Robust filtering for TDR traces

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***Abstract*—Time-domain reflectometry (TDR) is a valuable technique for the measurement of dielectric properties of soils. The filtering of noisy TDR traces is treated in this paper. The problem of biased estimation occurs when the permittivity of the soil varies with the depth. In practical issues, only the apparent permittivity of the soil along the TDR line is available. Hence, robust optimal filtering has to be developed to provide a robust and reliable estimation. This filtering step is essential for the estimation of the permittivity with the depth.**

***Index Terms*—Discrete time filters, Kalman filtering, robustness, time-domain reflectometry (TDR), uncertain systems.**

I. Introduction

Soil moisture estimation is based on time-domain reflectometry (TDR) measurements. The TDR technique consists of at least two rods that are "plugged" in the soil. At the inlet of the rods, we impose a step voltage and measure the variation of the current at the inlet of the line. This variation is related to the water content of the soil under consideration. Original works have shown that a relation exists between the permittivity of the soil and its water content [9], [11], [14]. The estimation of the apparent permittivity of the soil can be obtained from TDR traces, whereas the estimation of the variation of the permittivity along the rods is an open problem [5], [10], [12], [15]. In both cases, the estimation is based on the analysis of the recorded measurements. Consequently, as the data are corrupted by noise, they should be filtered before use.

In this context, a major problem is the lack of information on the permittivity profile in the soil. In fact, to filter the data, we should refer to the model of the TDR line. Since this model is uncertain because of the impreciseness of the estimation of the permittivity, we should develop an estimator that will ensure good estimation performance in the presence of modeling errors.

In this paper, the TDR line is described by means of a state-space representation after a centered finite-difference discretization. The development of this representation is given in detail in Section II. The effect of model mismatch on the signal estimation by a standard Kalman filter is also shown as a benchmark problem in Section II. The robust estimation technique is presented in Section III. It is based on the so-called compensated Kalman filter [3], [6]. This approach permits one to keep the computational burden at the same level as the standard Kalman filter. This point is essential as the discretization of the TDR line entails a large-scale model.

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P. Neveux is with the Faculté des Sciences, Unité Mixte de Recherche (UMR) 1114 EMMAH, Institut National de la Recherche Agronomique, Université d'Avignon et des Pays de Vaucluse, 84029 Avignon, France (e-mail: philippe.neveux@univ-avignon.fr).

E. Blanco is with the Laboratoire Ampère, UMR 5005 CNRS, Ecole Centrale de Lyon, 69134 Ecully, France (e-mail: eric.blanco@ec-lyon.fr).

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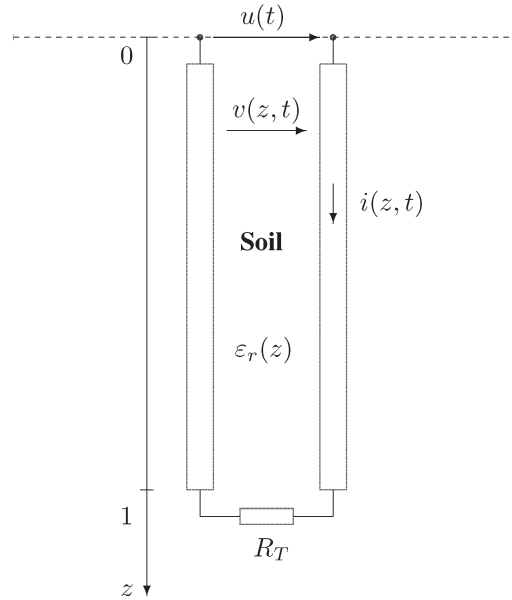

Fig. 1. Schematic view of the TDR line.

