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MARKOV PATHS, LOOPS AND FIELDS

(Preliminary version)

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Abstract

We study the Poissonian ensembles of Markov loops, the related occupation field and its renormalized powers in relation with the Gaussian free field.

1 Introduction

The purpose of these notes is to explore some simple relations between Markovian path and loop measures, spanning trees, determinants, and Markov fields such as the free field. They include the notes of the lecture I gave in St Flour in July 2008 with some additional material. I choose this opportunity to express my thanks to Jean Picard, to the audience and to the readers of the preliminary set of notes, in particular to Jürgen Angst, Cedric Bordenave, Cedric Boutiller, Antoine Dahlqvist, Michel Emery, Jacques Franchi, Liza Jones, Adrien Kassel, Rick Kenyon, Thierry Levy, Gregorio Moreno, Bruno Shapira, Vincent Vigon, Lorenzo Zambotti and Jean Claude Zambrini.

The main emphasis is put on the study of occupation fields defined by Poissonian ensembles of Markov loops. These were defined in [15] for planar Brownian motion in relation with SLE processes and in [16] for simple random walks. They appeared informally already in [39]. For half integral values $\frac{k}{2}$ of the intensity parameter α , these occupation fields can be identified with the sum of squares of k copies of the associated free field (i.e. the Gaussian field whose covariance is given by the Green function). This is related to Dynkin's isomorphism (cf [7], [26], [19]). We first present the results in the elementary framework of symmetric Markov chains on a finite space, proving also several interesting results such as the relation between loop ensembles and spanning trees and the reflection positivity property. Then we show that some results can be extended to more general Markov processes. There are no essential difficulties when points are not polar but other cases are more problematic. As for the square of the free field, cases for which the Green function is Hilbert Schmidt such as those corresponding to two and three dimensional Brownian motion can be dealt with through appropriate renormalization.

We will show that the renormalised powers of the occupation field (i.e. the self intersection local times of the loop ensemble) converge in the two dimensional case and that they can be identified with higher even Wick powers of the free field when α is a half integer.

2 Symmetric Markov processes on finite spaces

Notations: Functions and measures on finite (or countable) spaces are often denoted as vectors and covectors.

The multiplication operator defined by a function f acting on functions or on measures is in general simply denoted by f , but sometimes it will be denoted M_f . The function obtained as the density of a measure μ with respect to some other measure ν is simply denoted $\frac{\mu}{\nu}$.

2.1 Graphs

Our basic object will be a finite space X and a set of non negative conductances $C_{x,y} = C_{y,x}$, indexed by pairs of distinct points of X . This situation allows to define a kind of *discrete topology and geometry* we will briefly study in this section and in the following ones.

We say $\{x, y\}$ is a link or an edge iff $C_{x,y} > 0$ and an oriented edge (x, y) is defined by the choice of an ordering in an edge. We set $-(x, y) = (y, x)$ and if $e = (x, y)$, we denote it also (e^-, e^+) . The degree d_x of a vertex x is by definition the number of edges incident at x .

The points of X together with the set of non oriented edges E define a graph (X, E) . We assume it is connected. The set of oriented edges is denoted E^o . It will always be viewed as a subset of X^2 , without reference to any imbedding.

The associated line graph is the oriented graph defined by E^o as set of vertices and in which oriented edges are pairs (e_1, e_2) such that $e_1^+ = e_2^-$. The mapping $e \rightarrow -e$ is an involution of the line graph.

An important example is the case in which conductances are equal to zero or one. Then the conductance matrix is the adjacency matrix of the graph: $C_{x,y} = 1_{\{x,y\} \in E}$

A complete graph is defined by all conductances equal to one.

The complete graph with n vertices is denoted K_n . The complete graph K_4 is the graph defined by the tetrahedron. K_5 is not planar (i.e. cannot be imbedded in a plane), but K_4 is.



A finite discrete path on X , say (x_0, x_1, \dots, x_n) is called a (discrete) *geodesic arc* iff $\{x_i, x_{i+1}\} \in E$ (path segment on the graph) and $x_{i-1} \neq x_{i+1}$ (without backtracking). Geodesic arcs starting at x_0 form a *marked tree* \mathfrak{T}_{x_0} rooted in x_0 (The marks are the points of X). Oriented edges are defined by pairs of geodesic arcs of the form:

$((x_0, x_1, \dots, x_n), (x_0, x_1, \dots, x_n, x_{n+1}))$ (the orientation is defined in reference to the root). \mathfrak{T}_{x_0} is a *universal cover* of X [25].

On the space \mathfrak{L}_{x_0} of discrete loops based at some point x_0 , we can define an operation of concatenation, which provides a monoid structure, i.e. is associative with a neutral element (the empty loop). The concatenation of two closed geodesics based at x_0 is not directly a closed geodesic. It can involve backtracking "in the middle" but then after cancellation of the two inverse subarcs, we get a closed geodesic, possibly empty if the two closed geodesics are identical up to reverse order. With this operation, *closed geodesics based at x_0 define a group Γ_{x_0}* . The structure of Γ_{x_0} does not depend on the base point and defines the *fundamental group Γ* (as the graph is connected: see for example [25]). Any geodesic arc γ_1 from x_0 to another point y_0 of X defines an isomorphism between Γ_{x_0} and Γ_{y_0} . It associates to a closed geodesic γ based in x_0 the closed geodesic $[\gamma_1]^{-1}\gamma\gamma_1$ (here $[\gamma_1]^{-1}$ denotes the backward arc. In the case where $x_0 = y_0$, it is an interior isomorphism (conjugation by γ_1).

There is a natural left action of Γ_{x_0} on \mathfrak{T}_{x_0} . It can be interpreted as a change of root in the tree (with the same mark). Besides, any geodesic arc between x_0 and another point y_0 of X defines an isomorphism between \mathfrak{T}_{x_0} and \mathfrak{T}_{y_0} (change of root, with possibly different marks) .

We have just seen that the universal covering of the finite graph (X, E) at x_0 is a tree \mathfrak{T}_{x_0} projecting on X . The fiber at x_0 is Γ_{x_0} . These groups are conjugated in a non canonical way. Note that $X = \Gamma_{x_0} \backslash \mathfrak{T}_{x_0}$ (here the use of the quotient on the left corresponds to the left action).

Example 1 *Among graphs, the simplest ones are r -regular graphs, in which each point has r neighbours. A universal covering of any r -regular graph is isomorphic to the r -regular tree $\mathfrak{T}^{(r)}$.*

Example 2 *Cayley graphs: A finite group with a set of generators $S = \{g_1, ..g_k\}$ such that $S \cap S^{-1}$ is empty yields an oriented $2k$ -regular graph .*

A *spanning tree T* is by definition a subgraph of (X, E) which is a tree and covers all points in X . It has necessarily $|X| - 1$ edges, See for example two spanning trees of K_4 .



Two spanning trees of K_4

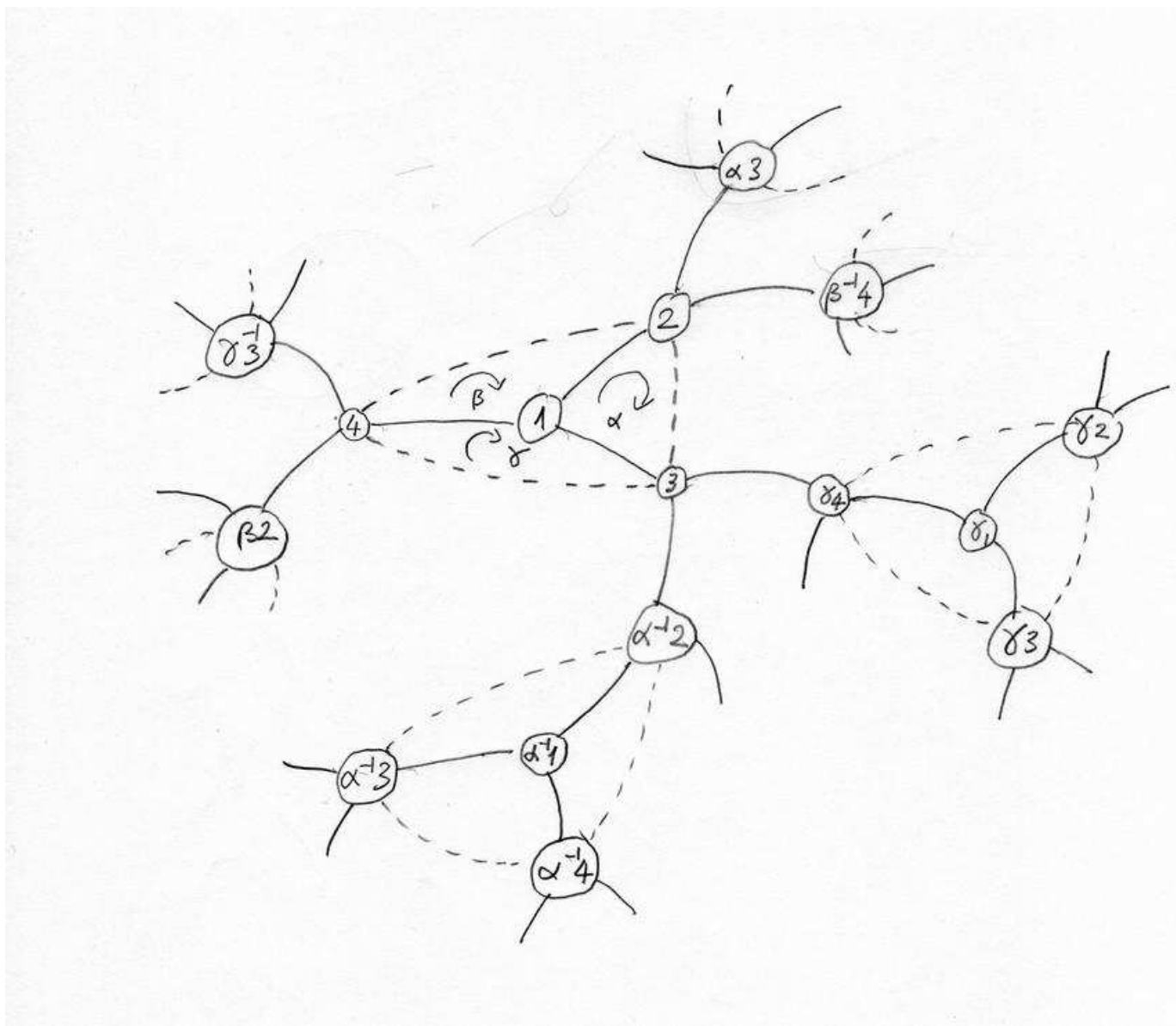
Its inverse images by the canonical projection from a universal cover \mathfrak{T}_{x_0} onto X form a tessellation on \mathfrak{T}_{x_0} , i.e. a partition of \mathfrak{T}_{x_0} in identical subtrees, which are fundamental domains for the action of Γ_{x_0} . Conversely, a section of the canonical projection from the universal cover defines a spanning tree.

Fixing a spanning tree determines a unique geodesic between two points of X . Therefore, it determines the conjugation isomorphisms between the various groups Γ_{x_0} and the isomorphisms between the universal covers \mathfrak{T}_{x_0} .

The fundamental group Γ is a free group with $|E| - |X| + 1 = r$ generators. To construct a set of generators, one considers a spanning tree T of the graph, and choose an orientation on each of the r remaining links. This defines r oriented cycles on the graph and a system of r generators for the fundamental group. (See [25] or Serres ([36]) in a more general context).

Example 3 *Consider K_3 and K_4 .*

Here is a picture of the universal covering of K_4 , and of the action of the fundamental group with the tessellation defined by a spanning tree.



Universal cover and tessellation of K_4

There are various non ramified coverings, intermediate between (X, E) and the universal covering. Non ramified means that locally, the covering space is identical to the graph (same incident edges). Then each oriented path segment on X can be lifted to the covering in a unique way, given a lift of its starting point.

Each non ramified covering is (up to an isomorphism) associated with a subgroup H of Γ , defined up to conjugation. More precisely, given a non ramified covering \tilde{X} , a point x_0 of X and a point \tilde{x}_0 in the fiber above x_0 , the closed geodesics based at x_0 whose lift to the covering starting at \tilde{x}_0 are closed form a subgroup $H_{\tilde{x}_0}$ of Γ_{x_0} , canonically isomorphic to the fundamental group of \tilde{X} represented by closed geodesics based at \tilde{x}_0 . If we consider a different point \tilde{y}_0 , any geodesic path segment $\tilde{\gamma}_1$ between \tilde{x}_0 and \tilde{y}_0 defines an isomorphism between Γ_{x_0} and Γ_{y_0} which exchanges $H_{\tilde{x}_0}$ and $H_{\tilde{y}_0}$. Denoting γ_1 the projection of $\tilde{\gamma}_1$ on X , it associates to a closed geodesic γ based in x_0 whose lift to the covering is closed the closed geodesic $[\gamma_1]^{-1}\gamma\gamma_1$ whose lift to the covering is also closed.

Conversely, if H is a subgroup of Γ_{x_0} , the covering is defined as the quotient graph (Y, F) with $Y = H \backslash \mathfrak{T}_{x_0}$ and F the set of edges defined by the canonical projection from \mathfrak{T}_{x_0} onto Y . H can be interpreted as the group of closed geodesics on the quotient graph, based at H_{x_0} , i.e. as the fundamental group of Y .

If H is a normal subgroup, the quotient group (also called the covering group) $H \backslash \Gamma_{x_0}$ acts faithfully on the fiber at x_0 . An exemple is the commutator subgroup $[\Gamma_{x_0}, \Gamma_{x_0}]$. The associate covering is the maximal Abelian covering at x_0

Example 4 *By central symmetry, the cube is a two fold covering of the tetrahedron associated with the group $\mathbb{Z}/2\mathbb{Z}$.*

Exercise 5 *Determine the maximal Abelian cover of the tetrahedron.*

2.2 Energy

Let us consider a nonnegative function κ on X . Set $\lambda_x = \kappa_x + \sum_y C_{x,y}$ and $P_y^x = \frac{C_{x,y}}{\lambda_x}$. P is a λ -symmetric (sub) stochastic transition matrix: $\lambda_x P_y^x = \lambda_y P_x^y$ with $P_x^x = 0$ for all x in X .

It defines a symmetric irreducible Markov chain ξ_n .

We can define above it a continuous time λ -symmetric irreducible Markov chain x_t , with exponential holding times of parameter 1. We have $x_t = \xi_{N_t}$, where N_t denotes a Poisson process of intensity 1. The infinitesimal generator is given by $L_y^x = P_y^x - \delta_y^x$.

We denote by P_t its (sub) Markovian semigroup $\exp(Lt) = \sum \frac{t^k}{k!} L^k$. L and P_t are λ -symmetric.

We will use the Markov chain associated with C, κ , sometimes in discrete time, sometimes in continuous time (with exponential holding times).

Recall that for any complex function $z^x, x \in X$, the “energy”

$$e(z) = \langle -Lz, \bar{z} \rangle_\lambda = \sum_{x \in X} -(Lz)^x \bar{z}^x \lambda_x$$

is nonnegative as it can be written (easy exercise)

$$e(z) = \frac{1}{2} \sum_{x,y} C_{x,y} (z^x - z^y) (\bar{z}^x - \bar{z}^y) + \sum_x \kappa_x z^x \bar{z}^x = \sum_x \lambda_x z^x \bar{z}^x - \sum_{x,y} C_{x,y} z^x \bar{z}^y$$

The Dirichlet space ([9]) is the space of real functions equipped with the energy scalar product

$$e(f, g) = \frac{1}{2} \sum_{x,y} C_{x,y} (f^x - f^y) (g^x - g^y) + \sum_x \kappa_x f^x g^x = \sum_x \lambda_x f^x g^x - \sum_{x,y} C_{x,y} f^x g^y$$

defined by polarization of e .

Note that the non negative symmetric ”conductance matrix” C and the non negative equilibrium or “killing” (or “equilibrium”) measure κ are the free parameters of the model. The eigenfunction associated with the lowest eigenvalue of $-L$ has constant sign by the well known argument based on the fact that the map $z \rightarrow z^+$ lowers the energy.

We have a dichotomy between:

- the recurrent case where 0 is the lowest eigenvalue of $-L$, and the corresponding eigenspace is formed by constants. Equivalently, $P1 = 1$ and κ vanishes.
- the transient case where the lowest eigenvalue is positive which means there is a ”Poincaré inequality”: For some positive ε , the energy $e(f, f)$ dominates $\varepsilon \langle f, f \rangle_\lambda$ for all f . Equivalently, (as we are on a finite space) κ does not vanish.

In the transient case, we denote by V the associated potential operator $(-L)^{-1} = \int_0^\infty P_t dt$. It can be expressed in terms of the spectral resolution of L .

We denote by G the Green function defined on X^2 as $G^{x,y} = \frac{V_y^x}{\lambda_y} = \frac{1}{\lambda_y} [(I - P)^{-1}]_y^x$ i.e. $G = (M_\lambda - C)^{-1}$. It induces a linear bijection from measures into functions. We will denote $\sum_y G^{x,y} \mu_y$ by $(G\mu)^x$ or $G\mu(x)$.

Note that $G\mu$ is characterized by the identity $e(f, G\mu) = \langle f, \mu \rangle$ (i.e. $\sum_x f^x \mu_x$) valid for all functions f and measures μ . In particular $G\kappa = 1$ as $e(1, f) = \sum f^x \kappa_x = \langle f, 1 \rangle_\kappa$.

See ([9]) for a development of this theory in a more general setting.

In the recurrent case, the potential operator V operates on the space λ^\perp of functions f such that $\langle f, 1 \rangle_\lambda = 0$ as the inverse of the restriction of $I - P$ to λ^\perp . The Green operator G maps the space of measures of total charge zero onto λ^\perp : Setting for any signed measure ν of total charge zero $G\nu = V\frac{\nu}{\lambda}$, we have for any function f , $\langle \nu, f \rangle = e(G\nu, f)$ (as $e(G\nu, 1) = 0$) and in particular $f^x - f^y = e(G(\delta_x - \delta_y), f)$.

Exercise 6 Compute the Green operator in the case of the complete graph K_n

In quantum mechanics, $-L$ is called the Hamiltonian and its eigenvalues are the energy levels.

One can learn more on graphs and eigenvalues in [2]

2.3 Feynman-Kac formula

A discrete analogue of the Feynman-Kac formula can be given as follows: Let s be any function on X taking values in $(0, 1]$. Then, for the discrete Markov chain ξ_n associated with P , it is a straightforward consequence of the Markov property that:

$$\mathbb{E}_x\left(\prod_{j=0}^{n-1} s(\xi_j) 1_{\{\xi_n=y\}}\right) = [(M_s P)^n]_y^x$$

Similarly, for the continuous time Markov chain x_t (with exponential holding times), if $k(x)$ is a nonnegative function defined on X , we have:

$$\mathbb{E}_x\left(e^{-\int_0^t k(x_s) ds} 1_{\{x_t=y\}}\right) = [\exp(t(L - M_k))]_y^x.$$

It can be shown easily by differentiating the first member $V(t)$ with respect to t , to check $V'(t) = (L - M_k)V(t)$.

For any nonnegative measure χ , set $V_\chi = (-L + M_\chi)^{-1}$ and $G_\chi = V_\chi M_\chi = (M_\chi + M_\lambda - C)^{-1}$. It is a symmetric nonnegative function on $X \times X$. G_0 is the Green function G , and G_χ can be viewed as the Green function of the energy form $e_\chi = e + \|\cdot\|_{L^2(\chi)}^2$.

Note that e_χ has the same conductances C as e , but χ is added to the killing measure. Note also that V_χ is not the potential of the Markov chain associated with e_χ when one takes exponential holding times of parameter 1: the holding time parameter at x becomes

$1 + \chi(x)$. But the Green function is intrinsic i.e. invariant under a change of time scale. Still, we have by Feynman Kac formula

$$\int_0^\infty \mathbb{E}_x(e^{-\int_0^t \chi(x_s) ds} \mathbf{1}_{\{x_t=y\}}) dt = [V_\chi]_y^x.$$

We have also the "resolvent" equation $V - V_\chi = VM_\chi V_\chi = V_\chi M_\chi V$. Then,

$$G - G_\chi = GM_\chi G_\chi = G_\chi M_\chi G \quad (1)$$

Note that the recurrent Green operator G defined on signed measures of zero charge is the limit of the transient Green operator G_χ , as $\chi \rightarrow 0$.

2.4 Recurrent extension of a transient chain

It will be convenient to add a cemetery point Δ to X , and extend C , λ and G to $X^\Delta = \{X \cup \Delta\}$ by setting $\lambda_\Delta = \sum_{x \in X} \kappa_x$, $C_{x,\Delta} = \kappa_x$ and $G^{x,\Delta} = G^{\Delta,x} = G^{\Delta,\Delta} = 0$ for all $x \in X$. Note that $\lambda(X^\Delta) = \sum_{X \times X} C_{x,y} + 2 \sum_X \kappa_x$

One can consider the recurrent "resurrected" Markov chain defined by the extensions the conductances to X^Δ . An energy e^Δ is defined by the formula

$$e^\Delta(z) = \frac{1}{2} \sum_{x,y \in X^\Delta} C_{x,y} (z^x - z^y)(\bar{z}^x - \bar{z}^y)$$

From the irreducibility assumption, it follows that e^Δ vanishes only on constants. We denote by P^Δ the transition kernel on X^Δ defined by

$$e^\Delta(f, g) = \langle f - P^\Delta f, g \rangle_\lambda$$

or equivalently by

$$[P^\Delta]_y^x = \frac{C_{x,y}}{\sum_{y \in X^\Delta} C_{x,y}} = \frac{C_{x,y}}{\lambda_x}$$

Note that $P^\Delta \mathbf{1} = \mathbf{1}$ so that λ is now an invariant measure. with $\lambda_x [P^\Delta]_y^x = \lambda_y [P^\Delta]_x^y$ on X^Δ . Denote V^Δ and G^Δ the associated potential and Green operators.

Note that for μ carried by X , for all $x \in X$, denoting by ε_Δ the unit point mass at Δ ,

$$\begin{aligned} \mu_x &= e^\Delta(G^\Delta(\mu - \mu(X)\varepsilon_\Delta), \mathbf{1}_x) = \lambda_x((I - P^\Delta)G^\Delta(\mu - \mu(X)\varepsilon_\Delta))(x) \\ &= \lambda_x((I - P)G^\Delta(\mu - \mu(X)\varepsilon_\Delta))(x) - \kappa_x G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(\Delta). \end{aligned}$$

Hence, applying G , it follows that on X^Δ ,

$$G\mu = G^\Delta(\mu - \mu(X)\varepsilon_\Delta) - G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(\Delta)G\kappa = G^\Delta(\mu - \mu(X)\varepsilon_\Delta) - G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(\Delta).$$

Moreover, as $G^\Delta(\mu - \mu(X)\varepsilon_\Delta)$ is in λ^\perp ,

$$\sum_{x \in X} \lambda_x G(\mu)^x = -G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(\Delta)\lambda(X^\Delta).$$

Therefore, $G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(\Delta) = \frac{-\langle \lambda, G\mu \rangle}{\lambda(X^\Delta)}$ and $G^\Delta(\mu - \mu(X)\varepsilon_\Delta) = -\frac{\langle \lambda, G\mu \rangle}{\lambda(X^\Delta)} + G\mu$.

This type of extension can be done in a more general context (See [21] and Dellacherie-Meyer [5])

Remark 7 *Conversely, a recurrent chain can be killed at any point x_0 of X , defining a Green function $G^{X-\{x_0\}}$ on $X - \{x_0\}$. Then, for any μ carried by $X - \{x_0\}$,*

$$G^{X-\{x_0\}}\mu = G(\mu - \mu(X)\varepsilon_{x_0}) - G(\mu - \mu(X)\varepsilon_{x_0})(x_0).$$

This transient chain allows to recover the recurrent one by the above procedure.

Exercise 8 *Consider a transient process which is killed with probability p at each passage in Δ . Determine the associated energy and Green operator.*

2.5 Transfer matrix

Let us suppose we are in the recurrent case: We can define a scalar product on the space \mathbb{A} of functions on E^o (oriented edges) as follows

$\langle \omega, \eta \rangle_{\mathbb{A}} = \frac{1}{2} \sum_{x,y} C_{x,y} \omega^{x,y} \eta^{x,y}$. Denoting as in [23] $df^{u,v} = f^v - f^u$, we note that $\langle df, dg \rangle_{\mathbb{A}} = e(f, g)$ In particular

$$\langle df, dG(\delta_y - \delta_x) \rangle_{\mathbb{A}} = df^{x,y}$$

Denote \mathbb{A}_- , (\mathbb{A}_+) the space of real valued functions on E^o odd (even) for orientation reversal. Note that the spaces \mathbb{A}_+ and \mathbb{A}_- are orthogonal for the scalar product defined on \mathbb{A} . The space \mathbb{A}_- should be viewed as the space of "discrete differential forms".

For any α in \mathbb{A}_- , define $d^*\alpha$ by $(d^*\alpha)^x = -\sum_{y \in X} P_y^x \alpha^{x,y}$. Note it belongs to λ^\perp as $\sum_{x,y} C_{x,y} \alpha^{x,y}$ vanishes.

We have

$$\begin{aligned} \langle \alpha, df \rangle_{\mathbb{A}} &= \frac{1}{2} \sum_{x,y} \lambda_x P_y^x \alpha^{x,y} (f^y - f^x) \\ &= \frac{1}{2} \sum_{x \in X} (d^*\alpha)^x f^x \lambda_x - \frac{1}{2} \sum_{x,y} \lambda_x P_y^x \alpha^{x,y} f^y = \sum_{x \in X} (d^*\alpha)^x f^x \lambda_x \end{aligned}$$

as the two terms of the sum are in fact equal since α is skew symmetric. The image of d and the kernel of d^* are therefore orthogonal in \mathbb{A}_- . We say α in \mathbb{A}_- is harmonic iff $d^*\alpha = 0$.

Moreover, $e(f, f) = \langle df, df \rangle_{\mathbb{A}} = \sum_{x \in X} (d^*df)^x f^x \lambda_x$.

Note that for any function f , $d^*df = -Pf + f = -Lf$.

The projection of any α in \mathbb{A}_- on the image of d is easily obtained as $dVd^*(\alpha)$. Indeed, for any function g , $\langle \alpha, dg \rangle_{\mathbb{A}} = \langle d^*\alpha, g \rangle_{\lambda} = e(Vd^*\alpha, g) = \langle dVd^*(\alpha), dg \rangle_{\mathbb{A}}$.

d is the discrete analogue of the differential and d^* the analogue of its adjoint, depending on the metric which is here defined by the conductances.

Set $\alpha_{(u,v)}^{x,y} = \pm \frac{1}{C_{u,v}}$ if $(x,y) = \pm(u,v)$ and 0 elsewhere. Then $\lambda_x d^*\alpha_{(u,v)}(x) = \delta_v^x - \delta_u^x$ and $dVd^*(\alpha_{(u,v)}) = dG(\delta_v - \delta_u)$. Note that given any orientation of the graph, the family $\{\alpha_{(u,v)}^* = \sqrt{C_{u,v}} \alpha_{(u,v)}, (u,v) \in E^+\}$ is an orthonormal basis of \mathbb{A}_- (here E^+ denotes the set of positively oriented edges).

The symmetric transfer matrix $K^{(x,y),(u,v)}$, indexed by pairs of oriented edges, is defined to be

$$K^{(x,y),(u,v)} = [dG(\delta_v - \delta_u)]^{x,y} = G(\delta_v - \delta_u)^y - G(\delta_v - \delta_u)^x = \langle dG(\delta_y - \delta_x), dG(\delta_v - \delta_u) \rangle_{\mathbb{A}}$$

for $x,y,u,v \in X$, with $C_{x,y}C_{u,v} > 0$. For every oriented edge $e = (x,y)$ in X , set $K^e = dG(\delta^y - \delta^x)$.

We have $\langle K^e, K^g \rangle_{\mathbb{A}} = K^{e,g}$. We can view dG as a linear operator mapping the space measures of total charge zero into \mathbb{A}_- . As measures of the form $\delta_y - \delta_x$ span the space of measures of total charge zero, it is determined by the transfer matrix.

Note that $d^*dGv = v/\lambda$ for any v of total charge zero, that for all α in \mathbb{A}_- , $(d^*\alpha)\lambda$ has total charge zero and that $dG((d^*\alpha)\lambda) = dVd^*(\alpha)$ is the projection $\Pi(\alpha)$ of α on the image of d in \mathbb{A}_- .

In particular

$$\langle \alpha_{(u,v)}, \Pi(\alpha_{(u',v')}) \rangle_{\mathbb{A}} = \langle \alpha_{(u,v)}, dG(\delta_{v'} - \delta_{u'}) \rangle_{\mathbb{A}} = K^{(u,v),(u',v')}$$

Consider now, in the transient case, the transfer matrix associated with G^Δ . We see that for x and y in X , $G^\Delta(\delta_x - \delta_y)^u - G^\Delta(\delta_x - \delta_y)^v = G(\delta_x - \delta_y)^u - G(\delta_x - \delta_y)^v$. We can see also that $G^\Delta(\delta_x - \delta_\Delta) = G\delta_x - \frac{\langle \lambda, G\delta_x \rangle}{\lambda(X^\Delta)}$. So the same identity holds in X^Δ . Therefore, as $G^{x,\Delta} = 0$, in all cases,

$$K^{(x,y),(u,v)} = G^{x,u} + G^{y,v} - G^{x,v} - G^{y,u}$$

Exercise 9 *Cohomology and complex transition matrices.*

Consider, in the recurrent case, $\omega \in \mathbb{A}_-$ such that $d^*\omega = 0$. Note that the space H^1 of such ω 's is isomorphic to the first cohomology space, i.e. the quotient $\mathbb{A}_- / \text{Im}(d)$. Prove that $P(I + i\omega)$ is λ -self adjoint on X , maps 1 onto 1 and that we have $\mathbb{E}_x(\prod_{j=0}^{n-1} 1 + \omega(\xi_j, \xi_{j+1})1_{\{\xi_n=y\}}) = [(P(I + i\omega))^n]_y^x$.

