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Olivier Guibé

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# UNIQUENESS OF THE SOLUTION TO QUASILINEAR ELLIPTIC EQUATIONS UNDER A LOCAL CONDITION ON THE DIFFUSION MATRIX

OLIVIER GUIBÉ

ABSTRACT. We prove the uniqueness of the renormalized solution to the elliptic equation  $-\operatorname{div}(\mathbf{A}(x, u)Du) = f + \operatorname{div}(g)$ . The data  $f + \operatorname{div}(g)$  belongs to  $L^1 + H^{-1}$  and we assume a local condition on the diffusion matrix  $\mathbf{A}(x, s)$  with respect to  $s$ .

## 1. INTRODUCTION

The present paper is concerned with the uniqueness of the solution to the quasilinear elliptic boundary-value problem on  $\Omega$

$$(1.1) \quad \begin{cases} -\operatorname{div}(\mathbf{A}(x, u)Du) = f + \operatorname{div}(g) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a bounded open subset of  $\mathbb{R}^N$ ,  $f \in L^1(\Omega)$ ,  $g \in (L^2(\Omega))^N$  and  $\mathbf{A}(x, s)$  is a Carathéodory function with matrix values.

When  $f$  belongs to  $L^2(\Omega)$  (i.e. the right-hand side of (1.1) lies in  $H^{-1}(\Omega)$ ) the variational solution of (1.1) is unique under a global Lipschitz condition on the function  $\mathbf{A}(x, s)$  with respect to the variable  $s$  (or a global and strong control of the modulus of continuity), see Artola [1986], Carrillo and Chipot [1985] and for more general and nonlinear operators Boccardo et al. [1992], Chipot and Michaille [1989]. Moreover in Carrillo and Chipot [1985], Chipot and Michaille [1989] the authors show that if  $\mathbf{A}(x, s)$  is Hölder continuous in  $s$  with a Hölder exponent greater or equal to  $1/2$  and if  $\mathbf{A}(x, s)$  is Lipschitz continuous in  $x$  then the solution is unique. For this last result the quasilinear character of the equation and the regularity of  $\mathbf{A}(x, s)$  in  $x$  are crucial.

In the case where  $f$  lies in  $L^1(\Omega)$  and if  $\mathbf{A}$  is uniformly coercive we cannot expect to have a solution of (1.1) in the sense of distributions without any growth condition on  $\mathbf{A}(x, s)$  with respect to  $s$ . Moreover it is well known that, in the simple case where  $\mathbf{A}$  does not depend on  $s$ , a solution in the sense of distribution exists (see e.g. Boccardo and Gallouët [1989]) but it is not unique in general (see the counter example in Serin [1964]). In the present paper we use the framework of renormalized solution (see Dal Maso et al. [1999], Murat [1993, 1994]) which insures the existence of such a solution when  $f$  belongs to  $L^1(\Omega)$ ,  $\mathbf{A}$  is uniformly coercive and  $\mathbf{A} \in L^\infty(\Omega \times ]-K, K])^{N \times N}$  for any  $K > 0$ .

Uniqueness results have been recently obtained in Blanchard et al. [2005] in the framework of renormalized solutions and in Porretta [2004] in the very close framework of entropy solutions for equations (1.1) with  $f$  belonging to  $L^1$  with very general

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and global conditions on the matrix field  $\mathbf{A}$ . Roughly speaking the modulus of continuity of  $\mathbf{A}$  with respect to  $s$  has to be controlled by  $\exp(c|s|)$  ( $c > 0$ ) in Porretta [2004] and by a function which satisfies an appropriate differential inequality in Blanchard et al. [2005].

In the present paper we state in Theorem 3.2 that the renormalized solution of (1.1) is unique if  $\mathbf{A}$  is locally Hölder continuous in  $s$  with a Hölder exponent greater or equal to  $1/2$  and under a global control of the modulus of continuity of  $\mathbf{A}$  with respect to the space variable  $x$ . The main novelty between our and known uniqueness results is the very local condition on  $\mathbf{A}$ , i.e. we do not assume any control on the growth of the modulus of continuity of  $\mathbf{A}$  in  $s$  as in the above cited papers. The price to pay to get rid of this global behavior is to assume a regularity with respect to  $x$ . The results obtained in the present paper rely on the mixing of the assumptions and the techniques developed in Carrillo and Chipot [1985] (see also Chipot and Michaille [1989]) with those used to study  $L^1$ -problems with the help of renormalized solutions (see Lemma 3.3 and Remark 3.4).

At last the question of the uniqueness under a local condition in  $s$  remains still open in general.

The paper is organized as follows. In Section 2 we give the assumptions on the data and we recall the definition of a renormalized solution of (1.1). Section 3 is devoted to a comparison result stated in Theorem 3.1 which implies the uniqueness of the solution given in Theorem 3.2.

