$

a decreasing function of a , which tends to infinity when a tends to zero (*i.e.* when d tends to 1_+) and to $\pi/(2n-2)$ when a (or d) tends to infinity. (Hints. When a tends to zero, use the fact that $T(a)$ is bigger than some constant times the integral $\int_1^2 ((t-1)(\cosh(a)t-1))^{-1/2} dt$ which can be computed explicitly. When a tends to infinity, use the fact that $\int (t^N - 1)^{-1/2} t^{-1} dt = \frac{2}{N} \arctan \sqrt{t^N - 1}$.) The assertions on the asymptotic boundary are clear.

Remark. Observe that the limiting value $T(a)$ when a tends to infinity is the same the limiting value $R(a)$ for the family of catenoids.

$d = 1$ It follows from Equation (3.54) that

$$\dot{\mu}(\rho) = (\cosh^{2n-2}(\rho) - 1)^{-1/2}$$

so that, when $d = 1$, the solution is given by

$$(3.58) \quad \mu_0(\rho) = \int_b^\rho (\cosh^{2n-2}(r) - 1)^{-1/2} dr$$

for some constant $b > 0$, and $\mu_0(\rho)$ tends to $-\infty$ when ρ tends to zero and to a finite value when ρ tends to infinity. The corresponding hypersurface is complete. It is a vertical graph so that it is stable. The assertion on the asymptotic boundary is clear.

$0 < d < 1$ In this case, Equation (3.54) gives the following solution (up to a vertical translation),

$$(3.59) \quad \mu_-(d, \rho) = d \int_0^\rho (\cosh^{2n-2}(r) - d^2)^{-1/2} dr.$$

The corresponding curve can be extended by symmetry and we get a complete hypersurface in a vertical slab with finite height. This surface is an entire vertical graph (hence stable). The assertion on the asymptotic boundary is clear. \square

The generating curves for translation invariant minimal hypersurfaces are given in Figure 7. Note that they cannot meet tangentially at finite distance.

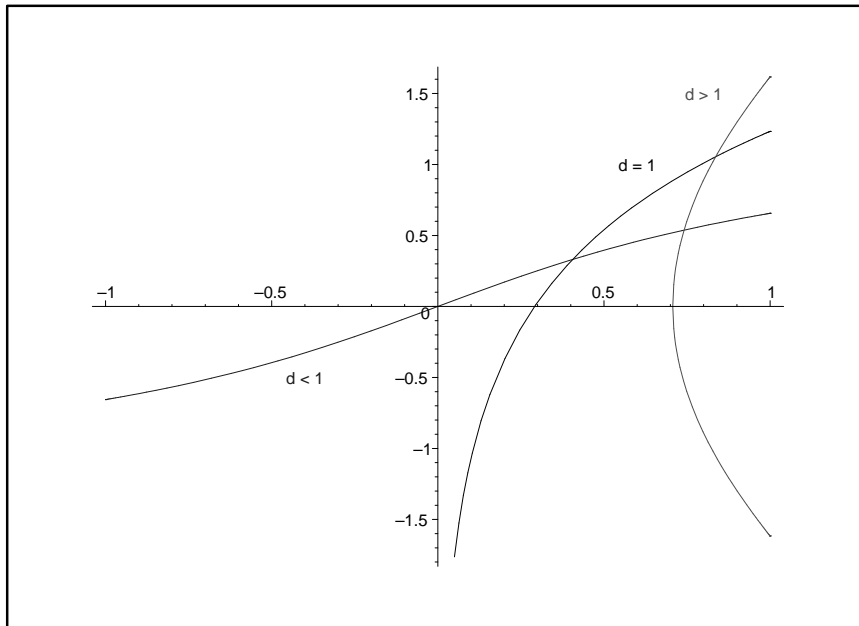


Figure 6: Generatrices for translation invariant hypersurfaces, $n = 2$

Remark. Using the catenoids and the minimal translations hypersurfaces, the second author and E. Toubiana have extended the 2-dimensional results of their paper [25] to higher dimensions, see [24].