3 Loop measures

3.1 A measure on based loops

We denote by \mathbb{P}^x the family of probability laws on piecewise constant paths defined by P_t .

$$\mathbb{P}^x(\gamma(t_1) = x_1, \dots, \gamma(t_h) = x_h) = P_{t_1}(x, x_1)P_{t_2-t_1}(x_1, x_2) \dots P_{t_h-t_{h-1}}(x_{h-1}, x_h)$$

Denoting by $p(\gamma)$ the number of jumps and T_i the jump times, we have:

$$\begin{aligned} \mathbb{P}_x(p(\gamma) = k, \gamma_{T_1} = x_1, \dots, \gamma_{T_{k-1}} = x_{k-1}, T_1 \in dt_1, \dots, T_k \in dt_k) \\ = \frac{C_{x,x_1} \dots C_{x_{k-1},x_k} K_{x_k}}{\lambda_x \lambda_{x_1} \dots \lambda_{x_k}} 1_{\{0 < t_1 < \dots < t_k\}} e^{-t_k} dt_1 \dots dt_k \end{aligned}$$

For any integer $p \geq 2$, let us define a based loop with p points in X as a couple $l = (\xi, \tau) = ((\xi_m, 1 \leq m \leq p), (\tau_m, 1 \leq m \leq p+1),)$ in $X^p \times \mathbb{R}_+^{p+1}$, and set $\xi_1 = \xi_{p+1}$ (equivalently, we can parametrize the associated discrete based loop by $\mathbb{Z}/p\mathbb{Z}$). The integer p represents the number of points in the discrete based loop $\xi = (\xi_1, \dots, \xi_{p(\xi)})$ and will be denoted $p(\xi)$.

Note two time parameters are attached to the base point since the based loops do not in general end or start with a jump.

Based loops with one point ($p = 1$) are simply given by a pair (ξ, τ) in $X \times \mathbb{R}_+$.

Based loops have a natural time parametrization $l(t)$ and a time period $T(\xi) = \sum_{i=1}^{p(\xi)+1} \tau_i$. If we denote $\sum_{i=1}^m \tau_i$ by T_m : $l(t) = \xi_{m-1}$ on $[T_{m-1}, T_m)$ (with by convention $T_0 = 0$ and $\xi_0 = \xi_p$).

Let $\mathbb{P}_t^{x,y}$ denote the (non normalized) "bridge measure" on piecewise constant paths from x to y of duration t constructed as follows:

If $t_1 < t_2 \dots < t_h < t$,

$$\mathbb{P}_t^{x,y}(l(t_1) = x_1, \dots, l(t_h) = x_h) = [P_{t_1}]_{x_1}^{x_h} [P_{t_2-t_1}]_{x_2}^{x_1} \dots [P_{t-t_h}]_y^{x_h} \frac{1}{\lambda_y}$$

Its mass is $p_t^{x,y} = \frac{[P_t]_y^x}{\lambda_y}$. For any measurable set A of piecewise constant paths indexed by $[0, t]$, we can also write

$$\mathbb{P}_t^{x,y}(A) = \mathbb{P}_x(A \cap \{x_t = y\}) \frac{1}{\lambda_y}.$$

A σ -finite measure μ is defined on based loops by

$$\mu = \sum_{x \in X} \int_0^\infty \frac{1}{t} \mathbb{P}_t^{x,x} \lambda_x dt$$

From the expression of the bridge measure, we see that by definition of μ , if $t_1 < t_2 \dots < t_h < t$,

$$\mu(l(t_1) = x_1, \dots, l(t_h) = x_h, T \in dt) = [P_{t_1+t-t_h}]_{x_1}^{x_h} [P_{t_2-t_1}]_{x_2}^{x_1} \dots [P_{t_h-t_{h-1}}]_{x_h}^{x_{h-1}} \frac{1}{t} dt. \quad (2)$$

Note also that for $k > 1$, using the second expression of $\mathbb{P}_t^{x,x}$ and the fact that conditionally on $N_t = k$, the jump times are distributed like an increasingly reordered k -uniform sample of $[0, t]$

$$\begin{aligned} \lambda_x \mathbb{P}_t^{x,x}(p = k, \xi_2 = x_2, \dots, \xi_k = x_k, T_1 \in dt_1, \dots, T_k \in dt_k) \\ &= e^{-t} \frac{t^k}{k!} P_{x_2}^x P_{x_3}^{x_2} \dots P_x^{x_k} 1_{\{0 < t_1 < \dots < t_k < t\}} \frac{k!}{t^k} dt_1 \dots dt_k \\ &= P_{x_2}^x P_{x_3}^{x_2} \dots P_x^{x_k} 1_{\{0 < t_1 < \dots < t_k < t\}} e^{-t} dt_1 \dots dt_k \end{aligned}$$

Therefore,

$$\mu(p = k, \xi_1 = x_1, \dots, \xi_k = x_k, T_1 \in dt_1, \dots, T_k \in dt_k, T \in dt) \quad (3)$$

$$= P_{x_2}^{x_1} \dots P_{x_1}^{x_k} \frac{1_{\{0 < t_1 < \dots < t_k < t\}}}{t} e^{-t} dt_1 \dots dt_k dt \quad (4)$$

for $k > 1$.

Moreover, for one point-loops, $\mu\{p(\xi) = 1, \xi_1 = x_1, \tau_1 \in dt\} = \frac{e^{-t}}{t} dt$.

It is clear on these formulas that for any positive constant c , the energy forms e and ce define the same loop measure.

3.2 First properties

Note that the loop measure is invariant under time reversal.

If D is a subset of X , the restriction of μ to loops contained in D , denoted μ^D is clearly the loop measure induced by the Markov chain killed at the exit of D . This can be called the *restriction property*.

Let us recall that this killed Markov chain is defined by the restriction of λ to D and the restriction P^D of P to D^2 (or equivalently by the restriction e_D of the Dirichlet norm e to functions vanishing outside D).

As $\int \frac{t^{k-1}}{k!} e^{-t} dt = \frac{1}{k}$, it follows from (3) that for $k > 1$, on based loops,

$$\mu(p(\xi) = k, \xi_1 = x_1, \dots, \xi_k = x_k) = \frac{1}{k} P_{x_2}^{x_1} \dots P_{x_1}^{x_k}. \quad (5)$$

In particular, we obtain that, for $k \geq 2$

$$\mu(p = k) = \frac{1}{k} Tr(P^k)$$

and therefore, as $Tr(P) = 0$,

$$\mu(p > 1) = \sum_2^\infty \frac{1}{k} Tr(P^k) = -\log(\det(I - P)) = \log(\det(G) \prod_x \lambda_x) \quad (6)$$

since (denoting M_λ the diagonal matrix with entries λ_x), we have

$$\det(I - P) = \frac{\det(M_\lambda - C)}{\det(M_\lambda)}$$

Note that $\det(G)$ is defined as the determinant of the matrix $G^{x,y}$. It is the determinant of the matrix representing the scalar product defined on $\mathbb{R}^{|X|}$ (more precisely, on the space of measures on X) by G in any basis, orthonormal with respect to the natural euclidean scalar product on $\mathbb{R}^{|X|}$.

Moreover

$$\int p(l)1_{\{p>1\}}\mu(dl) = \sum_2^\infty Tr(P^k) = Tr((I - P)^{-1}P) = Tr(GC)$$

3.3 Loops and pointed loops

It is clear on formula 2 that μ is invariant under the time shift that acts naturally on based loops.

A loop is defined as an equivalence class of based loops for this shift. Therefore, μ induces a *measure on loops also denoted by μ* .

A loop is defined by the discrete loop ξ° formed by the ξ_i in circular order, (i.e. up to translation) and the associated holding times. We clearly have:

$$\mu(\xi^\circ = (x_1, x_2, \dots, x_k)^\circ) = P_{x_2}^{x_1} \dots P_{x_1}^{x_k}$$

However, loops are not easy to parametrize, that is why we will work mostly with based loops or *pointed loops*. These are defined as based loops ending with a jump, or as loops with a starting point. They can be parametrized by a based discrete loop and by the holding times at each point. Calculations are easier if we work with based or pointed loops, even though we will deal only with functions independent of the base point.

The parameters of the pointed loop naturally associated with a based loop are ξ_1, \dots, ξ_p and

$$\tau_1 + \tau_{p+1} = \tau_1^*, \tau_i = \tau_i^*, 2 \leq i \leq p$$

An elementary change of variables, shows the expression of μ on pointed loops can be written:

$$\mu(p = k, \xi_i = x_i, \tau_i^* \in dt_i) = P_{x_2}^{x_1} \dots P_{x_1}^{x_k} \frac{t_1}{\sum t_i} e^{-\sum t_i} dt_1 \dots dt_k. \quad (7)$$

Trivial ($p = 1$) pointed loops and trivial based loops coincide.

Note that loop functionals can be written

$$\Phi(l^\circ) = \sum 1_{\{p=k\}} \Phi_k((\xi_i, \tau_i^*), i = 1, \dots, k)$$

with Φ_k invariant under circular permutation of the variables (ξ_i, τ_i^*) .

Then, for non negative Φ_k

$$\int \Phi_k(l^\circ) \mu(dl) = \int \sum_i \Phi_k(x_i, t_i) P_{x_2}^{x_1} \dots P_{x_1}^{x_k} e^{-\sum t_i} \frac{t_1}{\sum t_i} dt_1 \dots dt_k$$

and by invariance under circular permutation, the term t_1 can be replaced by any t_i . Therefore, adding up and dividing by k , we get that

$$\int \Phi_k(l^\circ) \mu(dl) = \int \frac{1}{k} \sum_i \Phi_k(x_i, t_i) P_{x_2}^{x_1} \dots P_{x_1}^{x_k} e^{-\sum t_i} dt_1 \dots dt_k.$$

The expression on the right side, applied to any pointed loop functional defines a different measure on pointed loops, we will denote by μ^* . It induces the same measure as μ on loops.

We see on this expression that conditionally on the discrete loop, the holding times of the loop are independent exponential variables.

$$\mu^*(p = k, \xi_i = x_i, \tau_i^* \in dt_i) = \frac{1}{k} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} \frac{C_{\xi_i, \xi_{i+1}}}{\lambda_{\xi_i}} e^{-t_i} dt_i \quad (8)$$

Conditionally on $p(\xi) = k$, T is a gamma variable of density $\frac{t^{k-1}}{(k-1)!} e^{-t}$ on \mathbb{R}_+ and $(\frac{\tau_i^*}{T}, 1 \leq i \leq k)$ an independent ordered k -sample of the uniform distribution on $(0, T)$ (whence the factor $\frac{1}{t}$). Both are independent, conditionally on p of the discrete loop. We see that μ , on based loops, is obtained from μ on the loops by choosing the based point uniformly. On the other hand, it induces a choice of ξ_1 biased by the size of the τ_i^* 's, different of μ^* (whence the factor $\frac{1}{k}$). But we will consider only loop functionals for which μ and μ^* coincide.

It will be convenient to rescale the holding time at each ξ_i by λ_{ξ_i} and set

$$\widehat{\tau}_i = \frac{\tau_i^*}{\lambda_{\xi_i}}.$$

The discrete part of the loop is the most important, though we will see that to establish a connection with Gaussian fields it is necessary to consider occupation times. The simplest variables are the number of jumps from x to y , defined for every oriented edge (x, y)

$$N_{x,y} = \#\{i : \xi_i = x, \xi_{i+1} = y\}$$

(recall the convention $\xi_{p+1} = \xi_1$) and

$$N_x = \sum_y N_{x,y}$$

Note that $N_x = \#\{i \geq 1 : \xi_i = x\}$ except for trivial one point loops for which it vanishes.

Then, the measure on pointed loops (7) can be rewritten as:

$$\mu^*(p = 1, \xi = x, \widehat{\tau} \in dt) = e^{-\lambda_x t} \frac{dt}{t} \quad \text{and} \quad (9)$$

$$\mu^*(p = k, \xi_i = x_i, \widehat{\tau}_i \in dt_i) = \frac{1}{k} \prod_{x,y} C_{x,y}^{N_{x,y}} \prod_x \lambda_x^{-N_x} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} \lambda_{\xi_i} e^{-\lambda_{\xi_i} t_i} dt_i. \quad (10)$$

Another *bridge measure* $\mu^{x,y}$ can be defined on paths γ from x to y : $\mu^{x,y}(d\gamma) = \int_0^\infty \mathbb{P}_t^{x,y}(d\gamma) dt$.

Note that the mass of $\mu^{x,y}$ is $G^{x,y}$. We also have, with similar notations as the one defined for loops, p denoting the number of jumps

$$\begin{aligned} \mu^{x,y}(p(\gamma) = k, \gamma_{T_1} = x_1, \dots, \gamma_{T_{k-1}} = x_{k-1}, T_1 \in dt_1, \dots, T_{k-1} \in dt_{k-1}, T \in dt) \\ = \frac{C_{x,x_2} C_{x_2,x_3} \dots C_{x_{k-1},y}}{\lambda_x \lambda_{x_2} \dots \lambda_y} \mathbf{1}_{\{0 < t_1 < \dots < t_k < t\}} e^{-t} dt_1 \dots dt_k dt. \end{aligned}$$

For any $x \neq y$ in X and $s \in [0, 1]$, setting $P_v^{(s),u} = P_v^u$ if $(u, v) \neq (x, y)$ and $P_y^{(s),x} = s P_y^x$, we can prove in the same way as (6) that:

$$\mu(s^{N_{x,y}} \mathbf{1}_{\{p > 1\}}) = -\log(\det(I - P^{(s)})).$$

Differentiating in $s = 1$, and remembering that for any invertible matrix function $M(s)$, $\frac{d}{ds} \log(\det(M(s))) = \text{Tr}(M'(s)M(s)^{-1})$, it follows that:

$$\mu(N_{x,y}) = [(I - P)^{-1}]_x^y P_y^x = G^{x,y} C_{x,y}$$

and $\mu(N_x) = \sum_y \mu(N_{x,y}) = \lambda_x G^{x,x} - 1$ (as $G(M_\lambda - C) = Id$).

Exercise 10 Show that more generally $\mu(N_{x,y}(N_{x,y}-1)\dots(N_{x,y}-k+1)) = (k-1)!(G^{x,y}C_{x,y})^k$

Exercise 11 Show that more generally, if x_i, y_i are n distinct oriented edges:

$$\mu\left(\prod N_{x_i, y_i}\right) = \prod C_{x_i, y_i} \frac{1}{n} \sum_{\sigma \in \mathcal{S}_n} \prod G^{y_{\sigma(i)}, x_{\sigma(i+1)}}$$

We finally note that if $C_{x,y} > 0$, any path segment on the graph starting at x and ending at y can be naturally extended into a loop by adding a jump from y to x . We have the following

Proposition 12 a) For $C_{x,y} > 0$, the natural extension of $\mu^{x,y}$ to loops coincides with $\frac{N_{y,x}(l)}{C_{x,y}} \mu(dl)$.

b) More generally, if x_i, y_i are n oriented edges, $\prod \frac{N_{x_i, y_i}(l)}{C_{x_i, y_i}} \mu(dl)$ can be obtained as the sum of the images, by concatenation in all circular orders, of the product of the bridge measures $\mu^{y_{\sigma(i)}, x_{\sigma(i+1)}}(dl)$.

Proof. The first assertion follows from the formulas, noticing that a loop l can be associated to $N_{y,x}(l)$ distinct bridges from x to y , obtained by "cutting" one jump from y to x . ■

3.4 Occupation field

To each loop l° we associate local times, i.e. an occupation field $\{\widehat{l}_x, x \in X\}$ defined by

$$\widehat{l}_x = \int_0^{T(l)} 1_{\{\xi(s)=x\}} \frac{1}{\lambda_{\xi(s)}} ds = \sum_{i=1}^{p(l)} 1_{\{\xi_i=x\}} \widehat{\tau}_i$$

for any representative $l = (\xi_i, \tau_i^*)$ of l° .

For a path γ , $\widehat{\gamma}$ is defined in the same way.

Note that

$$\mu((1 - e^{-\alpha \widehat{l}_x}) 1_{\{p=1\}}) = \int_0^\infty e^{-t} (1 - e^{-\frac{\alpha}{\lambda_x} t}) \frac{dt}{t} = \log\left(1 + \frac{\alpha}{\lambda_x}\right) \quad (11)$$

(by expanding $1 - e^{-\frac{\alpha}{\lambda_x} t}$ before the integration, assuming first that α is small and then by analyticity of both members, or more elegantly, noticing that $\int_a^b (e^{-cx} - e^{-dx}) \frac{dx}{x}$ is symmetric in (a, b) and (c, d)).

In particular, $\mu(\widehat{l}_x 1_{\{p=1\}}) = \frac{1}{\lambda_x}$.

From formula 7, we get easily that the joint conditional distribution of $(\widehat{l}^x, x \in X)$ given $(N_x, x \in X)$ is a product of gamma distributions. In particular, from the expression of the moments of a gamma distribution, we get that for any function Φ of the discrete loop and $k \geq 1$,

$$\mu((\widehat{l}^x)^k 1_{\{p>1\}} \Phi) = \lambda_x^{-k} \mu((N_x + k - 1) \dots (N_x + 1) N_x \Phi)$$

In particular, $\mu(\widehat{l}^x) = \frac{1}{\lambda_x} [\mu(N_x) + 1] = G^{x,x}$.

Note that functions of \widehat{l} are not the only functions naturally defined on the loops. Other such variables of interest are, for $n \geq 2$, the multiple local times, defined as follows:

$$\widehat{l}^{x_1, \dots, x_n} = \sum_{j=0}^{n-1} \int_{0 < t_1 < \dots < t_n < T} 1_{\{\xi(t_1)=x_{1+j}, \dots, \xi(t_{n-j})=x_n, \dots, \xi(t_n)=x_j\}} \prod \frac{1}{\lambda_{x_i}} dt_i$$

It is easy to check that, when the points x_i are distinct,

$$\widehat{l}^{x_1, \dots, x_n} = \sum_{j=0}^{n-1} \sum_{1 \leq i_1 < \dots < i_n \leq p(l)} \prod_{l=1}^n 1_{\{\xi_{i_l}=x_{l+j}\}} \widehat{T}_{i_l}. \quad (12)$$

Note that in general $\widehat{l}^{x_1, \dots, x_k}$ cannot be expressed in terms of \widehat{l} .

If $x_1 = x_2 = \dots = x_n$, $\widehat{l}^{x_1, \dots, x_n} = \frac{1}{(n-1)!} [\widehat{l}^x]^n$. It can be viewed as a n -th self intersection local time.

One can deduce from the definitions of μ the following:

Proposition 13 $\mu(\widehat{l}^{x_1, \dots, x_n}) = G^{x_1, x_2} G^{x_2, x_3} \dots G^{x_n, x_1}$

In particular, $\mu(\widehat{l}^{x_1} \dots \widehat{l}^{x_n}) = \frac{1}{n} \sum_{\sigma \in \mathcal{S}_n} G^{x_{\sigma(1)}, x_{\sigma(2)}} G^{x_{\sigma(2)}, x_{\sigma(3)}} \dots G^{x_{\sigma(n)}, x_{\sigma(1)}}$

Proof. Let us denote $\frac{1}{\lambda_y} [P_t]_y^x$ by $p_t^{x,y}$ or $p_t(x, y)$. From the definition of $\widehat{l}^{x_1, \dots, x_n}$ and μ , $\mu(\widehat{l}^{x_1, \dots, x_n})$ equals:

$$\sum_x \lambda_x \sum_{j=0}^{n-1} \int \int_{\{0 < t_1 < \dots < t_n < t\}} \frac{1}{t} p_{t_1}(x, x_{1+j}) \dots p_{t-t_n}(x_{n+j}, x) \prod dt_i dt$$

where sums of indices $k + j$ are computed mod(n). By the semigroup property, it equals

$$\sum_{j=0}^{n-1} \int \int_{\{0 < t_1 < \dots < t_n < t\}} \frac{1}{t} p_{t_2-t_1}(x_{1+j}, x_{2+j}) \dots p_{t_1+t-t_n}(x_{n+j}, x_{1+j}) \prod dt_i dt.$$

Performing the change of variables $v_2 = t_2 - t_1, \dots, v_n = t_n - t_{n-1}, v_1 = t_1 + t - t_n$, and $v = t_1$, we obtain:

$$\begin{aligned}
& \sum_{j=0}^{n-1} \int_{\{0 < v < v_1, 0 < v_i\}} \frac{1}{v_1 + \dots + v_n} p_{v_2}(x_{1+j}, x_{2+j}) \dots p_{v_1}(x_{n+j}, x_{1+j}) \prod dv_i dv \\
&= \sum_{j=0}^{n-1} \int_{\{0 < v_i\}} \frac{v_1}{v_1 + \dots + v_n} p_{v_2}(x_{1+j}, x_{2+j}) \dots p_{v_1}(x_{n+j}, x_{1+j}) \prod dv_i \\
&= \sum_{j=1}^n \int_{\{0 < v_i\}} \frac{v_j}{v_1 + \dots + v_n} p_{v_2}(x_1, x_2) \dots p_{v_1}(x_n, x_1) \prod dv_i \\
&= \int_{\{0 < v_i\}} p_{v_2}(x_1, x_2) \dots p_{v_1}(x_n, x_1) \prod dv_i \\
&= G^{x_1, x_2} G^{x_2, x_3} \dots G^{x_n, x_1}.
\end{aligned}$$

Note that another proof can be derived from formula (12) ■

Exercise 14 (*Shuffle product*) Given two positive integers $n > k$, let $\mathcal{P}_{n,k}$ be the family of partitions of into k consecutive non empty intervals $I_l = (i_l, i_{l+1}, \dots, i_{l+1} - 1)$ with $i_1 = 1 < i_2 \dots < i_k < i_{k+1} = n + 1$.

Show that

$$\widehat{j}^{x_1, \dots, x_n} \widehat{j}^{y_1, \dots, y_m} = \sum_{j=0}^{m-1} \sum_{k=1}^{\inf(n,m)} \sum_{I \in \mathcal{P}_{n,k}} \sum_{J \in \mathcal{P}_{m,k}} \widehat{j}^{x_{I_1}, y_{j+J_1}, x_{I_2}, \dots, y_{j+J_k}}$$

Similarly, we can define $N_{(x_1, y_1), \dots, (x_n, y_n)}$ to be $\sum_{j=0}^{n-1} \sum_{1 \leq i_1 < \dots < i_n \leq p(l)} \prod_{l=1}^n \mathbf{1}_{\{\xi_{i_l} = x_{l+j}, \xi_{i_l+1} = y_{l+j}\}}$. If $(x_i, y_i) = (x, y)$ for all i , it equals $\frac{N_{x,y}(N_{x,y}-1) \dots (N_{x,y}-n+1)}{(n-1)!}$.

Notice that

$$\prod N_{(x_i, y_i)} = \frac{1}{n} \sum_{\sigma \in \mathcal{S}_n} N_{(x_{\sigma(1)}, y_{\sigma(1)}), \dots, (x_{\sigma(n)}, y_{\sigma(n)})}$$

Then we have the following:

Proposition 15 $\int N_{(x_1, y_1), \dots, (x_n, y_n)}(l) \mu(dl) = (\prod C_{x_i, y_i}) G^{y_1, x_2} G^{y_2, x_3} \dots G^{y_n, x_1}$

The proof is left as exercise.

Exercise 16 For $x_1 = x_2 = \dots = x_k$, we could define different self intersection local times

$$\widehat{l}^{x,(k)} = \sum_{1 \leq i_1 < \dots < i_k \leq p(l)} \prod_{l=1}^k 1_{\{\xi_{i_l}=x\}} \widehat{\tau}_{i_l}$$

which vanish on $N_x < k$. Note that

$$\widehat{l}^{x,(2)} = \frac{1}{2}((\widehat{l}^x)^2 - \sum_{i=1}^{p(l)} 1_{\{\xi_i=x\}} (\widehat{\tau}_i)^2).$$

1. For any function Φ of the discrete loop, show that

$$\mu(\widehat{l}^{x,2}\Phi) = \lambda_x^{-2} \mu\left(\frac{N_x(N_x - 1)}{2} 1_{\{N_x \geq 2\}} \Phi\right).$$

2. More generally prove in a similar way that

$$\mu(\widehat{l}^{x,(k)}\Phi) = \lambda_x^{-k} \mu\left(\frac{N_x(N_x - 1)\dots(N_x - k + 1)}{k!} 1_{\{N_x \geq k\}} \Phi\right).$$

Let us come back to the occupation field to compute its Laplace transform. From the Feynman-Kac formula, it comes easily that, denoting $M_{\frac{\chi}{\lambda}}$ the diagonal matrix with coefficients $\frac{\chi_x}{\lambda_x}$

$$\mathbb{P}_t^{x,x}(e^{-\langle \widehat{l}, \chi \rangle} - 1) = \frac{1}{\lambda_x} (\exp(t(P - I - M_{\frac{\chi}{\lambda}}))_x^x - \exp(t(P - I))_x^x).$$

Integrating in t after expanding, we get from the definition of μ (first for χ small enough):

$$\begin{aligned} \int (e^{-\langle \widehat{l}, \chi \rangle} - 1) d\mu(l) &= \sum_{k=1}^{\infty} \int_0^{\infty} [Tr((P - M_{\frac{\chi}{\lambda}})^k) - Tr((P)^k)] \frac{t^{k-1}}{k!} e^{-t} dt \\ &= \sum_{k=1}^{\infty} \frac{1}{k} [Tr((P - M_{\frac{\chi}{\lambda}})^k) - Tr((P)^k)] \\ &= -Tr(\log(I - P + M_{\frac{\chi}{\lambda}})) + Tr(\log(I - P)) \end{aligned}$$

Hence, as $Tr(\log) = \log(\det)$

$$\int (e^{-\langle \widehat{l}, \chi \rangle} - 1) d\mu(l) = \log[\det(-L(-L + M_{\chi/\lambda})^{-1})] = -\log \det(I + VM_{\frac{\chi}{\lambda}})$$

which now holds for all non negative χ as both members are analytic in χ . Besides, by the "resolvent" equation (1):

$$\det(I + GM_\chi)^{-1} = \det(I - G_\chi M_\chi) = \frac{\det(G_\chi)}{\det(G)}. \quad (13)$$

Note that $\det(I + GM_\chi) = \det(I + M_{\sqrt{\chi}} G M_{\sqrt{\chi}})$ and $\det(I - G_\chi M_\chi) = \det(I - M_{\sqrt{\chi}} G_\chi M_{\sqrt{\chi}})$, so we can deal with symmetric matrices. Finally we have

Proposition 17 $\mu(e^{-\langle \hat{l}, \chi \rangle} - 1) = -\log(\det(I + M_{\sqrt{\chi}} G M_{\sqrt{\chi}})) = \log\left(\frac{\det(G_\chi)}{\det(G)}\right)$

Note that in particular $\mu(e^{-t\hat{l}^x} - 1) = -\log(1 + tG^{x,x})$. Consequently, the image measure of μ by \hat{l}^x is $1_{\{s>0\}} \frac{e^{-G^{x,x}s}}{s} ds$.

Note finally that if χ has support in D , by the restriction property

$$\mu(1_{\{\hat{l}(X \setminus D)=0\}}(e^{-\langle \hat{l}, \chi \rangle} - 1)) = -\log(\det(I + M_{\sqrt{\chi}} G^D M_{\sqrt{\chi}})) = \log\left(\frac{\det(G_\chi^D)}{\det(G^D)}\right).$$

Here the determinants are taken on matrices indexed by D and G^D denotes the Green function of the process killed on leaving D .

For paths we have $\mathbb{P}_t^{x,y}(e^{-\langle \hat{l}, \chi \rangle}) = \frac{1}{\lambda_y} \exp(t(L - M_{\frac{\chi}{\lambda}}))_{x,y}$. Hence

$$\mu^{x,y}(e^{-\langle \hat{\gamma}, \chi \rangle}) = \frac{1}{\lambda_y} ((I - P + M_{\chi/\lambda})^{-1})_{x,y} = [G_\chi]^{x,y}.$$

In particular, note that from the resolvent equation (1), we get that

$$G^{y,x} = [G_{\varepsilon\delta_x}]^{y,x} + \varepsilon [G_{\varepsilon\delta_x}]^{y,x} G^{x,x}$$

Hence $\frac{[G_{\varepsilon\delta_x}]^{y,x}}{G^{y,x}} = \frac{1}{1 + \varepsilon G^{x,x}}$ and under the probability $\frac{\mu^{y,x}}{G^{y,x}}$, \hat{l}_x follows an exponential distribution of mean $G^{x,x}$.

Also $\mathbb{E}^x(e^{-\langle \hat{\gamma}, \chi \rangle}) = \sum_y [G_\chi]^{x,y} \kappa_y$ i.e. $[G_\chi \kappa]^x$.

Finally, let us note that a direct calculation shows the following

Proposition 18 *On loops based in x , $\mu^{x,x}(dl) = \hat{l}^x \mu(dl)$*

More generally, if x_i are n points, $\hat{l}^{x_1, \dots, x_n} \mu(dl)$ can be obtained as the the image by circular concatenation of the product of the bridge measures $\mu^{x_i, x_{i+1}}(dl)$ and $\prod \hat{l}^{x_i} \mu(dl)$ can be obtained as the sum of the images, by concatenation in all circular orders, of the product of the excursions measures $\mu^{y_{\sigma(i)}, x_{\sigma(i+1)}}(dl)$.

Remark 19 Propositions 12 and 18 can be generalized: For example, one can evaluate expressions of the form $\prod \widehat{l}^{z_j} \prod \frac{N_{x_i, y_i}(l)}{C_{x_i, y_i}} \mu(dl)$ as a sum of images, by concatenation in all circular orders, of a product of bridge measures .

3.5 Wreath products

Associate to each point of X an integer n_x . Let Z be the product of the groups $\mathbb{Z}/n_x\mathbb{Z}$. On $X \times Z$, define a set of conductances $\widetilde{C}_{(x,z),(x',z')}$ by:

$$\widetilde{C}_{(x,z),(x',z')} = \frac{1}{n_x n_{x'}} C_{x,x'} \prod_{y \neq x, x'} 1_{\{z_y = z'_y\}}$$

and set $\widetilde{\kappa}_{(x,z)} = \kappa_x$. Let \widetilde{e} be the corresponding energy form. Note that $\widetilde{\lambda}_{(x,z)} = \lambda_x$.

Then, denoting $\widetilde{\mu}$ the loop measure and \widetilde{P} the transition matrix on $X \times Z$ defined by \widetilde{e} , we have the following

Proposition 20 $\int 1_{\{p>1\}} \prod_{x, N_x(l)>0} \frac{1}{n_x} \mu(dl) = \widetilde{\mu}(p > 1) = -\log(\det(I - \widetilde{P}))$. In particular, if $n_x = n$ for all x ,

$$\int 1_{\{p>1\}} n^{-\#\{x, N_x(l)>0\}} \mu(dl) = \widetilde{\mu}(p > 1) = -\log(\det(I - \widetilde{P}))$$

Proof. Each time the Markov chain on $X \times Z$ defined by \widetilde{e} jumps from a point above x to a point above y , z_x and z_y are resampled according to the uniform distribution on $\mathbb{Z}/n_x\mathbb{Z} \times \mathbb{Z}/n_y\mathbb{Z}$. It follows that $\sum_{z=1}^{n_x} [\widetilde{P}^k]_{(x,z)}^{(x,z)} = \sum_{x_1, \dots, x_{k-1}} P_{x_1}^x P_{x_2}^{x_1} \dots P_x^{x_{k-1}} \prod_{y \in \{x, x_1, \dots, x_{k-1}\}} \frac{1}{n_y}$. The detail of the proof is left as an exercise. ■

This construction therefore gives an interesting information about the number of distinct points visited by the loop, which is more difficult to evaluate than the occupation measure.