II. Position of the Problem

The TDR line (see Fig. 1) is described in 1-D by the telegraphist's equations [7]. The dimensionless form (Σ_∂) of the model is given by the following set of equations [2]:

$$(\Sigma_\partial)\begin{cases}\mu_r\partial_t i = -\partial_z v - ai\\ \varepsilon_r\partial_t v = -\partial_z i - bv\end{cases}$$

with the boundary conditions

$$v(z=0,t) = u(t) \tag{1}$$

$$i(z=1,t) = \frac{v(z=1,t)}{R_T} \tag{2}$$

where

- $i(z,t)$ and $v(z,t)$: intensity and voltage at time t and position $z \in [0,1]$;
- ε_r: relative permittivity of the soil evolving with depth z;
- μ_r: relative magnetic permeability of the soil;
- a and b: related to the resistivity of the rods and the permittivity of the soil, respectively;

$u(t)$ deterministic input signal that can be assimilated to the Heaviside function;
R_T resistivity of the soil at the end of the line;
∂_x partial derivative with respect to x.

The objective of this paper is to develop a formalism that permits one to estimate the inlet current $i(z = 0, t)$ from its noisy measurement when the permittivity ε_r evolves with the depth and the designer holds the apparent permittivity.

A. State-Space Modeling of the TDR Line

The discretization of the equations of (Σ_∂) will be done due to the centered finite-difference method, except for the boundary conditions [10]. The discretization in direction z with a step Δz permits one to define $N + 1$ interpolation nodes. The objective of this section is to write the model (Σ_∂) (considering that ε_r is constant) in the form (Σ_s)

$$(\Sigma_s) \begin{cases} X^{k+1} = FX^k + GU^k \\ Y^k = HX^k \end{cases}$$

where

k discrete time $k\Delta t$ with Δt as the time discretization step;
X^k, U^k, and Y^k state vector of the line, input, and measured output at time k, respectively;
F, G, and H constant real-valued matrices with appropriate dimensions.

The development of this representation will be made in three steps, namely

1) for the interior nodes;
2) for the boundary nodes;
3) for the overall transmission line.

In steps 1) and 2), we will present the results of the discretization by the appropriate finite-difference method, and then, we will give the structure of the matrices F, G, and H in step 3).

1) Interior Nodes: For each interpolation node with a node number n such that $0 < n < N$, we discretize the model (Σ_∂) with the centered finite difference. Thus, we obtain

$$i_n^{k+1} = i_n^{k-1} - \alpha i_n^k - \beta \left(v_{n+1}^k - v_{n-1}^k\right) \tag{3}$$

$$v_n^{k+1} = v_n^{k-1} - \gamma v_n^k - \delta \left(i_{n+1}^k - i_{n-1}^k\right) \tag{4}$$

with the parameters defined by

$$\alpha = \frac{2a\Delta t}{\mu_r} \quad \beta = \frac{\Delta t}{\mu_r \Delta z}$$
$$\gamma = \frac{2b\Delta t}{\varepsilon_r} \quad \delta = \frac{\Delta t}{\varepsilon_r \Delta z}.$$

2) Boundary Nodes: The derivation of the expression of the current and voltage at the boundary nodes will be done using the classical finite-difference method. From the conditions (1) and (2), we obtain the following.

- At the beginning of the line $(n = 0)$

$$\begin{aligned} i_0^{k+1} &= \left(1 - \frac{\alpha}{2}\right) i_1^k - \frac{\beta}{2} \left[v_1^k - u^k\right] \\ v_0^k &= u^k. \end{aligned} \tag{5}$$

- At the end of the line $(n = N)$

$$\begin{aligned} i_N^k &= \frac{v_N^k}{R_T} \\ v_N^{k+1} &= \left(1 - \frac{\alpha}{2} - R_T\beta\right) v_N^k + R_T\beta v_{N-1}^k. \end{aligned} \tag{6}$$

3) State-Space Model of the Line: To build the matrices F and G, we define the matrices

$$\tilde{F}_- = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{bmatrix}, \qquad \tilde{F}_0 = 1 - \frac{\alpha}{2}$$

$$\tilde{F}_+ = \begin{bmatrix} 0 & 0 & -\frac{\beta}{2} & 0 \end{bmatrix}$$

$$F_- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 \\ \delta & 0 & 0 & 0 \end{bmatrix}, \qquad F_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\gamma \end{bmatrix}$$

$$F_+ = -F_-$$

$$\overline{F}_- = [0 \quad 0 \quad R_T\beta \quad 0] \qquad \overline{F}_0 = 1 - \frac{\alpha}{2} - R_T\beta$$

$$\overline{F}_+ = \begin{bmatrix} 0 \\ -\beta \\ 0 \\ -\frac{\delta}{R_T} \end{bmatrix}$$

$$G_0 = \frac{\beta}{2}, \qquad G_1 = \begin{bmatrix} 0 \\ \beta \\ 0 \\ 0 \end{bmatrix}.$$

The state-space model of the line is given by the result.