4 Minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, total curvature and index

4.1 Index of catenoids in $\mathbb{H}^n \times \mathbb{R}$

In order to study the index of the catenoids \mathcal{C}_a constructed in Section 3.2, we consider the following parameterization, $Y(a, s, \xi)$ where s is the arc-length parameter along the meridian curves.

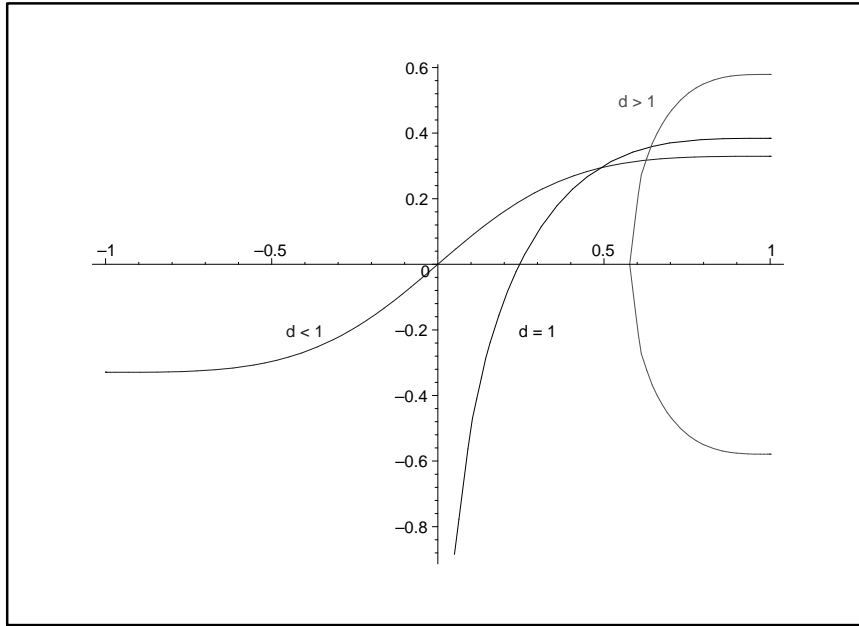


Figure 7: Generatrices for translation invariant hypersurfaces, $n = 4$

More precisely, we consider the parametrization

$$(4.60) \quad Y(a, s, \xi) = \left(\tanh(R(a, s)/2)\xi, \lambda(a, R(a, s)) \right)$$

such that $(R_s(a, s))^2(1 + \dot{\lambda}^2(a, R(a, s))) \equiv 1$, where R_s denotes $\frac{\partial R}{\partial s}$. Using (3.42), we find that

$$ds = \sinh^{n-1}(\rho) \left(\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a) \right)^{-1/2} d\rho$$

i.e.

$$(4.61) \quad S(a, \rho) = \sinh(a) \int_1^{\sinh(\rho)/\sinh(a)} v^{n-1} (\sinh^2(a)v^2 + 1)^{-1/2} (v^{2n-2} - 1)^{-1/2} dv.$$

This integral converges at $v = 1$ and behaves like ρ at infinity. The function $S(a, \rho)$, as a function of ρ , increases from 0 to infinity when ρ increases from a to infinity. For fixed a , we define $R(a, s)$ by

$$(4.62) \quad \begin{cases} R(a, s) \text{ is the inverse function of } S(a, \rho), \text{ i.e.} \\ S(a, R(a, s)) \equiv s \text{ and, in particular, } S_\rho(a, R(a, s)) \cdot R_s(a, s) \equiv 1. \end{cases}$$

Note, see (3.42) and (3.49), that

$$(4.63) \quad \begin{cases} R_s(a, s) &= \left(\sinh^{2n-2}(R(a, s)) - \sinh^{2n-2}(a) \right)^{1/2} / \sinh^{n-1}(R(a, s)), \\ R_s(a, s) &= v_{\mathcal{C}_a}(a, s) \text{ the component of } N_{\mathcal{C}_a} \text{ along } \partial_t. \end{cases}$$