## 2. ASSUMPTIONS AND DEFINITIONS

In the whole paper we assume that  $\mathbf{A} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{N \times N}$  is a Carathéodory function with  $\mathbf{A}(x, s) = (a_{ij}(x, s))_{1 \leq i, j \leq N}$  and such that

$$(2.1) \quad \exists \alpha > 0, \quad \mathbf{A}(x, s) \xi \cdot \xi \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \forall s \in \mathbb{R}, \text{ a.e. in } \Omega;$$

$$(2.2) \quad \forall K > 0, \quad \exists C_K > 0 \quad |\mathbf{A}(x, s)| \leq C_K, \quad \forall s \in [-K, K], \text{ a.e. in } \Omega;$$

for any  $r$  in  $\mathbb{R}$  and any  $1 \leq i, j \leq N$ , the function  $a_{ij}(r, \cdot)$  belongs to  $W^{1, \infty}(\Omega)$  and there exists  $M > 0$  such that

$$(2.3) \quad \left| \frac{\partial a_{ij}}{\partial x_k}(x, r) \right| \leq M \sum_{1 \leq i, j \leq N} a_{i, j}(x, r), \quad \forall r \in \mathbb{R}, \forall 1 \leq i, j \leq N, \text{ a.e. in } \Omega.$$

Moreover we assume that for any  $K > 0$  there exists a nonnegative, non decreasing continuous function  $\omega_K$  such that

$$(2.4) \quad |\mathbf{A}(x, s) - \mathbf{A}(x, r)| \leq \omega_K(|s - r|) \quad \forall r, s \in \mathbb{R} \text{ with } |s| \leq K, |r| \leq K, \text{ a.e. in } \Omega;$$

$$(2.5) \quad \int_{0^+} \frac{ds}{\omega_K^2(s)} = +\infty.$$

The data  $f$  and  $g$  are such that

$$(2.6) \quad f \in L^1(\Omega);$$

$$(2.7) \quad g \in (L^2(\Omega))^N.$$

**Remark 2.1.** Assumptions (2.1) and (2.2) are classical in the framework of renormalized solutions and allow to obtain the existence of such a solution with a data belonging to  $L^1 + H^{-1}$ . Conditions (2.4) and (2.5) concern a local condition on the modulus of continuity of the matrix field  $\mathbf{A}(x, s)$  in  $s$ . If the matrix field  $\mathbf{A}(x, s)$  does not depend on  $x$  and is locally Hölder continuous with an exponent greater or equal to  $1/2$  then assumptions (2.3), (2.4) and (2.5) are satisfied. Assumption (2.3) is crucial when  $\mathbf{A}(x, s)$  depends on  $x$ . As an example, if  $b$  is an element of  $W^{1,\infty}(\Omega)$  and  $h$  is a non negative locally Hölder continuous function with an exponent greater or equal to  $1/2$ , then  $\mathbf{A}(x, s) = (\exp(s^2) + b(x)h(s))I$  verifies (2.1)–(2.5).

**Remark 2.2.** Since  $\omega_K$  is a nonnegative, non decreasing continuous function satisfying (2.5) we can assume without loss of generality that there exists  $C_K$  such that

$$(2.8) \quad \forall 0 < r < 1, \quad \frac{r}{\omega_K^2(r)} \leq C_K.$$

Indeed it is sufficient to take  $\omega_K(r) + \sqrt{r}$  in place of  $\omega_K$  in (2.4) which also verifies condition (2.5).

For any  $K > 0$  we denote by  $T_K$  the truncation function at height  $\pm K$ ,  $T_K(s) = \max(-K, \min(K, s))$  for any  $s \in \mathbb{R}$  and we define the continuous function  $h_n$  by

$$(2.9) \quad h_n(s) = 1 - \left| \frac{T_{2n}(s) - T_n(s)}{n} \right|.$$

We now recall the definition of the gradient of functions whose truncates belong to  $H_0^1(\Omega)$  (see Bénénil et al. [1995]).

**Definition 2.3.** Let  $u : \Omega \rightarrow \mathbb{R}$  be a measurable function, finite almost everywhere in  $\Omega$ , such that  $T_K(u) \in H_0^1(\Omega)$  for any  $K > 0$ . Then there exists a unique measurable vector field  $v : \Omega \rightarrow \mathbb{R}^N$  such that

$$DT_K(u) = \mathbb{1}_{\{|u| < K\}} v \quad \text{a.e. in } \Omega.$$

This function  $v$  is called the gradient of  $u$  and is denoted by  $Du$ .

Following Dal Maso et al. [1999] (see also Murat [1993, 1994]) we now recall the definition of a renormalized solution to (1.1).

**Definition 2.4.** A measurable function  $u$  defined from  $\Omega$  into  $\mathbb{R}$  is called a renormalized solution of (1.1) if

$$(2.10) \quad \forall K > 0, \quad T_K(u) \in H_0^1(\Omega);$$

if for any function  $h \in W^{1,\infty}(\mathbb{R})$  such that  $\text{supp } h$  is compact,  $u$  satisfies the equation

$$(2.11) \quad -\text{div}[h(u)\mathbf{A}(x, u)Du] + h'(u)\mathbf{A}(x, u)Du \cdot Du \\ = fh(u) + \text{div}(gh(u)) - h'(u)g \cdot Du \quad \text{in } \mathcal{D}'(\Omega),$$

$$(2.12) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{n < |u| < 2n} \mathbf{A}(x, u)Du \cdot Dudx = 0$$