In the case where X is a group and P defines a random walk, \widetilde{P} is associated with a random walk on $X \times Z$ equipped with its wreath product structure (Cf [30]).

3.6 Countable spaces

The assumption of finiteness of X can of course be relaxed. On countable spaces, the previous results extend easily under spectral gap conditions. In the transient case, the Dirichlet space \mathbb{H} is the space of all functions f with finite energy $e(f)$ which are limits in energy norm of functions with finite support. The energy of a measure is defined as

$\sup_{f \in \mathbb{H}} \frac{\mu(f)^2}{e(f)}$. Finitely supported measures have finite energy. The potential $G\mu$ is well defined in for all finite energy measures μ , by the identity $e(f, G\mu) = \langle f, \mu \rangle$, valid for all f in the Dirichlet space. The energy of the measure μ equals $e(G\mu) = \langle G\mu, \mu \rangle$.

Most important examples are the non ramified covering of finite graphs (Recall that non ramified means that the projection is locally one to one, i.e. that the projection on X of each vertex v of the covering space has the same number of incident edges as v). Consider a non ramified covering graph (Y, F) defined by a normal subgroup H_{x_0} of Γ_{x_0} . The conductances C and the measure λ can be lifted in an obvious way to Y as $H_{x_0} \backslash \Gamma_{x_0}$ -periodic functions but the associated Green function \widehat{G} or semigroup are non trivial.

We have $G^{x,y} = \sum_{\gamma \in H_{x_0} \backslash \Gamma_{x_0}} \widehat{G}^{i(x), \gamma(i(y))}$ for any section i of the canonical projection from Y onto X .

Let us consider the universal covering (then H_{x_0} is trivial). It is easy to check it will be transient even in the recurrent case as soon as (X, E) is not circular.

The expression of the Green function \widehat{G} on a universal covering can be given exactly when it is a regular tree, i.e. in the regular graph case. In fact a more general result can be proved as follows:

Given a graph (X, E) , set $d_x = \sum_y 1_{\{x,y\} \in E}$ (degree or valency of the vertex x), $D_{x,y} = d_x \delta_{x,y}$ and denote $A_{x,y}$ the incidence matrix $1_E(\{x, y\})$.

Consider the Green function associated with $\lambda_x = (d_x - 1)u + \frac{1}{u}$, with $0 < u < \inf(\frac{1}{d_x - 1})$ and for $\{x, y\} \in E$, $C_{x,y} = 1$: Then, noting that $\sum_x A_{z,x} u^{d(x,y)} = (d_z - 1)u^{d(z,y)+1} + u^{d(z,y)-1}$, we have $G = [u^{-1}I + (D - I)u - A]^{-1}$.

Proposition 21 *On the universal covering \mathfrak{T}_{x_0} , $\widehat{G}^{x,y} = u^{d(x,y)}(\frac{1}{u} - u)^{-1}$.*

Indeed, $\sum_x (\lambda_z \delta_x^z - A_{z,x}) u^{d(x,y)} = 0$ for $z \neq y$ and equals $\frac{1}{u} - u$ for $z = y$.

It follows that for any section i of the canonical projection from \mathfrak{T}_{x_0} onto X ,

$$\sum_{\gamma \in \Gamma_{x_0}} u^{d(i(x), \gamma(i(y)))} = (\frac{1}{u} - u) G^{x,y}. \quad \square$$

3.7 Zeta functions for discrete loops

We present briefly the terminology of symbolic dynamics (see for example [28]) in this simple framework (setting $f(x_0, x_1, \dots, x_n, \dots) = \log(P_{x_0, x_1})$, P induces the Ruelle operator L_f associated with f).

The pressure is defined as the logarithm of the highest eigenvalue β of P . It is associated with a unique normalized positive eigenfunction h , by Perron Frobenius theorem. Note that $Ph = \beta h$ implies $\lambda h P = \beta \lambda h$ by duality and that in the recurrent case, the pressure vanishes and $h = \frac{1}{\sqrt{\lambda(X)}}$.

In continuous time, the lowest eigenvalue of $-L$ i.e. $1 - \beta$ plays the role of the pressure

The equilibrium measure associated with f , $m = h^2 \lambda$ is the law of the stationary Markov chain associated with transition probability $\frac{1}{\beta h_x} P_y^x h_y$.

If $P1 = 1$, i.e. $\kappa = 0$, we can consider a Feynman Kac type perturbation $P^{(\varepsilon\kappa)} = PM_{\frac{\lambda}{\lambda+\varepsilon\kappa}}$, with $\varepsilon \downarrow 0$ and κ a positive measure. Perturbation theory shows that $\beta^{(\varepsilon\kappa)} - 1 = \frac{1}{\lambda(X)} \sum_x \frac{\lambda_x}{1+\varepsilon\kappa_x} - 1 + o(\varepsilon) = -\frac{\varepsilon\kappa(X)}{\lambda(X)} + o(\varepsilon)$ and that $h^{(\varepsilon\kappa)} = \frac{1}{\sqrt{\lambda(X)}} + o(\varepsilon)$.

We deduce from that the asymptotic behaviour of $\int (e^{-\varepsilon \langle \widehat{l}, \chi \rangle} - 1) d\mu^{(\varepsilon\kappa)}(l) = \log(\det(I - P^{(\varepsilon\kappa)})) - \log(\det(I - P^{(\varepsilon(\kappa+\chi))}))$ which is equivalent to $-\log(1 - \beta^{(\varepsilon(\kappa+\chi))}) + \log(1 - \beta^{(\varepsilon\kappa)})$ and therefore to $\log(\frac{\kappa(X)}{\kappa(X)+\chi(X)})$.

The study of relations between the loop measure μ and the zeta function $(\det(I - sP))^{-1}$ and more generally $(\det(I - M_f P))^{-1}$ with f a function on $[0, 1]$ can be done in the context of discrete loops. $\exp(\sum_{discreteloops}^{based} \frac{1}{p(\xi)} s^{p(\xi)} \mu(\xi)) = (\det(I - sP))^{-1}$ can be viewed as a type of zeta function defined for $s \in [0, 1/\beta)$

Primitive (discrete) non trivial based loops are defined as (discrete) based loops which cannot be obtained by the concatenation of $n \geq 2$ identical based loops.

The zeta function has an Euler product expansion: if we denote by ξ° this discrete loop defined by the based discrete loop ξ , for $\xi = (\xi_1, \dots, \xi_k)$, $\mu(\xi^\circ) = P_{\xi_2}^{\xi_1} P_{\xi_3}^{\xi_2} \dots P_{\xi_1}^{\xi_k}$, we have:

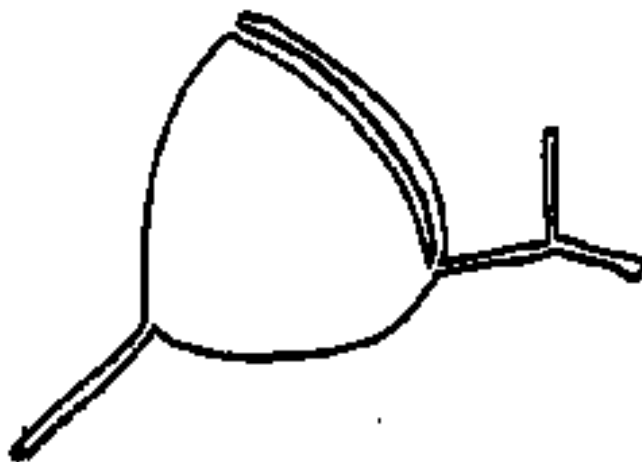
$$(\det(I - sP))^{-1} = \exp\left(\sum_{\substack{based \\ discreteloops}} \frac{1}{p(\xi)} s^{p(\xi)} \mu(\xi)\right) = \prod_{\substack{primitive \\ discrete \\ loops}} (1 - \int s^{p(\xi^\circ)} \mu(\xi^\circ))^{-1}$$

4 Geodesic loops

4.1 Reduction

Given any finite path ω with starting point x_0 , the reduced path ω^R is defined as the geodesic arc defined by the endpoint of the lift of ω to \mathfrak{T}_{x_0} .

Tree-contour like based loops can be defined as discrete based loops whose lift to the universal covering are still based loops. Each link is followed the same number of times in opposite directions (backtracking). The reduced path ω^R can equivalently be obtained by removing all tree-contour like based loops imbedded into it. In particular each loop l based at x_0 defines an element l^R in Γ_{x_0} .

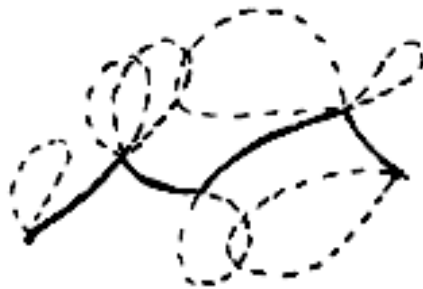


Based loop



Reduced based loop

This procedure is an example of loop erasure. In any graph, given a path ω , the loop erased path ω^{LE} is defined by removing successively all based loops imbedded in the path, starting from the origin. It produces a self avoiding path (and we see geodesics in \mathfrak{T}_{x_0} are self avoiding paths). Hence any non ramified covering defines a specific reduction operation by composition of lift, loop erasure, and projection.



Loop erasure

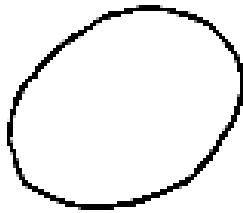
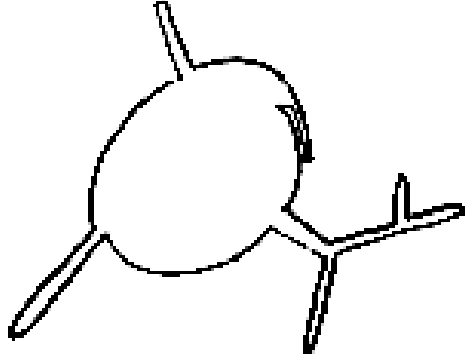
4.2 Geodesic loops and conjugacy classes

Then, we can consider loops i.e. equivalence classes of based loops under the natural shift.

Geodesic loops are of particular interest. Note their based loops representatives have to be "tailless": If γ is a geodesic based loop, with $|\gamma| = n$, the tail of γ is defined as $\gamma_1\gamma_2\dots\gamma_i$ if $i = \sup(j, \gamma_1\gamma_2\dots\gamma_j = \gamma_n\gamma_{n-1}\dots\gamma_{n-j+1})$. The associated geodesic loop is obtained by removing the tail.

The geodesic loops are clearly in bijection with the set of conjugacy classes of the fundamental group. Indeed, if we fix a base point x_0 , a geodesic loop defines the conjugation class formed of the elements of Γ_{x_0} obtained by choosing a base point on the loop and a geodesic segment linking it to x_0 . Any non trivial element of Γ_{x_0} can be obtained in this way.

Given a loop, there is a canonical geodesic loop associated with it. It is obtained by removing all tails imbedded in it. It can be done by removing one by one all tail edges (i.e. pairs of consecutive inverse oriented edges of the loop). Note that after removal of a tail edge, another tail edge cannot disappear, and that new tail edges appear during this process.



Loop and associated geodesic loop

A closed geodesic based at x_0 is called primitive if it cannot be obtained as the concatenation of several identical closed geodesic, i.e. if it is not a non trivial power in Γ_{x_0} . This property is clearly stable under conjugation. Let \mathfrak{P} be corresponding set of primitive geodesic loops. They represent conjugacy classes of primitive elements of Γ (see [37]).

4.3 Geodesics and boundary

Geodesics lines (half-lines) on a graph are defined as paths without backtracking indexed by \mathbb{Z} (\mathbb{N}).

Paths and geodesics can be defined on (X, E) or on a universal cover \mathfrak{T} and lifted or projected on any intermediate covering space.

Equivalence classes of geodesics half lines for the confluence relation define the boundary $\partial\mathfrak{T}$ of \mathfrak{T} . A geodesic half-line on \mathfrak{T} can be identified with its origin o and a boundary point θ . It projects on a geodesic half-line on X .

There is a natural σ -field on the boundary generated by cylinder sets B_g defined by half geodesics starting with a given oriented edge g .

Given any point x_0 in \mathfrak{T} , one can define a probability measure on the boundary called the harmonic measure and denoted ν_{x_0} : $\nu_{x_0}(B_g)$ is the probability that the lift of the P -Markov chain starting at x_0 hits g^+ after its last visit to g^- .

Note that Γ acts on the boundary in such a way that $\gamma^*(\nu_x) = \nu_{\gamma(x)}$, for all $\gamma \in \Gamma_{x_0}$.

Clearly, in the case of a regular graph, as the universal covering is a r -regular tree, $\nu_{x_0}(B_g) = \frac{1}{r} \frac{1}{(r-1)^{d(x_0, g^-)}}$ where d denotes the distance in the tree. When conductances are all equal, it can also be computed and is in general distinct from the visibility measure from x_0 , $\nu^{vis}(B_g)$, defined as $\frac{1}{d_{x_0}} \prod \frac{1}{d_{x_i}-1}$, x_1, x_2, \dots being the points of the geodesic segment linking x_0 to g^- .

Note that $\sum_{x \in \mathfrak{T}} \delta_x^o \nu_x^{vis}(d\theta)$ is a shift and Γ -invariant measure on the set of half-geodesics of \mathfrak{T} .

It induces a shift and Γ -invariant probability on half geodesics on X obtained by restricting the sum to any fundamental domain and normalizing by $|X|$.

4.4 Closed geodesics and associated Zeta function

Recall that \mathfrak{P} denotes the set of primitive geodesic loops.

Ihara's zeta function $IZ(u)$ is defined for $0 \leq u < 1$ as

$$IZ(u) = \prod_{\gamma \in \mathfrak{P}} (1 - u^{p(\gamma)})^{-1}$$

It depends only on the graph.

Note that $u \frac{d}{du} \frac{IZ(u)}{IZ(u)} = \sum_{\gamma \in \mathfrak{P}} \frac{p(\gamma) u^{p(\gamma)}}{1 - u^{p(\gamma)}} = \sum_{\gamma \in \mathfrak{P}} \sum_{n=1}^{\infty} p(\gamma) u^{np(\gamma)} = \sum N_m u^m$ where N_m denotes the number of *tailless* geodesic based loops of length m . Indeed, each primitive geodesic loop γ traversed n times still induces $p(\gamma)$ distinct tailless geodesic based loops. Therefore $IZ(u)$ can also be written as $\exp(\sum_{m=2}^{\infty} \frac{N_m u^m}{m})$

Similarly, one can define $\Pi\Gamma$ to be the set of primitive elements of the fundamental group Γ and the Γ -zeta function to be:

$$\Gamma Z(u) = \prod_{\gamma \in \text{III}\Gamma} (1 - u^{p(\gamma)})^{-1}$$

Note that $u \frac{d}{du} \frac{\Gamma Z(u)}{\Gamma Z(u)} = \sum_2^\infty L_m u^m$ where L_m denotes the number of geodesic based loops of length m . $\Gamma Z(u)$ can also be written as $\exp(\sum_{m=2}^\infty \frac{L_m u^m}{m})$. Recall that A denotes the adjacency matrix of the graph, and D the diagonal matrix whose entries are given by the degrees of the vertices.

Assume now that $u < \inf(\frac{1}{d_x - 1})$.

Theorem 22 a) $\sum L_m u^m = (1 - u^2) \text{Tr}([I + (D - I)u^2 - uA]^{-1}) - |V|$

b) $I Z(u) = (1 - u^2)^{-\chi} \det(I - uA + u^2(D - I))^{-1}$ where $\chi = \text{denotes the Euler number } |E| - |X|$ of the graph.

Proof. We adapt the approach of Stark-Terras ([37])

It follows from proposition 21 that $|V| + \sum_2^\infty L_m u^m = (\frac{1}{u} - u) \text{Tr}(G) = (1 - u^2) \text{Tr}([I + (D - I)u^2 - uA]^{-1})$

Given a geodesic loop l (possibly empty) and a base point y of l , let $S_{x,y,l}$ be the sum of the coefficients $u^{p(\delta)}$, where δ varies on all geodesic loops based at x composed with l and a tail ending at y . If $x = y$, we have $S_{y,y,l} = u^{p(l)}$. Set $S_{x,l} = \sum_y S_{x,y,l}$.

Clearly, for any section i of the canonical projection from \mathfrak{X}_x onto X ,

$$\sum_{y,l} S_{x,y,l} = \sum_{\gamma \in \Gamma_x - \{I\}} u^{d(i(x), \gamma i(x))} = (\frac{1}{u} - u) G^{x,x} - 1$$

Therefore, considering first the tailless case, then the case where the tail has length 1, and finally decomposing the case where the tail has length at least two according to the position of point next to x , we obtain the expression: $\sum_x S_{x,y,l} = u^{p(l)} + (d_y - 2)u^{p(l)+2} + \sum_{x \neq y} (d_x - 1)u^2 S_{x,y,l}$

$$= u^{p(l)} - u^{p(l)+2} + \sum_x (d_x - 1)u^2 S_{x,y,l}$$

summing in y , it comes that

$$\sum_x S_{x,l} = p(l)(u^{p(l)} - u^{p(l)+2}) + \sum_x (d_x - 1)u^2 S_{x,l}$$

Then, summing on all geodesic loops l

$$\sum_x ((\frac{1}{u} - u) G^{x,x} - 1) = (1 - u^2)(\sum N_m u^m) + \sum_x (d_x - 1)u^2 ((\frac{1}{u} - u) G^{x,x} - 1)$$

$$\begin{aligned} \text{Therefore, } \sum N_m u^m &= \text{Tr}((I - u^2(D - I))([I + (D - I)u^2 - uA]^{-1} - (1 - u^2)^{-1}I)) \\ &= \text{Tr}(I - (2u^2(D - I) - uA)[I + (D - I)u^2 - uA]^{-1} - (1 - u^2)^{-1}(I - u^2(D - I))) \end{aligned}$$

$$= \text{Tr}((2u^2(D - I) - uA)[I + (D - I)u^2 - uA]^{-1} + (1 - u^2)^{-1}(u^2(D - 2I)))$$

To conclude note that $\frac{d}{du} \log(\det(I + (D - I)u^2 - uA)) = \text{Tr}((2u(D - I) - A)[I + (D - I)u^2 - uA]^{-1})$ and that $\text{Tr}(u^2(D - 2I)) = 2u^2\chi$. ■

Other proofs can be found in the litterature, especially the following one due to Kotani-Sunada ([11]):

On the line graph, we define a transfer operator Q by $Q_{(x,y)}^{(y',z)} = \delta_y^{y'} 1_{\{z \neq x\}}$. Then

$$IZ(u) = \det((I - uQ)^{-1})$$

Define T and τ on \mathbb{A} by $T\alpha(x) = \sum_{y, \{x,y\} \in E} \alpha(x, y)$ and $\tau\alpha(e) = \alpha(-e)$.

Denote S the operator from functions on the graph to \mathbb{A} defined by $Sf(x, y) = f(y)$.

Note that $T\tau S = D$, $TS = A$, and $Q = -\tau + ST$.

Then, for any scalar u ,

$$(I - u\tau)(I - uQ) = (1 - u^2)I - (I - u\tau)uST$$

and

$$(I - uQ)(I - u\tau) = (1 - u^2)I - ST(I - u\tau) \quad (14)$$

Therefore $T(I - u\tau)(I - uQ) = ((1 - u^2)I - uT(I - u\tau)ST) = (I + u^2(D - I) - uA)T$ and

$$T(I - u\tau)(I - uQ)(I - u\tau) = (I + u^2(D - I) - uA)T(I - u\tau)$$

Moreover $(I - uQ)(I - u\tau)S = S((1 - u^2)I - uT(I - u\tau)S)$ and

$$(I - uQ)(I - u\tau)S = S((I + u^2(D - I) - uA)) \quad (15)$$

It follows from these two last identities that $\text{Im}(S)$ and $\text{Ker}(T(I - u\tau))$ are stable under $(I - uQ)(I - u\tau)$.

Note that S is the dual of $-T\tau$: Indeed, for any function f on vertices and α on oriented edges, $\sum_{(x,y) \in E^O} \alpha(x, y)Sf(x, y) = \sum_{(x,y) \in E^O} \alpha(x, y)f(y) = \sum_y T\tau\alpha(y)f(y)$.

Therefore, $\dim(\text{Im}(S)) + \dim(\text{Ker}(T\tau)) = 2|E|$

Note also that $\dim(\text{Ker}(T\tau)) = \dim(\text{Ker}(T)) = \dim(\text{Ker}(T(I - u\tau)))$ (as $u < 1$).

Moreover, except for a finite set of u 's, $\text{Im}(S) \cap \text{Ker}(T(I - u\tau)) = \{0\}$. Indeed $T(I - u\tau)S = A - uD$ which is invertible, except for a finite set of u 's.

Note that (14) implies that $(I - uQ)(I - u\tau)$ equals $(1 - u^2)I$ on $\text{Ker}(T(I - u\tau))$ and that (15) implies it equals $S((I + u^2(D - I) - uA))S^{-1}$ on $\text{Im}(S)$.

It comes that:

$$\det((I - u\tau)(I - uQ)) = (1 - u^2)^{2|E| - |X|} \det(I + u^2(D - I) - uA)$$

On the other hand, $\det((I - u\tau)) = (1 - u^2)^{|E|}$, which allows to conclude.

5 Poisson process of loops

5.1 Definition

Still following the idea of [15], which was already implicitly in germ in [39], define, for all positive α , the Poissonian ensemble of loops \mathcal{L}_α with intensity $\alpha\mu$. We denote by \mathbb{P} or $\mathbb{P}_{\mathcal{L}_\alpha}$ its distribution.

Recall it means that for any functional Φ on the loop space, vanishing on loops of arbitrary small length,

$$E(e^{i\sum_{l \in \mathcal{L}_\alpha} \Phi(l)}) = \exp(\alpha \int (e^{i\Phi(l)} - 1)\mu(dl))$$

Of course, \mathcal{L}_α includes trivial loops. The periods τ_l of the trivial loops based at any point x form a Poisson process of intensity $\alpha \frac{e^{-t}}{t}$. It follows (Cf [29] and references therein) the following

Remark 23 *The sum of these periods $\sum \tau_l$ and the set of "frequencies" $\frac{\tau_l}{\sum \tau_l}$ (in decreasing order) are independent and follow respectively a $\Gamma(\alpha)$ and a Poisson - Dirichlet(0, α) distribution.*

Note that by the restriction property, $\mathcal{L}_\alpha^D = \{l \in \mathcal{L}_\alpha, l \subseteq D\}$ is a Poisson process of loops with intensity μ^D , and that \mathcal{L}_α^D is independent of $\mathcal{L}_\alpha \setminus \mathcal{L}_\alpha^D$.

Remark 24 *Note also that these Poisson ensembles can be considered for fixed α or as a point process of loops indexed by the "time" α . In that case, \mathcal{L}_α is an increasing set of loops with stationnary increments. We will denote by \mathcal{LP} the associated Poisson point process of intensity $\mu \otimes d\alpha$. It is formed by a countable set of pairs (l_i, α_i) formed by a time and a loop*

We denote by \mathcal{DL}_α the set of non trivial discrete loops in \mathcal{L}_α . Then,

$$\mathbb{P}(\mathcal{DL}_\alpha = \{l_1, l_2, \dots, l_k\}) = e^{-\alpha\mu(p>1)} \alpha^k \mu(l_1) \dots \mu(l_k) = \alpha^k \left[\frac{\det(G)}{\prod_x \lambda_x} \right]^\alpha \prod_{x,y} C_{x,y}^{N_{x,y}^{(\alpha)}} \prod_x \lambda_x^{-N_x^{(\alpha)}}$$

with $N_x^{(\alpha)} = N_x(\mathcal{L}_\alpha) = \sum_{l \in \mathcal{L}_\alpha} N_x(l)$ and $N_{x,y}^{(\alpha)} = N_{x,y}(\mathcal{L}_\alpha) = \sum_{l \in \mathcal{L}_\alpha} N_{x,y}(l)$, when these loops are distinct.

We can associate to \mathcal{L}_α a σ -finite measure (in fact as we will see, finite when X is finite, and more generally if G is trace class) called local time or occupation field

$$\widehat{\mathcal{L}}_\alpha = \sum_{l \in \mathcal{L}_\alpha} \widehat{l}$$

Then, for any non-negative measure χ on X

$$\mathbb{E}(e^{-\langle \widehat{\mathcal{L}}_\alpha, \chi \rangle}) = \exp\left(\alpha \int (e^{-\langle \widehat{l}, \chi \rangle} - 1) d\mu(l)\right)$$

and therefore by proposition 17 we have

Corollary 25 $\mathbb{E}(e^{-\langle \widehat{\mathcal{L}}_\alpha, \chi \rangle}) = \det(I + M_{\sqrt{\chi}} G M_{\sqrt{\chi}})^{-\alpha} = \left(\frac{\det(G_\chi)}{\det(G)}\right)^\alpha.$

Many calculations follow from this result.

Note first that $\mathbb{E}(e^{-t\widehat{\mathcal{L}}_\alpha^x}) = (1 + tG^{x,x})^{-\alpha}$. Therefore $\widehat{\mathcal{L}}_\alpha^x$ follows a gamma distribution $\Gamma(\alpha, G^{x,x})$, with density $1_{\{x>0\}} \frac{e^{-\frac{x}{G^{x,x}}}}{\Gamma(\alpha)} \frac{x^{\alpha-1}}{(G^{x,x})^\alpha}$ (in particular, an exponential distribution of mean $G^{x,x}$ for $\alpha = 1$, as \widehat{l}^x under $\frac{\mu^{y,x}}{G^{y,x}}$). When we let α vary as a time parameter, we get a family of gamma subordinators, which can be called a "multivariate gamma subordinator"¹.

We check in particular that $\mathbb{E}(\widehat{\mathcal{L}}_\alpha^x) = \alpha G^{x,x}$ which follows directly from $\mu(\widehat{l}_x) = G^{x,x}$.

Exercise 26 If $\mathcal{L}_\alpha = \{l_i\}$, check that the set of "frequencies" $\frac{\widehat{l}_i^x}{\widehat{\mathcal{L}}_\alpha^x}$ follows a Poisson-Dirichlet distribution of parameters $(0, \alpha)$.

¹A subordinator is an increasing Levy process. See for example reference [1].

(Hint: use the μ -distribution of \widehat{l}_x)

Note also that for $\alpha > 1$,

$$\mathbb{E}\left(\left(1 - \exp\left(-\frac{\widehat{\mathcal{L}}_\alpha^x}{G^{x,x}}\right)\right)^{-1}\right) = \zeta(\alpha).$$

More generally, for two points:

$$\mathbb{E}\left(e^{-t\widehat{\mathcal{L}}_\alpha^x} e^{-s\widehat{\mathcal{L}}_\alpha^y}\right) = \left((1 + tG^{x,x})(1 + sG^{y,y}) - st(G^{x,y})^2\right)^{-\alpha}$$

This allows us to compute the joint density of $\widehat{\mathcal{L}}_\alpha^x$ and $\widehat{\mathcal{L}}_\alpha^y$ in terms of Bessel and Struve functions.

We can condition the loop configuration by the set of associated non trivial discrete loops by using the restricted σ -field $\sigma(\mathcal{DL}_\alpha)$ which contains the variables $N_{x,y}$. We see from 11 and 9 that

$$\mathbb{E}\left(e^{-\langle \widehat{\mathcal{L}}_\alpha, \chi \rangle} | \mathcal{DL}_\alpha\right) = \prod_x \left(\frac{\lambda_x}{\lambda_x + \chi_x}\right)^{N_x^{(\alpha)} + 1}$$

The distribution of $\{N_x^{(\alpha)}, x \in X\}$ follows easily, from corollary 25 in terms of generating functions:

$$\mathbb{E}\left(\prod_x s_x^{N_x^{(\alpha)} + 1}\right) = \det\left(\delta_{x,y} + \sqrt{\frac{\lambda_x(1-s_x)}{s_x}} G_{x,y} \sqrt{\frac{\lambda_y(1-s_y)}{s_y}}\right)^{-\alpha} \quad (16)$$

so that the vector of components $N_x^{(\alpha)}$ follows a multivariate negative binomial distribution (see for example [41]).

It follows in particular that $N_x^{(\alpha)}$ follows a negative binomial distribution of parameters $-\alpha$ and $\frac{1}{\lambda_x G^{xx}}$. Note that for $\alpha = 1$, $N_x^{(1)} + 1$ follows a geometric distribution of parameter $\frac{1}{\lambda_x G^{xx}}$.

Note finally that in the recurrent case, with the setting and the notations of subsection 3.7, denoting $\mathcal{L}_{\alpha\varepsilon}^{(\varepsilon\kappa)}$ the Poisson process of loops of intensity $\varepsilon\alpha\mu^{(\varepsilon\kappa)}$, we get that the associated occupation field converges towards a constant random variable following a Gamma distribution.

5.2 Moments and polynomials of the occupation field

It is easy to check (and well known from the properties of the gamma distributions) that the moments of $\widehat{\mathcal{L}}_\alpha^x$ are related to the factorial moments of $N_x^{(\alpha)}$:

$$\mathbb{E}((\widehat{\mathcal{L}}_\alpha^x)^k | \mathcal{DL}_\alpha) = \frac{(N_x^{(\alpha)} + k)(N_x^{(\alpha)} + k - 1) \dots (N_x^{(\alpha)} + 1)}{k! \lambda_x^k}$$

Exercise 27 Denoting \mathcal{L}_α^+ the set of non trivial loops in \mathcal{L}_α , define

$$\widehat{\mathcal{L}}_\alpha^{x,(k)} = \sum_{m=1}^k \sum_{k_1 + \dots + k_m = k} \sum_{l_1 \neq l_2 \dots \neq l_m \in \mathcal{L}_\alpha^+} \prod_{j=1}^m \widehat{l}_j^{x,(k_j)}.$$

Deduce from exercise 16 that $\mathbb{E}(\widehat{\mathcal{L}}_\alpha^{x,(k)} | \mathcal{DL}_\alpha) = \frac{1}{k! \lambda_x^k} 1_{\{N_x \geq k\}} (N_x^{(\alpha)} - k + 1) \dots (N_x^{(\alpha)} - 1) N_x^{(\alpha)}$

It is well known that Laguerre polynomials $L_k^{(\alpha-1)}$ with generating function

$$\sum_0^\infty t^k L_k^{(\alpha-1)}(u) = \frac{e^{-\frac{ut}{1-t}}}{(1-t)^\alpha}$$

are orthogonal for the $\Gamma(\alpha)$ distribution. They have mean zero and variance $\frac{\Gamma(\alpha+k)}{k!}$. Hence if we set $\sigma_x = G^{x,x}$ and $P_k^{\alpha,\sigma}(x) = (-\sigma)^k L_k^{(\alpha-1)}(\frac{x}{\sigma})$, the random variables $P_k^{\alpha,\sigma_x}(\widehat{\mathcal{L}}_\alpha^x)$ are orthogonal with mean 0 and variance $\sigma^{2k} \frac{\Gamma(\alpha+k)}{k!}$, for $k > 0$.