The expression for the induced metric in the coordinates (s, ξ) is

$$g_M = ds^2 + \sinh^2(R(a, s)) g_S.$$

Using Properties 3.3, we obtain the formulas

$$(4.64) \quad \begin{cases} |A_{\mathcal{C}_a}|^2(a, s) &= n(n-1) \sinh^{2n-2}(a) \frac{\cosh^2(R(a, s))}{\sinh^{2n}(R(a, s))}, \\ \widehat{\text{Ric}}(N_{\mathcal{C}_a}, N_{\mathcal{C}_a})(a, s) &= -(n-1) \sinh^{2n-2}(a) \frac{1}{\sinh^{2n-2}(R(a, s))}. \end{cases}$$

Remark. When $n = 2$ Equations (4.62) – (4.64) yield back the relations described in Section 2.

The quadratic form associated with the Jacobi operator

$$J_{\mathcal{C}_a} := -\Delta_{\mathcal{C}_a} - \left(|A_{\mathcal{C}_a}|^2 + \widehat{\text{Ric}}(N_{\mathcal{C}_a}, N_{\mathcal{C}_a}) \right)$$

on the catenoids is given by

$$(4.65) \quad \mathcal{J}_{\mathcal{C}_a}(f) = \iint \left[\left(\frac{\partial f}{\partial s} \right)^2 + \sinh^{-2}(R(s)) |d_S f|^2 - C(s) f^2 \right] \sinh^{n-1}(R(s)) ds d\mu_S$$

where we have written $R(s)$ instead of $R(a, s)$ for simplicity and where

$$C(s) = |A_{\mathcal{C}_a}|^2 + \widehat{\text{Ric}}(N_{\mathcal{C}_a}, N_{\mathcal{C}_a}).$$

Integrating by parts once in local coordinates, we find the formula (compare with Properties 7)

$$(4.66) \quad \mathcal{J}_{\mathcal{C}_a}(f) = - \iint f \left\{ \left[\frac{\partial}{\partial s} \left(\sinh^{n-1}(R(s)) \frac{\partial f}{\partial s} \right) \right] + C(s) \sinh^{n-1}(R(s)) f \cdots \right. \\ \left. \cdots + \sinh^{n-3}(R(s)) \Delta_S f \right\} ds d\mu_S$$

which gives us the expression of the Laplacian on \mathcal{C}_a in the (s, ξ) parameters (where Δ_S is the non-positive Laplacian on S^{n-1}). Decompose $f(s, \xi)$ into spherical harmonics with respect to ξ ,

$$f(s, \xi) = \sum_{k=0}^{\infty} f_k(s) Y_k(\xi).$$

Using this decomposition, we can study the eigenvalue problem for the Jacobi operator by considering the eigenvalue problem for the following family of Sturm-Liouville operators in the s -variable,

$$(4.67) \quad \begin{cases} L_k(u) = \lambda u, & \text{where} \\ L(u) := -\ddot{u}(s) - (n-1)\dot{R}(s)\coth(R(s))\dot{u}(s) - C(s)u(s), \\ L_k(u) := L(u) + \mu_k \sinh^{-2}(R(s)), \end{cases}$$

where the $\mu_k, k \geq 0$, are the eigenvalues of the Laplacian Δ_S on S^{n-1} .

Remarks.

- We can extend $R(a, s)$ as an *even* function of s and $\lambda(a, R(a, s))$ as an *odd* function of s so that we have a parametrization of the whole catenoids.
- The operators (4.67) are then defined on the whole line.

Natural Jacobi fields on the catenoids \mathcal{C}_a

As in dimension 2, see Section 2.2, the geometry provides natural Jacobi fields on \mathcal{C}_a .

- Let $v_{\mathcal{C}_a}(s)$ (extended to an *odd* function of s) be the vertical component of the unit normal vector along the catenoids. It is a solution of the Jacobi equation (because it is induced by the Killing field associated with vertical translations), *i.e.* $v_{\mathcal{C}_a}$ satisfies

$$L(v_{\mathcal{C}_a}) = 0, \quad v_{\mathcal{C}_a}(0) = 0, \quad \text{and} \quad v_{\mathcal{C}_a} > 0 \quad \text{on} \quad]0, \infty[$$

so that the upper (*resp.* lower) half of the catenoid \mathcal{C}_a is stable (this is not surprising because the half-catenoid is a vertical graph).