**Remark 2.5.** Condition (2.10) and Definition 2.3 allow to define  $Du$  almost everywhere in  $\Omega$ . In (2.11) which is formally obtained by the point-wise multiplication of (1.1) by  $h(u)$  every terms are well defined. Indeed since  $\text{supp}(h)$  is compact, we have  $\text{supp}(h) \subset [-K, K]$  for  $K > 0$  sufficiently large. It follows that  $h(u)\mathbf{A}(x, u)Du = h(u)\mathbf{A}(x, T_K(u))DT_K(u)$  almost everywhere in  $\Omega$  and then it belongs to  $(L^2(\Omega))^N$ . Similarly  $h'(u)\mathbf{A}(x, u)Du \cdot Du$  is identified to  $h'(u)\mathbf{A}(x, T_K(u))DT_K(u) \cdot DT_K(u)$  which belongs to  $L^1(\Omega)$ . The same arguments imply that the right hand side of (2.11) lies in  $L^1(\Omega) + H^{-1}(\Omega)$ . Condition (2.12) is classical in the framework of renormalized solutions and gives additional information on  $Du$  for large value of  $|u|$ .

It is well know that under assumptions (2.1), (2.2), (2.6) and (2.7) there exists at least one renormalized solution to equation (1.1), see e.g. Blanchard et al. [2005], Lions and Murat, Murat [1993, 1994].

### 3. MAIN RESULT

In Theorem 3.1 below we give a comparison result from which it follows a uniqueness result.

**Theorem 3.1.** *Assume that (2.1)–(2.5) hold true. Let  $f_1$  and  $f_2$  belong to  $L^1(\Omega)$  and let  $g_1$  and  $g_2$  belong to  $(L^2(\Omega))^N$  such that*

$$(3.1) \quad f_1 + \text{div}(g_1) \leq f_2 + \text{div}(g_2) \quad \text{in } \mathcal{D}'(\Omega)$$

*Let  $u_1$  be a renormalized solution of (1.1) with  $(f_1, g_1)$  in place of  $(f, g)$  and let  $u_2$  be a renormalized solution of (1.1) with  $(f_2, g_2)$  in place of  $(f, g)$ . Then  $u_1 \leq u_2$  almost everywhere in  $\Omega$ .*

An immediate consequence is the uniqueness of the solution for a fixed data  $f + \text{div}(g) \in L^1(\Omega) + H^{-1}(\Omega)$ .

**Theorem 3.2.** *Assume that (2.1)–(2.7) hold true. Then the renormalized solution of (1.1) is unique.*

To prove Theorem 3.1 we mix the methods developed by Chipot and Carrillo in Carrillo and Chipot [1985] (see also Chipot and Michaille [1989]) together with the techniques of renormalized solutions. The main tool is the following lemma which is a truncated version to Theorem 4 in Carrillo and Chipot [1985].

**Lemma 3.3.** *For any  $\varphi$  belonging to  $\mathcal{C}^1(\bar{\Omega})$*

$$(3.2) \quad \lim_{n \rightarrow +\infty} \int_{\{u_1 - u_2 > 0\}} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot D\varphi dx = 0.$$

**Remark 3.4.** In Carrillo and Chipot [1985] (see also Chipot and Michaille [1989]) when  $f + \text{div}(g)$  belongs to  $H^{-1}(\Omega)$  and under more restrictive conditions on the matrix field  $\mathbf{A}(x, s)$  (roughly speaking  $\mathbf{A}(x, s)$  is bounded) the authors state that

$$\int_{\{u_1 - u_2 > 0\}} (\mathbf{A}(x, u_1)Du_1 - \mathbf{A}(x, u_2)Du_2) \cdot D\varphi dx = 0.$$

In the  $L^1(\Omega) + H^{-1}(\Omega)$  case, since we do not have any growth assumption on  $\mathbf{A}(x, s)$  with respect to  $s$ , we cannot expect to have  $\mathbf{A}(x, u_1)Du_1$  in  $L^1_{\text{loc}}(\Omega)$ . It follows that the above equality does not have any sense or equivalently that the limit in (3.2) cannot be written in terms of  $u_1$  and  $u_2$ .

*Proof of Lemma 3.3.* Let  $\varphi$  belong to  $\mathcal{C}^1(\overline{\Omega})$  with  $\varphi \geq 0$  on  $\Omega$  and let  $n$  be a positive integer. Assume without loss of generality that  $\omega_{2n+1}(r) > 0 \forall r > 0$ . Following Chipot and Michaille [1989] let us define for any  $0 < \varepsilon < 1$

$$(3.3) \quad I_n(\varepsilon) = \int_{\varepsilon}^1 \frac{1}{\omega_{2n+1}^2(s)} ds,$$

$$(3.4) \quad F_n^\varepsilon(r) = \begin{cases} 1 & \text{if } r \geq 1, \\ \frac{1}{I_n(\varepsilon)} \int_{\varepsilon}^r \frac{1}{\omega_{2n+1}^2(s)} ds & \text{if } 1 > r > \varepsilon, \\ 0 & \text{if } r \leq \varepsilon. \end{cases}$$

From the definition of  $F_n^\varepsilon$  and the regularity of  $\omega_{2n+1}$  it follows that  $F_n^\varepsilon$  is a nonnegative Lipschitz continuous function such that  $F_n^\varepsilon(s) = 0 \forall s \leq \varepsilon$ .