Proposition 1: The 1-D state-space model of the TDR line is given by (Σ_s), where

- the dynamic matrix F is obtained by means of concatenation, i.e.,

$$F = \begin{bmatrix} \tilde{F}_0 & \tilde{F}_+ & 0 & \cdots & \cdots & 0 \\ \tilde{F}_- & F_0 & F_+ & \ddots & & \vdots \\ 0 & F_- & F_0 & F_+ & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & F_- & F_0 & \overline{F}_+ \\ 0 & \cdots & \cdots & 0 & \overline{F}_- & \overline{F}_0 \end{bmatrix}$$

- the input matrix G is given by

$$G = \begin{bmatrix} G_0 \\ G_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- the output matrix H is given by

$$H = [\,1 \quad 0 \quad \cdots \quad 0\,].$$

The proof of this result is given in the Appendix.

Remark 1: In this paper, we present the model for ε_r and b constant with z. The model for $\varepsilon_r(z)$ and $b(z)$ can be easily derived. In fact, in the space dicretization, we should consider $\varepsilon_{r,n}$ and b_n. Consequently, we will have γ_n and δ_n. To construct F and G, the concatenation would be the same, with the block matrices indexed with the number of nodes.

B. TDR Trace Filtering With the Kalman Filter

1) Standard Kalman Filter: The model (Σ_s) is a typical difference state-space model that is used in linear discrete-time systems in the control and estimation theory [1], [4]. In the context of filtering, the model is modified to take into account both input and output noises. Clearly, (Σ_s) is written as (Σ'_s), i.e.,

$$(\Sigma'_s)\begin{cases} X^{k+1} = FX^k + GU^k + GW^k \\ Y^k = HX^k + V^k \\ \Upsilon^k = HX^k \end{cases}$$

where W^k and V^k are zero-mean white processes, which are mutually uncorrelated with covariances Q and R, respectively.

The celebrated Kalman filter for the system (Σ'_s) is given by the following algorithm [4].

1) State estimate extrapolation:

$$\hat{X}_-^{k+1} = F\hat{X}_+^k + GU^k.$$

2) Error covariance extrapolation:

$$P_-^{k+1} = FP_+^kF' + GQG'.$$

3) Kalman gain matrix:

$$K^{k+1} = P_-^{k+1}H'\left(HP_-^{k+1}H' + R\right)^{-1}.$$

4) State estimate update:

$$\hat{X}_+^{k+1} = \hat{X}_-^{k+1} + K^{k+1}\left(Y^{k+1} - H\hat{X}_-^{k+1}\right).$$

5) Output estimate:

$$\hat{\Upsilon}^{k+1} = H\hat{X}^{k+1}.$$

6) Error covariance update:

$$P_+^{k+1} = (I - K^{k+1}H)P_-^{k+1}.$$

H' stands for the transpose of matrix H.

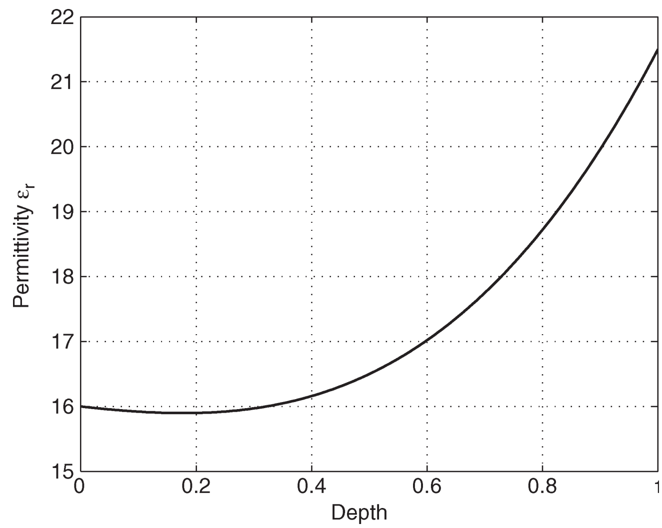

Fig. 2. Benchmark problem. Variation of ε_r with the depth.

