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# A high order purely frequential harmonic balance formulation for continuation of periodic solutions

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## Abstract

The combination of the harmonic balance method (HBM) with a continuation technique is a widely used approach to follow the periodic solutions of dynamical systems with respect to a bifurcation parameter. However, since the derivation of the algebraic system containing the Fourier coefficients can be very cumbersome, the classical HBM is often limited to simple nonlinearities (quadratic, cubic) and/or to a small number of harmonics. Several variations of the classical HBM have been introduced in the literature to overcome this shortcoming, for instance, the incremental HBM or the alternating frequency/time domain HBM. In this paper, we present an alternative approach that can be applied to a very large class of dynamical systems (autonomous or forced) with regular equation. The main idea is to systematically recast the dynamical system under a quadratic form before applying the HBM. Once the equations are quadratic, the derivation of the algebraic system and its resolution by the so-called ANM continuation technique become obvious. Several classical examples are presented to illustrate this numerical technique.

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## 1 Introduction

Computing the periodic solutions of a system of nonlinear differential equations is a traditional requirement in engineering applications, and particularly in the field of nonlinear vibrations ([1,2]). When the interest is in accurate capture of the periodic solutions and their bifurcations, numerical continuation methods are generally preferred to approximate analytical techniques. These numerical methods are usually classified into two main categories :

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time-domain approach and the frequency-domain approach. The first category refers to methods where a time integration algorithm, generally limited to a single period, is used to transform the original differential system into a system of algebraic equations, which are then solved by continuation. In these methods, the unknowns of the algebraic system are the values of the unknown original variables at grid points along the periodic orbit. The classical shooting technique ([3,4]) and the orthogonal collocation methods used in AUTO ([5,6]) fall into this category. The second category corresponds to the so-called harmonic balance method where the unknown variables are decomposed into truncated Fourier series. In that case, the unknowns of the final algebraic system, which is obtained by balancing the harmonics in the differential equations, are simply the Fourier coefficients of the unknowns. Both categories of methods are widely used for many applications and there is still an important research effort to improve both types. Practically, the choice between the time-domain or the frequency-domain approach highly depends on whether the periodic solution may be represented by a few Fourier components. Hence, for a given problem, one or the other of the two methods may be preferred, depending on the operating point.

In this paper, we focus on the harmonic balance method (HBM) and some variations that are briefly recalled here. Though the name "harmonic balance" seems to first appear in 1936 ([7]), the method is widely used since the sixties only, especially for electrical and mechanical engineering purposes. Forced vibrations have first been studied ([8]), then self-sustained oscillations a decade later ([9]). The work from Nakhla & Vlach ([10]) is often reported as being a milestone in the modern formulation of HBM. The classical HBM is simple in its principle, but it can be cumbersome or even impracticable as stated by Peng et al. ([11]) when the system has complex nonlinearities and when the required number of harmonics is high, says, more than 5 or 10. The first difficulty is to write down the algebraic system for the Fourier coefficients and the second, to solve efficiently this strongly nonlinear system. To overcome these shortcomings, many variations of the basic HBM have been proposed in the literature. Here, we limit ourselves to give the main idea of the incremental harmonic balance (IHB) and the alternating frequency/time domain harmonic balance (AFT). In the IHB ([12]), the incremental-iterative method used for the continuation is closely combined with the harmonic principle, so as to apply the HB principle to the incremental linear problem instead of the nonlinear original system. Any kind of nonlinearity can be treated by this technique [13,14]. In the AFT variant, the harmonic balance of the Fourier coefficients is not done explicitly. At each increment (or iteration) of the continuation procedure, the unknowns are transferred to the time domain (inverse FFT), so as to use the original system of equations. The nonlinear responses are then transferred back to the frequency domain using FFT. The use of FFT/FFT<sup>-1</sup> procedures seems to first appear in [15]. This technique has been used, for instance, to analyse the nonlinear vibrations of mechanical systems

with contact and dry friction [16]. Since the pioneer work on IHB and AFT, many variations have been proposed to extend the applications, and also to improve the computational cost (see [17] or [18] for recent examples).

In this paper, we propose an alternative strategy for applying the classical HBM with a large number of harmonics and without avoiding the explicit balancing. The idea is to apply a transformation to the original system of differential equations before the application of the HBM. The goal of the procedure is to transform the non-linearities of the original system into non-linear quadratic terms only. Then, the application of the classical HBM becomes obvious. This transformation could appear as a limitation of the method since not every system can be recasted under a quadratic form. However, it will be shown in this paper that a very large class of systems with regular equations do can be recasted quadratic with simple algebraic manipulations and a few addition of equations and auxiliary variables. The idea of using a quadratic formulation to make easy the (Fourier) series expansion is taken from another numerical method, the so-called Asymptotic Numerical Method (ANM) ([20,21]) that has been proposed first in 1990 ([19]). ANM is a continuation technique ([22]) that relies on high order powers series expansion of the branches of solutions. For reasons that will be given latter on, this ANM continuation is the ideal companion to solve the algebraic system coming from the proposed HB method.

The paper is organised as follows : the details of the transformation into a quadratic form, the HBM application and the principle of continuation are given in section 2 for autonomous systems. Three simple examples are used for illustration. Section 3 is devoted to periodically forced systems, and two more examples are presented. Section 4 shows the results of all the examples and we finish by concluding remarks and perspectives.

## 2 Presentation of the method for periodic solutions of autonomous systems

We consider an autonomous system of differential equations

$$\dot{Y} = f(Y, \lambda) \tag{1}$$

where  $Y$  is a vector of unknowns,  $f$  a nonlinear vector valued function and  $\lambda$  a real paramater. The dot stands for the derivative with respect to time  $t$ . We assume that this system has branches of periodic solutions when  $\lambda$  varies, and we want to find and follow them by applying the harmonic balance method and a continuation procedure (path following technique). The important case of forced system (non-autonomous) will be treated separately in section 3.

## 2.1 The harmonic balance principle

As recalled in the introduction, the basic principle of the HBM consists in decomposing  $Y(t)$  under the form of a truncated Fourier series :

$$Y(t) = Y_0 + \sum_{k=1}^H Y_{c,k} \cos(k\omega t) + \sum_{k=1}^H Y_{s,k} \sin(k\omega t) \quad (2)$$

This ansatz is put into (1) and  $f(Y, \lambda)$  is expanded into Fourier series. By balancing the first  $2H + 1$  harmonic terms, one obtains an algebraic system with  $2H + 1$  vector equations for the  $2H + 1$  vector unknowns  $Y_i$ , the unknown pulsation  $\omega$ , and the parameter  $\lambda$ . Adding a phase condition ([3]) leads to a system with  $N$  equations for  $N + 1$  variables. The branches of solutions of this algebraic system are then followed by a continuation technique. This procedure provides only approximate periodic solutions, since in the expansion of (1), all the harmonic terms greater than  $H$  remain unbalanced. However, if the number of harmonic  $H$  is large enough, accurate solutions can be expected.

The very crucial point is the expansion of the vector  $f(Y, \lambda)$  in Fourier series. It can be rather straightforward if  $f(Y, \lambda)$  is quadratic or cubic with respect to  $Y$ , and if the number of harmonics  $H$  is small. In that case, the expansion can be made by hand. But for the general situation, where  $f(Y, \lambda)$  presents any kind of nonlinearities, this computation can be very cumbersome, even with the help of a symbolic software. The only alternative is then the use of numerical procedures to estimate the Fourier serie of  $f(Y, \lambda)$ , for example through iterative FFT/FFT<sup>-1</sup> steps. This drawback has been the starting point for the developpement of many variants of the HBM as it has been noticed in the introduction.