Let us consider the test function  $W_n^\varepsilon = F_n^\varepsilon(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2)))\varphi$  which belongs to  $L^\infty(\Omega) \cap H_0^1(\Omega)$  due to (2.10) and the regularities of  $F_n^\varepsilon$  and  $\varphi$ . Moreover we have

$$\begin{aligned} DW_n^\varepsilon &= F_n^\varepsilon(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2)))D\varphi + \varphi T_1'(T_{2n+1}(u_1) - T_{2n+1}(u_2)) \\ &\quad \times (F_n^\varepsilon)'(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2)))(DT_{2n+1}(u_1) - DT_{2n+1}(u_2)) \end{aligned}$$

almost everywhere in  $\Omega$ .

Choosing  $h = h_n$  in (2.11) written in  $u_1$  yields

$$(3.5) \quad \begin{aligned} &\int_{\Omega} h_n(u_1)\mathbf{A}(x, u_1)Du_1 \cdot DW_n^\varepsilon dx + \int_{\Omega} h_n'(u_1)\mathbf{A}(x, u_1)Du_1 \cdot Du_1 W_n^\varepsilon dx \\ &= \int_{\Omega} h_n(u_1)f_1 W_n^\varepsilon dx - \int_{\Omega} h_n(u_1)g_1 \cdot DW_n^\varepsilon dx - \int_{\Omega} h_n'(u_1)g_1 \cdot Du_1 W_n^\varepsilon dx. \end{aligned}$$

Since  $\text{supp}(h_n) = [-2n, 2n]$  we have  $h_n(u_1)F_n^\varepsilon(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2))) = h_n(u_1)F_n^\varepsilon(T_1(u_1 - u_2))$  almost everywhere in  $\Omega$  and

$$h_n(u_1)\mathbb{1}_{\{|T_{2n+1}(u_1) - T_{2n+1}(u_2)| < 1\}}(DT_{2n+1}(u_1) - DT_{2n+1}(u_2)) = h_n(u_1)\mathbb{1}_{\{|u_1 - u_2| < 1\}}(Du_1 - Du_2)$$

almost everywhere in  $\Omega$ . It follows that (3.5) can be rewritten as

$$\begin{aligned} &\int_{\{|u_1 - u_2| < 1\}} h_n(u_1)\mathbf{A}(x, u_1)Du_1 \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx \\ &+ \int_{\Omega} h_n(u_1)\mathbf{A}(x, u_1)Du_1 \cdot D\varphi F_n^\varepsilon(T_1(u_1 - u_2))dx + \int_{\Omega} h_n'(u_1)\mathbf{A}(x, u_1)Du_1 \cdot Du_1 F_n^\varepsilon(T_1(u_1 - u_2))\varphi dx \\ &= \int_{\Omega} h_n(u_1)f_1 W_n^\varepsilon dx - \int_{\Omega} h_n(u_1)g_1 \cdot DW_n^\varepsilon dx - \int_{\Omega} h_n'(u_1)g_1 \cdot Du_1 F_n^\varepsilon(T_1(u_1 - u_2))\varphi dx. \end{aligned}$$

Subtracting the equivalent equality written in  $u_2$  gives

$$\begin{aligned}
& \int_{\{|u_1-u_2|<1\}} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx \\
& \quad + \int_{\Omega} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot D\varphi F_n^\varepsilon(T_1(u_1 - u_2)) dx \\
& \quad + \int_{\Omega} (h_n'(u_1)\mathbf{A}(x, u_1)Du_1 \cdot Du_1 - h_n'(u_2)\mathbf{A}(x, u_2)Du_2 \cdot Du_2) F_n^\varepsilon(T_1(u_1 - u_2))\varphi dx \\
& \quad = \int_{\Omega} (h_n(u_1)f_1 - h_n(u_2)f_2)W_n^\varepsilon dx - \int_{\Omega} (h_n(u_1)g_1 - h_n(u_2)g_2) \cdot DW_n^\varepsilon dx \\
& \quad \quad - \int_{\Omega} (h_n'(u_1)g_1 \cdot Du_1 - h_n'(u_2)g_2 \cdot Du_2) F_n^\varepsilon(T_1(u_1 - u_2))\varphi dx,
\end{aligned}$$

which reads as

$$(3.6) \quad A_{n,\varepsilon} + B_{n,\varepsilon} + C_{n,\varepsilon} = D_{n,\varepsilon} + E_{n,\varepsilon} + F_{n,\varepsilon}.$$

In the following we pass to the limit in (3.6) as  $\varepsilon$  tends to 0, and then as  $n$  tends to  $+\infty$ . We claim that

$$(3.7) \quad \liminf_{\varepsilon \rightarrow 0} A_{n,\varepsilon} \geq 0,$$

$$(3.8) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} |C_{n,\varepsilon}| = 0,$$

$$(3.9) \quad \limsup_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} (D_{n,\varepsilon} + E_{n,\varepsilon}) \leq 0,$$

$$(3.10) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} |F_{n,\varepsilon}| = 0.$$