Note that $P_1^{\alpha,\sigma_x}(\widehat{\mathcal{L}}_\alpha^x) = \widehat{\mathcal{L}}_\alpha^x - \alpha \sigma_x = \widehat{\mathcal{L}}_\alpha^x - \mathbb{E}(\widehat{\mathcal{L}}_\alpha^x)$. It will be denoted $\widetilde{\mathcal{L}}_\alpha^x$.

Moreover, we have $\sum_0^\infty t^k P_k^{\alpha,\sigma}(u) = \sum (-\sigma t)^k L_k^{(\alpha-1)}(\frac{u}{\sigma}) = \frac{e^{-\frac{ut}{1+\sigma t}}}{(1+\sigma t)^\alpha}$

Note that by corollary 25,

$$\begin{aligned} \mathbb{E}\left(\frac{e^{\frac{\widetilde{\mathcal{L}}_\alpha^x t}{1+\sigma_x t}}}{(1+\sigma_x t)^\alpha} \frac{e^{\frac{\widetilde{\mathcal{L}}_\alpha^y s}{1+\sigma_y s}}}{(1+\sigma_y s)^\alpha}\right) \\ = \frac{1}{(1+\sigma_x t)^\alpha (1+\sigma_y s)^\alpha} \left(\left(1 - \frac{\sigma_x t}{1+\sigma_x t}\right) \left(1 - \frac{\sigma_y s}{1+\sigma_y s}\right) - \frac{t}{1+\sigma_x t} \frac{s}{1+\sigma_y s} (G^{x,y})^2 \right)^{-\alpha} \\ = (1 - st(G^{x,y})^2)^{-\alpha}. \end{aligned}$$

Therefore, we get, by developping in entire series in (s, t) and identifying the coefficients:

$$\mathbb{E}(P_k^{\alpha, \sigma_x}(\widehat{\mathcal{L}}_\alpha^x), P_l^{\alpha, \sigma_y}(\widehat{\mathcal{L}}_\alpha^y)) = \delta_{k,l} (G^{x,y})^{2k} \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!} \quad (17)$$

Let us stress the fact that $G^{x,x}$ and $G^{y,y}$ do not appear on the right hand side of this formula. This is quite important from the renormalisation point of view, as we will consider in the last section the two dimensional Brownian motion for which the Green function diverges on the diagonal.

More generally one can prove similar formulas for products of higher order.

Note that since $G_\chi M_\chi$ is a contraction, from determinant expansions given in [40] and [41], we have

$$\det(I + M_{\sqrt{\chi}} G M_{\sqrt{\chi}})^{-\alpha} = 1 + \sum_{k=1}^{\infty} (-1)^k \sum \chi_{i_1} \dots \chi_{i_k} Per_\alpha(G_{i_l, i_m}, 1 \leq l, m \leq k) \quad (18)$$

and then, from corollary 25, it follows that:

$$\mathbb{E}(\langle \widehat{\mathcal{L}}_\alpha, \chi \rangle^k) = \sum \chi_{i_1} \dots \chi_{i_k} Per_\alpha(G_{i_l, i_m}, 1 \leq l, m \leq k)$$

Here the α -permanent Per_α is defined as $\sum_{\sigma \in \mathcal{S}_k} \alpha^{m(\sigma)} G_{i_1, i_{\sigma(1)}} \dots G_{i_k, i_{\sigma(k)}}$ with $m(\sigma)$ denoting the number of cycles in σ .

Note that an explicit form for the multivariate negative binomial distribution, and therefore, a series expansion for the density of the multivariate gamma distribution, follows directly (see [41]) from this determinant expansion.

It is actually not difficult to give a direct proof of this result. Thus, the Poisson process of loops provides a natural probabilistic proof and interpretation of this combinatorial identity (see [41] for an historical view of the subject).

We can show in fact that:

Proposition 28 *For any (x_1, \dots, x_k) in X^k , $\mathbb{E}(\widehat{\mathcal{L}}_\alpha^{x_1} \dots \widehat{\mathcal{L}}_\alpha^{x_k}) = Per_\alpha(G^{x_l, x_m}, 1 \leq l, m \leq k)$*

Proof. The cycles of the permutations in the expression of Per_α are associated with point configurations on loops. We obtain the result by summing the contributions of all possible partitions of the points $i_1 \dots i_k$ into a finite set of distinct loops. We can then decompose again the expression according to ordering of points on each loop. We can conclude by using the formula $\mu(\widehat{l}^{x_1, \dots, x_m}) = G^{x_1, x_2} G^{x_2, x_3} \dots G^{x_m, x_1}$ and the following

property of Poisson measures (Cf formula 3-13 in [12]): For any system of non negative or μ -integrable loop functionals F_i

$$\mathbb{E}\left(\sum_{l_1 \neq l_2 \dots \neq l_k \in \mathcal{L}_\alpha} \prod F_i(l_i)\right) = \prod \alpha \mu(F_i) \quad (19)$$

■

Remark 29 We can actually, in the special case $i_1 = i_2 = \dots = i_k = x$, check this formula in a different way. From the moments of the Gamma distribution, we have that $\mathbb{E}((\widehat{\mathcal{L}}_\alpha^x)^n) = (G^{x,x})^n \alpha(\alpha+1)\dots(\alpha+n-1)$ and the α -permanent can be written $\sum_1^n d(n,k) \alpha^k$ where the coefficients $d(n,k)$ are the numbers of n -permutations with k cycles (Stirling numbers of the first kind). One checks that $d(n+1,k) = nd(n,k) + d(n,k-1)$.

Proposition 30 Given any bounded functionals Φ on loops configurations and F on loops, we have:

$$\mathbb{E}\left(\sum_{l \in \mathcal{L}_\alpha} F(l) \Phi(\mathcal{L}_\alpha)\right) = \int \mathbb{E}(\Phi(\mathcal{L}_\alpha \cup \{l\})) \alpha F(l) \mu(dl)$$

Proof. This is proved by considering first for $\Phi(\mathcal{L}_\alpha)$ the functionals of the form $\sum_{l_1 \neq l_2 \dots \neq l_q \in \mathcal{L}_\alpha} \prod_1^q G_j(l_j)$ (with G_j bounded and μ -integrable) which span an algebra separating distinct configurations and applying formula (19): Then, the common value of both members is $\alpha^q \sum_1^q \mu(FG_j) \prod_{l \neq j} \mu(G_l) + \alpha^{q+1} \mu(F) \prod_1^q \mu(G_j)$ ■

The above proposition applied to $F(l) = \widehat{l}^x, N_{x,y}^{(\alpha)}$ and propositions 12 and 18 yield the following:

Corollary 31 $\mathbb{E}(\Phi(\mathcal{L}_\alpha) \widehat{\mathcal{L}}_\alpha^x) = \alpha \int \mathbb{E}(\Phi(\mathcal{L}_\alpha \cup \{\gamma\})) \widehat{\gamma}^x \mu(d\gamma) = \alpha \int \mathbb{E}(\Phi(\mathcal{L}_\alpha \cup \{\gamma\})) \mu^{x,x}(d\gamma)$
and

$\mathbb{E}(\Phi(\mathcal{L}_\alpha) N_{x,y}^{(\alpha)}) = \alpha \int \mathbb{E}(\Phi(\mathcal{L}_\alpha \cup \{\gamma\})) N_{x,y}(\gamma) \mu(d\gamma) = \alpha C_{x,y} \int \mathbb{E}(\Phi(\mathcal{L}_\alpha \cup \{\gamma\})) \mu^{x,y}(d\gamma)$
if $x \neq y$.

Let \mathcal{S}_k^0 be the set of permutations of k elements without fixed point. They correspond to configurations without isolated points.

Set $Per_\alpha^0(G^{i_l, i_m}, 1 \leq l, m \leq k) = \sum_{\sigma \in \mathcal{S}_k^0} \alpha^{m(\sigma)} G^{i_1, i_{\sigma(1)}} \dots G^{i_k, i_{\sigma(k)}}$. Then an easy calculation shows that:

Corollary 32 $\mathbb{E}(\widetilde{\mathcal{L}}_\alpha^{i_1} \dots \widetilde{\mathcal{L}}_\alpha^{i_k}) = Per_\alpha^0(G^{i_l, i_m}, 1 \leq l, m \leq k)$

Proof. Indeed, the expectation can be written

$$\sum_{p \leq k} \sum_{I \subseteq \{1, \dots, k\}, |I|=p} (-1)^{k-p} \prod_{l \in I^c} G^{i_l, i_l} Per_\alpha(G^{i_a, i_b}, a, b \in I)$$

and

$$Per_\alpha(G^{i_a, i_b}, a, b \in I) = \sum_{J \subseteq I} \prod_{j \in I \setminus J} G^{j, j} Per_\alpha^0(G^{i_a, i_b}, a, b \in J).$$

Then, expressing $\mathbb{E}(\widetilde{\mathcal{L}}_\alpha^{i_1} \dots \widetilde{\mathcal{L}}_\alpha^{i_k})$ in terms of Per_α^0 's, we see that if $J \subseteq \{1, \dots, k\}$, $|J| < k$, the coefficient of $Per_\alpha^0(G^{i_a, i_b}, a, b \in J)$ is $\sum_{I, I \supseteq J} (-1)^{k-|I|} \prod_{j \in J^c} G^{i_j, i_j}$ which vanishes as $(-1)^{-|I|} = (-1)^{|I|} = (-1)^{|J|} (-1)^{|I \setminus J|}$ and $\sum_{I \supseteq J} (-1)^{|I \setminus J|} = (1-1)^{k-|J|} = 0$. ■

Set $Q_k^{\alpha, \sigma}(u) = P_k^{\alpha, \sigma}(u + \alpha\sigma)$ so that $P_k^{\alpha, \sigma}(\widehat{\mathcal{L}}_\alpha^x) = Q_k^{\alpha, \sigma}(\widetilde{\mathcal{L}}_\alpha^x)$. This quantity will be called the n -th renormalized self intersection local time or the n -th renormalized power of the occupation field and denoted $\widetilde{\mathcal{L}}_\alpha^{x, n}$.

From the recurrence relation of Laguerre polynomials

$$nL_n^{(\alpha-1)}(u) = (-u + 2n + \alpha - 2)L_{n-1}^{(\alpha-1)} - (n + \alpha - 2)L_{n-2}^{(\alpha-1)},$$

we get that

$$nQ_n^{\alpha, \sigma}(u) = (u - 2\sigma(n-1))Q_{n-1}^{\alpha, \sigma}(u) - \sigma^2(\alpha + n - 2)Q_{n-2}^{\alpha, \sigma}(u)$$

In particular $Q_2^{\alpha, \sigma}(u) = \frac{1}{2}(u^2 - 2\sigma u - \alpha\sigma^2)$.

We have also, from (17)

$$\mathbb{E}(Q_k^{\alpha, \sigma_x}(\widetilde{\mathcal{L}}_\alpha^x), Q_l^{\alpha, \sigma_y}(\widetilde{\mathcal{L}}_\alpha^y)) = \delta_{k, l} (G^{x, y})^{2k} \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!} \quad (20)$$

The comparison of the identity (20) and corollary 32 yields a combinatorial result which will be fundamental in the renormalizing procedure presented in the last section.

The identity (20) can be considered as a polynomial identity in the variables σ_x , σ_y and $G^{x, y}$.

Set $Q_k^{\alpha, \sigma_x}(u) = \sum_{m=0}^k q_m^{\alpha, k} u^m \sigma_x^{k-m}$, and denote $N_{n, m, r, p}$ the number of ordered configurations of n black points and m red points on r non trivial oriented cycles, such that only $2p$ links are between red and black points. We have first by corollary 32:

$$\mathbb{E}((\widetilde{\mathcal{L}}_\alpha^x)^n (\widetilde{\mathcal{L}}_\alpha^y)^m) = \sum_r \sum_{p \leq \inf(m,n)} \alpha^r N_{n,m,r,p} (G^{x,y})^{2p} (\sigma_x)^{n-p} (\sigma_y)^{m-p}$$

and therefore

$$\sum_r \sum_{p \leq m \leq k} \sum_{p \leq n \leq l} \alpha^r q_m^{\alpha,k} q_n^{\alpha,l} N_{n,m,r,p} = 0 \text{ unless } p = l = k. \quad (21)$$

$$\sum_r \alpha^r q_k^{\alpha,k} q_k^{\alpha,k} N_{k,k,r,k} = \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!} \quad (22)$$

Note that one can check directly that $q_k^{\alpha,k} = \frac{1}{k!}$, and $N_{k,k,1,k} = k!(k-1)!$, $N_{k,k,k,k} = k!$ which confirms the identity (22) above.

5.3 Hitting probabilities

Denote by $[H^F]_y^x = \mathbb{P}_x(x_{T_F} = y)$ be the hitting distribution of F by the Markov chain starting at x (H^F is called the balayage or Poisson kernel). Set $D = F^c$ and denote by e^D , $P^D = P|_{D \times D}$, $V^D = [(I - P^D)]^{-1}$ and $G^D = [(M_\lambda - C)|_{D \times D}]^{-1}$ the energy, the transition matrix, the potential and the Green function of the Markov chain killed at the hitting time of F .

Denote by \mathbb{P}_x^D the law of the killed Markov chain starting at x .

Recall that

$[H^F]_y^x = 1_{\{x=y\}} + \sum_0^\infty \sum_{z \in D} [(P^D)^k]_z^x P_y^z = 1_{\{x=y\}} + \sum_{z \in D} [V^D]_z^x P_y^z$. Moreover we have by the strong Markov property, $V = V^D + H^F V$ and therefore $G = G^D + H^F G$. (Here we extend V^D and G^D to $X \times X$ by adding zero entries outside $D \times D$).

As G and G^D are symmetric, we have $[H^F G]_y^x = [H^F G]_x^y$ so that for any measure ν , $H^F(G\nu) = G(\nu H^F)$.

Therefore we see that for any function f and measure ν , $e(H^F f, G^D \nu) = e(H^F f, G\nu) - e(H^F f, H^F G\nu) = \langle H^F f, \nu \rangle - e(H^F f, G(H^F \nu)) = 0$ as $(H^F)^2 = H^F$.

Equivalently, we have the following:

Proposition 33 *For any g vanishing on F , $e(H^F f, g) = 0$ so that $I - H^F$ is the e -orthogonal projection on the space of functions supported in D .*

$H^F 1$ is called the capacitary potential of F and $e(H^F 1, H^F 1)$ the capacity of F .

Note that these results extend without difficulty to the recurrent case. In particular, for any measure ν supported in D , $G^D \nu = G(\nu - \nu H^F)$ and $e(H^F f, G^D \nu) = 0$ for all f . For further developments see for exemple ([18]) and its references.

The restriction property holds for \mathcal{L}_α as it holds for μ . The set \mathcal{L}_α^D of loops inside D is associated with μ^D and is independent of $\mathcal{L}_\alpha - \mathcal{L}_\alpha^D$. Therefore, we see from corollary 25 that

$$\mathbb{E}(e^{-\langle \widehat{\mathcal{L}}_\alpha - \widehat{\mathcal{L}}_\alpha^D, \chi \rangle}) = \left(\frac{\det(G_\chi)}{\det(G)} \frac{\det(G^D)}{\det(G_\chi^D)} \right)^\alpha.$$

From the support of the the Gamma distribution, we see that $\mu(\widehat{l}(F) > 0) = \infty$. But this is clearly due to trivial loops as it can be seen directly from the definition of μ that in this simple framework they cover the whole space X .

Note however that

$$\begin{aligned} \mu(\widehat{l}(F) > 0, p > 1) &= \mu(p > 1) - \mu(\widehat{l}(F) = 0, p > 1) = \mu(p > 1) - \mu^D(p > 1) \\ &= -\log\left(\frac{\det(I - P)}{\det_{D \times D}(I - P)}\right) = -\log\left(\frac{\det(G^D)}{\prod_{x \in F} \lambda_x \det(G)}\right). \end{aligned}$$

It follows that the probability that no non trivial loop (i.e. a loop which is not reduced to a point) in \mathcal{L}_α intersects F equals

$$\exp(-\alpha \mu(\{l, p(l) > 1, \widehat{l}(F) > 0\})) = \left(\frac{\det(G^D)}{\prod_{x \in F} \lambda_x \det(G)} \right)^\alpha.$$

Recall that for any $(n + p, n + p)$ invertible matrix A , denoting e_i the canonical basis,

$$\begin{aligned} \det(A^{-1}) \det(A_{ij}, 1 \leq i, j \leq n) &= \det(A^{-1}) \det(Ae_1, \dots, Ae_n, e_{n+1}, \dots, e_{n+p}) \\ &= \det(e_1, \dots, e_n, A^{-1}e_{n+1}, \dots, A^{-1}e_{n+p}) \\ &= \det((A^{-1})_{k,l}, n \leq k, l \leq n + p). \end{aligned}$$

In particular, $\det(G^D) = \frac{\det(G)}{\det(G|_{F \times F})}$, so we have the

Proposition 34 *The probability that no non-trivial loop in \mathcal{L}_α intersects F equals*

$$\left[\prod_{x \in F} \lambda_x \det_{F \times F}(G) \right]^{-\alpha}.$$

Moreover $\mathbb{E}(e^{-\langle \widehat{\mathcal{L}}_\alpha - \widehat{\mathcal{L}}_\alpha^D, \chi \rangle}) = \left(\frac{\det_{F \times F}(G_\chi)}{\det_{F \times F}(G)} \right)^\alpha$

In particular, it follows that the probability that no non-trivial loop in \mathcal{L}_α visits x equals $(\frac{1}{\lambda_x G^{x,x}})^\alpha$ which is also a consequence of the fact that N_x follows a negative binomial distribution of parameters $-\alpha$ and $\frac{1}{\lambda_x G^{x,x}}$

Also, if F_1 and F_2 are disjoint,

$$\begin{aligned} \mu(\widehat{l}(F_1)\widehat{l}(F_2) > 0) &= \mu(\widehat{l}(F_1) > 0, p > 1) + \mu(\widehat{l}(F_2) > 0, p > 1) - \mu(\widehat{l}(F_1 \cup F_2) > 0, p > 1) \\ &= \log\left(\frac{\det(G) \det(G^{D_1 \cap D_2})}{\det(G^{D_1}) \det(G^{D_2})}\right). \end{aligned}$$

Therefore the probability that no non-trivial loop in \mathcal{L}_α intersects F_1 and F_2 equals

$$\exp(-\alpha \mu(\{l, p(l) > 1, \prod \widehat{l}(F_i) > 0\})) = \left(\frac{\det(G) \det(G^{D_1 \cap D_2})}{\det(G^{D_1}) \det(G^{D_2})}\right)^{-\alpha}$$

It follows that the probability no non-trivial loop in \mathcal{L}_α visits two distinct points x and y equals $(\frac{G^{x,x}G^{y,y} - (G^{x,y})^2}{G^{x,x}G^{y,y}})^\alpha$ and in particular $1 - \frac{(G^{x,y})^2}{G^{x,x}G^{y,y}}$ if $\alpha = 1$.

Exercise 35 Generalize this formula to n disjoint sets:

$$\mathbb{P}(\#l \in \mathcal{L}_\alpha, \prod \widehat{l}(F_i) > 0) = \left(\frac{\det(G) \prod_{i < j} \det(G^{D_i \cap D_j})}{\prod \det(G^{D_i}) \prod_{i < j < k} \det(G^{D_i \cap D_j \cap D_k})}\right)^{-\alpha}$$

Note this yields an interesting determinant product inequality.

6 The Gaussian free field

6.1 Dynkin's Isomorphism

By a well known calculation, if X is finite, for any $\chi \in \mathbb{R}_+^X$,

$$\frac{\sqrt{\det(M_\lambda - C)}}{(2\pi)^{|X|/2}} \int e^{-\frac{1}{2}\langle z, \chi \rangle} e^{-\frac{1}{2}e(z)} \prod_{u \in X} dz^u = \sqrt{\frac{\det(G_\chi)}{\det(G)}}$$

and

$$\frac{\sqrt{\det(M_\lambda - C)}}{(2\pi)^{|X|/2}} \int z^x z^y e^{-\frac{1}{2}\langle z^2, \chi \rangle} e^{-\frac{1}{2}e(z)} \prod_{u \in X} dz^u = (G_\chi)^{x,y} \sqrt{\frac{\det(G_\chi)}{\det(G)}}$$

This can be easily reformulated by introducing on an independent probability space the Gaussian field ϕ defined by the covariance $\mathbb{E}_\phi(\phi^x \phi^y) = G^{x,y}$ (this reformulation cannot be dispensed when X becomes infinite)

So we have $\mathbb{E}_\phi(e^{-\frac{1}{2}\langle \phi^2, \chi \rangle}) = \det(I + GM_\chi)^{-\frac{1}{2}} = \sqrt{\det(G_\chi G^{-1})}$ and

$\mathbb{E}_\phi((\phi^x \phi^y e^{-\frac{1}{2}\langle \phi^2, \chi \rangle}) = (G_\chi)^{x,y} \sqrt{\det(G_\chi G^{-1})}$ Then since sums of exponentials of the form $e^{-\frac{1}{2}\langle \cdot, \chi \rangle}$ are dense in continuous functions on \mathbb{R}_+^X the following holds:

Theorem 36 a) The fields $\widehat{\mathcal{L}}_{\frac{1}{2}}$ and $\frac{1}{2}\phi^2$ have the same distribution.

b) $\mathbb{E}_\phi((\phi^x \phi^y F(\frac{1}{2}\phi^2))) = \int \mathbb{E}(F(\widehat{\mathcal{L}}_{\frac{1}{2}} + \widehat{\gamma})) \mu^{x,y}(d\gamma)$ for any bounded functional F of a non negative field.

Remarks:

a) This is a version of Dynkin's isomorphism (Cf [7]). It can be extended to non-symmetric generators (Cf [20]).

b) By corollary 31, if $C_{x,y} \neq 0$, b) implies that

$$\mathbb{E}_\phi((\phi^x \phi^y F(\frac{1}{2}\phi^2))) = \frac{2}{C_{x,y}} \int \mathbb{E}(F(\widehat{\mathcal{L}}_{\frac{1}{2}})) N_{x,y}^{(\frac{1}{2})}$$

c) An analogous result can be given when α is any positive half integer, by using real vector valued Gaussian field, or equivalently complex fields for integral values of α (in particular $\alpha = 1$): If $\vec{\phi} = (\phi_1, \phi_2 \dots \phi_k)$ are k independent copies of the free field, the fields $\widehat{\mathcal{L}}_{\frac{k}{2}}$ and $\frac{1}{2} \|\vec{\phi}\|^2 = \frac{1}{2} \sum_1^k \phi_j^2$ have the same law and $\mathbb{E}_{\vec{\phi}}(\langle \vec{\phi}^x, \vec{\phi}^y \rangle F(\frac{1}{2} \|\phi\|^2)) = k \int \mathbb{E}(F(\widehat{\mathcal{L}}_{\frac{k}{2}} + \widehat{\gamma})) \mu^{x,y}(d\gamma)$

d) Note it implies immediately that the process ϕ^2 is infinitely divisible. See [8] and its references for a converse and earlier proofs of this last fact.

Remark 47 and theorem 36 suggest the following:

Exercise 37 Show any bounded functional F of a non negative field, if x_i are $2k$ points:

$$\mathbb{E}(F(\phi^2) \prod \phi^{x_i}) = \int \mathbb{E}(F(\widehat{\mathcal{L}}_\alpha + \sum_1^k \widehat{\gamma}_j)) \sum_{\text{pairings}} \prod \mu^{y_j, z_j}(d\gamma_j)$$

where \sum_{pairings} means that the k pairs y_j, z_j are formed with all the $2k$ points x_i , in all $\frac{(2k)!}{2k!}$ possible ways.

Hint: As in the proof of theorem 36, we take F of the form $e^{-\frac{1}{2}\langle \cdot, \chi \rangle}$ and prove the identities from formula 29 by perturbation and differentiation.

Exercise 38 For any f in the Dirichlet space \mathbb{H} of functions of finite energy (i.e. all functions if X is finite), the law of $f + \phi$ is absolutely continuous with respect to the law of ϕ , with density $\exp(\langle -Lf, \phi \rangle_m - \frac{1}{2}e(f))$

Exercise 39 a) Using proposition 33, show (it was observed by Nelson in the context of the classical (or Brownian) free field) that the Gaussian field ϕ is Markovian: Given any subset F of X , denote \mathcal{H}_F the Gaussian space spanned by $\{\phi^y, y \in F\}$. Then, for $x \in D = F^c$, the projection of ϕ^x on \mathcal{H}_F (i.e. the conditional expectation of ϕ^x given $\sigma(\phi^y, y \in F)$) is $\sum_{y \in F} [H^F]_y^x \phi^y$.

b) Moreover, show that $\phi^D = \phi - H^F \phi$ is the Gaussian field associated with the process killed at the exit of D .

6.2 Fock spaces and Wick product

The Gaussian space \mathcal{H} spanned by $\{\phi^x, x \in X\}$ is isomorphic to the Dirichlet space \mathbb{H} by the linear map mapping ϕ^x on $G^{x,\cdot}$. This isomorphism extends into an isomorphism between the space of square integrable functionals of the Gaussian fields and the symmetric Fock space $\Gamma(\mathbb{H}) = \overline{\oplus \mathbb{H}^{\otimes n}}$ obtained as the closure of the sum of all symmetric tensor powers of \mathbb{H} . This is Bose second quantization: See [34], [27] for a complete description of this isomorphism.

We have seen in theorem 36 that L^2 functionals of $\widehat{\mathcal{L}}_1$ can be represented in this space of Gaussian functionals. In order to prepare the extension of this representation to the more difficult framework of continuous spaces (which can often be viewed as scaling limits of discrete spaces), including especially the planar Brownian motion considered in [15], we shall introduce the renormalized (or Wick) powers of ϕ . We set $:(\phi^x)^n := (G^{x,x})^{\frac{n}{2}} H_n(\phi^x / \sqrt{G^{x,x}})$ where H_n is the n -th Hermite polynomial (characterized by $\sum \frac{t^n}{n!} H_n(u) = e^{tu - \frac{t^2}{2}}$). It is the inverse image of the n -th tensor power of $G^{x,\cdot}$ in the Fock space.

Setting as before $\sigma_x = G^{x,x}$, from the relation between Hermite polynomials H_{2n} and Laguerre polynomials $L_n^{-\frac{1}{2}}$,

$$H_{2n}(x) = (-2)^n n! L_n^{-\frac{1}{2}}\left(\frac{x^2}{2}\right)$$

it follows that:

$$: (\phi^x)^{2n} := 2^n n! P_n^{\frac{1}{2}, \sigma}\left(\left(\frac{(\phi^x)^2}{2}\right)\right).$$

More generally, if $\phi_1, \phi_2, \dots, \phi_k$ are k independent copies of the free field, we can define $: \prod_{j=1}^k \phi_j^{n_j} := \prod_{j=1}^k : \phi_j^{n_j} :$. Then it follows that:

$$: \left(\sum_1^k \phi_j^2\right)^n := \sum_{n_1 + \dots + n_k = n} \frac{n!}{n_1! \dots n_k!} \prod_{j=1}^k : \phi_j^{2n_j} : .$$

From the generating function of the polynomials $P_n^{\frac{k}{2}, \sigma}$,

$$P_n^{\frac{k}{2}, \sigma}\left(\sum_1^k u_j\right) = \sum_{n_1 + \dots + n_k = n} \frac{n!}{n_1! \dots n_k!} \prod_{j=1}^k P_{n_j}^{\frac{1}{2}, \sigma}(u_j).$$

Therefore,

$$P_n^{\frac{k}{2}, \sigma}\left(\frac{\sum(\phi_j)^2}{2}\right) = \frac{1}{2^n n!} : \left(\sum_1^k \phi_j^2\right)^n : . \quad (23)$$

Note that $: \sum_1^k \phi_j^2 := \sum_1^k \phi_j^2 - \sigma$. These variables are orthogonal in L^2 . Let $\tilde{l}^x = \widehat{l}^x - \sigma$ be the centered occupation field. Note that an equivalent formulation of theorem 36 is that the fields $\frac{1}{2} : \sum_1^k \phi_j^2 :$ and $\tilde{\mathcal{L}}_{\frac{k}{2}}$ have the same law.

Let us now consider the relation of higher Wick powers with self intersection local times.

Recall that the renormalized n -th self intersections field $\tilde{\mathcal{L}}_1^{x, n} = P_n^{\alpha, \sigma}(\widehat{\mathcal{L}}_\alpha^x) = Q_n^{\alpha, \sigma}(\tilde{\mathcal{L}}_\alpha^x)$ have been defined by orthonormalization in L^2 of the powers of the occupation time.

Then comes the

Proposition 40 *The fields $\tilde{\mathcal{L}}_{\frac{k}{2}}^{x, n}$ and $:\left(\frac{1}{n! 2^n} \sum_1^k \phi_j^2\right)^n:$ have the same law.*

This follows directly from (23).

Remark 41 *As a consequence, it can be shown that:*

$$\mathbb{E}\left(\prod_{j=1}^r Q_{k_j}^{\alpha, \sigma_{x_j}}(\widetilde{\mathcal{L}}_\alpha^{x_j})\right) = \sum_{\sigma \in \mathcal{S}_{k_1, k_2, \dots, k_j}} (2\alpha)^{m(\sigma)} G^{i_1, i_{\sigma(1)}} \dots G^{i_k, i_{\sigma(k)}}$$

where $\mathcal{S}_{k_1, k_2, \dots, k_j}$ is the set of permutations σ of $k = \sum k_j$ such that $\sigma(\{\sum_1^{j-1} k_l + 1, \dots, \sum_1^{j-1} k_l + k_j\}) \cap \{\sum_1^{j-1} k_l + 1, \dots, \sum_1^{j-1} k_l + k_j\}$ is empty for all j .

The identity follows from Wick's theorem when α is a half integer, then extends to all α since both members are polynomials in α . The condition on σ indicates that no pairing is allowed inside the same Wick power.