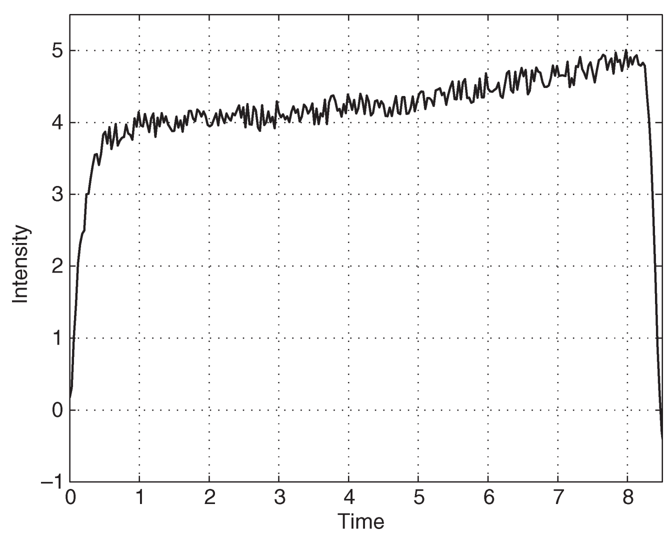

Fig. 3. Benchmark problem. Measured signal Y^k.

2) Benchmark Problem: This modeling procedure has been developed for soil with permittivity varying with depth (see Fig. 2), according to the trends given in Remark 1. The state-space model has been implemented with the apparent permittivity ($\bar{\varepsilon}_r = 17.25$). The simulated measured output signal Y^k is given in Fig. 3. The result of the estimation is given in Fig. 4, together with the TDR trace. It clearly appears that the Kalman filter based on the apparent permittivity model fails to estimate the TDR trace. To overcome this problem, we should use a filter that permits one to estimate the TDR trace, even if the model is erroneous.

III. Robust Filtering Technique

A. Filtering Procedure

Different approaches have been developed to estimate the signal Υ^k in the presence of a model error. In this paper, as the discretized system (Σ'_s) is a large-scale system, we have decided to develop a so-called robust filter with memory cost

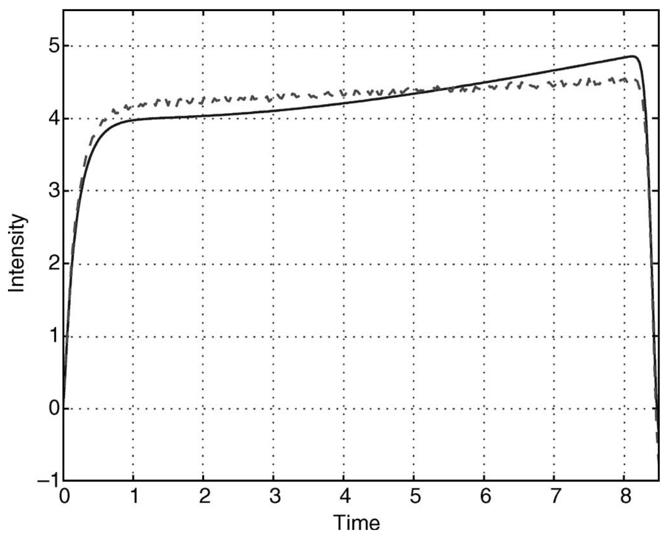

Fig. 4. Benchmark problem. Signal Υ^k (solid line) and Kalman filter estimate (dotted line).

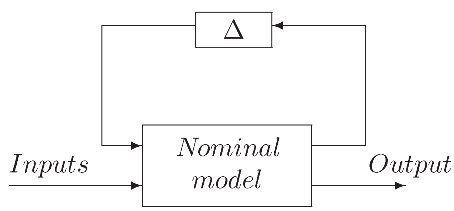

Fig. 5. Feedback representation of the uncertainties.

and computational burden equivalent to those of the standard Kalman filter presented in Section II-B1.