*Proof of (3.7).* We split  $A_{n,\varepsilon}$  into

$$(3.11) \quad A_{n,\varepsilon} = A_{n,\varepsilon}^1 + A_{n,\varepsilon}^2 + A_{n,\varepsilon}^3$$

with

$$\begin{aligned}
A_{n,\varepsilon}^1 &= \int_{\{|u_1-u_2|<1\}} h_n(u_1)\mathbf{A}(x, u_1)(Du_1 - Du_2) \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx, \\
A_{n,\varepsilon}^2 &= \int_{\{|u_1-u_2|<1\}} h_n(u_1)(\mathbf{A}(x, u_1) - \mathbf{A}(x, u_2))Du_2 \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx, \\
A_{n,\varepsilon}^3 &= \int_{\{|u_1-u_2|<1\}} (h_n(u_1) - h_n(u_2))\mathbf{A}(x, u_2)Du_2 \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx.
\end{aligned}$$

Since  $h_n$ ,  $\varphi$  and  $(F_n^\varepsilon)'$  are nonnegative functions the coercivity of the matrix field  $\mathbf{A}(x, s)$  yields that

$$(3.12) \quad \liminf_{\varepsilon \rightarrow 0} A_{n,\varepsilon}^1 \geq \liminf_{\varepsilon \rightarrow 0} \alpha \int_{\{|u_1-u_2|<1\}} h_n(u_1)|Du_1 - Du_2|^2 dx \geq 0.$$

Recalling that  $\text{supp}(h_n) = [-2n, 2n]$ , assumption (2.4) implies that

$$\begin{aligned}
(3.13) \quad \mathbb{1}_{\{|u_1-u_2|<1\}} h_n(u_1)|\mathbf{A}(x, u_1) - \mathbf{A}(x, u_2)| &\leq \mathbb{1}_{\{|u_1-u_2|<1\}} h_n(u_1) \mathbb{1}_{\{|u_1|<2n+1\}} \\
&\quad \times \mathbb{1}_{\{|u_2|<2n+1\}} |\mathbf{A}(x, u_1) - \mathbf{A}(x, u_2)| \\
&\leq \mathbb{1}_{\{|u_1-u_2|<1\}} h_n(u_1) \mathbb{1}_{\{|u_1|<2n+1\}} \\
&\quad \times \mathbb{1}_{\{|u_2|<2n+1\}} \omega_{2n+1}(|u_1 - u_2|),
\end{aligned}$$

almost everywhere in  $\Omega$ . Young's inequality and (3.13) then lead to

$$\begin{aligned} |A_{n,\varepsilon}^2| &\leq \frac{\alpha}{2} \int_{\{|u_1 - u_2| < 1\}} h_n(u_1) |Du_1 - Du_2|^2 (F_n^\varepsilon)'(T_1(u_1 - u_2)) \varphi dx \\ &\quad + \frac{1}{2\alpha} \int_{\substack{\{|u_1 - u_2| < 1\} \cap \\ \{|u_1| < 2n+1\} \cap \{|u_2| < 2n+1\}}} |Du_2|^2 \omega_{2n+1}^2(|u_1 - u_2|) (F_n^\varepsilon)'(T_1(u_1 - u_2)) \varphi dx. \end{aligned}$$

Assumption (2.1), the definition (3.4) of  $F_n^\varepsilon$  and (3.12) give that

$$|A_{n,\varepsilon}^2| \leq \frac{1}{2} A_{n,\varepsilon}^1 + \frac{\|\varphi\|_{L^\infty(\Omega)}}{2\alpha I_n(\varepsilon)} \int_{\Omega} |DT_{2n+1}(u_2)|^2 dx.$$

Since  $\lim_{\varepsilon \rightarrow 0} I_n(\varepsilon) = +\infty$ , from (3.12) and the above inequality we get

$$(3.14) \quad \liminf_{\varepsilon \rightarrow 0} (A_{n,\varepsilon}^1 + A_{n,\varepsilon}^2) \geq \frac{1}{2} \liminf_{\varepsilon \rightarrow 0} A_{n,\varepsilon}^1 \geq 0.$$

As far as  $A_{n,\varepsilon}^3$  is concerned, due to Remark 2.2 we can choose  $\omega_{2n+1}$  such that

$$\forall 0 < r < 1 \quad \frac{r}{\omega_{2n+1}(r)} \leq C, \quad \text{with } C > 0.$$

Because the function  $h_n$  is Lipschitz continuous we deduce that

$$\|A_{n,\varepsilon}^3\| \leq \frac{C}{n I_n(\varepsilon)} \int_{\substack{\{|u_1| < 2n+1\} \cap \{|u_2| < 2n+1\}}} |\mathbf{A}(x, u_2) Du_2| (|Du_1| + |Du_2|) \varphi dx$$

from which it follows that

$$(3.15) \quad \lim_{\varepsilon \rightarrow 0} |A_{n,\varepsilon}^3| = 0.$$

From (3.11), (3.12), (3.14) and (3.15) we conclude that (3.7) holds.