7 Energy variation and representations

The loop measure μ depends on the energy e which is defined by the free parameters C, κ . It will sometimes be denoted μ_e . We shall denote \mathcal{Z}_e the determinant $\det(G) = \det(M_\lambda - C)^{-1}$. Then $\mu(p > 1) = \log(\mathcal{Z}_e) + \sum \log(\lambda_x)$.

\mathcal{Z}_e^α is called the partition function of \mathcal{L}_α .

The following result is suggested by an analogy with quantum field theory (Cf [10]).

Proposition 42 *i) $\frac{\partial \mu}{\partial \kappa_x} = -\widehat{l}^x \mu$*

ii) If $C_{x,y} > 0$, $\frac{\partial \mu}{\partial C_{x,y}} = -T_{x,y} \mu$ with $T_{x,y}(l) = (\widehat{l}^x + \widehat{l}^y) - \frac{N_{x,y}}{C_{x,y}}(l) - \frac{N_{y,x}}{C_{x,y}}(l)$.

Proof. Recall that by formula (9): $\mu^*(p = 1, \xi = x, \widehat{\tau} \in dt) = e^{-\lambda_x t \frac{dt}{t}}$ and $\mu^*(p = k, \xi_i = x_i, \widehat{\tau}_i \in dt_i) = \frac{1}{k} \prod_{x,y} C_{x,y}^{N_{x,y}} \prod_x \lambda_x^{-N_x} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} \lambda_{\xi_i} e^{-\lambda_{\xi_i} t_i dt_i}$

Moreover we have $C_{x,y} = C_{y,x} = \lambda_x P_y^x$ and $\lambda_x = \kappa_x + \sum_y C_{x,y}$

The two formulas follow by elementary calculation. ■

Recall that $\mu(\widehat{l}^x) = G^{x,x}$ and $\mu(N_{x,y}) = G^{x,y} C_{x,y}$.

So we have $\mu(T_{x,y}) = G^{x,x} + G^{y,y} - 2G^{x,y}$.

Then, the above proposition allows us to compute all moments of T and \widehat{l} relative to μ_e (they could be called Schwinger functions).

Exercise 43 *Use the proposition above to show that:*

$$\int \widehat{l}_x \widehat{l}_y \mu(dl) = (G^{x,y})^2$$

$$\int \widehat{l}_x T_{y,z}(l) \mu(dl) = (G^{x,y} - G^{x,z})^2$$

and

$$\int T_{x,y}(l) T_{u,v}(l) \mu(dl) = (G_{x,u} + G_{y,v} - G_{x,v} - G_{y,u})^2$$

The calculations are done noticing that for any invertible matrix function $M(s)$, $\frac{d}{ds} M(s)^{-1} = -M(s)^{-1} M'(s) M(s)^{-1}$. The formula is applied to $M = M_\lambda - C$ and $s = \kappa_x$ or $C_{x,y}$.

We can apply these results to the Poissonian loop ensembles, to get the following

Corollary 44 *For any bounded functional Φ on loop configurations*

$$\begin{aligned} i) \frac{\partial}{\partial \kappa_x} \mathbb{E}(\Phi(\mathcal{L}_\alpha)) &= -\mathbb{E}(\Phi(\mathcal{L}_\alpha) \widehat{\mathcal{L}}_\alpha^x) \\ ii) \text{ If } C_{x,y} > 0, \frac{\partial}{\partial C_{x,y}} \mathbb{E}(\Phi(\mathcal{L}_\alpha)) &= \mathbb{E}(\widetilde{T}_{x,y}^{(\alpha)} \Phi(\mathcal{L}_\alpha)) \text{ with } T_{x,y}^{(\alpha)} = \sum_{l \in \mathcal{L}_\alpha} T_{x,y}(l) = (\widehat{\mathcal{L}}_\alpha^x + \\ \widehat{\mathcal{L}}_\alpha^y) - \frac{N_{x,y}^{(\alpha)}}{C_{x,y}} - \frac{N_{y,x}^{(\alpha)}}{C_{x,y}} \text{ and } \widetilde{T}_{x,y}^{(\alpha)} &= T_{x,y}^{(\alpha)} - \mathbb{E}(T_{x,y}^{(\alpha)}) = T_{x,y}^{(\alpha)} - \alpha(G^{x,x} + G^{y,y} - 2G^{x,y}). \end{aligned}$$

The proof is easily performed, taking first Φ of the form $\sum_{l_1 \neq l_2 \dots \neq l_q \in \mathcal{L}_\alpha} \prod_1^q G_j(l_j)$.

The proposition 42 is in fact the infinitesimal form of the following formula.

Proposition 45 *Consider another energy form e' defined on the same graph. Then we have the following identity:*

$$\frac{\partial \mu_{e'}}{\partial \mu_e} = e^{\sum N_{x,y} \log(\frac{C'_{x,y}}{C_{x,y}}) - \sum (\lambda'_x - \lambda_x) \widehat{l}^x}$$

Consequently

$$\mu_e((e^{\sum N_{x,y} \log(\frac{C'_{x,y}}{C_{x,y}}) - \sum (\lambda'_x - \lambda_x) \widehat{l}^x} - 1)) = \log\left(\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}\right) \quad (24)$$

Proof. The first formula is a straightforward consequence of (7). The proof of (24) goes by evaluating separately the contribution of trivial loops, which equals $\sum_x \log(\frac{\lambda_x}{\lambda'_x})$. Indeed,

$$\mu_e((e^{\sum N_{x,y} \log(\frac{C'_{x,y}}{C_{x,y}}) - \sum (\lambda'_x - \lambda_x) \widehat{l}^x} - 1)) = \mu_{e'}(p > 1) - \mu_e(p > 1) + \mu_e(1_{\{p=1\}}(e^{\sum (\lambda'_x - \lambda_x) \widehat{l}^x} - 1)).$$

The difference of the first two terms equals $\log(\mathcal{Z}_{e'}) + \sum \log(\lambda'_x) - (\log(\mathcal{Z}_e) - \sum \log(\lambda_x))$. The last term equals $\sum_x \int_0^\infty (e^{-\frac{\lambda'_x - \lambda_x}{\lambda_x} t} - 1) \frac{e^{-t}}{t} dt$ which can be computed as before:

$$\mu_e(1_{\{p=1\}}(e^{\sum(\lambda'_x - \lambda_x)\widehat{l}^x} - 1)) = - \sum \log\left(\frac{\lambda'_x}{\lambda_x}\right) \quad (25)$$

■

Remark 46 (*h-transforms*) Note that if $C'_{x,y} = h^x h^y C_{x,y}$ and $\kappa'_x = -h^x (Lh)^x \lambda_x$ for some positive function h on E such that $Lh \leq 0$, as $\lambda' = h^2 \lambda$ and $[P']_y^x = \frac{1}{h^x} P_y^x h^y$, we have $[G']^{x,y} = \frac{G^{x,y}}{h^x h^y}$ and $\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e} = \frac{1}{\prod(h^x)^2}$.

Remark 47 Note also that $[\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}]^{\frac{1}{2}} = \mathbb{E}(e^{-\frac{1}{2}[e' - e](\phi)})$, if ϕ is the Gaussian free field associated with e .

Integrating out the holding times, formula (24) can be written equivalently:

$$\mu_e\left(\prod_{(x,y)} \left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y}} \prod_x \left[\frac{\lambda_x}{\lambda'_x}\right]^{N_x+1} - 1\right) = \log\left(\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}\right) \quad (26)$$

and therefore

$$\mathbb{E}\left(\prod_{(x,y)} \left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y}(\mathcal{L}_\alpha)} \prod_x \left[\frac{\lambda_x}{\lambda'_x}\right]^{N_x(\mathcal{L}_\alpha)+1}\right) = \mathbb{E}\left(\prod_{(x,y)} \left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y}(\mathcal{L}_\alpha)} e^{-\langle \lambda' - \lambda, \widehat{\mathcal{L}}_\alpha \rangle}\right) = \left(\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}\right)^\alpha \quad (27)$$

Note also that $\prod_{(x,y)} \left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y}} = \prod_{\{x,y\}} \left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y} + N_{y,x}}$.

Remark 48 These $\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}$ determine, when e' varies with $\frac{C'}{C} \leq 1$ and $\frac{\lambda'}{\lambda} = 1$, the Laplace transform of the distribution of the traversal numbers of non oriented links $N_{x,y} + N_{y,x}$.

Other variables of interest on the loop space are associated with elements of the space \mathbb{A}^- of odd real valued functions ω on oriented links : $\omega^{x,y} = -\omega^{y,x}$. Let us mention a few elementary results.

The operator $[P^{(\omega)}]_y^x = P_y^x \exp(i\omega^{x,y})$ is also self adjoint in $L^2(\lambda)$. The associated loop variable can be written $\sum_{x,y} \omega^{x,y} N_{x,y}(l)$. We will denote it $\int_l \omega$. This notation will be used even when ω is not odd. Note it is invariant if ω is replaced by $\omega + dg$ for some g . Set $[G^{(\omega)}]_{x,y} = \frac{[(I-P^{(\omega)})^{-1}]_y^x}{\lambda_y}$. By an argument similar to the one given above for the occupation field, we have:

$$\mathbb{P}_{x,x}^t(e^{i \int_l \omega} - 1) = \exp(t(P^{(\omega)} - I))_x^x - \exp(t(P - I))_x^x.$$

Integrating in t after expanding, we get from the definition of μ :

$$\int (e^{i \int_l \omega} - 1) d\mu(l) = \sum_{k=1}^{\infty} \frac{1}{k} [Tr((P^{(\omega)})^k) - Tr((P)^k)].$$

Hence $\int (e^{i \int_l \omega} - 1) d\mu(l) = \log[\det(-L(I - P^{(\omega)})^{-1})] = \log(\det(G^{(\omega)}G^{-1}))$

We can now extend the previous formulas (26) and (27) to obtain, setting $\det(G^{(\omega)}) = \mathcal{Z}_{e,\omega}$

$$\int (e^{\sum N_{x,y} \log(\frac{C'_{x,y}}{C_{x,y}}) - \sum (\lambda'_x - \lambda_x) \widehat{l}_x + i \int_l \omega} - 1) \mu_e(dl) = \log(\frac{\mathcal{Z}_{e',\omega}}{\mathcal{Z}_e}) \quad (28)$$

and

$$\mathbb{E}\left(\prod_{x,y} \left[\frac{C'_{x,y}}{C_{x,y}} e^{i\omega_{x,y}}\right]^{N_{x,y}^{(\alpha)}} e^{-\sum (\lambda'_x - \lambda_x) \widehat{\mathcal{L}}_x^\alpha}\right) = \left(\frac{\mathcal{Z}_{e',\omega}}{\mathcal{Z}_e}\right)^\alpha \quad (29)$$

Remark 49 *The α -th power of a complex number is a priori not univoquely defined as a complex number. But $\log[\det(I - P^{(\omega)})]$ and therefore $\log(\mathcal{Z}_{e,\omega})$ are well defined as $P^{(\omega)}$ is a contraction. Then $\mathcal{Z}_{e,\omega}^\alpha$ is taken to be $\exp(\alpha \log(\mathcal{Z}_{e,\omega}))$.*

To simplify the notations slightly, one could consider more general energy forms with complex valued conductances so that the discrete one form is included in e' . But it is more interesting to generalize the notion of perturbation of P into $P^{(\omega)}$ as follows:

Definition 50 *A unitary representation of the graph (X, E) is a family of unitary matrices $[U^{x,y}]$, with common rank d_U , indexed by E^O , such that $[U^{y,x}] = [U^{x,y}]^{-1}$.*

We set $P^{(U)} = P \otimes U$ (more explicitly $[P^{(U)}]_{x,i}^{y,j} = P_x^y [U^{x,y}]_i^j$)

Similarly, we can define $C^{(U)} = \frac{\lambda}{d_U} P^{(U)}$, $V^{(U)} = (I - P^{(U)})^{-1}$, $G^{(U)} = \frac{d_U V^{(U)}}{\lambda}$.

One forms define one-dimensional representations. The sum and tensor product of two unitary representations U and V are unitary representations are defined as usual, and their ranks are respectively $d_U + d_V$ and $d_U d_V$.

Definition 51 Given any based loop l , set $\tau_U(l) = \frac{1}{d_U} \text{Tr}(\prod U^{\xi_i, \xi_{i+1}})$ if $p(l) \geq 2$ and the associated discrete based loop is $(\xi_1, \xi_2, \dots, \xi_p)$, and $\tau_U(l) = 1$ if $p(l) = 1$.

For any set of loops \mathcal{L} , we set $\tau_U(\mathcal{L}) = \prod_{l \in \mathcal{L}} \tau_U(l)$.

Remark 52 a) $|\tau_U(l)| \leq 1$

b) τ_U is obviously a functional of the discrete loop ξ°

c) $\tau_U(l) = 1$ if ξ° is tree-like. In particular it is always the case when the graph is a tree.

d) If U and V are two unitary representations of the graph, $\tau_{U+V} = \tau_U + \tau_V$ and $\tau_{U \otimes V} = \tau_U \tau_V$.

From b) and c) above it is easy to get the first part of

Proposition 53 i) The trace $\tau_U(l)$ depends only on the canonical geodesic loop associated with the loop ξ° , i.e. of the conjugacy class of the fundamental group associated with the element of Γ_{ξ_1} defined by the based loop ξ .

ii) The variables $\tau_U(l)$ determine, as U , varies, the geodesic loop associated with l .

Proof. The second assertion follows from the fact that traces of unitary representations separate the conjugacy classes of finite groups (Cf) and from the so-called CS-property satisfied by free groups (Cf [38]): Given two elements belonging to different conjugacy classes, there exists a finite quotient of the group in which they are not conjugate. ■

Again, by an argument similar to the one given for the occupation field, we have:

$$\mathbb{P}_{x,x}^t(\tau_U - 1) = \frac{1}{d_U} \sum_{i=1}^{d_U} \exp(t(P^{(U)} - I)_{x,i}^{x,i}) - \exp(t(P - I)_x^x).$$

Integrating in t after expanding, we get from the definition of μ :

$$\int (\tau_U(l) - 1) d\mu(l) = \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{1}{d_U} \text{Tr}((P^{(U)})^k) - \text{Tr}((P)^k) \right].$$

We can extend P into a matrix $P^{(I_{d_U})} = P \otimes I_{d_U}$ indexed by $X \times \{1, \dots, d_U\}$ by taking its tensor product with the identity on \mathbb{R}^{d_U} .

Then:

$$\int (\tau_U(l) - 1) d\mu(l) = \frac{1}{d_U} \sum_{k=1}^{\infty} \frac{1}{k} [Tr((P^{(U)})^k) - Tr((P^{(I_{d_U})})^k)].$$

Hence, as in the case of the occupation field $\int (\tau_U(l) - 1) d\mu(l) = \frac{1}{d_U} \log(\det(V^{(U)})[V \otimes I_{d_U}]^{-1})$

$$= \frac{1}{d_U} \log(\det(G^{(U)})) - \log(\det(G)) \text{ as } \det(G \otimes I_{d_U}) = d_U \det(G)$$

Then, denoting $\mathcal{Z}_{e,U}$ the $\frac{1}{d_U}$ -th power of the determinant of the $(|X| d_U, |X| d_U)$ matrix $G^{(U)}$ (well defined by remark 49), the formulas (28) and (29) extend easily to give the following

Proposition 54 a) $\int (e^{\sum N_{x,y} \log(\frac{C'_{x,y}}{C_{x,y}}) - \sum (\lambda'_x - \lambda_x) \widehat{l}_x} \tau_U(l) - 1) \mu_e(dl) = \log(\frac{\mathcal{Z}_{e',U}}{\mathcal{Z}_e})$
b) $\mathbb{E}(\prod_{x,y} [\frac{C'_{x,y}}{C_{x,y}}]^{N_{x,y}(\mathcal{L}_\alpha)} e^{-\sum (\lambda'_x - \lambda_x) \widehat{\mathcal{L}}_\alpha^x} \tau_U(\mathcal{L}_\alpha)) = (\frac{\mathcal{Z}_{e',U}}{\mathcal{Z}_e})^\alpha$

Let us now introduce a new

Definition 55 We say that sets Λ_i of non-trivial loops are equivalent when the associated occupation fields are equal and when the total traversal numbers $\sum_{l \in \Lambda_i} N_{x,y}(l)$ are equal for all oriented edges (x, y) . Equivalence classes will be called loop networks on the graph. We denote $\overline{\Lambda}$ the loop network defined by Λ .

Similarly, a set L of non-trivial discrete loops defines a discrete network characterized by the total traversal numbers.

Note that these expectations determine the distribution of the network $\overline{\mathcal{L}}_\alpha$ defined by the loop ensemble \mathcal{L}_α . We will denote $B^{e,e',\omega}(l)$ the variables $e^{\sum N_{x,y}(l) \log(\frac{C'_{x,y}}{C_{x,y}}) - \sum (\lambda'_x - \lambda_x) \widehat{l}_x + i \int_l \omega}$ and $B^{e,e',\omega}(\mathcal{L}_\alpha)$ the variables

$$\prod_{l \in \mathcal{L}_\alpha} B^{e,e',\omega}(l) = \prod_{x,y} [\frac{C'_{x,y}}{C_{x,y}} e^{i\omega_{x,y}}]^{N_{x,y}(\mathcal{L}_\alpha)} e^{-\sum (\lambda'_x - \lambda_x) \widehat{\mathcal{L}}_\alpha^x}.$$

More generally, we can define $B^{e,e',U}(l)$ and $B^{e,e',U}(\mathcal{L}_\alpha)$ in a similar way as $B^{e,e',\omega}(l)$ and $B^{e,e',\omega}(\mathcal{L}_\alpha)$, using $\tau_U(l)$ instead of $e^{i \int_l \omega}$. Note that for each fixed e , linear combinations of the variables $B^{e,e',U}(\mathcal{L}_\alpha)$ form an algebra as $B^{e,e'_1,U_1} B^{e,e'_2,U_2} = B^{e,e'_1+e'_2,U_1 \otimes U_2}$, and in particular, $B^{e,e'_1,\omega_1} B^{e,e'_2,\omega_2} = B^{e,e'_1+e'_2,\omega_1+\omega_2}$.

Remark 56 Note that the expectations of the variables $B^{e,e',\omega}(\mathcal{L}_\alpha)$ determine the distribution of the network $\overline{\mathcal{L}}_\alpha$ defined by the loop ensemble \mathcal{L}_α . The expectations of the variables $B^{e,e',U}(\mathcal{L}_\alpha)$ determine in addition the law of the set of geodesic loops associated with \mathcal{L}_α .

To work with μ , we will rather consider linear combinations of the form $\sum \lambda_i (B^{e,e',U_i} - 1)$, with $\sum \lambda_i = 0$, which form also an algebra.

Remark 57 Formulas (28) and (29) apply to the calculation of loop indices: If we have for example a simple random walk on an oriented planar graph, and if z' is a point of the dual graph X' , $\omega^{(z')}$ can be chosen such that for any loop l , $\int_l \omega^{(z')}$ is the winding number of the loop around a given point z' of the dual graph X' . Then $e^{i\pi \sum_{l \in \mathcal{L}_\alpha} \int_l \omega^{(z')}$ is a spin system of interest. We then get for example that

$$\mu\left(\int_l \omega_{z'} \neq 0\right) = -\frac{1}{2\pi} \int_0^{2\pi} \log(\det(G^{(2\pi u \omega^{(z')})} G^{-1})) du$$

and hence

$$\mathbb{P}\left(\sum_{l \in \mathcal{L}_\alpha} \left|\int_l \omega^{(z')}\right| = 0\right) = e^{\frac{\alpha}{2\pi} \int_0^{2\pi} \log(\det(G^{(2\pi u \omega^{(z')})} G^{-1})) du}$$

Conditional distributions of the occupation field with respect to values of the winding number can also be obtained.

8 Decompositions

Note first that with the energy e , we can associate a rescaled Markov chain \widehat{x}_t in which holding times at any point x are exponential times of parameters λ_x : $\widehat{x}_t = x_{\tau_t}$ with $\tau_t = \inf(s, \int_0^s \frac{1}{\lambda_{x_u}} du = t)$. For the rescaled Markov chain, local times coincide with the time spent in a point and the duality measure is simply the counting measure. The Markov loops can be rescaled as well and we did it in fact already when we introduced pointed loops. More generally we may introduce different holding time parameters but it would essentially be useless as the random variables we are interested in are intrinsic, i.e. depend only on e .

If $D \subset X$ and we set $F = D^c$, the orthogonal decomposition of the energy $e(f, f) = e(f)$ into $e^D(f - H^F f) + e(H^F f)$ (see proposition 33) leads to the decomposition of the Gaussian free field mentioned above and also to a decomposition of the rescaled Markov chain into the rescaled Markov chain killed at the exit of D and the trace of the rescaled Markov chain on F , i.e. $\widehat{x}_t^{\{F\}} = \widehat{x}_{S_t^F}$, with $S_t^F = \inf(s, \int_0^s 1_F(\widehat{x}_u) du = t)$.

Proposition 58 *The trace of the rescaled Markov chain on F is the rescaled Markov chain defined by the energy functional $e^{\{F\}}(f) = e(H^F f)$, for which*

$$C_{x,y}^{\{F\}} = C_{x,y} + \sum_{a,b \in D} C_{x,a} C_{b,y} [G^D]^{a,b}$$

$$\lambda_x^{\{F\}} = \lambda_x - \sum_{a,b \in D} C_{x,a} C_{b,x} [G^D]^{a,b}$$

and

$$\mathcal{Z}_e = \mathcal{Z}_{e^D} \mathcal{Z}_{e^{\{F\}}}$$

Proof. For the second assertion, note first that for any $y \in F$,

$$[H^F]_y^x = 1_{x=y} + 1_D(x) \sum_{b \in D} [G^D]^{x,b} C_{b,y}.$$

Moreover, $e(H^F f) = e(f, H^F f)$ and therefore

$$\lambda_x^{\{F\}} = e^{\{F\}}(1_{\{x\}}) = e(1_{\{x\}}, H^F 1_{\{x\}}) = \lambda_x - \sum_{a \in D} C_{x,a} [H^F]_x^a = \lambda_x (1 - p_x^{\{F\}})$$

where $p_x^{\{F\}} = \sum_{a,b \in D} P_a^x [G^D]^{a,b} C_{b,x} = \sum_{a \in D} P_a^x [H^F]_x^a$ is the probability that the Markov chain starting at x will return to x after an excursion in D .

Then for distinct x and y in F ,

$$\begin{aligned} C_{x,y}^{\{F\}} &= -e^{\{F\}}(1_{\{x\}}, 1_{\{y\}}) = -e(1_{\{x\}}, H^F 1_{\{y\}}) \\ &= C_{x,y} + \sum_a C_{x,a} [H^F]_y^a = C_{x,y} + \sum_{a,b \in D} C_{x,a} C_{b,y} [G^D]^{a,b}. \end{aligned}$$

Note that the graph defined on F by the non-vanishing conductances $C_{x,y}^{\{F\}}$ has in general more edges than the restriction to F of the original graph.

For the third assertion, note also that $G^{\{F\}}$ is the restriction of G to F as for all $x, y \in F$, $e^{\{F\}}(G\delta_{y|F}, 1_{\{x\}}) = e(G\delta_y, [H^F 1_{\{x\}}]) = 1_{\{x=y\}}$. Hence the determinant decomposition already used in section 5.3 yields the final formula. The cases where F has one point was already treated in section 5.3.

For the first assertion note the transition matrix $[P^{\{F\}}]_y^x$ can be computed directly and equals

$P_y^x + \sum_{a,b \in D} P_a^x V^{D \cup \{x\}}_b^a P_y^b = P_y^x + \sum_{a,b \in D} P_a^x [G^{D \cup \{x\}}]^{a,b} C_{b,y}$. It can be decomposed according to whether the jump to y occurs from x or from D and the number of excursions from x to x :

$$\begin{aligned} [P^{\{F\}}]_y^x &= \sum_{k=0}^{\infty} \left(\sum_{a,b \in D} P_a^x [V^D]_b^a P_x^b \right)^k (P_y^x + \sum_{a,b \in D} P_a^x [V^D]_b^a P_y^b) \\ &= \sum_{k=0}^{\infty} \left(\sum_{a,b \in D} P_a^x [G^D]^{a,b} C_{b,x} \right)^k (P_y^x + \sum_{a,b \in D} P_a^x [G^D]^{a,b} C_{b,y}). \end{aligned}$$

The expansion of $\frac{C_{x,y}^{\{F\}}}{\lambda_x^{\{F\}}}$ in geometric series yields exactly the same result.

Finally, remark that the holding times of $\hat{x}_t^{\{F\}}$ at any point $x \in F$ are sums of a random number of independent holding times of \hat{x}_t . This random integer counts the excursions from x to x performed by the chain \hat{x}_t during the holding time of $\hat{x}_t^{\{F\}}$. It follows a geometric distribution of parameter $1 - p_x^{\{F\}}$. Therefore, $\frac{1}{\lambda_x^{\{F\}}} = \frac{1}{\lambda_x(1-p_x^{\{F\}})}$ is the expectation of the holding times of $\hat{x}_t^{\{F\}}$ at x . ■

If χ is carried by D and if we set $e_\chi = e + \|\cdot\|_{L^2(\chi)}$ and denote $[e_\chi]^{\{F\}}$ by $e^{\{F,\chi\}}$ we have

$$C_{x,y}^{\{F,\chi\}} = C_{x,y} + \sum_{a,b} C_{x,a} C_{b,y} [G_\chi^D]^{a,b}, \quad p_x^{\{F,\chi\}} = \sum_{a,b \in D} P_a^x [G_\chi^D]^{a,b} C_{b,x}$$

and $\lambda_x^{\{F,\chi\}} = \lambda_x(1 - p_x^{\{F,\chi\}})$.

More generally, if $e^\#$ is such that $C^\# = C$ on $F \times F$, and $\lambda = \lambda^\#$ on F we have:

$$C_{x,y}^{\#\{F\}} = C_{x,y} + \sum_{a,b} C_{x,a}^\# C_{b,y}^\# [G^{\#D}]^{a,b}, \quad p_x^{\#\{F\}} = \sum_{a,b \in D} P_a^{\#x} [G^{\#D}]^{a,b} C_{b,x}^\#$$

and $\lambda_x^{\#\{F\}} = \lambda_x(1 - p_x^{\#\{F\}})$.

A loop in X which hits F can be decomposed into a loop $l^{\{F\}}$ in F and its excursions in D which may come back to their starting point. Let $\mu_D^{a,b}$ denote the bridge measure

(with mass $[G^D]^{a,b}$) associated with e^D .

Set

$$\nu_{x,y}^D = \frac{1}{C_{x,y}^{\{F\}}} [C_{x,y} \delta_\emptyset + \sum_{a,b \in D} C_{x,a} C_{b,y} \mu_D^{a,b}], \quad \rho_x^D = \sum_{n=1}^{\infty} \frac{1}{\lambda_x p_x^{\{F\}}} \left(\sum_{a,b \in D} C_{x,a} C_{b,x} \mu_D^{a,b} \right)$$

and $\nu_x^D = \frac{1}{1-p_x^{\{F\}}} [\delta_\emptyset + \sum_{n=1}^{\infty} [p_x^{\{F\}} \rho_x^D]^{\otimes n}]$.

Note that $\rho_x^D(1) = \nu_{x,y}^D(1) = \nu_x^D(1) = 1$.

A loop l hitting F can be decomposed into its restriction $l^{\{F\}} = (\xi_i, \widehat{\tau}_i)$ in F (possibly a one point loop), a family of excursions $\gamma_{\xi_i, \xi_{i+1}}$ attached to the jumps of $l^{\{F\}}$ and systems of i.i.d. excursions $(\gamma_{\xi_i}^h, h \leq n_{\xi_i})$ attached to the points of $l^{\{F\}}$. Note the set of excursions can be empty.

Let μ^D be the restriction of μ to loops contained in D . It is the loop measure associated to the process killed at the exit of D . We get a decomposition of $\mu - \mu^D$ in terms of the loop measure $\mu^{\{F\}}$ defined on loops of F by the trace of the Markov chain on F , probability measures $\nu_{x,y}^D$ on excursions in D indexed by pairs of points in F and measures ρ_x^D on excursions in D indexed by points of F . Moreover, the integers n_{ξ_i} follow a Poisson distribution of parameter $\lambda_{\xi_i}^{\{F\}} \widehat{\tau}_i$ (the total holding time in ξ_i before another point of F is visited) and the conditional distribution of the rescaled holding times in ξ_i before each excursion $\gamma_{\xi_i}^l$ is the distribution $\beta_{n_{\xi_i}, \tau_i^*}$ of the increments of a uniform sample of n_{ξ_i} points in $[0, \widehat{\tau}_i]$ put in increasing order. We denote these holding times by $\widehat{\tau}_{i,h}$ and set $l = \Lambda(l^{\{F\}}, (\gamma_{\xi_i, \xi_{i+1}}, (n_{\xi_i}, \gamma_{\xi_i}^h, \widehat{\tau}_{i,h}))$.

Then $\mu - \mu^D$ is the image measure by Λ of

$$\mu^{\{F\}}(dl^{\{F\}}) \prod (\nu_{\xi_i, \xi_{i+1}}^D)(d\gamma_{\xi_i, \xi_{i+1}}) \prod e^{-\lambda_{\xi_i}^{\{F\}} \widehat{\tau}_i} \sum_k \frac{[\lambda_{\xi_i}^{\{F\}} \widehat{\tau}_i]^k}{k!} \delta_{n_{\xi_i}}^k [\rho_x^D]^{\otimes k} (d\gamma_{\xi_i}^h) \beta_{k, \tau_i^*}(d\widehat{\tau}_{i,h}).$$

Note that for x, y belonging to F , the bridge measure $\mu^{x,y}$ can be decomposed in the same way, with the same excursion measures.

The one point case

If F is reduced to a point x , the decomposition is simpler. $l^{\{x\}}$ is a one point loop with $\widehat{\tau} = \frac{\widehat{l}^x}{1-p_x^{\{x\}}}$ and the number of excursions (all independent with the same distribution $\rho_x^{\{x\}^c}$) follows a Poisson distribution of parameter $\lambda_x^{\{x\}} \widehat{\tau} = \lambda_x \widehat{l}^x$.

The law $\frac{\mu^{x,x}}{G^{x,x}}$ can of course be decomposed in the same way, with the same conditional distribution given \widehat{l}^x . Recall it follows an exponential distribution with mean $G^{x,x}$.

$\widehat{\mathcal{L}}_\alpha^x$ follows a $\Gamma(\alpha, G^{x,x})$ distribution, in particular an exponential distribution with mean $G^{x,x}$ for $\alpha = 1$. Moreover, the union of the excursions of all loops of \mathcal{L}_α outside x has obviously the same Poissonian conditional distribution, given $\widehat{\mathcal{L}}_\alpha^x = s$ than μ and $\frac{\mu^{x,x}}{G^{x,x}}$. The set of excursions outside x defined by the $\frac{\mu^{x,x}}{G^{x,x}}$ -distributed bridge and by \mathcal{L}_1 are therefore identically distributed, as the total holding time in x .