- As in dimension 2, we can also consider the “vertical half catenoids” as horizontal graphs which are stable.
- We can finally look at the variation field, $\partial_a Y(a, s, \xi)$, which comes from the family \mathcal{C}_a of catenoids. It is given by

$$\partial_a Y(a, s, \xi) = \left(\frac{R_a(a, s)\xi}{2 \cosh^2(R(a, s)/2)}, \lambda_a(a, R(a, s)) + R_a(a, s)\lambda_\rho(a, R(a, s)) \right),$$

and the corresponding Jacobi field is given by $y(a, s) := \widehat{g}(\partial_a Y, N)$ (this function does not depend on ξ). Using (3.46) we find that

$$y(a, s) = R_s(a, s)\lambda_a(a, R(a, s))$$

see Equation (4.63).

Recall that $\widehat{g}(\partial_a Y, N)$ is a Jacobi field, *i.e.* $J_a(\widehat{g}(\partial_a Y, N)) = 0$. Computing $Y(a, 0, \xi)$ and $Y_s(a, 0, \xi)$ we find that

$$(4.68) \quad \begin{cases} L(y(a, \cdot)) & = 0, \\ y(a, 0) & = -1, \\ y_s(a, 0) & = 0. \end{cases}$$

• **Geometric interpretation.** Consider the meridian curves (catenaries)

$$Y_1(a, s) := \left(\tanh(R(a, s)/2), \lambda(a, R(a, s)) \right) =: \left(A(a, s), B(a, s) \right)$$

of the catenoids \mathcal{C}_a . The envelope of this family of curves is given by

$$E(a, s) := \begin{vmatrix} A_a & B_a \\ A_s & B_s \end{vmatrix} = 0.$$

It follows easily that

$$E(a, s) = 0 \Leftrightarrow R_a B_s - R_s B_a = -R_s \lambda_a = 0.$$

We can easily deduce from this fact that the zeroes of the Jacobi field $y(a, \cdot)$ correspond to the points at which the catenary C_a (or the catenoid \mathcal{C}_a) touches the envelope of the family.

As in dimension 2, we can prove that this function has exactly one positive zero.

Summarizing the above analysis, we obtain, as in dimension 2, the following result.

Theorem 4.1 *The compact connected domain of \mathcal{C}_a , whose boundary consists of the two spheres where the catenoid \mathcal{C}_a touches the envelope of the family is a maximal weakly stable domain. The catenoid \mathcal{C}_a has index 1.*

Remark. A result similar to Proposition 2.5 in Section 2.2 can also be established in higher dimension. We also point out the paper [27] in which L.F. Tam and D. Zhou, using other methods, show that the index of the n -dimensional catenoid in \mathbb{R}^{n+1} is one.

4.2 Minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ with finite total curvature

Recall the formula for the Jacobi operator,

$$J_M := -\Delta_M - (|A_M|^2 + \widehat{\text{Ric}}(N_M, N_M))$$

where N_M is a unit normal field along M and A_M the second fundamental form of M with respect to N_M .

Let $v_M := \hat{g}(N_M, \partial_t)$ be the vertical component of the unit normal vector N_M . A simple computation gives that $\widehat{\text{Ric}}(N_M, N_M) = -(n-1)(1-v_M^2)$. It follows that the Jacobi operator of M is given by

$$(4.69) \quad J_M := -\Delta_M + (n-1)(1-v_M^2) - |A_M|^2.$$

We have the following result.