## 2.2 A key point : the quadratic recast

The main goal of this paper is to promote a simple but powerful procedure that permits to remove the drawback mentioned above. The idea is to recast the original system (1) into a new system where the nonlinearities are at most quadratic. Afterward, the application of the harmonic balance method becomes straightforward. This quadratic new system will be written as

$$m(\dot{Z}) = c + l(Z) + q(Z, Z) \quad (3)$$

and it may contains both differential and algebraic equations. The number of equations is denoted by  $N_e$ . The unknown vector  $Z$  (size  $N_e$ ) contains the

original components of the vector  $Y$ , and some new variables that are added to transform (1) into a quadratic form. The right hand side of (3) is organized as follows :  $c$  is a constant vector with respect to the unknown  $Z$ ,  $l(\cdot)$  is a linear vector valued operator with respect to the vector entry, and  $q(\cdot, \cdot)$  is a quadratic vector valued operator, it is linear with respect to both entries. At this stage, the three vectors  $c$ ,  $l(Z)$  and  $q(Z, Z)$  may depend on the real parameter  $\lambda$ . However, it will be convenient that the quadratic operator  $q(\cdot, \cdot)$  does not depend on  $\lambda$ , which may require the introduction of a new variable in  $Z$  (see example 1 below). In the left hand side,  $m(\cdot)$  is a linear vector valued operator with respect to the vector entry. Algebraic equations correspond to zero values of  $m(\cdot)$ .

Since the nonlinearities are quadratic, it is clear that the HBM can be applied on (3) in a systematic and easy way, even with a large number of harmonics. The question is now: does any system can be easily put under the form (3) ? We will see that the answer is yes for a very large class of nonlinear systems.

The idea to recast a nonlinear system into one with a quadratic polynomial nonlinearity is not new. It has been systematically used by the authors of the so-called ANM continuation method to compute power series expansion of solution branches, see ([20,21,22,23]) for the basic algorithms and ([24,25,26]) for some applications in mechanics. Indeed, the quadratic recast allows to have a very simple and systematic algebra for the series determination, even for high order of truncature. The same benefits hold for Fourier series expansions.

It is now worth illustrating this main idea on elementary examples where the recast from (1) to (3) will be made explicitly. We consider the classical Van der Pol oscillator, the Rössler system and a model of clarinet-like musical instruments hereafter. More examples, with other kinds of nonlinearities such as rational fraction, will be given later on.

**Example 1 :** Van der Pol oscillator. We consider the following second order autonomous system, known as the Van der Pol oscillator. Here, the parameter  $\lambda$  governs the amplitude of the nonlinear damping term.

$$\ddot{u} - \lambda(1 - u^2) \dot{u} + u = 0 \tag{4}$$

This equation can be classically recasted into a first order system as (1) by introducing the velocity  $v(t) = \dot{u}(t)$  as a new unknown. We get

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= \lambda(1 - u^2) v - u \end{aligned} \tag{5}$$

If we now introduce the auxilliary variables  $w(t) = 1 - u^2(t)$  and  $r(t) = v(t)w(t)$ , the system can be put under the form (3) with  $Z = [u, v, w, r]^t$

$$\begin{aligned}
\dot{u} &= 0 && +v \\
\dot{v} &= 0 && -u + \lambda r \\
0 &= 1 && -w && -u^2 \\
\underbrace{\dot{0}}_{m(\dot{Z})} &= \underbrace{0}_c && \underbrace{+r}_{l(Z,\lambda)} && \underbrace{-vw}_{q(Z,Z)}
\end{aligned} \tag{6}$$

We finally get two first order differential equations and two algebraic equations, each of them being quadratic. Here, only the operator  $l$  depends on  $\lambda$ .

**Example 2 :** The Rössler system. We consider the following first order autonomous system of three equations known as the Rössler system.

$$\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - \lambda)
\end{aligned} \tag{7}$$

where  $x, y, z$  are functions of time and  $a, b, c$  are float parameters. This system can already be written in the form of (3) without any auxiliary variable (i.e.  $Z = [x, y, z]^t$ ):

$$\begin{aligned}
\dot{x} &= 0 && -y - z \\
\dot{y} &= 0 && +x + ay \\
\underbrace{\dot{z}}_{m(\dot{Z})} &= \underbrace{b}_c && \underbrace{-\lambda z}_{l(Z,\lambda)} && \underbrace{+zx}_{q(Z,Z)}
\end{aligned} \tag{8}$$

Here again, only the operator  $l$  depends on  $\lambda$ . This example is considered hereafter to demonstrate the ability of the approach presented in this paper to find period-doubling and period-4ing bifurcations.

**Example 3:** A model of clarinet-like musical instruments. In the case of small amplitudes of oscillation, a simple model of reed instruments (clarinet,

saxophone ...) can be written according to a modal formulation (see [27]):

$$\begin{aligned}
\ddot{x} + q_r \omega_r \dot{x} + \omega_r^2 x &= \omega_r^2 p \\
\ddot{p}_n + 2\alpha_n c \dot{p}_n + \omega_n^2 p_n &= \frac{2c}{l} \dot{u} \quad \forall n \in [1 \dots N] \\
u &= \zeta(1 - \lambda + x) \sqrt{\lambda - p} \\
p &= \sum_{n=1}^N p_n
\end{aligned} \tag{9}$$

where the unknowns  $x$ ,  $p_{n=1..N}$ ,  $p$ ,  $u$  are functions of time, and  $q_r$ ,  $\omega_r$ ,  $\alpha_n$ ,  $\omega_n$ ,  $c$ ,  $l$ ,  $\zeta$  are given parameters describing either the physics or the player action.  $N$  is the number of acoustic modes. Here  $\lambda$  stands for the blowing pressure.

It is interesting to note that even if this system contains a square root, it can be recasted in the form of (3). Indeed, if we introduce the auxiliary variables  $y = \dot{x}$ ,  $z_n = \dot{p}_n$  and  $v = \sqrt{\lambda - p}$ , system (9) can be rewritten, with  $Z = [x, y, p_1, \dots, p_N, z_1 \dots z_N, u, v]^t$ :

$$\begin{aligned}
\dot{x} &= 0 && +y \\
\dot{y} &= 0 && +\omega_r^2 p - q_r \omega_r y - \omega_r^2 x \\
\dot{p}_n &= 0 && +z_n \quad \forall n \in [1 \dots N] \\
\dot{z}_n - \frac{2c}{l} \dot{u} &= 0 && -2\alpha_n c z_n - \omega_n^2 p_n \quad \forall n \in [1 \dots N] \\
0 &= 0 && -u + \zeta(1 - \lambda)v \quad +\zeta xv \\
0 &= 0 && -p + p_1 + \dots + p_N \\
\underbrace{0}_{m(\dot{Z})} &= \underbrace{\lambda}_{c(\lambda)} \underbrace{-p}_{l(Z, \lambda)} \underbrace{-v^2}_{q(Z, Z)}
\end{aligned} \tag{10}$$

Here, not only the linear term  $l$  is a function of  $\lambda$ , but also the constant term  $c$ .