In fact, the celebrated techniques of robust filtering are intimately related to the notion of exogenous perturbation (see [8] and references therein) that enters the nominal model in a feedback manner (see Fig. 5). This exogenous perturbation is, by construction, correlated with the state of the system. Hence, in addition to the error covariance matrix, we should also compute a correlation matrix with the same dimensions. These matrices are obtained by means of two distinct Riccati equations. In the context of this paper, this solution is not convenient.

The elegant solution developed in this paper is based on the compensated Kalman filter theory [3], [6]. When using a compensated Kalman filter, we have the following representation:

1) state estimation extrapolation, i.e.,

$$\hat{X}_-^{k+1} = F\hat{X}_+^k + GU^k$$
$$\hat{\xi}_-^{k+1} = \hat{\xi}_+^k$$

2) state estimate update, i.e.,

$$\hat{X}_+^{k+1} = \hat{X}_-^{k+1} + \kappa_1\hat{\xi}_-^{k+1} + K^{k+1}\left(Y^k - H\hat{X}_-^{k+1}\right)$$
$$\hat{\xi}_+^{k+1} = \hat{\xi}_-^{k+1} + \tilde{\kappa}_2\left(Y^k - H\hat{X}_-^{k+1}\right)$$
$$\hat{\Upsilon}^{k+1} = H\hat{X}_+^{k+1}$$

where the state variable $\hat{\xi}_+^k$ is, by construction, the integral of the error signal $(Y^k - H\hat{X}_-^{k+1})$. This integration term plays the same role as the integration term used in the proportional–integral controller in the control theory. It permits one to ensure that the tracking error will vanish to zero. κ_1 and $\tilde{\kappa}_2$ are real-valued matrices with appropriate dimensions. They should be chosen such that the estimator is stable.

This estimator can be thought of as being designed for the following system:

$$\overline{X}^{k+1} = F\overline{X}^k + GU^k + GW^k + \kappa_1\xi^k$$
$$\xi^{k+1} = \xi^k + \kappa_2\overline{V}^k$$
$$Y^k = H\overline{X}^k + V^k$$
$$\Upsilon^k = H\overline{X}^k$$

where $\overline{V}^k$ is a zero-mean white process with unit covariance, which is uncorrelated with W^k and V^k.

Considering the augmented system with the state vector, we have a new representation (Σ_c) of the model, i.e.,

$$(\Sigma_c)\begin{cases}\chi^{k+1} = A\chi^k + BU^k + M\omega^k\\ Y^k = C\chi^k + V^k\\ \Upsilon^k = C\chi^k\end{cases}$$

with

$$A = \begin{bmatrix} I & 0\\ \kappa_1 & F\end{bmatrix} \quad B = \begin{bmatrix}0\\ G\end{bmatrix} \quad M = \begin{bmatrix}0 & \kappa_2\\ G & 0\end{bmatrix}$$
$$\chi^k = \begin{bmatrix}\xi^k\\ \overline{X}^k\end{bmatrix} \quad \omega^k = \begin{bmatrix}W^k\\ \overline{V}^k\end{bmatrix} \quad C = [0 \quad H]$$
$$E\{\omega^{k'}\omega^k\} = \begin{bmatrix}Q & 0\\ 0 & 1\end{bmatrix} = \mathcal{Q}.$$

Remark 2: The solution that we have presented can be viewed as a compensation of the model by adding a fictitious signal ζ^k, as follows:

$$\overline{X}^{k+1} = F\overline{X}^k + GU^k + GW^k + \zeta^k.$$

The compensation signal is considered as a Wiener process [13] by

$$\zeta^k = \kappa_1\xi^k$$
$$\xi^{k+1} = \xi^k + \kappa_2\overline{V}^k.$$

Hence, the robust filtering technique can also be considered as a deconvolution technique for the signal ζ^k.