*Proof of (3.8).* Due to the definitions of  $I_n(\varepsilon)$  and  $F_n^\varepsilon$  we have  $0 \leq F_n^\varepsilon(T_1(u_1 - u_2)) \leq 1$  almost everywhere in  $\Omega$ . Therefore we have

$$\left| \int_{\Omega} h_n'(u_1) \mathbf{A}(x, u_1) Du_1 \cdot Du_1 F_n^\varepsilon(T_1(u_1 - u_2)) \varphi dx \right| \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{n} \int_{\{|n < |u_1| < 2n\}} \mathbf{A}(x, u_1) Du_1 \cdot Du_1 dx$$

and condition (2.12) allows to obtain (3.8).

*Proof of (3.9).* Since  $W_n^\varepsilon$  belongs to  $H_0^1(\Omega) \cap L^\infty(\Omega)$  and since  $h_n(u_1)$  belongs to  $H^1(\Omega) \cap L^\infty(\Omega)$  we have

$$\begin{aligned} (3.16) \quad &\int_{\Omega} (h_n(u_1) f_1 - h_n(u_2) f_2) W_n^\varepsilon dx - \int_{\Omega} (g_1 h_n(u_1) - g_2 h_n(u_2)) \cdot DW_n^\varepsilon dx \\ &= \int_{\Omega} (f_1 - f_2) h_n(u_1) W_n^\varepsilon dx - \int_{\Omega} (g_1 - g_2) \cdot D(h_n(u_1) W_n^\varepsilon) dx \\ &+ \int_{\Omega} f_2 (h_n(u_1) - h_n(u_2)) W_n^\varepsilon dx - \int_{\Omega} (h_n(u_1) - h_n(u_2)) g_2 \cdot DW_n^\varepsilon dx + \int_{\Omega} h_n'(u_1) W_n^\varepsilon (g_1 - g_2) \cdot Du_1 dx. \end{aligned}$$

Due to the definitions of  $h_n$  and  $W_n^\varepsilon$ , the field  $h_n(u_1) W_n^\varepsilon$  is a non negative element of  $H_0^1(\Omega) \cap L^\infty(\Omega)$ . Assumption (3.1) on  $f_1 + \operatorname{div}(g_1)$  and  $f_2 + \operatorname{div}(g_2)$  leads to

$$(3.17) \quad \int_{\Omega} (f_1 - f_2) h_n(u_1) W_n^\varepsilon dx - \int_{\Omega} (g_1 - g_2) \cdot D(h_n(u_1) W_n^\varepsilon) dx \leq 0.$$

We now prove that the third, fourth and fifth terms of (3.16) tend to zero as  $\varepsilon$  goes to zero and then as  $n$  goes to infinity.



Recalling the definition of  $W_n^\varepsilon$  we have

$$(3.18) \quad \left| \int_{\Omega} f_2(h_n(u_1) - h_n(u_2)) W_n^\varepsilon dx \right| \leq \|\varphi\|_{L^\infty(\Omega)} \|F_n^\varepsilon\|_{L^\infty(\Omega)} \int_{\Omega} |f_2| |h_n(u_1) - h_n(u_2)| dx.$$

Since  $h_n \rightarrow 1$  in  $L^\infty$  weak-\* and almost everywhere in  $\Omega$  as  $n$  goes to infinity and since  $f_2 \in L^1(\Omega)$  the Lebesgue convergence theorem and the fact that  $|F_n^\varepsilon| \leq 1$  uniformly with respect to  $\varepsilon$  and  $n$  imply that

$$(3.19) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \left| \int_{\Omega} f_2(h_n(u_1) - h_n(u_2)) W_n^\varepsilon dx \right| = 0.$$

We now turn to the fourth term of (3.16). Since  $h_n$  is a Lipschitz continuous function we obtain (recalling the definition of  $W_n^\varepsilon$ )

$$(3.20) \quad \left| \int_{\Omega} (h_n(u_1) - h_n(u_2)) g_2 \cdot DW_n^\varepsilon dx \right| \\ \leq \frac{1}{n} \int_{\{|u_1| < 2n+1\} \cap \{|u_2| < 2n+1\}} |u_1 - u_2| |g_2| |(F_n^\varepsilon)'(T_1(u_1 - u_2))| |Du_1 - Du_2| \varphi dx \\ + \int_{\Omega} |h_n(u_1) - h_n(u_2)| |g_2| |D\varphi| |F_n^\varepsilon(T_1(u_1 - u_2))| dx.$$

On the one hand, due to arguments already used we know that

$$(3.21) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} |h_n(u_1) - h_n(u_2)| |g_2| |D\varphi| |F_n^\varepsilon(T_1(u_1 - u_2))| dx = 0.$$

On the other hand, Remark 2.2 yields that

$$\frac{1}{n} \int_{\substack{\{|u_1 - u_2| < 1\} \cap \\ \{|u_1| < 2n+1\} \cap \{|u_2| < 2n+1\}}} |u_1 - u_2| |g_2| \frac{|Du_1 - Du_2| |\varphi|}{I_n(\varepsilon) \omega_{2n+1}(|u_1 - u_2|)} dx \\ \leq \frac{C}{n I_n(\varepsilon)} \|g_2\|_{(L^2(\Omega))^N} [\|DT_{2n+1}(u_1)\|_{(L^2(\Omega))^N} + \|DT_{2n+1}(u_2)\|_{(L^2(\Omega))^N}].$$