### 2.3 The harmonic balance method applied to a quadratic system

In this section the harmonic balance method is applied to the system (3). The unknown (column) vector  $Z$  is decomposed into Fourier series with  $H$  harmonics:

$$Z(t) = Z_0 + \sum_{k=1}^H Z_{c,k} \cos(k\omega t) + \sum_{k=1}^H Z_{s,k} \sin(k\omega t) \tag{11}$$



The components of the Fourier series are collected into a large (column) vector  $U$  of size  $(2H + 1) \times N_e$ , where  $N_e$  is the number of equations of (3).

$$U = [Z_0^t, Z_{c,1}^t, Z_{s,1}^t, Z_{c,2}^t, Z_{s,2}^t, \dots, Z_{c,H}^t, Z_{s,H}^t]^t \quad (12)$$

By introducing the expansion (11) into the set of equation (3), collecting the terms of the same harmonic index, and neglecting the higher order harmonics, one obtains a large system of  $(2H + 1) \times N_e$  equations for the unknown vector  $U$ ,

$$\omega M(U) = C + L(U) + Q(U, U) \quad (13)$$

The new operators  $M(\cdot)$ ,  $C$ ,  $L(\cdot)$ , and  $Q(\cdot, \cdot)$  that apply to  $U$  only depend on the operators  $m(\cdot)$ ,  $c$ ,  $l(\cdot)$  and  $q(\cdot, \cdot)$  of (3) and on the number of harmonics  $H$ . The explicit formulas have been reported in annexe 1 for a sake of brevity. It should be noticed that these expressions of  $M(\cdot)$ ,  $L(\cdot)$  and  $Q(\cdot, \cdot)$  are the same for all the examples. The change from one particular system to another is only concerned with the operators  $m(\cdot)$ ,  $l(\cdot)$  and  $q(\cdot, \cdot)$ .

The final system (13) contains  $(2H + 1) \times N_e$  equations for the  $(2H + 1) \times N_e$  unknowns  $U$  plus the angular frequency  $\omega$  and the continuation parameter  $\lambda$ . Since the original system is autonomous, a phase condition has to be added to (13) to define a unique orbit. Indeed, the time  $t$  does not appear in (3) and if  $U(t)$  is a solution of (13) then  $U(t + \tau)$  is also a solution, for any  $\tau$ . We refer here to textbooks dealing with periodic solution continuation ([3,6,4]) for the choice of such a phase condition. Typically  $Z_{s,1} = 0$  is an often encountered additional phase condition.

## 2.4 The continuation procedure

### 2.4.1 Framework

As a result of the harmonic balance operation, we are now faced to solve an algebraic system

$$\mathbf{R}(\mathbf{U}) = 0 \quad (14)$$

with  $\mathbf{R} \in \mathbb{R}^N$  and  $\mathbf{U} \in \mathbb{R}^{N+1}$  ( $N = (2H + 1) \times N_e + 1$ ). Any numerical continuation method ([28,29,3]) and software (see ([30] for an overview) could be used to achieve this goal. However, in this paper we shall use the so-called Asymptotic-Numerical-Method (ANM), from which, the spirit of the quadratic recast has been taken. It should be noticed that the final system

(13) is already quadratic with respect to the Fourier coefficients  $U$  and  $\omega$ , and so, the application of the ANM is straightforward.

In the continuation procedure, we shall use a pseudo-arc length parametrization in order to be able to pass the limit points with respect to  $\lambda$ . Hence, the parameter  $\lambda$  will now become an unknown as it is for  $U$  and  $\omega$ . In that perspective, we have to give details on how the operators  $m$ ,  $c$ ,  $l$  and  $q$  of (3) depend on  $\lambda$ . In the three examples, we have organised the terms of equations so that the parameter  $\lambda$  only appears in the operator  $c$  and  $l$ . The idea is to avoid to put  $\lambda$  in the operator  $q$ , for which the algebra of the HBM is more complex than for  $c$  and  $l$ . Moreover, we manage to have this dependence at most linear with respect to  $\lambda$ .

Once again, this writing can be obtained for a very large class of systems provided that suitable additional variables are introduced. For instance, a term  $\lambda u^2$  should be rewritten as  $\lambda v$  with  $v = u^2$ , and put into  $l$  instead of  $q$ .

So, for the following, we assume that  $c$  and  $l$  can be written as:

$$\begin{aligned} c &= c_0 + \lambda c_1 \\ l(\cdot) &= l_0(\cdot) + \lambda l_1(\cdot) \end{aligned} \tag{15}$$

where  $c_0$ ,  $c_1$ ,  $l_0$  and  $l_1$  are independent of  $\lambda$ . Under this hypothesis, the operators  $C$  and  $L$  simply become

$$\begin{aligned} C &= C_0 + \lambda C_1 \\ L(\cdot) &= L_0(\cdot) + \lambda L_1(\cdot) \end{aligned} \tag{16}$$

and the final algebraic system (14)

$$\mathbf{R}(\mathbf{U}) = \mathbf{L}0 + \mathbf{L}(\mathbf{U}) + \mathbf{Q}(\mathbf{U}, \mathbf{U}) \tag{17}$$

with  $\mathbf{U} = [U^t, \lambda, \omega]^t$  and

$$\begin{aligned} \mathbf{L}0 &= C_0 \\ \mathbf{L}(\mathbf{U}) &= L_0(U) + \lambda C_1 \\ \mathbf{Q}(\mathbf{U}, \mathbf{U}) &= Q(U, U) + \lambda L_1(U) - \omega M(U) \end{aligned} \tag{18}$$

In (17),  $\mathbf{L}0$  is a constant vector,  $\mathbf{L}(\cdot)$  a linear vector valued operator and  $\mathbf{Q}(\cdot, \cdot)$  a bilinear vector valued operator. In the following, we make a brief review of the ANM continuation technique for solving such quadratic system.

### 2.4.2 The ANM continuation

The particularity of the ANM is to give access to branches of solution as power series. Suppose we know a regular solution point  $\mathbf{U}_0$ , the branch of solution passing through this point is computed as a power serie expansion (truncated at order  $n$ ) of the pseudo-arclength path parameter  $a = (\mathbf{U} - \mathbf{U}_0)^t \mathbf{U}_1$ , where  $\mathbf{U}_1$  is the tangent vector at  $\mathbf{U}_0$ :

$$\mathbf{U}(a) = \mathbf{U}_0 + a\mathbf{U}_1 + a^2\mathbf{U}_2 + a^3\mathbf{U}_3 + \dots + a^n\mathbf{U}_n. \quad (19)$$

The serie is replaced in equation (17) and each power of  $a$  is equated to zero, leading to a serie of linear systems :

- **order 0** :  $\mathbf{L}\mathbf{0} + \mathbf{L}(\mathbf{U}_0) + \mathbf{Q}(\mathbf{U}_0, \mathbf{U}_0) = 0$ , which is obvious since  $\mathbf{U}_0$  is a solution of (17).
- **order 1** :  $\mathbf{L}(\mathbf{U}_1) + \mathbf{Q}(\mathbf{U}_0, \mathbf{U}_1) + \mathbf{Q}(\mathbf{U}_1, \mathbf{U}_0) = 0$ , which can also be written  $\mathbf{J}_{\mathbf{U}_0} \mathbf{U}_1 = 0$  with  $\mathbf{J}_{\mathbf{U}_0} \in \mathbb{R}^{N \times N+1}$  the jacobian matrix of  $\mathbf{R}$  evaluated at  $\mathbf{U}_0$ .
- **order  $2 \leq p \leq n$**  :  $\mathbf{J}_{\mathbf{U}_0} \mathbf{U}_p + \sum_{i=1}^{p-1} \mathbf{Q}(\mathbf{U}_i, \mathbf{U}_{p-i}) = 0$