Theorem 4.2 *Let $M^n \looparrowright \mathbb{H}^n \times \mathbb{R}$ a complete oriented minimal immersion. Assume that M has finite total curvature, i.e. $\int_M |A|^n d\mu_M < \infty$.*

1. *For $n \geq 2$, the second fundamental form A_M tends to zero uniformly at infinity.*
2. *For $n \geq 3$, the Jacobi operator of the immersion has finite index and, more precisely, there exists a universal constant $C(n)$ such that*

$$(4.70) \quad \text{Ind}(J_M) \leq C(n) \int_M |A|^n d\mu_M.$$

Remarks. (i) There are examples of minimal hypersurfaces proving that the converse statements in the previous theorems are not true in general, see Section 3.4. (ii) Note that the second assertion of Theorem 4.2 only holds for $\dim(M) \geq 3$ (see [3, 8]).

Proof. As in the proof of Theorem 2.7, the manifold M satisfies a Sobolev inequality of the form (2.22), namely

$$\|f\|_{n/(n-1)} \leq S(M) \|df\|_1 \quad \text{for all } f \in C_0^1(M)$$

for some constant $S(M)$. Furthermore, A satisfies the following Simons' equation (compare with (2.18)),

$$-\Delta|A_M| \leq |A_M|^3 + C(n)|A_M|$$

for some constant $C(n)$ which only depends on the dimension (this follows from the expression of the term $\widehat{R}(A)$ as given in [26, 2]).

It follows from Theorem 2.8 that $|A_M|$ tends to zero uniformly at infinity.

Since $|v_M| \leq 1$, the operator J_M is bounded from below and essentially self-adjoint. Furthermore, its index is less than or equal to the index of the operator $-\Delta - |A_M|^2$ which is also bounded from below and essentially self-adjoint. Since the manifold M satisfies the above Sobolev inequality, we can control the minimal heat kernel $k_M(t, x, y)$ on M (see [28, 8]) by

$$k(t, x, x) \leq D(n)t^{-n/2}.$$

The estimate (4.70) then follows by applying Theorem 37 in [3] with $\rho_- = -|A_M|^2$, *i.e.* to the operator $-\Delta - |A_M|^2$. \square

Remark. The preceding results can be generalized to minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}^k$ or $\mathbb{H}^n \times \mathbb{H}^k$.

5 Applications, embedded minimal hypersurfaces with boundary contained in a slice

In this section, we use the examples constructed in Section ?? as barriers to prove some general results on minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$. These results generalize results obtained in [20] for dimension 2. Similar results hold for H -hypersurfaces as well, see [20, 6].

Theorem 5.1 *Let $\Gamma \subset \mathbb{H}^n$ be a compact embedded hypersurface and consider two copies of Γ in different slices, $\Gamma_- = \Gamma \times \{-a\}$ and $\Gamma_+ = \Gamma \times \{a\} \subset \mathbb{H}^n \times \mathbb{R}$, for some $a > 0$. Assume that Γ is convex.*

Let $M \subset \mathbb{H}^n \times \mathbb{R}$ be a compact immersed minimal hypersurface such that $\partial M = \Gamma_- \cup \Gamma_+$. Then,

$$2a < \frac{\pi}{n-1} \quad (\text{the height of the family of catenoids}).$$

Furthermore, if M is embedded,

1. M is symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$.
2. The parts of M above and below the slice of symmetry are vertical graphs.

3. If Γ is a horizontal graph and symmetric with respect to a hyperbolic hyperplane P , then M is a horizontal graph and symmetric with respect to the vertical hyperplane $P \times \mathbb{R}$. In particular, if Γ is an $(n-1)$ -sphere, then M is part of a catenoid.