Remark 59 Note finally that by exercise 26, the distribution of $\mathcal{L}_1/\mathcal{L}_1^{\{x\}^c}$ can be recovered from a unique sample of $\frac{\mu^{x,x}}{G^{x,x}}$ by splitting the bridge according to an independent sample U_i of Poisson – Dirichlet $(0, \alpha)$, more precisely, by splitting the bridge (in fact a based loop) l into based subloops $l|_{[\sigma_i, \sigma_{i+1}]}$, with $\sigma_i = \inf(s, \frac{1}{\lambda_x} \int_0^s \mathbf{1}_{\{x\}}(l_s) ds = U_i \widehat{l}^x)$.

Conversely, a sample of the bridge could be recovered from a sample of the loop set $\mathcal{L}_1/\mathcal{L}_1^{\{x\}^c}$ by concatenation in random order. This random ordering can be defined by taking a projective limit of the randomly ordered finite subset of loops $\{l_{i,n}\}$ defined by assuming for example that $\widehat{l}_{i,n}^x > \frac{1}{n}$.

Coming back to the general case, the Poisson process $\mathcal{L}_\alpha^{\{F\}} = \{l^{\{F\}}, l \in \mathcal{L}_\alpha\}$ has intensity $\mu^{\{F\}}$ and is independent of \mathcal{L}_α^D .

Note that $\widehat{\mathcal{L}}_\alpha^{\{F\}}$ is the restriction of $\widehat{\mathcal{L}}_\alpha$ to F .

If χ is a measure carried by D , we have:

$$\begin{aligned} \mathbb{E}(e^{-\langle \widehat{\mathcal{L}}_\alpha, \chi \rangle} | \mathcal{L}_\alpha^{\{F\}}) &= \mathbb{E}(e^{-\langle \widehat{\mathcal{L}}_\alpha^D, \chi \rangle}) \left(\prod_{x,y \in F} \left[\int e^{-\langle \widehat{\gamma}, \chi \rangle} \nu_{x,y}^D(d\gamma) \right]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \right) \\ &\quad \times \prod_{x \in F} e^{\lambda_x^{\{F\}} [\widehat{\mathcal{L}}_\alpha^{\{F\}}]^x \int (e^{-\langle \widehat{\gamma}, \chi \rangle} - 1) \rho_x^D(d\gamma)} \\ &= \left[\frac{\mathcal{Z}_{e^\chi}^D}{\mathcal{Z}_{e^D}} \right]^\alpha \left(\prod_{x,y \in F} \left[\frac{C_{x,y}^{\{F,\chi\}}}{C_{x,y}^{\{F\}}} \right]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \prod_{x \in F} e^{[\lambda_x^{\{F,\chi\}} - \lambda_x^{\{F\}}] \widehat{\mathcal{L}}_\alpha^x} \right). \end{aligned}$$

(recall that $\widehat{\mathcal{L}}_\alpha^{\{F\}}$ is the restriction of $\widehat{\mathcal{L}}_\alpha$ to F). Also, if we condition on the set of discrete loops $\mathcal{DL}_\alpha^{\{F\}}$

$$\mathbb{E}(e^{-\langle \widehat{\mathcal{L}}_\alpha, \chi \rangle} | \mathcal{DL}_\alpha^{\{F\}}) = \left[\frac{\mathcal{Z}_{e^\chi}^D}{\mathcal{Z}_{e^D}} \right]^\alpha \left(\prod_{x,y \in F} \left[\frac{C_{x,y}^{\{F,\chi\}}}{C_{x,y}^{\{F\}}} \right]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \prod_{x \in F} \left[\frac{\lambda_x^{\{F\}}}{\lambda_x^{\{F,\chi\}}} \right]^{N_x(\mathcal{L}_\alpha^{\{F\}})+1} \right)$$

where the last exponent $N_x + 1$ is obtained by taking into account the loops which have a trivial trace on F (see formula (25)).

More generally we can show in the same way the following

Proposition 60 *If $C^\# = C$ on $F \times F$, and $\lambda = \lambda^\#$ on F , we denote $B^{e,e^\#}$ the multiplicative functional $\prod_{x,y} \left[\frac{C_{x,y}^\#}{C_{x,y}} \right]^{N_{x,y}} e^{-\sum_{x \in D} \widehat{l}_x(\lambda_x^\# - \lambda_x)}$.*

Then,

$$\mathbb{E}(B^{e,e^\#} | \mathcal{L}_\alpha^{\{F\}}) = \left[\frac{\mathcal{Z}_{e^\#D}}{\mathcal{Z}_{eD}} \right]^\alpha \left(\prod_{x,y \in F} \left[\frac{C_{x,y}^{\#\{F\}}}{C_{x,y}^{\{F\}}} \right]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \prod_{x \in F} e^{[\lambda_x^{\#\{F\}} - \lambda_x^{\{F\}}] \widehat{\mathcal{L}}_\alpha^x} \right)$$

and

$$\mathbb{E}(B^{e,e^\#} | \mathcal{D}\mathcal{L}_\alpha^{\{F\}}) = \left[\frac{\mathcal{Z}_{e^\#D}}{\mathcal{Z}_{eD}} \right]^\alpha \left(\prod_{x,y \in F} \left[\frac{C_{x,y}^{\#\{F\}}}{C_{x,y}^{\{F\}}} \right]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \prod_{x \in F} \left[\frac{\lambda_x^{\{F\}}}{\lambda_x^{\#\{F\}}} \right]^{N_x(\mathcal{L}_\alpha^{\{F\}}) + 1} \right)$$

These decomposition and conditional expectation formulas extend to include a current ω in $C^\#$. Note that if ω is closed (i.e. vanish on every loop) in D , one can define ω^F such that $[C e^{i\omega}]^{\{F\}} = C^{\{F\}} e^{i\omega^F}$. Then

$$\mathcal{Z}_{e,\omega} = \mathcal{Z}_{eD} \mathcal{Z}_{e^{\{F\}},\omega^F}$$

The previous proposition implies the following *Markov property*:

Remark 61 *If $D = D_1 \cup D_2$ with D_1 and D_2 strongly disconnected, (i.e. such that for any $(x, y, z) \in D_1 \times D_2 \times F$, $C_{x,y}$ and $C_{x,z}C_{y,z}$ vanish), the restrictions of the network $\overline{\mathcal{L}}_\alpha$ to $D_1 \cup F$ and $D_2 \cup F$ are independent conditionally on the restriction of \mathcal{L}_α to F .*

Proof. It follows from the fact that as D_1 and D_2 are disconnected, any excursion measure $\nu_{x,y}^D$ or ρ_x^D from F into $D = D_1 \cup D_2$ is an excursion measure either in D_1 or in D_2 . ■

Branching processes with immigration An interesting example can be given after extending slightly the scope of the theory to countable transient symmetric Markov chains: We can take $X = \mathbb{N} - \{0\}$, $C_{n,n+1} = 1$ for all $n \geq 1$ and $\kappa_1 = 1$. P is the transfer matrix of the simple symmetric random walk killed at 0.

Then we can apply the previous considerations to check that $\widehat{\mathcal{L}}_\alpha^n$ is a branching process with immigration.

The immigration at level n comes from the loops whose infimum is n and the branching from the excursions of the loops existing at level n to level $n + 1$. Set $F_n = \{1, 2, \dots, n\}$ and $D_n = F_n^c$.

From the calculations of conditional expectations made above, we get that for any positive parameter γ ,

$$\mathbb{E}(e^{-\gamma \widehat{\mathcal{L}}_\alpha^n} | \mathcal{L}_\alpha^{\{F_{n-1}\}}) = \mathbb{E}(e^{-\gamma [\widehat{\mathcal{L}}_\alpha^{D_{n-1}}]^n} e^{\lambda_{n-1}^{\{F_{n-1}, \gamma \delta_n\}} - \lambda_{n-1}^{\{F_{n-1}\}}} | \widehat{\mathcal{L}}_\alpha^{n-1})$$

From this formula, it is clear that $\widehat{\mathcal{L}}_\alpha^n$ is a branching Markov chain. To be more precise, note that for any $n, m > 0$, $V_m^n = 2(n \wedge m)$ and $\lambda_n = 2$ and that $G^{1,1} = 1$. Moreover, $G_{\gamma \delta_1}^{1,n} = G^{1,n} - G^{1,1} \gamma G_{\gamma \delta_1}^{1,n}$ so that $G_{\gamma \delta_1}^{1,n} = \frac{1}{1+\gamma}$ and for any $n > 0$, the restriction of the Markov chain to D_n is isomorphic to the original Markov chain. Then it comes that for all n , $p_n^{\{F_n\}} = \frac{1}{2}$, $\lambda_n^{\{F_n\}} = 1$, and $\lambda_n^{\{F_n, \gamma \delta_{n+1}\}} = 2 - \frac{1}{1+\gamma} = \frac{2\gamma+1}{1+\gamma}$ so that the Laplace exponent of the convolution semigroup ν_t defining the branching mechanism $\lambda_{n-1}^{\{F_{n-1}, \gamma \delta_n\}} - \lambda_{n-1}^{\{F_{n-1}\}}$ equals $\frac{2\gamma+1}{1+\gamma} - 1 = \frac{\gamma}{1+\gamma} = \int (1 - e^{-\gamma s}) e^{-s} ds$. It is the semigroup of a compound Poisson process whose Levy measure is exponential.

The immigration law (on \mathbb{R}^+) is a Gamma distribution $\Gamma(\alpha, G^{1,1}) = \Gamma(\alpha, 1)$. It is the law of $\widehat{\mathcal{L}}_\alpha^1$ and also of $[\widehat{\mathcal{L}}_\alpha^{D_{n-1}}]^n$ (it means the occupation field of the trace of \mathcal{L}_α on D_{n-1} evaluated at n) for all $n > 1$.

The conditional law of $\widehat{\mathcal{L}}_\alpha^{n+1}$ given $\widehat{\mathcal{L}}_\alpha^n$ is the convolution of the immigration law $\Gamma(\alpha, 1)$ with $\nu_{\widehat{\mathcal{L}}_\alpha^n}$

Exercise 62 *Alternatively, we can consider the integer valued process $N_n(\mathcal{L}_\alpha^{\{F_n\}}) + 1$ which is a Galton Watson process with immigration. In our example, we find the reproduction law $\pi(n) = 2^{-n-1}$ for all $n \geq 0$ (critical binary branching).*

Exercise 63 *Show that more generally, if $C_{n,n+1} = [\frac{p}{1-p}]^n$, for $n > 0$ and $\kappa_1 = 1$, with $0 < p < 1$, we get all asymmetric simple random walks. Show that $\lambda_n = \frac{p^{n-1}}{(1-p)^n}$ and $G^{1,1} = 1$. Determine the distributions of the associated branching and Galton Watson process with immigration.*

If we consider the occupation field defined by the loops going through 1, we get a branching process without immigration: it is the classical relation between random walks local times and branching processes.

9 Loop erasure and spanning trees.

Recall that an oriented link g is a pair of points (g^-, g^+) such that $C_g = C_{g^-, g^+} \neq 0$. Define $-g = (g^+, g^-)$.

Let $\mu_{x,y}^\neq$ be the measure induced by C on discrete self-avoiding paths between x and y : $\mu_{x,y}^\neq(x, x_2, \dots, x_{n-1}, y) = C_{x,x_2} C_{x_1,x_3} \dots C_{x_{n-1},y}$.

Another way to define a measure on discrete self avoiding paths from x to y from a measure on paths from x to y is loop erasure (see [13], [31], [14] and [24]). In this context, the loops, which can be reduced to points, include holding times, and loop erasure produces a discrete path without holding times.

We have the following:

Proposition 64 *The image of $\mu_{x,y}^\neq$ by the loop erasure map $\gamma \rightarrow \gamma^{BE}$ is $\mu_{BE}^{x,y}$ defined on self avoiding paths by $\mu_{BE}^{x,y}(\eta) = \mu_{x,y}^\neq(\eta) \frac{\det(G)}{\det(G_{\{\eta\}^c})} = \mu_{x,y}^\neq(\eta) \det(G_{|\{\eta\} \times \{\eta\}})$ (Here $\{\eta\}$ denotes the set of points in the path η) and by $\mu_{BE}^{x,y}(\emptyset) = \delta_y^x G_{x,x}$*

Proof. If $\eta = (x_1 = x, x_2, \dots, x_n = y)$, and $\eta_m = (x, \dots, x_m)$, $m > 1$

$$\mu^{x,y}(\gamma^{BE} = \eta) = \sum_{k=0}^{\infty} [P^k]_x P_{x_2}^x \mu_{\{x\}^c}^{x_2,y}(\gamma^{BE} = \theta\eta)$$

where $\mu_{\{x\}^c}^{x_2,y}$ denotes the bridge measure for the Markov chain killed as it hits x and θ the natural shift on discrete paths. By recurrence, this clearly equals

$$V_x P_{x_2}^x [V^{\{x\}^c}]_{x_2}^{x_2} \dots [V^{\{\eta_{n-1}\}^c}]_{x_{n-1}}^{x_{n-1}} P_y^{x_{n-1}} [V^{\{\eta\}^c}]_y \lambda_y^{-1} = \mu_{x,y}^\neq(\eta) \frac{\det(G)}{\det(G_{\{\eta\}^c})}$$

as

$$[V^{\{\eta_{m-1}\}^c}]_{x_m}^{x_m} = \frac{\det([(I - P)]_{|\{\eta_m\}^c \times \{\eta_m\}^c})}{\det([(I - P)]_{|\{\eta_{m-1}\}^c \times \{\eta_{m-1}\}^c})} = \frac{\det(V^{\{\eta_{m-1}\}^c})}{\det(V^{\{\eta_m\}^c})} = \frac{\det(G_{\{\eta_{m-1}\}^c})}{\det(G_{\{\eta_m\}^c})} \lambda^{x_m}.$$

for all $m \leq n - 1$. ■

Also, by Feynman-Kac formula, for any self-avoiding path η :

$$\begin{aligned} \int e^{-\langle \widehat{\gamma}, \chi \rangle} 1_{\{\gamma^{BE} = \eta\}} \mu^{x,y}(d\gamma) &= \frac{\det(G_\chi)}{\det(G_\chi^{\{\eta\}^c})} \mu_{x,y}^\neq(\eta) = \det(G_\chi)_{|\{\eta\} \times \{\eta\}} \mu_{x,y}^\neq(\eta) \\ &= \frac{\det(G_\chi)_{|\{\eta\} \times \{\eta\}}}{\det(G_{|\{\eta\} \times \{\eta\}})} \mu_{BE}^{x,y}(\eta). \end{aligned}$$

Therefore, recalling that by the results of section 5.3 conditionally on η , $\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c}$ and $\mathcal{L}_1^{\{\eta\}^c}$ are independent, we see that under $\mu^{x,y}$, the conditional distribution of $\widehat{\gamma}$ given $\gamma^{BE} = \eta$ is the distribution of $\widehat{\mathcal{L}}_1 - \widehat{\mathcal{L}}_1^{\{\eta\}^c}$ i.e. the occupation field of the loops of \mathcal{L}_1 which intersect η .

More generally, it can be shown that

Proposition 65 *The conditional distribution of the network $\overline{\mathcal{L}}_\gamma$ defined by the loops of γ , given that $\gamma^{BE} = \eta$, is identical to the distribution of the network defined by $\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c}$ i.e. the loops of \mathcal{L}_1 which intersect η .*

Proof. Recall the notation $\mathcal{Z}_e = \det(G)$. First an elementary calculation using (9) shows that $\mu_e^{x,y}(e^{i \int_\gamma \omega} 1_{\{\gamma^{BE}=\eta\}})$ equals

$$\mu_e^{x,y} \left(1_{\{\gamma^{BE}=\eta\}} \prod \left[\frac{C'_{\xi_i, \xi_{i+1}}}{C_{\xi_i, \xi_{i+1}}} e^{i\omega_{\xi_i, \xi_{i+1}}} \frac{\lambda_{\xi_i}}{\lambda'_{\xi_i}} \right] \right. \\ \left. \frac{C'_{x,x_2} C'_{x_1,x_3} \dots C'_{x_{n-1},y}}{C_{x,x_2} C_{x_1,x_3} \dots C_{x_{n-1},y}} e^{i \int_\eta \omega} \mu_e^{x,y} \left(\prod_{u \neq v} \left[\frac{C'_{u,v}}{C_{u,v}} e^{i\omega_{u,v}} \right]^{N_{u,v}(\mathcal{L}_\gamma)} e^{-\langle \lambda' - \lambda, \widehat{\gamma} \rangle} 1_{\{\gamma^{BE}=\eta\}} \right) \right).$$

(Note the term $e^{-\langle \lambda' - \lambda, \widehat{\gamma} \rangle}$ can be replaced by $\prod_u (\frac{\lambda_u}{\lambda'_u})^{N_u(\gamma)}$).

Moreover, by the proof of the previous proposition, applied to the Markov chain defined by e' perturbed by ω , we have also $\mu_e^{x,y}(e^{i \int_\gamma \omega} 1_{\{\gamma^{BE}=\eta\}}) = C'_{x,x_2} C'_{x_1,x_3} \dots C'_{x_{n-1},y} e^{i \int_\eta \omega} \frac{\mathcal{Z}_{e',\omega}}{\mathcal{Z}_{[e']\{\eta\}^c,\omega}}$.

Therefore,

$$\mu_e^{x,y} \left(\prod_{u \neq v} \left[\frac{C'_{u,v}}{C_{u,v}} e^{i\omega_{u,v}} \right]^{N_{u,v}(\mathcal{L}_\gamma)} e^{-\langle \lambda' - \lambda, \widehat{\gamma} \rangle} \middle| \gamma^{BE} = \eta \right) = \frac{\mathcal{Z}_{e\{\eta\}^c} \mathcal{Z}_{e',\omega}}{\mathcal{Z}_e \mathcal{Z}_{[e']\{\eta\}^c,\omega}}.$$

Moreover, by (29) and the properties of the Poisson processes,

$$\mathbb{E} \left(\prod_{u \neq v} \left[\frac{C'_{u,v}}{C_{u,v}} e^{i\omega_{u,v}} \right]^{N_{u,v}(\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c})} e^{-\langle \lambda' - \lambda, \widehat{\mathcal{L}}_1 - \widehat{\mathcal{L}}_1^{\{\eta\}^c} \rangle} \right) = \frac{\mathcal{Z}_{e\{\eta\}^c} \mathcal{Z}_{e',\omega}}{\mathcal{Z}_e \mathcal{Z}_{[e']\{\eta\}^c,\omega}}.$$

It follows that the joint distribution of the traversal numbers and the occupation field are identical for the set of erased loops and $\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c}$. ■

The general study of loop erasure which is done in this chapter yields the following result when applied to a universal covering \widehat{X} . Let \widehat{G} be the Green function associated with the lift of the Markov chain.

Corollary 66 *The image of $\mu^{x,y}$ under the reduction map is given as follows: If c is a geodesic arc between x and y : $\mu^{x,y}(\{\xi, \xi^R = c\}) = \prod C_{c_i, c_{i+1}} \det(\widehat{G}_{|\{c\} \times \{c\}})$*

Besides, if \widehat{x} and \widehat{y} are the endpoints of the lift of c to a universal covering, $\mu^{x,y}(\{\xi, \xi^R = c\}) = \widehat{G}_{\widehat{x}, \widehat{y}}$

Note this yields an interesting identity on the Green function \widehat{G} .

Exercise 67 *Check it in the special case treated in proposition 21.*

Similarly one can define the image of \mathbb{P}^x by BE which is given by

$$\mathbb{P}_{BE}^x(\eta) = C_{x_1, x_2} \dots C_{x_{n-1}, x_n} \kappa_{x_n} \det(G_{|\{\eta\} \times \{\eta\}}),$$

for $\eta = (x_1, \dots, x_n)$, and get the same results.

Note that in particular, $\mathbb{P}_{BE}^x(\emptyset) = V_x^x(1 - \sum_y P_y^x) = \kappa_x G^{x,x}$.

Wilson's algorithm (see [23]) iterates this construction, starting with x 's in arbitrary order. Each step of the algorithm reproduces the first step except it stops when it hits the already constructed tree of self avoiding paths. It provides a construction of a random spanning tree. Its law is a probability measure \mathbb{P}_{ST}^e on the set $ST_{X,\Delta}$ of spanning trees of X rooted at the cemetery point Δ defined by the energy e . The weight attached to each oriented link $g = (x, y)$ of $X \times X$ is the conductance and the weight attached to the link (x, Δ) is κ_x which we can also denote by $C_{x,\Delta}$. As the determinants simplify, the probability of a tree Υ is given by a simple formula:

$$\mathbb{P}_{ST}^e(\Upsilon) = \mathcal{Z}_e \prod_{\xi \in \Upsilon} C_\xi \tag{30}$$

It is clearly independent of the ordering chosen initially. Now note that, since we get a probability

$$\mathcal{Z}_e \sum_{\Upsilon \in ST_{X,\Delta}} \prod_{(x,y) \in \Upsilon} C_{x,y} \prod_{x,(x,\Delta) \in \Upsilon} \kappa_x = 1$$

or equivalently

$$\sum_{\Upsilon \in ST_{X,\Delta}} \prod_{(x,y) \in \Upsilon} P_y^x \prod_{x,(x,\Delta) \in \Upsilon} P_\Delta^x = \frac{1}{\prod_{x \in X} \lambda_x \mathcal{Z}_e}$$

Then, it follows that, for any e' for which conductances (including κ') are positive only on links of e ,

$$\mathbb{E}_{ST}^e \left(\prod_{(x,y) \in \Upsilon} \frac{P_y^{e'}}{P_y^x} \prod_{x,(x,\Delta) \in \Upsilon} \frac{P_\Delta^{e'}}{P_\Delta^x} \right) = \frac{\prod_{x \in X} \lambda_x \mathcal{Z}_e}{\prod_{x \in X} \lambda_x' \mathcal{Z}_{e'}}$$

and

$$\mathbb{E}_{ST}^e \left(\prod_{(x,y) \in \Upsilon} \frac{C'_{x,y}}{C_{x,y}} \prod_{x,(x,\Delta) \in \Upsilon} \frac{\kappa'_x}{\kappa_x} \right) = \frac{\mathcal{Z}_e}{\mathcal{Z}_{e'}}. \quad (31)$$

Note also that in the case of a graph (i.e. when all conductances are equal to 1), all spanning trees have the same probability. The expression of their cardinal as the determinant \mathcal{Z}_e is Cayley's theorem (see for exemple [23]).

The formula (31) shows a kind of duality between random spanning trees and \mathcal{L}_1 . It can be extended to \mathcal{L}_k for any integer k if we consider the sum (in terms of number of transitions) of k independent spanning trees.

Exercise 68 Show that more generally, for any tree T rooted in Δ ,

$$\mathbb{P}_{ST}^e(\{\Upsilon, T \subseteq \Upsilon\}) = \det(G_{\{T\} \times \{T\}}) \prod_{\xi \in T} C_\xi, \quad \{T\} \text{ denoting the vertex set of } T.$$

Corollary 69 The network defined by the random set of loops \mathcal{L}_W constructed in this algorithm is independent of the random spanning tree, and independent of the ordering. It has the same distribution as the network defined by the loops of \mathcal{L}_1 .

This result follows easily from proposition 65.

Remark 70 Note that proposition 65 and its corollary can be made more precise with the help of remark 59. The splitting procedure used there with the help of an auxiliary independent set of Poisson Dirichlet variables allows to reconstruct the set of loops $\mathcal{L}_1 / \mathcal{L}_1^{\{x\}^c}$ by splitting the first erased loop in the proof of the proposition. Iterating the procedure we can successively reconstruct all sets $\mathcal{L}_1^{\{\eta_m\}^c} / \mathcal{L}_1^{\{\eta_{m+1}\}^c}$ and finally $\mathcal{L}_1 / \mathcal{L}_1^{\{\eta\}^c}$. Then, by Wilson algorithm, we can reconstruct \mathcal{L}_1 .

Let us now consider the *recurrent case*.

A probability is defined on the non oriented spanning trees by the conductances: $\mathbb{P}_{ST}^e(\mathcal{T})$ is defined by the product of the conductances of the edges of \mathcal{T} normalized by the sum of these products on all spanning trees.

Note that any non oriented spanning tree of X along edges of E defines uniquely an oriented spanning tree $I_\Delta(\mathcal{T})$ if we choose a root Δ . The orientation is taken towards the

root which can be viewed as a cemetery point. Then, if we consider the associated Markov chain killed as it hits Δ defined by the energy form $e^{\{\Delta\}^c}$, the previous construction yields a probability $\mathbb{P}_{ST}^{e^{\{\Delta\}^c}}$ on spanning trees rooted at Δ which by (30) coincides with the image of \mathbb{P}_{ST}^e by I_Δ . This implies in particular that the normalizing factor $\mathcal{Z}_{e^{\{\Delta\}^c}}$ is independent of the choice of Δ as it has to be equal to $(\sum_{T \in ST_X} \prod_{\{x,y\} \in T} C_{x,y})^{-1}$. We denote it by \mathcal{Z}_e^0 . This factor can also be expressed in terms of the recurrent Green operator G . Recall it is defined as a scalar product on measures of zero mass. The determinant of G is defined as the determinant of its matrix in any orthonormal basis of this hyperplane, with respect to the natural Euclidean scalar product.

Recall that for any $x \neq \Delta$, $G(\varepsilon_x - \varepsilon_\Delta) = -\frac{\langle \lambda, G^{\{\Delta\}^c} \varepsilon_x \rangle}{\lambda(X)} + G^{\{\Delta\}^c} \varepsilon_x$. Therefore, for any $y \neq \Delta$, $\langle \varepsilon_y - \varepsilon_\Delta, G(\varepsilon_x - \varepsilon_\Delta) \rangle = [G^{\{\Delta\}^c}]^{x,y}$.

The determinant of the matrix $[G^{\{\Delta\}^c}]$, equal to \mathcal{Z}_e^0 , is therefore also the determinant of G in the basis $\{\delta_x - \delta_\Delta, x \neq \Delta\}$ which is not orthonormal with respect to the natural euclidean scalar product. An easy calculation shows it equals $\det(\langle \delta_y - \delta_\Delta, \delta_x - \delta_\Delta \rangle_{\mathbb{R}^{|X|}}, x, y \neq \Delta) \det(G) = |X| \det(G)$. Note also that if we set $\alpha_{x_0}(\mathcal{T}) = \prod_{(x,y) \in I_{x_0}(\mathcal{T})} P_y^x$, $\sum_{T \in ST_X} \alpha_{x_0}(\mathcal{T})$ is proportional to λ_{x_0} as x_0 varies in X .

Exercise 71 *Prove that if we set $\alpha_{x_0}(\mathcal{T}) = \prod_{(x,y) \in I_{x_0}(\mathcal{T})} P_y^x$, $\sum_{T \in ST_X} \alpha_{x_0}(\mathcal{T})$ is proportional to λ_{x_0} as x_0 varies in X . More precisely, it equals $K \lambda_{x_0}$, with $K = \frac{\mathcal{Z}_e^0}{\prod_{x \in X} \lambda_x}$. This fact is known as the matrix- tree theorem ([23]).*

Exercise 72 *Check directly. that $\mathcal{Z}_{e^{\{x_0\}^c}}$ is independent of the choice of x_0 .*

Let us come back briefly to the transient case by choosing some root x_0 . As by the strong Markov property, $V_x^y = \mathbb{P}_y(T_x < \infty) V_x^x$, we have $\frac{G^{y,x}}{G^{x,x}} = \frac{V_x^y}{V_x^x} = \mathbb{P}_y(T_x < \infty)$, and therefore

$$\mathbb{P}_{ST}^e((x, y) \in \Upsilon) = \mathbb{P}_x(\gamma_1^{BE} = y) = V_x^x P_y^x \mathbb{P}_y(T_x = \infty) = C_{x,y} G^{x,x} (1 - \frac{G^{x,y}}{G^{x,x}}).$$

Directly from the above, we recover Kirchhoff's theorem:

$$\begin{aligned} \mathbb{P}_{ST}^e(\pm(x, y) \in \Upsilon) &= C_{x,y} [G^{x,x} (1 - \frac{G^{x,y}}{G^{x,x}}) + G^{y,y} (1 - \frac{G^{y,x}}{G^{y,y}})] \\ &= C_{x,y} (G^{x,x} + G^{y,y} - 2G^{x,y}) = C_{x,y} K^{x,y}, (x,y) \end{aligned}$$

with the notations of section 2.5, and this is clearly independent of the choice of the root.

Exercise 73 Give an alternative proof of Kirchhoff's theorem by using (31), taking $C'_{x,y} = sC_{x,y}$ and $C'_{u,v} = C_{u,v}$ for $\{u, v\} \neq \{x, y\}$.

More generally we can show the transfer current theorem (see for exemple [22], [23]):

Theorem 74 $\mathbb{P}_{ST}^e(\pm\xi_1, \dots, \pm\xi_k \in \Upsilon) = (\prod_1^k C_{\xi_i}) \det(K^{\xi_i, \xi_j} \ 1 \leq i, j \leq k)$

Note this determinant does not depend on the orientation of the links.

Proof. Note first that if Υ is a spanning tree rooted in $x_0 = \Delta$ and $\xi_i = (x_{i-1}, x_i)$, $1 \leq i \leq |X| - 1$ are its oriented edges, the measures $\delta_{x_i} - \delta_{x_{i-1}}$ form another basis of the euclidean hyperplane of signed measures with zero charge, which has the same determinant as the basis $\delta_{x_i} - \delta_{x_0}$.

Therefore, \mathcal{Z}_e^0 is also the determinant of the matrix of G in this basis, i.e.

$$\mathcal{Z}_e^0 = \det(K^{\xi_i, \xi_j} \ 1 \leq i, j \leq |X| - 1)$$

$$\begin{aligned} \text{and } \mathbb{P}_{ST}^e(\Upsilon) &= (\prod_1^{|X|-1} C_{\xi_i}) \det(K^{\xi_i, \xi_j} \ 1 \leq i, j \leq |X| - 1) \\ &= \det(\sqrt{C_{\xi_i}} K^{\xi_i, \xi_j} \sqrt{C_{\xi_j}} \ 1 \leq i, j \leq |X| - 1). \end{aligned}$$

Recall that $\sqrt{C_{\xi_i}} K^{\xi_i, \xi_j} \sqrt{C_{\xi_j}} = \left\langle \alpha_{\xi_i}^* | \Pi | \alpha_{\xi_j}^* \right\rangle_{\mathbb{A}_-}$, where Π denotes the projection on the space of differentials and that $\alpha_{(\eta)}^{*x,y} = \pm \frac{1}{\sqrt{C_\eta}}$ if $(x, y) = \pm(\eta)$ and $= 0$ elsewhere.