The original nonlinear problem has therefore been reduced to a serie of  $n$  linear systems of  $N_e$  equations. However, for each order, the linear systems are under-dimensioned since they have  $N_e + 1$  unknowns. The additional equations required are given by inserting (19) into the definition of the path parameter  $a$  given above. This leads to:

- **order 1** :  $\mathbf{U}_1^t \mathbf{U}_1 = 1$
- **order  $2 \leq p \leq n$**  :  $\mathbf{U}_1^t \mathbf{U}_p = 0$

### 2.4.3 Remarks:

- The original nonlinear system of  $N_e$  equations has been transformed in  $n$  linear systems of  $N_e$  equations which can be solved successively (like in classical perturbation methods) i.e.  $\mathbf{U}_p$  is deduced from terms found at lower orders.
- Once each  $\mathbf{U}_p$  has been found, the area of validity of the series expansion is evaluated by calculating the value  $a_{max}$  such that

$$\forall a \in [0, a_{max}], \|\mathbf{R}(\mathbf{U}(a))\| \leq \epsilon_r, \quad \text{where } \epsilon_r \text{ is a user-defined parameter} \quad (20)$$

Using the approximation (see ([31])  $\mathbf{R}(\sum_{i=0}^n a^i \mathbf{U}_i) \simeq a^{n+1} F_{n+1}$  where  $F_{n+1}$  is the rhs vector of the linear system at order  $n + 1$ ,  $a_{max}$  is simply given by

$$a_{max} = \left( \frac{\epsilon_r}{\|F_{n+1}\|} \right)^{1/(n+1)} \quad (21)$$

It should be noticed that the range of validity  $a_{max}$  is generally closed to the radius of convergence of the series ([32]).

- The serie expansion and the associated range of validity  $a_{max}$  define a piece of the branch of solution, named *section*. The next section of the branch can be found by initializing the algorithm with the upper bound of this section of branch. The ANM is therefore able to find the entire branch as a succession of different sections of branch, the length of each being automatically given by the range of validity  $a_{max}$ .
- The ANM approach has some interesting properties that are recalled hereafter:
  - the solution branch is known analytically, section by section.
  - because all the linear systems to solve have the same jacobian matrix, the computational cost of the series (19) is low.
  - since the size of the section is given by the convergence properties of the current step, ie, the values of  $a_{max}$ , the ANM continuation algorithm does not require any strategy for the step-length control.
  - the ANM continuation algorithm is very robust, even when the branch has sharp turns. As a consequence, branch switching can be easily performed by using small pertubation in the system.

## 2.5 Implementation in the MANLAB software

### 2.5.1 A brief presentation of MANLAB

MANLAB is an interactive software for the continuation and the bifurcation analysis of algebraic system with the ANM continuation. The last version is programmed in Matlab using an objet-oriented approach ([33]). MANLAB has a Graphical User Interface (GUI) with buttons, on-line inputs and graphical windows for generating, visualising and analysing the bifurcation diagram and the solution of the system. To input the system of equation, the user has to provide three vector valued Matlab functions corresponding to the constant, linear and quadratic operators  $\mathbf{L0}$ ,  $\mathbf{L}$  and  $\mathbf{Q}$ . As an illustrative example, we consider the bio-chimical reaction system used by Doedel et Al in ([5])

$$\begin{aligned}
 r_1(u_1, u_2, \lambda) &= 2u_1 - u_2 + 100\frac{u_1}{1+u_1+u_1^2} - \lambda &= 0 \\
 r_2(u_1, u_2, \lambda) &= 2u_2 - u_1 + 100\frac{u_2}{1+u_2+u_2^2} - (\lambda + \mu) &= 0.
 \end{aligned}
 \tag{22}$$

Introducing the following additional variables  $v_1 = u_1 + u_1^2$ ,  $v_2 = u_2 + u_2^2$ ,  $v_3 = \frac{1}{1+v_1}$  and  $v_4 = \frac{1}{1+v_2}$ , the system is rewritten as,

$$\begin{aligned}
0 \quad +2u_1 - u_2 - \lambda + 100u_1v_3 &= 0 \\
-\mu \quad +2u_2 - u_1 - \lambda + 100u_2v_4 &= 0 \\
0 \quad +v_1 - u_1 \quad -u_1^2 &= 0 \\
0 \quad +v_2 - u_2 \quad -u_2^2 &= 0 \\
-1 \quad +v_3 &+ v_1v_3 = 0 \\
\underbrace{-1}_{\mathbf{L0}} \quad \underbrace{+v_4}_{\mathbf{L(U)}} &+ \underbrace{v_2v_4}_{\mathbf{Q(U,U)}} = 0
\end{aligned} \tag{23}$$

with  $U = [u_1, u_2, v_1, v_2, v_3, v_4, \lambda]$ . The three Matlab functions read when  $\mu = 0.05$ :

function [L0] = L0	function [L] = L(U)	function [Q] = Q(U,V)
L0=zeros(6,1);	L=zeros(6,1);	Q=zeros(6,1);
L0(1)= 0;	L(1)=2*U(1)- U(2) -U(7);	Q(1)=100*U(1)*V(5);
L0(2)= -0.05;	L(2)=2*U(2)- U(1) -U(7);	Q(2)=100*U(2)*V(6);
L0(3)= 0;	L(3)=U(3)-U(1);	Q(3)= -U(1)*V(1);
L0(4)= 0;	L(4)=U(4)-U(2);	Q(4)= -U(2)*V(2);
L0(5)= -1;	L(5)=U(5);	Q(5)= U(3)*V(5);
L0(6)= -1;	L(6)=U(6);	Q(6)= U(4)*V(6);

### 2.5.2 Implementation of the periodic solution continuation in MANLAB

To compute the branches of periodic solutions of a system, the user has to first recast his system under the form (3) with account of the additional decomposition (15). This is probably the most unusual and difficult task, for a new user. Afterward, following the same philosophy as above, the user has only to provide the operator  $m(\cdot)$ ,  $c_0$ ,  $c_1$ ,  $l(\cdot)$   $l_1(\cdot)$  and  $q(\cdot, \cdot)$  to the MANLAB software. The functions **L0**, **L** and **Q** which are the actual entry of MANLAB have been programmed once for all, using the expression (18) and the formulas of annexe 1. The examples demonstrated in this paper are available at <http://www.lma.cnrs-mrs.fr/~manlab>.

## 3 The case of a periodically forced system

We now focus on periodically forced (non-autonomous) systems:

$$\dot{Y} = f(t, Y, \lambda) \tag{24}$$

where  $f$  is periodic in  $t$ , with period  $T$  (forcing period). We look for periodic solutions (response) with a period  $pT$  or  $\frac{T}{p}$  where  $p$  is an integer. A classical strategy is to transform (24) into an augmented autonomous system ([3]), for instance, by adding an oscillator with the desired forcing period in the system of equations ([6]).