In the original formulation of the compensated Kalman filter, the choice of the matrices κ_1 and $\tilde{\kappa}_2$ was critical with regard to the stability of the estimator. The proposed technique permits one to limit the stability condition to the matrix κ_1. In fact, κ_1 should be chosen such that the eigenvalues of A lie in the unit circle. It should be noticed that if the tuning parameter κ_1 is set to zero, the state vector of the line and the compensation signal will be decoupled; hence, the filter will be the standard Kalman filter. The parameter κ_2 appears in the structure of

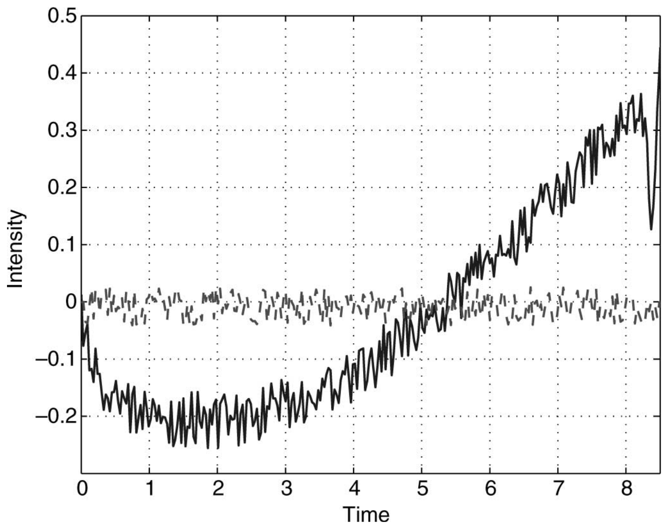

Fig. 6. Benchmark problem. Estimation error for the (solid line) standard Kalman filter and the (dashed line) proposed filter.

the noise effect on the system. Hence, the parameter κ_2 will permit one to tune the amount of noise in the estimation. If κ_2 is set to zero, the compensation signal will be considered as a constant and will stay at its initial value. If the initial value is null, then the estimator will be the standard Kalman filter.

The estimation algorithm is the same as the standard Kalman filter presented in the previous section, in which the corresponding matrices are replaced. In the context of TDR trace estimation, the output signal Y^k is a scalar signal. Thus, we only add a single state in the augmented model with respect to the original model (Σ'_s). This constant allows us to state that due to the large-scale model, the proposed method has fairly the same memory and computational costs as the standard Kalman filter.

B. Our Benchmark Problem

In this section, the proposed method has been applied to the case treated in Section II-B. The matrices κ_1 and κ_2 are chosen as follows:

$$\kappa_1 = 0.1\mathcal{U}, \quad \kappa_2 = 10^{-6}$$

where $\mathcal{U}$ is a unit vector of appropriate dimension.

To evaluate the performance of the proposed technique, we define the relative root-mean-square error (RRMSE) as follows:

$$\text{RRMSE} = \sqrt{\frac{E\left\{(\Upsilon^k - \hat{\Upsilon}^k)^2\right\}}{E\left\{(\Upsilon^k)^2\right\}}}$$

where $E\{\cdot\}$ is the mathematical expectation. The estimation error of the proposed technique is compared with the estimation error of the standard Kalman filter in Fig. 6. It clearly appears that the estimation error of the proposed technique has a magnitude that varies in a restricted area around zero, whereas the estimation error of the standard Kalman filter presents an important bias. This fact is strengthened by the value of the mean of the estimation error and the RRMSE given in Table I.

TABLE I
BENCHMARK PROBLEM: COMPARISON OF THE MEAN AND RRMSE OF THE ESTIMATION ERROR

	Mean	$RRMSE$
Standard Kalman filter	$78.7 \cdot 10^{-3}$	$74.8 \cdot 10^{-3}$
Proposed technique	$-8.5 \cdot 10^{-3}$	$5.5 \cdot 10^{-3}$

TABLE II
EXPERIMENTAL DATA: GRAVIMETRIC ESTIMATION OF ε_r

z	0.150	0.275	0.500	0.675	0.850
ε_r	5.511	5.572	5.760	6.832	6.796

It should be noticed that the optimal choice of the pair (κ_1, κ_2) is not unique. In fact, as κ_2 permits one to adjust the width of the bandpass of the estimator, the critical point lies in the choice of κ_1 such that the augmented system is stable. Another important point as to the choice of the tuning parameters is that we can concentrate the integral compensation on the states that are subject to an uncertain parameter. For example, in the benchmark problem, it appears that only the equations corresponding to the voltage are directly concerned with the parametric uncertainty. Hence, we could set to zero the components of κ_1 that do not correspond to states directly subject to uncertainties and set to 0.1 all the other components. In this case, the result is similar to the previous setting.