Proof. We reason ab absurdo. Suppose that the height of M is greater than or equal to $\frac{\pi}{n-1}$, that is $2a \geq \frac{\pi}{n-1}$. We recall that the height of the family of n -dimensional catenoids $\{\mathcal{C}_\rho, \rho \in (0, \infty)\}$ is bounded from above by $\frac{\pi}{n-1}$, but each catenoid \mathcal{C}_ρ has height strictly less than $\frac{\pi}{n-1}$. Now as M is compact, there is a (hyperbolic) radius ρ_0 big enough such that M is strictly contained inside the vertical cylinder \mathbb{M}_{ρ_0} of radius ρ_0 (where \mathbb{M}_{ρ_0} is a cylinder over a $n-1$ sphere $\mathbb{S}_{\rho_0} \subset \mathbb{H}^2 \times \{0\}$ of radius ρ_0) containing M in its mean convex side. Recall that, by the geometry of the catenoids, the catenoid \mathcal{C}_{ρ_0} whose distance to the t -axis is ρ_0 is contained in the closure of the non mean convex side of \mathbb{M}_{ρ_0} touching \mathbb{M}_{ρ_0} just along the $n-1$ sphere \mathbb{S}_{ρ_0} . Hence, M is strictly contained in the connected component of $\mathbb{H}^2 \times \mathbb{R} \setminus \mathcal{C}_{\rho_0}$ that contains the t axis of \mathcal{C}_{ρ_0} . Notice that the whole family of catenoids \mathcal{C}_ρ is strictly contained in the slab of $\mathbb{H}^2 \times \mathbb{R}$ with boundary $\Gamma_- \cup \Gamma_+$. Starting from $\rho = \rho_0$, making $\rho \rightarrow 0$, that is moving the family of catenoids $\{\mathcal{C}_\rho, \rho \leq \rho_0\}$ towards M , we will find a first interior point of contact with some \mathcal{C}_ρ and M , since the family of catenoid cannot touch the boundary of M . We arrive at a contradiction, by the the maximum principle. The proof of the first part of the Statement is completed.

Now using the family of slices $\mathbb{H} \times \{t\} \cup \mathbb{H} \times \{-t\}$ coming from the infinity towards M we get, by maximum principle, that M is entirely contained in the closed slab whose boundary is the slices $\mathbb{H} \times \{a\} \cup \mathbb{H} \times \{-a\}$ and $(\mathbb{H} \times \{a\} \cup \mathbb{H} \times \{-a\}) \cap M = \partial M$.

In the same way, considering the family of vertical hyperplanes, we get that M is contained in the mean convex side of the vertical cylinder \mathbb{M}_Γ over Γ and $\mathbb{M}_\Gamma \cap M = \partial M$. Now using Alexandrov Reflection Principle on the slices, moving the slices from $t = a$ towards $t = 0$, by vertical reflections, we get that the reflection of $M^+ = M \cap \{t \geq 0\}$, the part of M above $t = 0$, on the horizontal slice $t = 0$, is above $M^- = M \cap \{t \leq 0\}$. Moreover, we find that M^+ is a vertical graph. In the same way, moving the slices from $t = -a$ towards $t = 0$, doing vertical reflections, we get that the reflection of $M^- = M \cap \{t \leq 0\}$, the part of M above $t = 0$, on the horizontal slice $t = 0$, is below $M^+ = M \cap \{t \geq 0\}$. We conclude that $M^- = M^+$, hence both M^+ and M^- are vertical graphs and M is symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$. Therefore, the proof of the second part of the Statement is completed.

Let us assume now that $P \subset \mathbb{H} \times \{0\}$ is a hyperplane of symmetry of Γ . Consider the vertical hyperplane $\mathbb{P} = P \times \mathbb{R}$ and the family of hyperplanes \mathbb{P}_t at signed distance t from \mathbb{P} obtained from \mathbb{P} by horizontal translations along an oriented geodesic γ orthogonal to \mathbb{P} at the origin. Choosing $|t|$ big enough, we move the family \mathbb{P}_t towards

\mathbb{P} (in the two sides of $\mathbb{H}^2 \times \mathbb{R} \setminus \mathbb{P}$), doing Alexandrov Reflection Principle on \mathbb{P}_t , taking into account that Γ is a horizontal graph and that the symmetric of ∂M on \mathbb{P}_t stays on the slices $t = \pm a$, so that it does not touch the interior of M . We can argue as before to conclude that \mathbb{P} is a hyperplane of symmetry of M . Of course, if Γ is rotationally symmetric then M is a minimal hypersurface of revolution. Henceforth, by the classification theorem, M is part of a catenoid. This completes the proof of the theorem. \square

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