In the following, we use a direct approach which consists in expanding the forcing term into harmonics, and to take them into account in the balance of individual harmonics. It is illustrated below on two more examples.

**Example 4 :** Forced Duffing oscillator.

The normalized forced Duffing oscillator is the non-autonomous equation:

$$\ddot{u} + 2\mu\dot{u} + u + u^3 = f \cos(\lambda t) \quad (25)$$

We take the damping coefficient  $\mu$  and the force amplitude  $f$  constant, and we use the forcing angular frequency as the varying parameter  $\lambda$ . By using  $v(t) = \dot{u}(t)$  and  $w(t) = u^2(t)$ , this equation can be recasted as

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= f \cos(\lambda t) - 2\mu v - u - uw \\ \dot{w} &= 2uv \end{aligned} \quad (26)$$

$$\underbrace{\dot{0}}_{m(\dot{Z})} = \underbrace{0}_{c(t,\lambda)} \underbrace{w}_{l(Z)} \underbrace{-uu}_{q(Z,Z)}$$

with  $Z = [u, v, w]^t$ , and the forcing term deliberately put into  $c$ . We now link the forcing frequency with the response frequency by putting  $\lambda = \omega$  or possibly,  $\lambda = p\omega$  ( $p$  integer). Then, the term  $c(t)$  is expanded into harmonics with respect to  $\omega$ .

This results in a slight modification of the procedure:

- because of the synchronization of the response and the forcing, the phase condition has to be removed. Notice that the parameter  $\lambda$  is no more an unknown since it is linked to  $\omega$ . In short, we have one less equation and one less unknown than for the case of autonomous systems.
- for the final system (17)(18), the operators  $C_1$  and  $L_1$  disappear and the forcing amplitude  $f$  should be accounted for in  $L_0$

As a final illustration, we consider the Rayleigh-Plesset equation that models the large amplitude vibrations of a gaz bubble in a fluid. The treatment of the forcing is a little different and it permits also to show how to deal with a power  $-3$ .

**Example 5 :** Forced Rayleigh-Plesset equation ([34]).

Let  $R$  be the radius of the vibrating bubble, and  $R_0$  the radius at rest, the equation is

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = A\left\{\left(\frac{R_0}{R}\right)^3 - 1\right\} + B \cos(\lambda t) \quad (27)$$

with  $A$  and  $B$  fixed. By using  $u = \frac{R}{R_0}$ ,  $v = \frac{\dot{R}}{R_0}$ ,  $x = \frac{1}{u}$ ,  $y = x^2$ , and  $z = v^2$  this equation can be recasted as

$$\dot{v} = -\frac{A}{R_0^2}x + \frac{A}{R_0^2}y^2 - \frac{3}{2}xz + \frac{B}{R_0^2}x \cos(\lambda t) \quad (28)$$

After this transformation the forcing term is now multiplied by the variable  $x(t)$  and cannot be put directly into the operator  $c$  (the same situation occurs for parametrically forced system such as the Mathieu model equation). In that case, we introduce another variable  $r(t) = \cos(\lambda t)$ , and finally, we get

$$\begin{aligned} \dot{u} &= & v \\ \dot{v} &= & -\frac{A}{R_0^2}x & + \frac{A}{R_0^2}y^2 - \frac{3}{2}xz + \frac{B}{R_0^2}xr \\ 0 &= & 1 & -ux \\ 0 &= & y & -xx \\ 0 &= & z & -vv \\ \underbrace{0}_{m(\dot{Z})} &= & \underbrace{-\cos(\lambda t)}_{c(t)} & \underbrace{r}_{l(Z)} & \underbrace{\hspace{10em}}_{q(Z,Z)} \end{aligned} \quad (29)$$

with  $Z = [u, v, x, y, z, r]^t$ .

#### 4 Numerical results for selected examples

For the selected examples, numerical results obtained following the approach detailed in this paper are compared with time-domain simulations to validate our approach, or used to highlight particular features: influence of the number of harmonics for the Van der Pol oscillator, ability to follow period-doubling bifurcations for the Rössler system, ability to follow a direct or inverse Hopf bifurcation for the clarinet model and finally an illustration of a forced system with the Duffing oscillator.

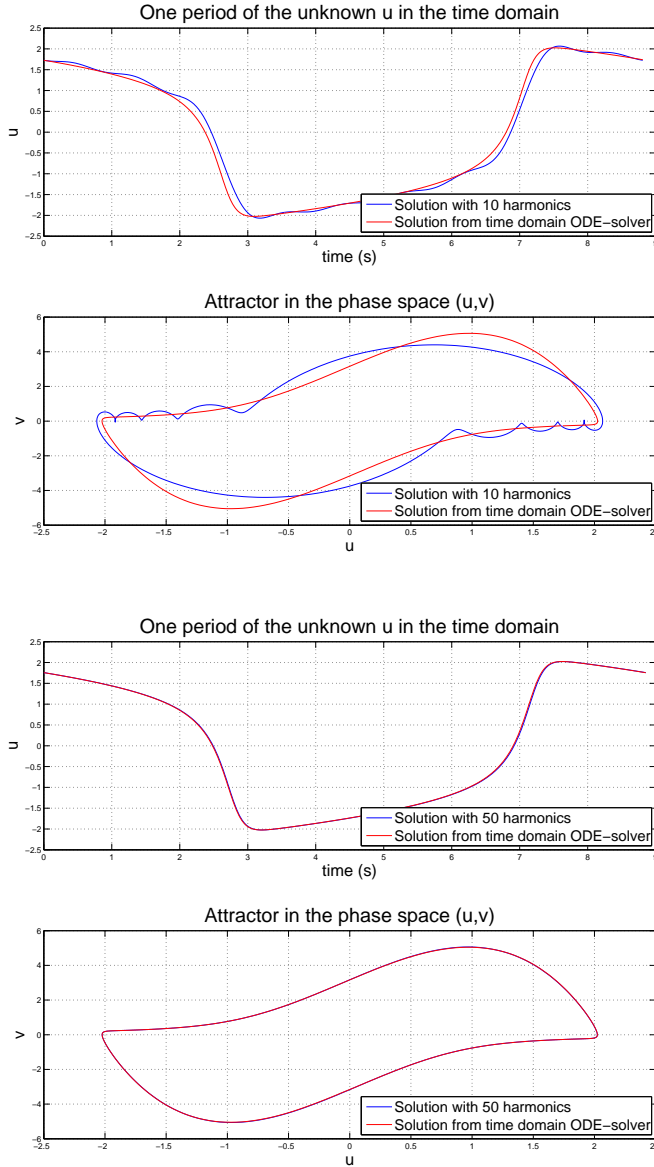


Figure 1. Van der Pol oscillator (Eq. (4)): validation of the harmonic balance results by comparison with a time domain simulation. Influence of the number of harmonics  $H$  taken into account (top  $H=10$ , bottom  $H=50$ ).

**Example 1: Van der Pol oscillator.** Numerical results related to the Van der Pol oscillator (Eq. (4)) are displayed in figure 1. Confrontation with a time domain simulation, using Matlab ODE solvers, shows a good agreement. As expected, the number of harmonics  $H$  of the solution sought by harmonic balance must be adapted to span the bandwidth of the solution found by direct integration. In the case of figure 1,  $\lambda = 3$ , which requires a relatively high  $H$  to match the reference solution.