Because  $I_n(\varepsilon) \rightarrow +\infty$  as  $\varepsilon$  goes to zero, from (3.20), (3.21) and the above inequality it follows that

$$(3.22) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \left| \int_{\Omega} (h_n(u_1) - h_n(u_2)) g_2 \cdot DW_n^\varepsilon dx \right| = 0.$$

For the last term in the right-hand side of (3.16), Hölder's inequality gives

$$\left| \int_{\Omega} h_n'(u_1) W_n^\varepsilon (g_1 - g_2) \cdot Du_1 dx \right| \leq \|W_n^\varepsilon\|_{L^\infty(\Omega)} \|g_1 - g_2\|_{(L^2(\Omega))^N} \left( \frac{1}{n^2} \int_{\{|u_1| < 2n\}} |Du_1|^2 dx \right)^{1/2}$$

and condition (2.12) then implies

$$(3.23) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \left| \int_{\Omega} h_n'(u_1) W_n^\varepsilon (g_1 - g_2) \cdot Du_1 dx \right| = 0.$$

Gathering (3.16), (3.17), (3.19), (3.22) and (3.23) we conclude that (3.9) holds true.

*Proof of (3.10).* We have

$$\begin{aligned} & \left| \int_{\Omega} (h'_n(u_1)g_1 \cdot Du_1 - h'_n(u_2)g_2 \cdot Du_2) W_n^\varepsilon dx \right| \\ & \leq \|W_n^\varepsilon\|_{L^\infty(\Omega)} \left( \|g_1\|_{(L^2(\Omega))^N} \frac{1}{2n} \|DT_{2n}(u_1)\|_{(L^2(\Omega))^N} + \|g_2\|_{(L^2(\Omega))^N} \frac{1}{2n} \|DT_{2n}(u_2)\|_{(L^2(\Omega))^N} \right). \end{aligned}$$

Recalling that  $W_n^\varepsilon$  is bounded in  $L^\infty(\Omega)$  uniformly with respect to  $n$  and  $\varepsilon$ , condition (2.12) implies (3.10).

We are now in a position to prove Lemma 3.3. With arguments already used we know that

$$(3.24) \quad \lim_{\varepsilon \rightarrow 0} B_{n,\varepsilon} = \int_{\{u_1 - u_2 > 0\}} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot D\varphi dx.$$

From equality (3.6) together with (3.7)–(3.10) and (3.24) it follows that

$$(3.25) \quad \limsup_{n \rightarrow +\infty} \int_{\{u_1 - u_2 > 0\}} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot D\varphi dx \leq 0.$$

Taking  $M - \varphi$  in place of  $\varphi$  in (3.25), with  $M$  sufficiently large so that  $M - \varphi \geq 0$ , gives

$$(3.26) \quad \liminf_{n \rightarrow +\infty} \int_{\{u_1 - u_2 > 0\}} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot D\varphi dx \geq 0.$$

At last (3.25) and (3.26) allow to conclude that (3.2) holds true. The proof of Lemma 3.3 is complete.  $\square$

With the help of Lemma 3.3 we now turn to Theorem 3.1.

*Proof of Theorem 3.1.* We use Lemma 3.3 with  $\varphi(x) = \exp(c \sum_{i=1}^N x_i)$ , where  $c > 0$ .

Since  $h_n(s) = 0$ ,  $\forall |s| \geq 2n$ , we have

$$h_n(u_1)\mathbf{A}(x, u_1)Du_1 \mathbb{1}_{\{u_1 - u_2 > 0\}} = h_n(T_{2n}(u_1))\mathbf{A}(x, T_{2n}(u_1))DT_{2n}(u_1) \mathbb{1}_{\{T_{2n}(u_1) - T_{2n}(u_2) > 0\}}$$

almost everywhere in  $\Omega$ . To shorten the notations we denote by  $u_1^{2n}$  the field  $T_{2n}(u_1)$  and by  $u_2^{2n}$  the field  $T_{2n}(u_2)$ . It follows that (3.2) can be rewritten as

$$(3.27) \quad \lim_{n \rightarrow +\infty} \int_{\{u_1^{2n} - u_2^{2n} > 0\}} (h_n(u_1^{2n})\mathbf{A}(x, u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x, u_2^{2n})Du_2^{2n}) \cdot D\varphi dx = 0.$$

Let us define

$$\tilde{a}_{i,j}^n(x, r) = \int_0^r a_{i,j}(x, s)h_n(s)ds.$$

Due to the regularity (2.10) of  $T_K(u_1)$  and  $T_K(u_2)$ , assumption (2.3) implies that both  $\tilde{a}_{i,j}^n(u_1^{2n})$  and  $\tilde{a}_{i,j}^n(u_2^{2n})$  belong to  $H_0^1(\Omega)$  and for  $l = 1, 2$

$$(3.28) \quad \frac{\partial \tilde{a}_{i,j}^n(u_l^{2n})}{\partial x_k} = h_n(u_l^{2n})a_{i,j}(u_l^{2n}) \frac{\partial u_l^{2n}}{\partial x_k} + \int_0^{u_l^{2n}} h_n(s) \frac{\partial a_{i,j}}{\partial x_k}(x, s)ds.$$