C. Experimental Data

The proposed filtering technique has been used on experimental data obtained from the Campbell TDR100 data acquisition device. The soil is a homogeneous sand medium. The data acquisition system has given an apparent permittivity $\bar{\varepsilon}_r = 6.08$. To verify whether the soil moisture profile is constant or not, a gravimetric study has been completed. The results of this study are given in Table II, where z is the dimensionless depth, and the permittivity is obtained by applying the Topp polynomial function [14]. In Table II, it appears that the permittivity evolves with the depth. Consequently, to filter the data, we have to apply the proposed method. As in the benchmark problem, we have compared our approach with the standard Kalman filter both developed with $\bar{\varepsilon}_r$. The result is given in Fig. 7. It appears that while the Kalman filter is biased, the proposed filter gives a good filtered estimate.

IV. CONCLUDING REMARKS

The estimation problem of TDR traces has been treated in this paper. The problem encountered is the lack of information that the designer holds on the permittivity profile along the TDR rods. Empirical techniques permit one to estimate an apparent value of this permittivity. We have shown on a benchmark problem that this value entails a biased estimation of the TDR traces. We have proposed an estimator based on the compensated Kalman filter theory that permits efficient estimation of the signal with the same computational burden as the standard Kalman filter.

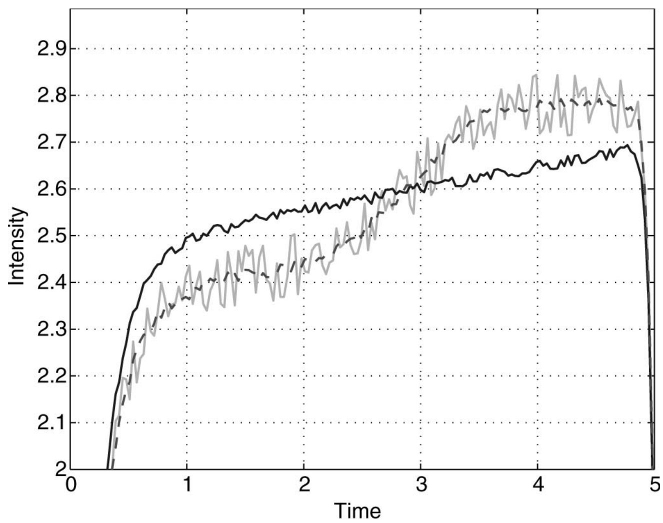

Fig. 7. Experimental data. (Solid gray line) Measured signal versus (solid dark line) standard Kalman filter estimate and (dashed line), proposed filter estimate.

Appendix
Proof of Proposition 1

To obtain (Σ_s), we should define the state variables as follows:

$$\begin{aligned} x_{n,1}^k &= i_n^k \\ x_{n,1}^{k+1} &= x_{n,2}^k \\ x_{n,3}^k &= v_n^k \\ x_{n,3}^{k+1} &= x_{n,4}^k. \end{aligned}$$

Then, (3) and (4) become

$$\begin{aligned} x_{n,2}^{k+1} &= x_{n,1}^k - \alpha x_{n,2}^k - \beta \left[x_{n+1,3}^k - x_{n-1,3}^k\right] \\ x_{n,4}^{k+1} &= x_{n,3}^k - \gamma x_{n,4}^k - \delta \left[x_{n+1,1}^k - x_{n-1,1}^k\right]. \end{aligned}$$

At the inlet $(n = 0)$, as the voltage is imposed by the user and considering (5), we set $x_{0,1}^k = i_0^k$. Thus

$$x_{0,1}^{k+1} = \left(1 - \frac{\alpha}{2}\right) x_{0,1}^k - \frac{\beta}{2} \left[x_{1,3}^k - u^k\right].$$

From (3), it is clear that the equation $i_1^{k+1} = x_{1,2}^{k+1}$ depends on v_0^k. Consequently, it should be modified as follows:

$$x_{1,2}^{k+1} = x_{1,1}^k - \alpha x_{1,2}^k - \beta \left[x_{2,3}^k - u^k\right].$$