**Example 2: Rössler system.** The ability to follow period-doubling bifurcations is illustrated in figure 2. Following such bifurcations is more difficult in the case of autonomous systems than in forced systems, since the period of oscillation is also an unknown. Moreover, since a  $T$ -periodic solution is also a  $2T$ -periodic solution, one cannot expect the period as found by the harmonic balance process to double when crossing a period-doubling bifurcation. Therefore, the trick used here is to introduce subharmonics complex amplitudes as additional unknowns. Then, to allow the detection of  $K$  period-doubling bifurcations, the sought solution is written:

$$Z(t) = Z_0 + \sum_{k=1}^H Z_{c,k} \cos\left(\frac{k}{2^K}\omega t\right) + \sum_{k=1}^H Z_{s,k} \sin\left(\frac{k}{2^K}\omega t\right) \quad (30)$$

When the solution belongs to the branch of the  $T$ -periodic solution,  $Z_{c,k}|_{k=1..K-1} = 0$  and  $Z_{s,k}|_{k=1..K-1} = 0$ . When the solution belongs to the branch of the  $2T$ -periodic solution,  $Z_{c,k}|_{k=1..K-1, k \neq 2^{K-1}} = 0$  and  $Z_{s,k}|_{k=1..K-1, k \neq 2^{K-1}} = 0$ . A practical consequence in term of computational cost is that in order to span the same bandwidth, the number of harmonics must be multiplied by  $2^K$ . This is illustrated in figure 2 where  $H = 10$  from the  $T$ -periodic solution to  $2^2 \times 10 = 40$  for the solution after two period-doubling bifurcations.

### Example 3: The clarinet model.

Two typical examples of bifurcation diagrams obtained with the clarinet model are shown on figure 3. These pictures focus around the oscillation threshold in order to stress the robustness of the method even at singular points. As seen on both pictures, the radius of convergence of power series expansion decreases when approaching a bifurcation point (and hence the length of each section of a branch lying between two consecutive points on figure 3). This is a typical behavior of ANM ([32]).

### Example 4: The forced Duffing oscillator

For this example, we show the frequency-amplitude diagram of the response obtained with the proposed method. We can see a classical bent resonance curve with some additional peaks corresponding to superharmonic resonances.

It should be noted that we have plotted the individual amplitude  $A_i = \sqrt{(u_{c_i}^2 + u_{s_i}^2)}$  of each odd harmonics, the even harmonics ( $A_0, A_2, A_4, \dots$ ) being zero. The computation of this branch of periodic orbits has required 25 steps of MAN-continuation when 5 harmonics are included, and 35 steps when 9 harmonics are included. We can see that the curves for harmonics 1 and 3 are slightly modified when passing from  $H = 5$  to  $H = 9$  harmonics in (11). The curve for  $A_5$  is more modified, and this confirms the logical result that more than 5 harmonics has to be used to obtain an accurate result for the harmonic 5.

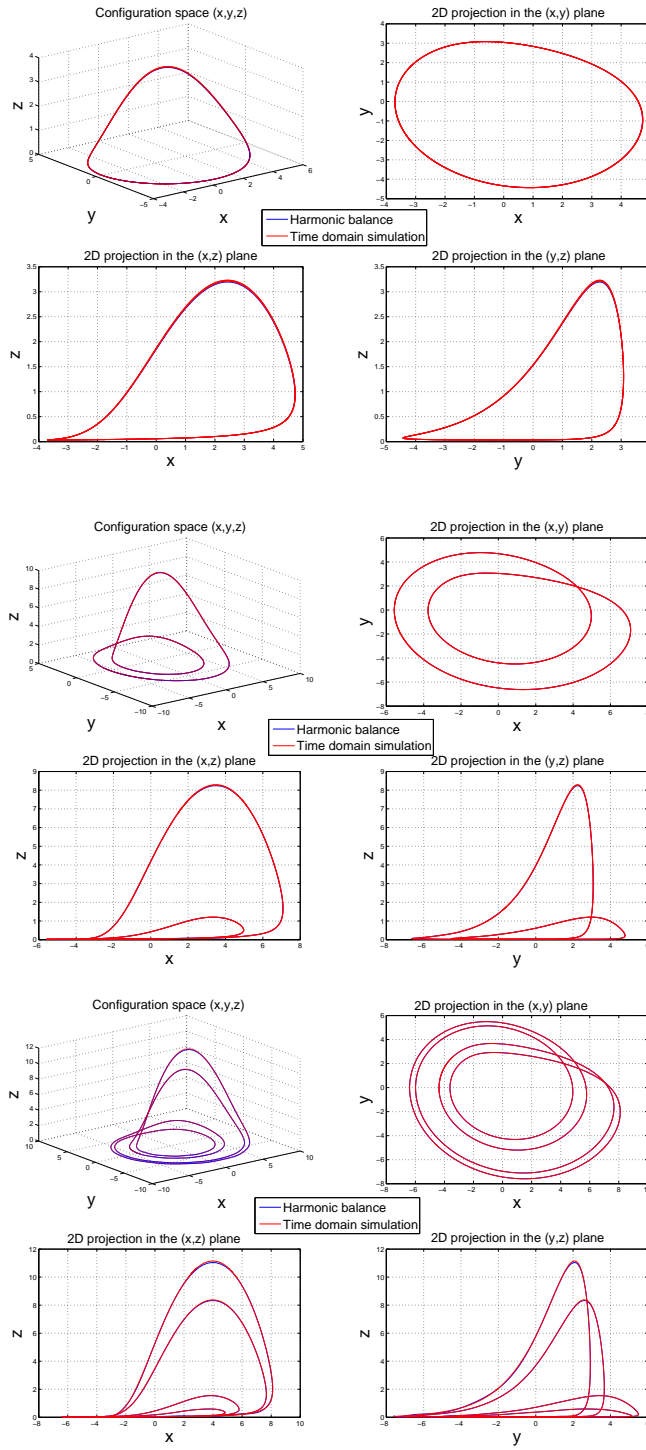


Figure 2. Rössler system (Eq. (7) with  $a = b = 0.2$ ): validation of the harmonic balance results by comparison with a time domain simulation. These three figures demonstrate the ability of the method to follow the sub-harmonic cascade, from  $\lambda = 2.5$  (period  $T$ ), to  $\lambda = 3.5$  (period  $2T$  after a first period-doubling bifurcation) and  $\lambda = 4$  (period  $4T$  after a second period-doubling bifurcation)

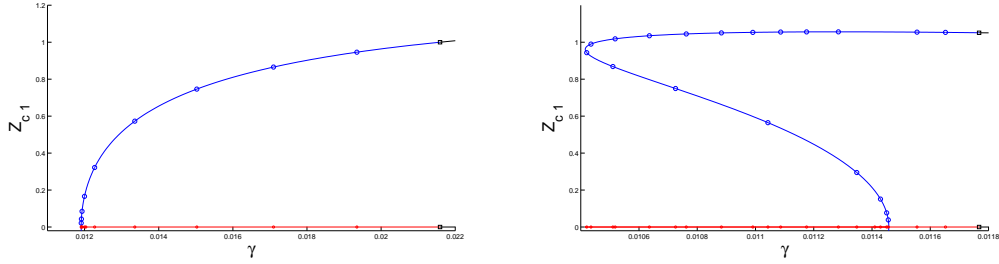


Figure 3. Clarinet model (Eq. (9) with  $q_r = 0.01$ ,  $\omega_r = 6597$ ,  $H = 3$ ,  $N = 5$ ,  $c = 340$ ,  $\zeta = 0.2$ ,  $l_t = 5.6e - 08$ ,  $l_v = 4e - 08$ ,  $C_{pv} = 1.4$ ,  $r = 0.007$ ,  $w_1 = 2426.5$ ,  $w_2 = 7279.5$ ,  $w_3 = 12132.5$ ,  $w_4 = 16985.4$ ,  $w_5 = 21838.4$ ,  $\alpha_1 = 0.11$ ,  $\alpha_2 = 0.19$ ,  $\alpha_3 = 0.25$ ,  $\alpha_4 = 0.3$ ,  $\alpha_5 = 0.34$ ): depending on the length of the resonator, the bifurcation is a direct Hopf bifurcation (left picture,  $L = 0.22$ ) or an inverse Hopf bifurcation (right picture,  $L = 0.2$ ).