To finish the proof of the theorem, it is helpful to use the exterior algebra. Note first that for any ONB $e_1, \dots, e_{|X|-1}$ of the space of differentials, $\Pi \alpha_\xi^* = \sum \langle \alpha_\xi^* | e_j \rangle e_j$ and $\mathbb{P}_{ST}^e(\Upsilon) = \det(\langle \alpha_{\xi_i}^* | e_j \rangle)^2 = \left\langle \alpha_{\xi_1}^* \wedge \dots \wedge \alpha_{\xi_{|X|-1}}^* | e_1 \wedge \dots \wedge e_{|X|-1} \right\rangle_{\wedge^{|X|-1} \mathbb{A}_-}^2$. Therefore

$$\mathbb{P}_{ST}^e(\xi_1, \dots, \xi_k \in \Upsilon) = \sum_{\eta_{k+1}, \dots, \eta_{|X|-1}} \left\langle \alpha_{\xi_1}^* \wedge \dots \wedge \alpha_{\xi_k}^* \wedge \alpha_{\eta_{k+1}}^* \wedge \dots \wedge \alpha_{\eta_{|X|-1}}^* | e_1 \wedge \dots \wedge e_{|X|-1} \right\rangle_{\wedge^{|X|-1} \mathbb{A}_-}^2$$

where the sum is on all edges $\eta_{k+1}, \dots, \eta_{|X|-1}$ completing ξ_1, \dots, ξ_k into a spanning tree. It can be extended to all systems of $|X| - 1 - k$ edges as all the additional term vanish. Indeed, the projection on an exterior product of differentials of an exterior product of $\alpha_{\xi_i}^*$ vanishes as soon as they form a loop. Hence the expression above equals:

$$\sum_{i_1 < i_2 < \dots < i_k} \left\langle \alpha_{\xi_{i_1}}^* \wedge \dots \wedge \alpha_{\xi_{i_k}}^* | e_{i_1} \wedge \dots \wedge e_{i_k} \right\rangle_{\wedge^k \mathbb{A}_-}^2 = \det(\sqrt{C_{\xi_i}} K^{\xi_i, \xi_j} \sqrt{C_{\xi_j}} \ 1 \leq i, j \leq k) \blacksquare$$

It follows that given any function g on non oriented links,

$$\begin{aligned}
\mathbb{E}_{ST}^e(e^{-\sum_{\xi \in \Upsilon} g(\xi)}) &= \mathbb{E}_{ST}^e\left(\prod_{\xi} (1 + (e^{-g(\xi)} - 1)1_{\xi \in \Upsilon})\right) \\
&= \sum_k \sum_{\pm \xi_1 \neq \pm \xi_2 \dots \neq \pm \xi_k} \prod (e^{-g(\xi_i)} - 1) \mathbb{P}_{ST}^e(\pm \xi_1, \dots, \pm \xi_k \in \Upsilon) \\
&= \sum_k \sum_{\pm \xi_1 \neq \pm \xi_2 \dots \neq \pm \xi_k} \prod (e^{-g(\xi_i)} - 1) \det(K^{\xi_i, \xi_j} \ 1 \leq i, j \leq k) \\
&= \sum_k \text{Tr}((M_{C(e^{-g}-1)}K)^{\wedge k}) = \det(I + KM_{C(e^{-g}-1)})
\end{aligned}$$

and we have proved the following

Proposition 75 $\mathbb{E}_{ST}^e(e^{-\sum_{\xi \in \Upsilon} g(\xi)}) = \det(I - M\sqrt{C(1-e^{-g})}KM\sqrt{C(1-e^{-g})})$

Here determinants are taken on matrices indexed by E .

This is an exemple of the Fermi point processes discussed in [35].

As a consequence, for any spanning tree T , if π_T denotes $M_{1_{\{T\}}}$ (the multiplication by the indicator function of T), it follows from the above, by letting g be $m1_{\{T^c\}}$, $m \rightarrow \infty$ that $\mathbb{P}_{ST}^e(T) = \det((I - KM_C)(I - \pi_T) + \pi_T) = \det((I - KM_C)_{T^c \times T^c})$

Note also that if e' is another energy form on the same graph, $\mathbb{E}_{ST}^e(\prod_{(x,y) \in \Upsilon} \frac{C'_{x,y}}{C_{x,y}}) = \det(I - M\sqrt{\frac{C-C'}{C}}KM\sqrt{\frac{C-C'}{C}})$

On the other hand, from (31), it also equals $(\sum_{T \in ST_X} \prod_{\{x,y\} \in T} C_{x,y}) \mathcal{Z}_e^0 = \frac{\mathcal{Z}_e^0}{\mathcal{Z}_{e'}^0}$ so that finally

$$\frac{\mathcal{Z}_e^0}{\mathcal{Z}_{e'}^0} = \det(I - M\sqrt{\frac{C-C'}{C}}KM\sqrt{\frac{C-C'}{C}})$$

Note that indicators of distinct individual edges are negatively correlated. More generally:

Theorem 76 (*Negative association*) *Given any sets disjoint of edges E_1 and E_2 , $\mathbb{P}_{ST}^e(E_1 \cup E_2 \subseteq \Upsilon) \leq \mathbb{P}_{ST}^e(E_1 \subseteq \Upsilon)\mathbb{P}_{ST}^e(E_2 \subseteq \Upsilon)$*

Proof. Denote by $K^\#(i, j)$ the restriction of $K^\# = (\sqrt{C_\xi}K^{\xi, \eta}\sqrt{C_\eta}, \ \xi, \eta \in E)$ to $E_1 \times E_2$. Then, $\frac{\mathbb{P}_{ST}^e(E_1 \cup E_2 \subseteq \Upsilon)}{\mathbb{P}_{ST}^e(E_1 \subseteq \Upsilon)\mathbb{P}_{ST}^e(E_2 \subseteq \Upsilon)} = \frac{\det(K^\#)}{\det(K^\#(2,2))\det(K^\#(1,1))} = \det\left(\begin{bmatrix} I & F \\ F^* & I \end{bmatrix}\right)$ with $F = K^\#(1, 2)K^\#(2, 1)^{-\frac{1}{2}}$.

Note that the matrix F is of the form $\langle e_i, f_j \rangle$ where e_i and f_j are systems of orthonormal vectors in \mathbb{R}^{E_1} and \mathbb{R}^{E_2} for the scalar product defined by $K^\#$. It follows that the determinant of $\begin{bmatrix} I & F \\ F^* & I \end{bmatrix}$ is the square of the sine of the angle between the two subspaces \mathbb{R}^{E_1} and \mathbb{R}^{E_2} . ■

10 Reflection positivity

In this section, we assume there exists a partition of X : $X = X^+ \cup X^-$, $X^+ \cap X^- = \emptyset$ and an involution ρ on X such that:

- a) e is ρ -invariant.
- b) ρ exchanges X^+ and X^- .
- c) The $X^+ \times X^+$ matrix $C_{x,y}^\pm = C_{x,\rho(y)}$, is nonnegative definite.

Then the following holds:

Theorem 77 *i) For any positive integer d and square integrable function Φ in $\sigma(\widehat{\mathcal{L}}_d^x, x \in X^+) \vee \sigma(N_{x,y}^{(d)}, x, y \in X^+)$,*

$$\mathbb{E}(\Phi(\mathcal{L}_d)\overline{\Phi}(\rho(\mathcal{L}_d))) \geq 0$$

ii) For any square integrable function Σ of the free field ϕ restricted to X^+ ,

$$\mathbb{E}_\phi(\Sigma(\phi)\overline{\Sigma}(\rho(\phi))) \geq 0$$

iii) For any set of edges $\{\xi_i\}$ in $X^+ \times X^+$ the matrix,

$$K_{i,j} = \mathbb{P}_{ST}(\xi_i \in T, \rho\xi_j \in T) - \mathbb{P}_{ST}(\xi_i \in T)\mathbb{P}_{ST}(\xi_j \in T)$$

is nonpositive definite

Proof. The property ii) is well known in a slightly different context: Cf for exemple [[34]], [[10]] and their references. Reflection positivity is a keystone in the bridge between statistical and quantum mechanics.

To prove i), we use the fact that the σ -algebra is generated by the algebra of random variables of the form $\Phi = \sum \lambda_j B_{(d)}^{e,e_j,\omega_j}$ with $C^{(e_j)} = C$ and $\omega_j = 0$ on $X^- \times X^-$, $C^{(e_j)} \leq C$, $\lambda^{(e_j)} = \lambda$ on X^- and $\lambda^{(e_j)} \geq \lambda$.

Then

$$\mathbb{E}(\Phi(\mathcal{L}_d)\overline{\Phi}(\rho(\mathcal{L}_d))) = \mathbb{E}(\sum \lambda_j \overline{\lambda}_q B_{(d)}^{e, e_j, q, \omega_j - \rho(\omega_q)}) = \sum \lambda_j \overline{\lambda}_q \left(\frac{\mathcal{Z}_{e_j, q, \omega_j - \rho(\omega_q)}}{\mathcal{Z}_e} \right)^d \text{ with } \lambda^{(e_j, q)} = \lambda^{(e_q)} + \lambda^{(e_j)} - \lambda, C^{(e_j, q)} = \frac{C^{(e_j)} C^{(e_j)}}{C}.$$

We have to prove this is non negative. It is enough to prove it for $d = 1$, as the Hadamard product of two nonnegative definite Hermitian matrices is nonnegative definite.

Let us first assume that the nonnegative definite matrix C^\pm is positivedefinite. We will see the general case can be reduced to this one.

Now note that $\mathcal{Z}_{e_j + \rho(e_q) - e, \omega_j - \rho(\omega_q)}$ is the inverse of the determinant of a positive definite matrix of the form: $D(j, q) = \begin{bmatrix} A(j) & -C^{\pm, i, j} \\ -C^{\pm, i, j} & A(q)^* \end{bmatrix}$ with $[A(j)]_{u, v} = \lambda_u^{(e_j)} \delta_{u, v} - C_{u, v}^{(e_j)} e^{i\omega_j^{u, v}}$ and $C^{\pm, i, j} = D^{(j)} \overline{D}^{(q)}$ with $D_{u, u}^{(j)} = \frac{1}{\sqrt{C_{u, u}^{+, -}}} C_{u, \rho(u)}^{(e_j)} e^{i\omega_j^{u, \rho(u)}}$.

It is enough to show that $\det(D(j, k))^{-1}$ can be expanded in series of products $\sum q_n(j) \overline{q}_n(k)$ with $\sum |q_n(j)|^2 < \infty$.

As

$$D(j, q) = \begin{bmatrix} D^{(j)} & 0 \\ 0 & D^{(q)} \end{bmatrix} \begin{bmatrix} [D^{(j)}]^{-1} A(j) [D^{(j)}]^{-1} & -I \\ -I & [\overline{D}^{(j)}]^{-1} A(q)^* [\overline{D}^{(q)}]^{-1} \end{bmatrix} \begin{bmatrix} \overline{D}^{(j)} & 0 \\ 0 & \overline{D}^{(q)} \end{bmatrix}$$

the $-\alpha$ -power of this determinant can be written

$$\det(D^{(j)})^{-2} \det(\overline{D}^{(q)})^{-2} \det(F(j)) \det(F(q)^*) \det\left(I - \begin{bmatrix} 0 & F(j) \\ F(q)^* & 0 \end{bmatrix}\right)^{-1}$$

with $F(j) = D^{(j)} A(j)^{-1} D^{(j)}$, or more simply:

$$F(j) = \det(A(j))^{-1} \det(A(q)^*)^{-1} \det\left(I - \begin{bmatrix} 0 & F(j) \\ F(q)^* & 0 \end{bmatrix}\right)^{-1}.$$

Note that $A(j)^{-1}$ is also the Green function of the restriction to X^+ of the Markov chain associated with e_j , twisted by ω_j . Therefore $A(j)^{-1} D^{(j)} = \frac{C^\pm}{D^{(j)}} [C^{\pm, i, j}]^{-\frac{1}{2}} F(j) [C^\pm]^{\frac{1}{2}}$ is the balayage kernel on X^- defined by this Markov chain with an additional phase under the expectation produced by ω_j . It is therefore clear from Frobenius theorem that the eigenvalues of the matrices $A(j)^{-1} D^{(j)}$ and $F(j) = \frac{D^{(j)}}{C^\pm} A(j)^{-1} D^{(j)}$ are of modulus less than one and it follows that $\begin{bmatrix} 0 & F(j) \\ F(q)^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} F(q)^* & 0 \\ 0 & F(j) \end{bmatrix}$ is a contraction.

If X^+ has only one point, $(1 - F(j)F(q)^*)^{-1} = \sum F(j)^{-n} \overline{F(q)}^{-n}$ which allows to conclude. Let us now treat the general case.

For any (n, m) matrix N , and $k = (k_1, \dots, k_m) \in \mathbb{N}^m$, $l = (l_1, \dots, l_n) \in \mathbb{N}^n$, let $N^{\{k, l\}}$ denote the $(|k|, |l|)$ matrix obtained from by repeting k_i times each line i ; then l_j times each column j .

We use the expansion

$$\det(I - M)^{-1} = 1 + \sum Per(M^{\{k, k\}})$$

valid for any strict contraction M (Cf [40] and [41]).

Note that if X has $2d$ points, if we denote (k_1, \dots, k_{2d}) by (k^+, k^-) , with $k^+ = (k_1, \dots, k_d)$ and $k^- = (k_{d+1}, \dots, k_{2d})$, $\begin{bmatrix} 0 & F(j) \\ F(q)^* & 0 \end{bmatrix}^{\{k, k\}} = \begin{bmatrix} 0 & F(j)^{\{k^+, k^-\}} \\ [F(q)^*]^{\{k^-, k^+\}} & 0 \end{bmatrix}$.

But the all terms in the permanent of a $(2n, 2n)$ matrix of the form $\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}$ vanish unless the submatrices A and B are square matrices (not necessarily of equal ranks). Hence in our case, we necessary have $|k^+| = |k^-|$, so that, A and B are (n, n) matrices.

Then, the non zero terms in the permanent come from permutations exchanging $\{1, 2..n\}$ and $\{n + 1, \dots, 2n\}$, which therefore decompose into a pair of permutations of $\{1, 2..n\}$. Therefore:

$$Per\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) = Per(A)Per(B^*)$$

which concludes the proof in the positive definite case as

$$Per(B^*) = \sum_{\tau \in \mathcal{S}_n} \prod_1^n B_{i, \tau(i)}^* = \sum_{\tau \in \mathcal{S}_n} \prod_1^n \overline{B_{\tau(i), i}} = \overline{Per(B)}.$$

To treat the general nonnegative case, we can use use a passage to the limit or alternatively, the proposition 60 (or more precisely its extension including a current) to reduce the sets X^+ and X^- to the support of C^\pm .

To prove iii) let us first show the assumptions imply that the $X^+ \times X^+$ matrix $G_{x, y}^\pm = G_{x, \rho(y)}$ is also nonnegative definite. Let us write G in the form $\begin{bmatrix} A & -C^\pm \\ -C^\pm & A \end{bmatrix}^{-1}$ with $A = M_\lambda - C$. Then

$$G = \begin{bmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & A^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} I & -A^{-\frac{1}{2}} C^\pm A^{-\frac{1}{2}} \\ -A^{-\frac{1}{2}} C^\pm A^{-\frac{1}{2}} & I \end{bmatrix}^{-1} \begin{bmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & A^{-\frac{1}{2}} \end{bmatrix}$$

$A^{-\frac{1}{2}} C^\pm A^{-\frac{1}{2}}$ is non negative definite and as before, we can check it is a contraction since $A^{-1} C^\pm$ is a balayage kernel.

Note that if a symmetric nonnegative definite matrix K has eigenvalues μ_i , the eigenvalues of the symmetric matrix E defined by $\begin{bmatrix} I & -K \\ -K & I \end{bmatrix}^{-1} = \begin{bmatrix} D & E \\ E & D \end{bmatrix}$ are easily seen (Exercise) to be $\frac{\mu_i}{1-\mu_i^2}$. Taking $K = A^{-\frac{1}{2}}C^\pm A^{-\frac{1}{2}}$, it follows that the symmetric matrix E , (and in our particular case $G^\pm = A^{-\frac{1}{2}}EA^{-\frac{1}{2}}$) is nonnegative definite.

To finish the proof, let us take Σ of the form $\sum \lambda_j e^{\langle \phi, \chi_j \rangle}$. Then $\mathbb{E}_\phi(\Sigma(\phi)\overline{\Sigma}(\rho(\phi))) = \sum \lambda_j \lambda_q \mathbb{E}_\phi(e^{\langle \phi, \chi_j \rangle + \langle \phi, \rho(\chi_q) \rangle}) = \sum \lambda_j e^{\frac{1}{2}\langle \chi_j, G^{++}\chi_j \rangle} \lambda_q e^{\frac{1}{2}\langle \chi_q, G^{++}\chi_q \rangle} e^{\frac{1}{2}\langle \chi_j, G^\pm \chi_q \rangle}$ and as G^\pm is positive definite, we can conclude since $e^{\frac{1}{2}\langle \chi_j, G^\pm \chi_q \rangle} = \mathbb{E}_w(e^{\langle w, \chi_j \rangle} e^{\langle w, \chi_q \rangle})$, w denoting the Gaussian field on X^+ with covariance G^\pm .

To prove iii), note that $K_{i,j} = -(G_{i,j}^\pm)^2$. Then, using again the Gaussian vector w , and the Wick squares of its components; $(G_{i,j}^\pm)^2 = E(: w_i^2 :: w_j^2 :)$. ■

Remark 78 a) If U_j are unitary representations such that $U_j^{x,y}$ is the identity outside $X^+ \times X^+$, i) can be extended to variables of the form $\sum \lambda_j B_{(\alpha)}^{e_i, e_j, U_j}$ and to the σ -field they generate.

b) In the case where α is a half integer, by remark 47, the reflection positivity of the free field i), implies i) holds also for any half integer α provided that $\Phi \in \sigma(\widehat{\mathcal{L}}_\alpha^x, x \in X^+) \vee \sigma(N_{x,y}^{(\alpha)} + N_{y,x}^{(\alpha)}, x, y \in X^+)$.

Exercise 79 Prove the above remark.

Remark 80 If there exists a partition of X : $X = X^+ \cup X^- \cup X^0$, and an involution ρ on X such that:

- a) e and X^0 are ρ -invariant.
- b) $\rho(X^\pm) = X^\mp$
- c) X^+ and X^- are disconnected.

Then the assumptions of the previous theorem are satisfied for the trace on $X^+ \cup X^-$.

Moreover, if $X^0 \times X^0$ does not contain any edge of the graph, the assertion i) of theorem 77 hold for the non disjoint sets $X^+ \cup X^0$ and $X^- \cup X^0$. More precisely, i), holds for Φ in $\sigma(\widehat{\mathcal{L}}_d^x, x \in X^+) \vee \sigma(N_{x,y}^{(d)}, x, y \in X^+)$. It is enough to apply the theorem to the graph obtained by duplication of each point x_0 in X^0 into (x_0^+, x_0^-) and let all conductances $C_{x_0^+, x_0^-}$ equal to a constant increasing to infinity.

Physical Hilbert space and time shift: We will now work under the assumptions of the previous remark, namely, without assuming that $X = X^+ \cup X^-$.

Let \mathcal{H}^+ be the space of square integrable functions in $\sigma((\widehat{\mathcal{L}}_1^x, x \in X^+) \vee \sigma(N_{x,y}^{(1)}, x, y \in X^+)$, equipped with the Hermitian scalar product $\langle \Phi, \Psi \rangle_{\mathcal{H}} = \mathbb{E}(\Phi(\mathcal{L}_1)\overline{\Psi}(\rho(\mathcal{L}_1)))$. Note that $\langle \Phi, \Phi \rangle_{\mathcal{H}} \leq \mathbb{E}(\Phi\overline{\Phi}(\mathcal{L}_1))$.

Let \mathcal{N} be the subspace $\{\Psi \in \mathcal{H}^+, \mathbb{E}(\Psi(\mathcal{L}_1)\overline{\Psi}(\rho(\mathcal{L}_1))) = 0\}$ and \mathcal{H} the closure (for the topology induced by this scalar product) of the quotient space $\mathcal{H}^+/\mathcal{N}$ (which can be called the physical Hilbert space). We denote Φ^\sim the equivalence class of Φ . Note that $\overline{\Phi^\sim} = \overline{\Phi}$. \mathcal{H} is equipped with the hermitian scalar product defined unambiguously by $\langle \Phi^\sim, \Psi^\sim \rangle_{\mathcal{H}} = \langle \Phi, \Psi \rangle_{\mathcal{H}}$.

Assume X is of the form $X_0 \times \mathbb{Z}$ (space \times time) and let θ be the natural time shift. We assume θ preserves e , i.e. that conductances and κ are θ -invariant. We define ρ by $\rho(x_0, n) = (x_0, -n)$ and assume e is ρ -invariant. Note that $\theta(X^+) \subseteq X^+$ and $\rho\theta = \theta^{-1}\rho$. The transformations ρ and θ induce a transformations on loops that preserves μ , and θ induces a linear transformation of \mathcal{H}^+ . Moreover, given any F in \mathcal{N} , $F \circ \theta \in \mathcal{N}$, as

$$\begin{aligned} \langle F \circ \theta, F \circ \theta \rangle_{\mathcal{H}} & \text{ is nonnegative and equals } \mathbb{E}(F \circ \theta(\mathcal{L}_1)\overline{F \circ \theta}(\rho(\mathcal{L}_1))) = \mathbb{E}(F(\theta(\mathcal{L}_1))\overline{F}(\rho \circ \theta^{-1}(\mathcal{L}_1))) \\ & = \mathbb{E}(F(\theta^2(\mathcal{L}_1))\overline{F}(\rho(\mathcal{L}_1))) \\ & = \langle F \circ \theta^2, F \rangle_{\mathcal{H}} \leq \sqrt{\langle F \circ \theta^2, F \circ \theta^2 \rangle_{\mathcal{H}}} \langle F, F \rangle_{\mathcal{H}} \text{ which vanishes.} \end{aligned}$$

Proposition 81 *There exist a self adjoint contraction of \mathcal{H} , we will denote $\Pi^{(\theta)}$ such that $[\Phi \circ \theta]^\sim = \Pi^{(\theta)}(\Phi^\sim)$*

Proof. The existence of $\Pi^{(\theta)}$ follows from the last observation made above. As θ preserves $\mathbb{P}_{\mathcal{L}_1}$, it follows from the identity $\rho\theta = \theta^{-1}\rho$ that $\langle F \circ \theta, G \rangle_{\mathcal{H}} = \mathbb{E}(F(\theta(\mathcal{L}_1))\overline{G}(\rho(\mathcal{L}_1))) = \mathbb{E}(F(\mathcal{L}_1)\overline{G}(\rho \circ \theta^{-1}(\mathcal{L}_1))) = \mathbb{E}(F(\mathcal{L}_1)\overline{G}(\theta \circ \rho(\mathcal{L}_1))) = \langle F \circ \theta, G \rangle_{\mathcal{H}}$. Therefore, $\Pi^{(\theta)}$ is self adjoint on $\mathcal{H}^+/\mathcal{N}$. To prove that it is a contraction, it is enough to show that $\langle F \circ \theta, F \circ \theta \rangle_{\mathcal{H}} \leq \langle F, F \rangle_{\mathcal{H}}$ for all $F \in \mathcal{H}^+$.

$$\text{But as shown above, } \langle F \circ \theta, F \circ \theta \rangle_{\mathcal{H}} = \langle F \circ \theta^2, F \rangle_{\mathcal{H}} \leq \sqrt{\langle F \circ \theta^2, F \circ \theta^2 \rangle_{\mathcal{H}}} \langle F, F \rangle_{\mathcal{H}}$$

By recursion, it follows that:

$$\langle F \circ \theta, F \circ \theta \rangle_{\mathcal{H}} = \langle F \circ \theta^2, F \rangle_{\mathcal{H}} \leq \langle F \circ \theta^{2^n}, F \circ \theta^{2^n} \rangle_{\mathcal{H}}^{2^{-n}} \langle F, F \rangle_{\mathcal{H}}^{1-2^{-n}}$$

As $\langle F \circ \theta^{2^n}, F \circ \theta^{2^n} \rangle_{\mathcal{H}}^{2^{-n}} \leq (\mathbb{E}(F(\mathcal{L}_1)\overline{F}(\mathcal{L}_1)))^{2^{-n}}$ converges to 1 as $n \rightarrow \infty$, the inequality follows. ■

For all $n \in \mathbb{Z}$, the symmetry $\rho^{(n)} = \theta^{-n}\rho\theta^n$ allows to define spaces $\mathcal{H}^{(n)}$ isometric to \mathcal{H} . These isometries can be denoted by the shift θ^n . For $n > m$, $j_{n,m} = \theta^m[\Pi^{(\theta)}]^{n-m}\theta^{-n}$ is a contraction from $\mathcal{H}^{(n)}$ into $\mathcal{H}^{(m)}$.

11 The case of general Markov processes

We now explain briefly how some of the above results will be extended to a symmetric Markov process on an infinite space X . The construction of the loop measure as well as a lot of computations can be performed quite generally, using Markov processes or Dirichlet space theory (Cf for example [9]). It works as soon as the bridge or excursion measures $\mathbb{P}_t^{x,y}$ can be properly defined. The semigroup should have a locally integrable kernel $p_t(x, y)$.

Let us consider more closely the occupation field \widehat{l} . The extension is rather straightforward when points are not polar. We can start with a Dirichlet space of continuous functions and a measure m such that there is a mass gap. Let P_t denote the associated Feller semigroup. Then the Green function is well defined as the mutual energy of the Dirac measures δ_x and δ_y which have finite energy. It is the covariance function of a Gaussian free field $\phi(x)$, which will be associated to the field $\widehat{\mathcal{L}}_{\frac{1}{2}}^x$ of local times of the Poisson process of random loops whose intensity is given by the loop measure defined by the semigroup P_t . This will apply to examples related to one-dimensional Brownian motion or to Markov chains on countable spaces.

When we consider Brownian motion on the half line, the associated occupation field $\widehat{\mathcal{L}}_\alpha$ is a continuous branching process with immigration, as in the simple random walk case.

When points are polar, one needs to be more careful. We will consider only the case of the two and three dimensional Brownian motion in a bounded domain D killed at the boundary, i.e. associated with the classical energy with Dirichlet boundary condition. The Green function does not induce a trace class operator but it is still Hilbert-Schmidt which allows us to define renormalized determinants \det_2 (Cf [33]).

If A is a symmetric Hilbert Schmidt operator, $\det_2(I + A)$ is defined as $\prod(1 + \lambda_i)e^{-\lambda_i}$ where λ_i are the eigenvalues of A .

The Gaussian field (called free field) whose covariance function is the Green function is now a generalized field: Generalized fields are not defined pointwise but have to be smeared by a test function f . Still $\phi(f)$ is often denoted $\int \phi(x)f(x)dx$.

Wick powers : ϕ^n : of the free field can be defined as a generalized field by approximation as soon as the $2n$ -th power of the Green function, $G(x, y)^{2n}$ is locally integrable (Cf [34]). This is the case for all n for Brownian motion in dimension two, as the Green function has only a logarithmic singularity on the diagonal, and for $n = 2$ in dimension

three as the singularity is of the order of $\frac{1}{\|x-y\|}$. More precisely, taking for example $\pi_\varepsilon^x(dy)$ to be the normalized area measure on the sphere of radius ε around x , $\phi(\pi_\varepsilon^x)$ is a Gaussian field with variance $\sigma_\varepsilon^x = \int G(z, z') \pi_\varepsilon^x(dz) \pi_\varepsilon^x(dz')$. Its Wick powers are defined with Hermite polynomials as we did previously:

$:\phi(\pi_\varepsilon^x)^n := (\sigma_\varepsilon^x)^{\frac{n}{2}} H_n\left(\frac{\phi(\pi_\varepsilon^x)}{\sqrt{\sigma_\varepsilon^x}}\right)$. Then one can see that, $\int f(x) : \phi(\pi_\varepsilon^x)^n : dx$ converges in L^2 for any bounded continuous function f with compact support towards a limit called the n -th Wick power of the free field evaluated on f and denoted $:\phi^n : (f)$. Moreover, $\mathbb{E}(:\phi^n : (f) : \phi^n : (h)) = \int G^{2n}(x, y) f(x) h(y) dx dy$.

In these cases, we can extend the statement of theorem 36 to the renormalized occupation field $\tilde{\mathcal{L}}_{\frac{1}{2}}^x$ and the Wick square $:\phi^2 :$ of the free field.

Let us explain this in more detail in the Brownian motion case. Let D be an open subset of \mathbb{R}^d such that the Brownian motion killed at the boundary of D is transient and has a Green function. Let $p_t(x, y)$ be its transition density and $G(x, y) = \int_0^\infty p_t(x, y) dt$ the associated Green function. The loop measure μ was defined in [15] as

$$\mu = \int_D \int_0^\infty \frac{1}{t} \mathbb{P}_t^{x,x} dt$$

where $\mathbb{P}_t^{x,x}$ denotes the (non normalized) excursion measure of duration t such that if $0 \leq t_1 \leq \dots \leq t_h \leq t$,

$$\mathbb{P}_t^{x,x}(\xi(t_1) \in dx_1, \dots, \xi(t_h) \in dx_h) = p_{t_1}(x, x_1) p_{t_2-t_1}(x_1, x_2) \dots p_{t-t_h}(x_h, x) dx_1 \dots dx_h$$

(the mass of $\mathbb{P}_t^{x,x}$ is $p_t(x, x)$). Note that μ is a priori defined on based loops but it is easily seen to be shift-invariant.

For any loop l indexed by $[0, T(l)]$, define the measure $\widehat{l} = \int_0^{T(l)} \delta_{l(s)} ds$: for any Borel set A , $\widehat{l}(A) = \int_0^{T(l)} 1_A(l_s) ds$.