At the end of the line $(n = N)$, considering (6), we set $x_{N,3}^k = v_N^k$. Thus

$$\begin{aligned} x_{N,1}^k &= \frac{x_{N,3}^k}{R_T} \\ x_{N,3}^{k+1} &= \left(1 - \frac{\alpha}{2} - R_T \beta\right) x_{N,3}^k + R_T \beta x_{N-1,3}^k. \end{aligned}$$

From (4), it is clear that the equation $v_{N-1}^{k+1} = x_{N-1,4}^{k+1}$ depends on i_N^k. Consequently, it should be modified as follows:

$$x_{N-1,4}^{k+1} = x_{N-1,3}^k - \gamma x_{N-1,4}^k - \delta \left[\frac{x_{N,3}^k}{R_T} - x_{N-2,1}^k\right].$$

To obtain the matrices F, G, and H, we introduce the node state vector $\psi_n^k = [x_{n,1}^k \quad x_{n,2}^k \quad x_{n,3}^k \quad x_{n,4}^k]'$. For the boundary nodes, it reduces to $\psi_0^k = x_{0,1}^k$ and $\psi_N^k = x_{N,3}^k$.

In the following, we will present a technique that permits one to obtain the matrices F and G. We will operate in ascending order of n.

Considering the previously given state equations together with the special cases due to the boundary conditions, we have

- for $n = 0$:

$$\psi_0^{k+1} = \tilde{F}_0 \psi_0^k + \tilde{F}_+ \psi_1^k + G_0 u^k \tag{7}$$

- for $n = 1$:

$$\psi_1^{k+1} = \tilde{F}_- \psi_0^k + F_0 \psi_1^k + F_+ \psi_2^k + G_1 u^k \tag{8}$$

- for $1 < n < N - 1$:

$$\psi_n^{k+1} = F_- \psi_{n-1}^k + F_0 \psi_n^k + F_+ \psi_{n+1}^k \tag{9}$$

- for $n = N - 1$:

$$\psi_{N-1}^{k+1} = F_- \psi_{N-2}^k + F_0 \psi_{N-1}^k + \overline{F}_+ \psi_N^k \tag{10}$$

- for $n = N$:

$$\psi_N^{k+1} = \overline{F}_- \psi_{N-1}^k + \overline{F}_0 \psi_N^k. \tag{11}$$

Now, introduce the state vector X^k, the input U^k, and the output Y_k as follows:

$$X^k = \begin{bmatrix} \psi_0^k \\ \psi_1^k \\ \vdots \\ \psi_{N-1}^k \\ \psi_N^k \end{bmatrix} \quad U^k = u^k \quad Y^k = i_0^k = \psi_0^k$$

with $X^k \in \Re^{(4N-2)\times 1}$.

Then, the result given in Theorem 1 can be easily derived. This completes the proof. ■

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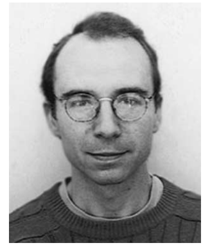

Philippe Neveux was born in 1970. He received the Ph.D. degree in automatic and signal processing from the Université de Lyon, Lyon, France, in 2000.

Since 2000, he has been with the Université d'Avignon et des Pays de Vaucluse, Avignon, France, where he works with the Institut National de la Recherche Agronomique on soil moisture estimation from TDR measurement. His main research activities are in the field of deconvolution, minimax optimization, and robust filtering.

Eric Blanco was born in 1975. He received the M.S. and Ph.D. degrees in automatic and signal processing from the Université de Lyon, Lyon, France, in 1998 and 2002, respectively.

From 1998 to 2003, he was with the Automatic and Process Engineering Laboratory (LAGEP), Université de Lyon, where he worked on robust estimation problems. In 2003, he obtained a chair in Automatic and Signal Processing with the Department of Electronics, Electrical and Control Engineering, Ecole Centrale of Lyon, Ecully, France. His current research interest at the Laboratoire Ampère includes control systems design, diagnosis, estimation, and prediction of complex systems.