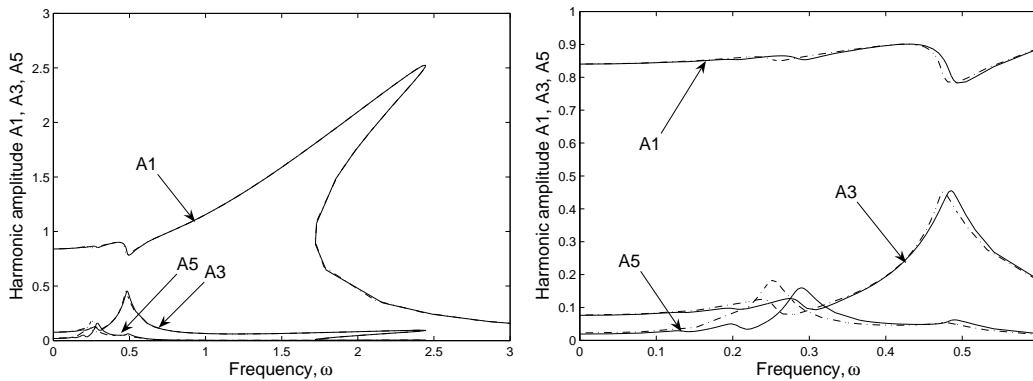


Figure 4. Duffing oscillator (25) with  $\mu = 0.1$  and  $f = 1.25$  as in ([18]). The figure shows the amplitude of the harmonics 1, 3 and 5 versus the frequency forcing  $\omega$ . Solid line : results when 5 harmonics are included ( $H=5$  in (11)). Dash line : results when 9 harmonics are included. The figure on the right is a zoom of the region  $\omega < 0.6$ .

We have also verified that the curve for  $A_5$  is only very slightly modified when passing from  $H = 9$  to  $H = 11$  harmonics in (11).

## 5 Discussion and Conclusion

A key idea in this work is the quadratic recast. In fact, it is worth pointing out that there are two quadratic recasts. The first one, in the time domain, transforms the original system (1) in (3), and is presented in section 2.2. The second one is more transparent for the user, and allows to formulate the continuation problem as the quadratic system (17) and (18). This point is at the origin of many characteristics of the method presented:

- All the procedure is in the frequency domain with an analytical expression

of the system to solve (like with classical harmonic balance) but allows to seek solutions with an arbitrarily high number of harmonics (like with AFT procedures).

- There is no iterative process to solve the system, hence no problems of convergence, and a computational cost always predictable (and rather low). On the contrary, convergence of the iterative process may be difficult with AFT procedures ([38, sect. 4.6, p117]).
- There is no need for FFT/FFT<sup>-1</sup> procedures, which increase the computational load of AFT approaches.
- There is no temporal discretization (since all the procedure lie in the frequency domain), and therefore no need to cope with aliasing.
- There is no need to use numerical schemes to compute derivatives. Indeed since the problem is quadratic, the jacobian is calculated analytically and very easily. This is a consequent advantage of the approach presented. Indeed, as stated by [38, p17], usually "analytical calculation of the jacobian involves considerable calculation and transformation" while on the other hand, when considering numerical calculation of the jacobian (for iterative approaches), "for very nonlinear circuits, the error introduced in the jacobian estimation results in inaccurate updates of the unknowns and a large number of iterations and possibly no convergence".

According to [11], to apply HB, "it is always necessary to write specific computation programs for different nonlinear models". On the contrary, another interesting feature of the method presented is the natural splitting between the automated part, which is independent of the problem (i.e. the transformation from operators  $c, l, q$  to operators  $\mathbf{L0}, \mathbf{L}, \mathbf{Q}$  and the ensuing solving by the ANM), and the part which depends on the particular problem to solve (the writting of operators  $c, l, q$ ) which is let to the user. From the point of view of the user, this means that the implication is relatively low and the software easy to use. Moreover, very few parameters have to be defined by the user : indeed, only two parameters have to be set, the number of harmonics of the solution and the parameter  $\epsilon_r$  in equation (20), which controls the precision of the ANM computation.

In this paper, we have presented only small size systems for simplicity of the presentation. The extension to large system of equations is obvious on the theoretical point of view, though, it may require additional technical work, and introduce limitation due to computer ressources. In [35] the proposed strategy has already been applied to compute the forced responses of geometrically nonlinear elastic structures discretized by the finite element method. Choosing the velocity  $v$ , the displacement  $u$  and the second Piola-Kirchhoff  $S$  as the unknown variables, the governing equation (equilibrium and constitutive law)

of a periodically forced elastic structural model reads (discrete form)

$$\begin{aligned}
\dot{u} &= v \\
\mathbf{M}\dot{v} &= -\int_V (B_L + B_{NL}(u))^t S dv + \mathbf{F} \cos(\lambda t) \\
S &= D(B_L + \frac{1}{2}B_{NL}(u))u
\end{aligned} \tag{31}$$

where  $\mathbf{M}$  is the mass matrix,  $D$  the Hooke operator and  $B_L, B_{NL}$  the linear and nonlinear strain-displacement operator [36]. The algebra given in annexe 1 has been directly introduced into the FEM code that supports the ANM continuation.

To finish, we discuss the limitations of the method and give some directions for future research. First of all, it is worth noting that when the original system is not directly under the quadratic form (3), the introduction of auxiliary variables as shown in the examples may add additional (possibly unphysical solutions) due to sign ambiguity. For example, in the case of the clarinet model (example 3), a variable  $v = \sqrt{\gamma - p}$  is introduced, and then further defined by  $v^2 = \gamma - p$ . Such a definition corresponds to  $v = \pm\sqrt{\gamma - p}$ . Practically, this means that the user has to check that the branch of solution computed corresponds to a positive  $v$ .