Lemma 82 *For any non-negative function f ,*

$$\mu(\langle \widehat{l}, f \rangle^n) = (n-1)! \int G(x_1, x_2) f(x_2) G(x_2, x_3) f(x_3) \dots G(x_n, x_1) f(x_1) \prod_1^n dx_i$$

Proof. From the definition of μ and \widehat{l} , $\mu(\langle \widehat{l}, f \rangle^n)$ equals:

$$\begin{aligned} n! \int \int_{\{0 < t_1 \dots < t_n < t\}} \frac{1}{t} f(x_1) \dots f(x_n) p_{t_1}(x, x_1) \dots p_{t-t_n}(x_n, x) \prod dt_i dx_i dt dx \\ = n! \int \int_{\{0 < t_1 \dots < t_n < t\}} \frac{1}{t} f(x_1) \dots f(x_n) p_{t_2-t_1}(x_1, x_2) \dots p_{t_1+t-t_n}(x_n, x_1) \prod dt_i dx_i dt \end{aligned}$$

Performing the change of variables $v_2 = t_2 - t_1, \dots, v_n = t_n - t_{n-1}, v_1 = t_1 + t - t_n$, and $v = t_1$, we obtain:

$$\begin{aligned} n! \int_{\{0 < v < v_1, 0 < v_i\}} \frac{1}{v_1 + \dots + v_n} f(x_1) \dots f(x_n) p_{v_2}(x_1, x_2) \dots p_{v_1}(x_n, x_1) \prod dv_i dx_i dv \\ = n! \int_{\{0 < v_i\}} \frac{v_1}{v_1 + \dots + v_n} f(x_1) \dots f(x_n) p_{v_2}(x_1, x_2) \dots p_{v_1}(x_n, x_1) \prod dv_i dx_i \\ = (n-1)! \int_{\{0 < v_i\}} f(x_1) \dots f(x_n) p_{v_2}(x_1, x_2) \dots p_{v_1}(x_n, x_1) \prod dv_i dx_i \\ \text{(as we get the same formula with any } v_i \text{ instead of } v_1) \\ = (n-1)! \int G(x_1, x_2) f(x_2) G(x_2, x_3) f(x_3) \dots G(x_n, x_1) f(x_1) \prod_1^n dx_i. \end{aligned}$$

■

One can define in a similar way the analogous of multiple local times, and get for their integrals with respect to μ a formula analogous to the one obtained in the discrete case.

Let G denote the operator on $L^2(D, dx)$ defined by G . Let f be a non-negative continuous function with compact support in D .

Note that $\langle \widehat{l}, f \rangle$ is μ -integrable only in dimension one as then, G is locally trace class. In that case, using for all x an approximation of the Dirac measure at x , local times \widehat{l}^x can be defined in such a way that $\langle \widehat{l}, f \rangle = \int \widehat{l}^x f(x) dx$.

$\langle \widehat{l}, f \rangle$ is μ -square integrable in dimensions one, two and three, as G is Hilbert-Schmidt if D is bounded, since $\int \int_{D \times D} G(x, y)^2 dx dy < \infty$, and otherwise locally Hilbert-Schmidt.

N.B.: Considering distributions χ such that $\int \int (G(x, y)^2 \chi(dx) \chi(dy)) < \infty$, we could see that $\langle \widehat{l}, \chi \rangle$ can be defined by approximation as a square integrable variable and $\mu(\langle \widehat{l}, \chi \rangle^2) = \int (G(x, y)^2 \chi(dx) \chi(dy))$.

Let z be a complex number such that $\operatorname{Re}(z) > 0$.

Note also that $e^{-z\langle \widehat{l}, f \rangle} + z\langle \widehat{l}, f \rangle - 1$ is bounded by $\frac{|z|^2}{2} \langle \widehat{l}, f \rangle^2$ and expands as an alternating series $\sum_2^\infty \frac{z^n}{n!} (-\langle \widehat{l}, f \rangle)^n$, with $\left| e^{-z\langle \widehat{l}, f \rangle} - 1 - \sum_1^N \frac{z^n}{n!} (-\langle \widehat{l}, f \rangle)^n \right| \leq \frac{|z\langle \widehat{l}, f \rangle|^{N+1}}{(N+1)!}$. Then, for $|z|$ small enough., it follows from the above lemma that

$$\mu(e^{-z\langle \widehat{l}, f \rangle} + z\langle \widehat{l}, f \rangle - 1) = \sum_2^\infty \frac{z^n}{n} \operatorname{Tr}(-(M_{\sqrt{\mathcal{F}}}GM_{\sqrt{\mathcal{F}}})^n)$$

As $M_{\sqrt{\mathcal{F}}}GM_{\sqrt{\mathcal{F}}}$ is Hilbert-Schmidt $\det_2(I + zM_{\sqrt{\mathcal{F}}}GM_{\sqrt{\mathcal{F}}})$ is well defined and the second member writes $-\log(\det_2(I + zM_{\sqrt{\mathcal{F}}}GM_{\sqrt{\mathcal{F}}}))$.

Then the identity

$$\mu(e^{-z\langle \widehat{l}, f \rangle} + z\langle \widehat{l}, f \rangle - 1) = -\log(\det_2(I + zM_{\sqrt{\mathcal{F}}}GM_{\sqrt{\mathcal{F}}}))$$

extends, as both sides are analytic as locally uniform limits of analytic functions, to all complex values with positive real part.

The renormalized occupation field $\widetilde{\mathcal{L}}_\alpha$ is defined as the compensated sum of all \widehat{l} in \mathcal{L}_α (formally, $\widetilde{\mathcal{L}}_\alpha = \widehat{\mathcal{L}}_\alpha - \int_0^{T(l)} \delta_{l_s} ds \mu(dl)$) By a standard argument used for the construction of Levy processes,

$$\langle \widetilde{\mathcal{L}}_\alpha, f \rangle = \lim_{\varepsilon \rightarrow 0} \langle \widetilde{\mathcal{L}}_{\alpha, \varepsilon}, f \rangle$$

with by definition

$$\langle \widetilde{\mathcal{L}}_{\alpha, \varepsilon}, f \rangle = \sum_{\gamma \in \mathcal{L}_\alpha} (1_{\{T > \varepsilon\}} \int_0^T f(\gamma_s) ds) - \alpha \mu(1_{\{T > \varepsilon\}} \int_0^T f(\gamma_s) ds)$$

The convergence holds a.s. and in L^2 , as

$$\mathbb{E}((\sum_{\gamma \in \mathcal{L}_\alpha} (1_{\{T > \varepsilon\}} \int_0^T f(\gamma_s) ds) - \alpha \mu(1_{\{T > \varepsilon\}} \int_0^T f(\gamma_s) ds))^2) = \alpha \int (1_{\{T > \varepsilon\}} \int_0^T f(\gamma_s) ds)^2 \mu(dl)$$

and $\mathbb{E}(\langle \widetilde{\mathcal{L}}_\alpha, f \rangle^2) = \operatorname{Tr}((M_{\sqrt{\mathcal{F}}}GM_{\sqrt{\mathcal{F}}})^2)$. Note that if we fix f , α can be considered as a time parameter and $\langle \widetilde{\mathcal{L}}_{\alpha, \varepsilon}, f \rangle$ as Levy processes with discrete positive jumps approximating a Levy process with positive jumps $\langle \widetilde{\mathcal{L}}_\alpha, f \rangle$. The Levy exponent $\mu(1_{\{T > \varepsilon\}}(e^{-\langle \widehat{l}, f \rangle} + \langle \widehat{l}, f \rangle - 1))$ of $\langle \widetilde{\mathcal{L}}_{\alpha, \varepsilon}, f \rangle$ converges towards the Lévy exponent of $\langle \widetilde{\mathcal{L}}_\alpha, f \rangle$ which is

$\mu((e^{-\langle \widehat{l}, f \rangle} + \langle \widehat{l}, f \rangle - 1))$ and, from the identity $E(e^{-\langle \widetilde{\mathcal{L}}_\alpha, f \rangle}) = e^{-\alpha \mu(e^{-\langle \widehat{l}, f \rangle} + \langle \widehat{l}, f \rangle - 1)}$, we get the

Theorem 83 *Assume $d \leq 3$. Denoting $\widetilde{\mathcal{L}}_\alpha$ the compensated sum of all \widehat{l} in \mathcal{L}_α , we have $\mathbb{E}(e^{-\langle \widetilde{\mathcal{L}}_\alpha, f \rangle}) = \det_2(I + M_{\sqrt{T}} G M_{\sqrt{T}})^{-\alpha}$*

Moreover $e^{-\langle \widetilde{\mathcal{L}}_{\alpha, \varepsilon}, f \rangle}$ converges a.s. and in L^1 towards $e^{-\langle \widetilde{\mathcal{L}}_\alpha, f \rangle}$.

Considering distributions of finite energy χ (i.e. such that $\int (G(x, y))^2 \chi(dx) \chi(dy) < \infty$), we can see that $\langle \widetilde{\mathcal{L}}_\alpha, \chi \rangle$ can be defined by approximation as $\lim_{\lambda \rightarrow \infty} (\langle \widetilde{\mathcal{L}}_\alpha, \lambda G_\lambda \chi \rangle)$ and

$$\mathbb{E}(\langle \widetilde{\mathcal{L}}_\alpha, \chi \rangle^2) = \alpha \int (G(x, y))^2 \chi(dx) \chi(dy).$$

Specializing to $\alpha = \frac{k}{2}$, k being any positive integer we have:

Corollary 84 *The renormalized occupation field $\widetilde{\mathcal{L}}_{\frac{k}{2}}$ and the Wick square $\frac{1}{2} : \sum_1^k \phi_l^2 :$ have the same distribution.*

If Θ is a conformal map from D onto $\Theta(D)$, it follows from the conformal invariance of the Brownian trajectories that a similar property holds for the Brownian "loop soup" (Cf [15]). More precisely, if $c(x) = \text{Jacobian}_x(\Theta)$ and, given a loop l , if $T^c(l)$ denotes the reparametrized loop l_{τ_s} , with $\int_0^{\tau_s} c(l_u) du = s$, the configuration $\Theta T^c(\mathcal{L}_\alpha)$ is a Brownian loop soup of intensity parameter α on $\Theta(D)$. Then we have the following:

Proposition 85 *$\Theta(c\widetilde{\mathcal{L}}_\alpha)$ is the renormalized occupation field on $\Theta(D)$.*

Proof. We have to show that the compensated sum is the same if we perform it after or before the time change. For this it is enough to check that

$$\begin{aligned} & \mathbb{E}([\sum_{\gamma \in \mathcal{L}_\alpha} (1_{\{\tau_T > \eta\}} 1_{\{T \leq \varepsilon\}} \int_0^T f(\gamma_s) ds - \alpha \int (1_{\{\tau_T > \eta\}} 1_{\{T \leq \varepsilon\}} \int_0^T f(\gamma_s) ds) \mu(d\gamma)]^2) \\ &= \alpha \int (1_{\{\tau_T > \eta\}} 1_{\{T \leq \varepsilon\}} \int_0^T f(\gamma_s) ds)^2 \mu(d\gamma) \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E}([\sum_{\gamma \in \mathcal{L}_\alpha} (1_{\{T>\varepsilon\}} 1_{\tau_T \leq \eta} \int_0^T f(\gamma_s) ds - \alpha \int (1_{\{T>\varepsilon\}} 1_{\tau_T \leq \eta} \int_0^T f(\gamma_s) ds) \mu(d\gamma)]^2) \\ & \alpha \int (1_{\{T>\varepsilon\}} 1_{\tau_T \leq \eta} \int_0^T f(\gamma_s) ds)^2 \mu(d\gamma) \end{aligned}$$

converge to zero as ε and η go to zero. It follows from the fact that:

$$\int [1_{\{T \leq \varepsilon\}} \int_0^T f(\gamma_s) ds]^2 \mu(d\gamma)$$

and

$$\int [1_{\tau_T \leq \eta} \int_0^T f(\gamma_s) ds]^2 \mu(d\gamma)$$

converge to 0. The second follows easily from the first if c is bounded away from zero. We can always consider the "loop soups" in an increasing sequence of relatively compact open subsets of D to reduce the general case to that situation. ■

As in the discrete case (see corollary 32), we can compute product expectations. In dimensions one and two, for f_j continuous functions with compact support in D :

$$\mathbb{E}(\langle \widetilde{\mathcal{L}}_\alpha, f_1 \rangle \dots \langle \widetilde{\mathcal{L}}_\alpha, f_k \rangle) = \int Per_\alpha^0(G(x_l, x_m), 1 \leq l, m \leq k) \prod f_j(x_j) dx_j. \quad (32)$$

12 Renormalized powers

In dimension one, as in the discrete case, powers of the occupation field can be viewed as integrated self intersection local times. In dimension two, renormalized powers of the occupation field, also called *renormalized self intersections local times* can be defined as follows:

Theorem 86 *Assume $d = 2$. Let $\pi_\varepsilon^x(dy)$ be the normalized arclength on the circle of radius ε around x , and set $\sigma_\varepsilon^x = \int G(y, z) \pi_\varepsilon^x(dy) \pi_\varepsilon^x(dz)$. Then, $\int f(x) Q_k^{\alpha, \sigma_\varepsilon^x}(\langle \widetilde{\mathcal{L}}_\alpha, \pi_\varepsilon^x \rangle) dx$ converges in L^2 for any bounded continuous function f with compact support towards a limit denoted $\langle \widetilde{\mathcal{L}}_\alpha^k, f \rangle$ and*

$$\mathbb{E}(\langle \widetilde{\mathcal{L}}_\alpha^k, f \rangle \langle \widetilde{\mathcal{L}}_\alpha^l, h \rangle) = \delta_{l,k} \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!} \int G^{2k}(x, y) f(x) h(y) dx dy.$$

Proof. The idea of the proof can be understood by trying to prove that

$\mathbb{E}((\int f(x)Q_k^{\alpha,\sigma_\varepsilon^x}(\langle \widetilde{\mathcal{L}}_\alpha, \pi_\varepsilon^x \rangle)dx)^2)$ remains bounded as ε decreases to zero. One will expand this expression in terms of sums of integrals of product of Green functions and check that the combinatorial identities (21) imply the cancelation of the logarithmic divergences.

This is done by showing (as done below in the proof of the theorem) that one can modify slightly the products of Green functions appearing in $\mathbb{E}(Q_k^{\alpha,\sigma_\varepsilon^x}(\langle \widetilde{\mathcal{L}}_\alpha, \pi_\varepsilon^x \rangle)Q_k^{\alpha,\sigma_\varepsilon^y}(\langle \widetilde{\mathcal{L}}_\alpha, \pi_\varepsilon^y \rangle))$ to replace them by products of the form $G(x, y)^j(\sigma_\varepsilon^x)^l(\sigma_\varepsilon^y)^h$. The cancelation of terms containing σ_ε^x and/or σ_ε^y then follows directly from the combinatorial identities.

Let us now prove the theorem. Consider first, for any x_1, x_2, \dots, x_n , ε small enough and $\varepsilon \leq \varepsilon_1, \dots, \varepsilon_n \leq 2\varepsilon$, with $\varepsilon_i = \varepsilon_j$ if $x_i = x_j$, an expression of the form:

$$\Delta = \left| \prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i)(\sigma_{\varepsilon_i}^{x_i})^{m_i} - \int G(y_1, y_2) \dots G(y_n, y_1) \pi_{\varepsilon_1}^{x_1}(dy_1) \dots \pi_{\varepsilon_n}^{x_n}(dy_n) \right|$$

in which we define m_i as $\sup(h, x_{i+h} = x_i)$.

In the integral term, we first replace progressively $G(y_{i-1}, y_i)$ by $G(x_{i-1}, x_i)$ whenever $x_{i-1} \neq x_i$, using triangle, then Schwartz inequalities, to get an upper bound of the absolute value of the difference made by this substitution in terms of a sum Δ' of expressions of the form

$$\prod_l G(x_l, x_{l+1}) \sqrt{\int (G(y_1, y_2) - G(x_1, x_2))^2 \pi_{\varepsilon_1}^{x_1}(dy_1) \pi_{\varepsilon_2}^{x_2}(dy_2) \int \prod G^2(y_k, y_{k+1}) \prod \pi_{\varepsilon_k}^{x_k}(dy_k)}.$$

The expression obtained after these substitutions can be written

$$W = \prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i) \int G(y_1, y_2) \dots G(y_{m_i-1}, y_{m_i}) \pi_{\varepsilon_i}^{x_i}(dy_1) \dots \pi_{\varepsilon_i}^{x_i}(dy_{m_i})$$

and we see the integral terms could be replaced by $(\sigma_\varepsilon^{x_i})^{m_i}$ if G was translation invariant. But as the distance between x and y tends to 0, $G(x, y)$ is equivalent to $G_0(x, y) = \frac{1}{\pi} \log(\|x - y\|)$ and moreover, $G(x, y) = G_0(x, y) - H^{D^c}(x, dz)G_0(z, y)$, H^{D^c} denoting the Poisson kernel on the boundary of D . As our points lie in a compact inside D , it follows that for some constant C , for $\|y_1 - x\| \leq \varepsilon$, $|\int (G(y_1, y_2) \pi_\varepsilon^x(dy_2) - \sigma_\varepsilon^x)| < C\varepsilon$.

Hence, the difference Δ'' between W and $\prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i)(\sigma_\varepsilon^{x_i})^{m_i}$ can be bounded by $\varepsilon W'$, where W' is an expression similar to W .

To get a good upper bound on Δ , using the previous observations, by repeated applications of Hölder inequality. it is enough to show that for ε small enough and $\varepsilon \leq \varepsilon_1, \varepsilon_2 \leq 2\varepsilon$, (with C and C' denoting various constants):

- 1) $\int (G(y_1, y_2) - G(x_1, x_2)^2 \pi_{\varepsilon_1}^{x_1}(dy_1) \pi_{\varepsilon_2}^{x_2}(dy_2)) < C(\varepsilon 1_{\{\|x_1 - x_2\| \geq \sqrt{\varepsilon}\}} + (G(x_1, x_2)^2 + \log(\varepsilon)^2) 1_{\{\|x_1 - x_2\| < \sqrt{\varepsilon}\}})$
- 2) $\int G(y_1, y_2)^k \pi_{\varepsilon}^x(dy_1) \pi_{\varepsilon}^x(dy_2) < C |\log(\varepsilon)|^k$ and more generally
- 3) $\int G(y_1, y_2)^k \pi_{\varepsilon_1}^{x_1}(dy_1) \pi_{\varepsilon_2}^{x_2}(dy_2) < C |\log(\varepsilon)|^k$

As the main contributions come from the singularities of G , they follow from the following simple inequalities:

1')

$$\begin{aligned} & \int |\log(\varepsilon^2 + 2R\varepsilon \cos(\theta) + R^2) - \log(R)|^2 d\theta \\ &= \int |\log((\varepsilon/R)^2 + 2(\varepsilon/R) \cos(\theta) + 1)|^2 d\theta < C((\varepsilon 1_{\{R \geq \sqrt{\varepsilon}\}}) + \log^2(R/\varepsilon) 1_{\{R < \sqrt{\varepsilon}\}}) \end{aligned}$$

(considering separately the cases where $\frac{\sqrt{\varepsilon}}{R}$ is large or small)

2') $\int |\log(\varepsilon^2(2 + 2 \cos(\theta)))|^k d\theta \leq C |\log(\varepsilon)|^k$

3') $\int |\log((\varepsilon_1 \cos(\theta_1) + \varepsilon_2 \cos(\theta_2) + r)^2 + (\varepsilon_1 \sin(\theta_1) + \varepsilon_2 \sin(\theta_2))^2)|^k d\theta_1 d\theta_2 \leq C (|\log(\varepsilon)|)^k$.

It can be proved by observing that for $r \leq \varepsilon_1 + \varepsilon_2$, we have near the singularities (i.e. the values $\theta_1(r)$ and $\theta_2(r)$ for which the expression under the log vanishes) to evaluate integrals bounded by $C \int_0^1 (-\log(\varepsilon u))^k du \leq C' (-\log(\varepsilon))^k$ for ε small enough.

Let us now show that for $\varepsilon \leq \varepsilon_1, \varepsilon_2 \leq 2\varepsilon$, we have, for some integer $N^{n,k}$

$$\begin{aligned} & \left| \mathbb{E}(Q_k^{\alpha, \sigma_{x_1}^{\varepsilon_1}}(\langle \widetilde{\mathcal{L}}_{\alpha}, \pi_{\varepsilon_1}^x \rangle) Q_l^{\alpha, \sigma_{y_2}^{\varepsilon_2}}(\langle \widetilde{\mathcal{L}}_{\alpha}, \pi_{\varepsilon_2}^y \rangle)) - \delta_{l,k} G(x, y)^{2k} \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!} \right| \\ & \leq C \log(\varepsilon)^{N_{l,k}} (\sqrt{\varepsilon} + G(x, y)^{2k}) 1_{\{\|x-y\| < \sqrt{\varepsilon}\}}. \quad (33) \end{aligned}$$

Indeed, developing the polynomials and using formula (32) we can express this expectation as a linear combination of integrals under $\prod_i \pi_{\varepsilon_1}^x(dx_i) \prod_j \pi_{\varepsilon_2}^y(dy_j)$ of products of $G(x_i, y_{i'})$, $G(x_i, x_j)$ and $G(y_j, y_{j'})$ as we did in the discrete case. If we replace each $G(x_i, y_j)$ by $G(x, y)$, each $G(x_i, x_{i'})$ by $\sigma_{\varepsilon_1}^x$ and each $G(y_j, y_{j'})$ by $\sigma_{\varepsilon_2}^y$, we can use the combinatorial identity (21) to get the value $\delta_{l,k} G(x, y)^{2k} \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!}$. Then, the above results allow us to bound the error made by this replacement.

The bound (33) is uniform in (x, y) only away from the diagonal as $G(x, y)$ can be arbitrarily large but we conclude from it that for any bounded integrable f and h ,

$$\left| \int (\mathbb{E}(Q_k^{\alpha, \sigma_{\varepsilon_1}^x}(\langle \widetilde{\mathcal{L}}_\alpha, \pi_{\varepsilon_1}^x \rangle) Q_l^{\alpha, \sigma_{\varepsilon_2}^y}(\langle \widetilde{\mathcal{L}}_\alpha, \pi_{\varepsilon_2}^y \rangle)) - \delta_{l,k} G(x, y)^{2k} \frac{\alpha \dots (\alpha + k - 1)}{k!}) f(x) h(y) dx dy \right| \leq C' \sqrt{\varepsilon} \log(\varepsilon)^{N_{l,k}}$$

(as $\int \int G(x, y)^{2k} 1_{\{\|x-y\| < \sqrt{\varepsilon}\}} dx dy$ can be bounded by $C\varepsilon^{\frac{2}{3}}$, for example).

Taking $\varepsilon_n = 2^{-n}$, it is then straightforward to check that $\int f(x) Q_k^{\alpha, \sigma_{\varepsilon_n}^x}(\langle \widetilde{\mathcal{L}}_\alpha, \pi_{\varepsilon_n}^x \rangle) dx$ is a Cauchy sequence in L^2 . The theorem follows. ■

Specializing to $\alpha = \frac{k}{2}$, k being any positive integer as before, Wick powers of $\sum_{j=1}^k \phi_j^2$ are associated with self intersection local times of the loops. More precisely, we have:

Proposition 87 *The renormalized self intersection local times $\widetilde{\mathcal{L}}_{\frac{k}{2}}^n$ and the Wick powers $\frac{1}{2^n n!} : (\sum_1^k \phi_l^2)^n :$ have the same joint distribution.*

The proof is similar to the one given in [19] and also to the proof of the above theorem, but simpler. It is just a calculation of the L^2 -norm of

$$\int [: (\phi^2)^n : (x) - Q_{\frac{1}{2}, \sigma_\varepsilon^x}^{\frac{1}{2}, \sigma_\varepsilon^x} (: \phi_x^2 : (\pi_\varepsilon^x))] f(x) dx$$

which converges to zero with ε .

Final remarks:

- a) These generalized fields have two fundamental properties:

Firstly they are local fields (or more precisely local functionals of the field $\widetilde{\mathcal{L}}_\alpha$ in the sense that their values on functions supported in an open set D depend only on the trace of the loops on D).

Secondly, noting we could use different regularizations to define $\widetilde{\mathcal{L}}_\alpha^k$, the action of a conformal transformation Θ on these fields is given by *the k -th power of the conformal factor $c = \text{Jacobian}(\Theta)$* . More precisely, $\Theta(c^k \widetilde{\mathcal{L}}_\alpha^k)$ is the renormalized k -th power of the occupation field in $\Theta(D)$.

- b) It should be possible to derive from the above remark the existence of exponential moments and introduce non trivial local interactions as in the constructive field theory derived from the free field (Cf [34]).
- c) Let us also briefly consider currents. We will restrict our attention to the one and two dimensional Brownian case, X being an open subset of the line or plane. Currents can be defined by vector fields, with compact support.

Then, if we now denote by ϕ the complex valued free field (its real and imaginary parts being two independent copies of the free field), $\int_l \omega$ and $\int_X (\bar{\phi} \partial_\omega \phi - \phi \partial_\omega \bar{\phi}) dx$ are well defined square integrable variables in dimension 1 (it can be checked easily by Fourier series). The distribution of the centered occupation field of the loop process "twisted" by the complex exponential $\exp(\sum_{l \in \mathcal{L}_\alpha} \int_l i\omega + \frac{1}{2} \widehat{l}(\|\omega\|^2))$ appears to be the same as the distribution of the field : $\phi \bar{\phi}$: "twisted" by the complex exponential $\exp(\int_X (\bar{\phi} \partial_\omega \phi - \phi \partial_\omega \bar{\phi}) dx)$ (Cf[20]).

In dimension 2, logarithmic divergences occur.

- d) There is a lot of related investigations. The extension of the properties proved here in the finite framework has still to be completed, though the relation with spanning trees should follow from the remarkable results obtained on SLE processes, especially [17]. Note finally that other essential relations between SLE processes, loops and free fields appear in [42], [32] and [6].

References

- [1] J. Bertoin Levy processes. Cambridge (1996)
- [2] N. Biggs Algebraic graph theory. Cambridge (1973)

- [3] N. Biggs Chip-Firing and the Critical group of a Graph. *J. of Algebraic combinatorics* 9 26-45 (1999)
- [4] S. Bochner Completely monotone functions on partially ordered spaces. *Duke Math. J.* 9 519-526 (1942).
- [5] C. Dellacherie, P.A. Meyer *Probabilités et Potentiel. Chapitres XII-XVI* Hermann. Paris. (1987)
- [6] J. Dubedat SLE and the free field: Partition functions and couplings. *ArXiv Math* 07123018
- [7] E.B.Dynkin Local times and Quantum fields. *Seminar on Stochastic processes, Gainesville 1982.* 69-84 *Progr. Prob. Statist.* 7 Birkhauser. (1984).
- [8] N. Eisenbaum, H. Kaspi A characterization of the infinitely divisible squared Gaussian processes. *Ann. Prob.* 34 728-742 (2006).
- [9] M. Fukushima, Y. Oshima, M. Takeda *Dirichlet forms and Markov processes.* De Gruyter. (1994)
- [10] K. Gawedzki *Conformal field theory. Lecture notes.* I.A.S. Princeton.
- [11] Kotani, M. , Sunada, T. Zeta functions of finite graphs. *J. Math. Sci. Univ. Tokyo* 7 7-25 (2000).
- [12] J.F.C. Kingman *Poisson processes.* Oxford (1993)
- [13] G. Lawler A self avoiding random walk. *Duke math. J.* 47 655-693 (1980)
- [14] G. Lawler Loop erased random walks. *H. Kesten Festshrift: Perplexing problems in probability.* *Progr.Prob.* 44 197-217 Birkhäuser (1999)
- [15] G. Lawler, W. Werner The Brownian loop soup. *PTRF* 128 565-588 (2004)
- [16] G. Lawler, J. Trujillo Ferreis Random walk loop soup. *TAMS* 359 767-787 (2007)
- [17] G. Lawler, O. Schramm, W. Werner Conformal invariance of planar loop erased random walks and uniform spanning trees. *Ann. Probability* 32, 939-995 (2004).

- [18] Y. Le Jan Mesures associées à une forme de Dirichlet. Applications. Bull. Soc. Math. Fr. 106 61-112 (1978).
- [19] Y. Le Jan On the Fock space representation of functionals of the occupation field and their renormalization. J;F.A. 80, 88-108 (1988)
- [20] Y. Le Jan Dynkin isomorphism without symmetry. Stochastic analysis in mathematical physics. ICM 2006 Satellite conference in Lisbon. 43-53 World Scientific. (2008)
- [21] Y. Le Jan Dual Markovian semigroups and processes. Functional analysis in Markov processes (Katata/Kyoto, 1981), Lecture Notes in Math. 923, 47-75 Springer (1982).
- [22] R. Lyons Determinantal Probability Measures. Publ. Math. Inst. Hautes Etudes Sci. 98, 167-212 (2003)
- [23] R. Lyons, Y. Peres Probability on trees and networks. Prepublication.
- [24] P. Marchal Loop erased Random Walks, Spanning Trees and Hamiltonian Cycles. E. Com. Prob. 5 39-50 (1999).
- [25] W.S. Massey Algebraic Topology: An Introduction Springer (1967)
- [26] M.B. Marcus, J. Rosen Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes. Ann. Prob. 20, 1603-1684 (1992)
- [27] J. Neveu Processus aléatoires gaussiens. Presses de l'Université de Montréal (1968)
- [28] Parry, W. , Pollicott, M. Zeta functions and the periodic orbit structure of hyperbolic dynamics. Asterisque 187-188 Société Mathématique de France (1990)
- [29] J. Pitman Combinatorial Stochastic Processes. St Flour Lecture Notes (2002)
- [30] C. Pittet, L. Saloff-Coste On random walks on wreath products. Annals of Probability 30 948-977 (2002)
- [31] Quian Minping, Quian Min Circulation for recurrent Markov chains. Z.F.Wahrsch. 59 205-210 (1982).
- [32] O. Schramm, S. Sheffield Contour lines of the two dimensional discrete Gaussian free field. Math. PR/0605337

- [33] B. Simon Trace ideals and their applications. London Math Soc Lect. Notes 35 Cambridge (1979)
- [34] B. Simon The $P(\phi_2)$ Euclidean (quantum) field theory. Princeton. (1974).
- [35] Shirai, T. , Takahashi, Y. Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes. J. Functional Analysis 205 414-463 (2003)
- [36] J.P. Serre Arbres, amalgames, SL_2 Asterisque 46 Société Mathématique de France (1977)
- [37] H.M. Stark, A.A. Terras Zeta functions on finite graphs and coverings. Advances in Maths 121 134-165 (1996)
- [38] P.F. Stebe A Residual Property of Certain Groups. Proc. American Math. Soc. 26 37-42 (1970)
- [39] K. Symanzik Euclidean quantum field theory. Scuola internazionale di Fisica "Enrico Fermi". XLV Corso. 152-223 Academic Press. (1969)
- [40] D. Vere Jones A generalization of permanents and determinants. Linear Algebra and Appl. 111 (1988)
- [41] D. Vere Jones Alpha permanents and their applications. New Zeland J. Math. 26 125-149 (1997)
- [42] W. Werner The conformally invariant measure on self-avoiding loops. J. American Math Soc. 21 137-169 (2008).