Since  $\frac{\partial \varphi}{\partial x_k} = c\varphi$ , using (3.28) we have

$$\begin{aligned} & \int_{\{u_1^{2n} - u_2^{2n} > 0\}} (h_n(u_1^{2n})\mathbf{A}(x, u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x, u_2^{2n})Du_2^{2n}) \cdot D\varphi dx \\ &= c \int_{\{u_1^{2n} - u_2^{2n} > 0\}} \sum_{1 \leq i, j \leq N} \left( \frac{\partial \bar{a}_{i,j}^n(u_1^{2n})}{\partial x_j} - \frac{\partial \bar{a}_{i,j}^n(u_2^{2n})}{\partial x_j} \right) \varphi dx \\ & \quad + c \int_{\{u_1^{2n} - u_2^{2n} > 0\}} \sum_{1 \leq i, j \leq N} \int_{u_1^{2n}}^{u_2^{2n}} h_n(s) \frac{\partial a_{i,j}}{\partial x_j}(x, s) ds \varphi dx. \end{aligned}$$

Let us define  $w_{2n} = (u_1^{2n} - u_2^{2n})^+$  which belongs to  $L^\infty(\Omega) \cap H_0^1(\Omega)$  and is such that  $u_1^{2n} = w_{2n} + u_2^{2n}$  almost everywhere on  $\{u_1^{2n} - u_2^{2n} > 0\}$ . Since  $\bar{a}_{i,j}^n(u_2^{2n} + w_{2n}) - \bar{a}_{i,j}^n(u_2^{2n})$  lies in  $L^\infty(\Omega) \cap H_0^1(\Omega)$ , a few computations and the integration by part formula give

$$\begin{aligned} & \int_{\{u_1^{2n} - u_2^{2n} > 0\}} (h_n(u_1^{2n})\mathbf{A}(x, u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x, u_2^{2n})Du_2^{2n}) \cdot D\varphi dx \\ &= c \int_{\Omega} \sum_{1 \leq i, j \leq N} \left( \frac{\partial \bar{a}_{i,j}^n(u_2^{2n} + w_{2n})}{\partial x_j} - \frac{\partial \bar{a}_{i,j}^n(u_2^{2n})}{\partial x_j} \right) \varphi dx \\ & \quad + c \int_{\Omega} \sum_{1 \leq i, j \leq N} \int_{u_2^{2n}}^{u_2^{2n} + w_{2n}} h_n(s) \frac{\partial a_{i,j}}{\partial x_j}(x, s) \varphi dx \\ &= -c^2 \int_{\Omega} \sum_{1 \leq i, j \leq N} \left( \bar{a}_{i,j}^n(u_2^{2n} + w_{2n}) - \bar{a}_{i,j}^n(u_2^{2n}) \right) \varphi dx \\ & \quad + c \int_{\Omega} \sum_{1 \leq i, j \leq N} \int_{u_2^{2n}}^{u_2^{2n} + w_{2n}} h_n(s) \frac{\partial a_{i,j}}{\partial x_j}(x, s) \varphi dx \\ &= -c \int_{\Omega} \int_{u_2^{2n}}^{u_2^{2n} + w_{2n}} h_n(s) \left( c \sum_{1 \leq i, j \leq N} a_{i,j}(x, s) + \sum_{1 \leq i, j \leq N} \frac{\partial a_{i,j}}{\partial x_j}(x, s) \right) ds \varphi dx. \end{aligned}$$

Because  $\varphi \geq 0$  in  $\Omega$ , from assumption (2.3) and the coercivity (2.1) of the matrix field  $\mathbf{A}$  we obtain for  $c$  sufficiently large independently of  $n$  ( $c > 2N^2M$  for example) that

$$(3.29) \quad \begin{aligned} & \int_{\{u_1^{2n} - u_2^{2n} > 0\}} (h_n(u_1^{2n})\mathbf{A}(x, u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x, u_2^{2n})Du_2^{2n}) \cdot D\varphi dx \\ & \leq -\frac{\alpha c}{2} \int_{\Omega} \int_{u_2^{2n}}^{u_2^{2n} + w_{2n}} h_n(s) ds \varphi dx. \end{aligned}$$

Since  $u_1$  and  $u_2$  are finite almost everywhere in  $\Omega$  while  $h_n$  converges to 1 almost everywhere in  $\mathbb{R}$  and is bounded by 1 we obtain

$$(3.30) \quad \lim_{n \rightarrow +\infty} \int_{u_2^{2n}}^{u_2^{2n} + w_{2n}} h_n(s) ds = \int_{u_2}^{u_2 + w} ds = w \quad \text{almost everywhere in } \Omega,$$

where  $w = (u_1 - u_2)^+$ .

Finally from (3.27), (3.29), (3.30) and Fatou lemma it follows that

$$\int_{\Omega} w dx \leq 0,$$

which leads to  $u_1 \leq u_2$  almost everywhere in  $\Omega$ .

The proof of Theorem 3.1 is complete.  $\square$

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LABORATOIRE DE MATHÉMATIQUES RAPHAËL SALEM UMR 6085 CNRS – UNIVERSITÉ DE ROUEN, AVENUE DE L'UNIVERSITÉ, BP.12, F76801 SAINT-ÉTIENNE DU ROUVRAY  
*E-mail address:* Olivier.Guibe@univ-rouen.fr