Today, the main limitation of the proposed method is that not all the original systems can be put under the quadratic form (3). For that reason, problems where one or several relations are given in the frequency domain (typically, when some variables are linked through an impedance) cannot be treated. Similar difficulties arise when the original system contains functions (trigonometric, exponential, non-integer powers,...) that are applied to the unknowns. The classical pendulum model  $\ddot{\theta} + \sin(\theta) = 0$  is a good example of such a system. In that cases, we can generally introduce new variables and new differential equations, so that the function is generated by the system itself. For the pendulum example, we introduce the velocity  $v = \dot{\theta}$  and the two new variables  $s(t) = \sin(\theta(t))$  and  $c(t) = \cos(\theta(t))$ . Taking the derivative with respect to time for the last two equations, we arrive at

$$\begin{aligned}
\dot{\theta} &= v \\
\dot{v} &= -s \\
\dot{s} &= cv \\
\dot{c} &= -sv
\end{aligned} \tag{32}$$

This quadratic system corresponds to the pendulum model provided that the two additional initial conditions are added

$$\begin{aligned} s(0) &= \sin(\theta(0)) \\ c(0) &= \cos(\theta(0)) \end{aligned} \tag{33}$$

The application of the HBM to (32) is now straightforward, but, because of (33), the final algebraic system is not quadratic. In that cases, the application of the ANM continuation is still feasible but it requires a more elaborate algebra for the computation of the powers series, as it has been explained in ([37,31]). The efficiency will be tested in a near future. Another interesting direction will be to test the capacity of the proposed method to solve regularized non-smooth dynamic problems, and for instance, the case of vibrating systems with contact conditions and frictions laws.

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## 6 Annexe 1

We give here the expression of the operators  $M$ ,  $C$ ,  $L$  and  $Q$  of (13) with respect to  $m$ ,  $c$ ,  $l$ ,  $q$  and  $H$ . We recall that (13) is obtained by inserting (11) into (3) and by collecting the term with the same harmonic index (cosine and sine). The entry of  $M$ ,  $L$  and  $Q$  is the vector  $U$ , containing the Fourier coefficients of  $Z(t)$

$$U = [Z_0^t, Z_{c,1}^t, Z_{s,1}^t, Z_{c,2}^t, Z_{s,2}^t, \dots, Z_{c,H}^t, Z_{s,H}^t]^t \quad (34)$$

### 6.0.3 Constant term

The constant vector  $c$  is grouped with the harmonic zero. The operator  $C$  is simply

$$C = [c^t, 0^t, 0^t, \dots]^t \quad (35)$$

### 6.0.4 Linear term

For the linear operator  $l$ , we have

$$\begin{aligned} l(Z(t)) &= l(Z_0 + \sum_{k=1}^H Z_{c,k} \cos(k\omega t) + Z_{s,k} \sin(k\omega t)) \\ &= l(Z_0) + l(Z_{c,1}) \cos(\omega t) + l(Z_{s,1}) \sin(\omega t) + l(Z_{c,2}) \cos(2\omega t) + \dots \end{aligned} \quad (36)$$

The operator  $L$  is

$$L(U) = [l(Z_0)^t, l(Z_{c,1})^t, l(Z_{s,1})^t, l(Z_{c,2})^t, l(Z_{s,2})^t, \dots, l(Z_{c,H})^t, l(Z_{s,H})^t]^t \quad (37)$$

For the linear operator  $m$ , we have

$$\begin{aligned} m(\dot{Z}(t)) &= m(\sum_{k=1}^H Z_{s,k} k\omega \cos(k\omega t) - Z_{c,k} k\omega \sin(k\omega t)) \\ &= 0 + m(Z_{s,1})\omega \cos(\omega t) - m(Z_{c,1})\omega \sin(\omega t) + 2m(Z_{s,2})\omega \cos(2\omega t) + \dots \end{aligned} \quad (38)$$

The operator  $M$  is

$$M(U) = [0^t, m(Z_{s,1})^t, -m(Z_{c,1})^t, 2m(Z_{s,2})^t, -2m(Z_{c,2})^t, \dots, Hm(Z_{s,H})^t, -Hm(Z_{c,H})^t]^t \quad (39)$$

### 6.0.5 Quadratic term

With the notation,  $Z^{(0)} = Z_0$  and  $Z^{(i)} = Z_{c,i} \cos(\omega t) + Z_{s,i} \sin(\omega t)$ , the decomposition of  $Z(t)$  is conveniently rewritten as

$$Z(t) = \sum_{i=0}^H Z^{(i)} \quad (40)$$

For the operator  $q$ , we have

$$\begin{aligned} q(X(t), Y(t)) &= \sum_{i=0}^H \sum_{j=0}^H q(X^{(i)}, Y^{(j)}) \\ &= q(X^{(0)}, Y^{(0)}) + \sum_{i=1}^H q(X^{(i)}, Y^{(0)}) + \sum_{j=1}^H q(X^{(0)}, Y^{(j)}) + \sum_{i=1}^H \sum_{j=1}^H q(X^{(i)}, Y^{(j)}) \end{aligned}$$

When  $i \geq 1$  and  $j \geq 1$ ,  $q(X^{(i)}, Y^{(j)})$  give harmonics  $i + j$  and  $i - j$  as follows,

$$\begin{aligned} q(X^{(i)}, Y^{(j)}) &= \frac{1}{2} \{q(X_{c,i}, Y_{c,j}) - q(X_{s,i}, Y_{s,j})\} \cos((i + j)\omega t) \\ &\quad + \frac{1}{2} \{q(X_{c,i}, Y_{s,j}) + q(X_{s,i}, Y_{c,j})\} \sin((i + j)\omega t) \\ &\quad + \frac{1}{2} \{q(X_{c,i}, Y_{c,j}) + q(X_{s,i}, Y_{s,j})\} \cos((i - j)\omega t) \\ &\quad + \frac{1}{2} \{q(X_{s,i}, Y_{c,j}) - q(X_{c,i}, Y_{s,j})\} \sin((i - j)\omega t) \end{aligned} \quad (42)$$

By grouping the term with the same harmonics index, and canceling the harmonics higher than index  $H$ , we have

$$q(X(t), Y(t)) = q_0 + \sum_{k=1}^H q_{c,k} \cos(k\omega t) + q_{s,k} \sin(k\omega t) \quad (43)$$

with

$$q_0 = q(X_0, Y_0) + \sum_{j=1}^H \frac{1}{2} \{q(X_{c,j}, Y_{c,j}) + q(X_{s,j}, Y_{s,j})\} \quad (44)$$

and for  $i \geq 1$

$$q_{c,i} = \{q(X_{c,i}, Y_0) + q(X_0, Y_{c,i})\} + \sum_{j=1}^{i-1} \frac{1}{2} \{q(X_{c,j}, Y_{c,i-j}) - q(X_{s,j}, Y_{s,i-j})\} \\ + \sum_{j=i+1}^H \frac{1}{2} \{q(X_{c,j}, Y_{c,j-i}) + q(X_{s,j}, Y_{s,j-i})\} + \frac{1}{2} \{q(X_{c,j-i}, Y_{c,j}) + q(X_{s,j-i}, Y_{s,j})\} \quad (45)$$

$$q_{s,i} = \{q(X_{s,i}, Y_0) + q(X_0, Y_{s,i})\} + \sum_{j=1}^{i-1} \frac{1}{2} \{q(X_{c,j}, Y_{s,i-j}) + q(X_{s,j}, Y_{c,i-j})\} \\ + \sum_{j=i+1}^H \frac{1}{2} \{-q(X_{c,j}, Y_{s,j-i}) + q(X_{s,j}, Y_{c,j-i})\} + \frac{1}{2} \{q(X_{c,j-i}, Y_{s,j}) - q(X_{s,j-i}, Y_{c,j})\} \quad (46)$$

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