Markov paths, loops and fields

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MARKOV PATHS, LOOPS AND FIELDS
(Preliminary version)

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Abstract

We study the Poissonnian ensembles of Markov loops and the associated renormalized self intersection local times.

1 Introduction

The purpose of these notes is to explore some simple relations between Markovian path and loop measures, spanning trees, determinants, and Markov fields such as the free field. The main emphasis is put on the study of occupation fields defined by Poissonian ensembles of Markov loops. These were defined in [14] for planar Brownian motion in relation with SLE processes and in [15] for simple random walks. They appeared informally already in [33]. For half integral values $\frac{k}{2}$ of the intensity parameter α, these occupation fields can be identified with the sum of squares of k copies of the associated free field (i.e. the Gaussian field whose covariance is given by the Green function). This is related to Dynkin's isomorphism (cf [6], [24], [18]). We first present the results in the elementary

framework of symmetric Markov chains on a finite space, proving also several interesting results such as the relation between loop ensembles and spanning trees and the reflection positivity property. Then we show some results can be extended to more general Markov processes. There are no essential difficulties when points are not polar but other cases are more problematic. As for the square of the free field, cases for which the Green function is Hilbert Schmidt such as two and three dimensional Brownian motion can be dealt with through appropriate renormalization.

We can show that the renormalised powers of the occupation field (i.e. the self intersection local times of the loop ensemble) converge in the two dimensional case and that they can be identified with higher even Wick powers of the free field when α is a half integer.

2 Symmetric Markov processes on finite spaces

Notations: Functions and measures on finite (or countable) spaces are often denoted as vectors and covectors.

The multiplication operator defined by a function f acting on functions or on measures is in general simply denoted by f, but sometimes it will be denoted M_f. The function obtained as the density of a measure μ with respect to some other measure ν is simply denoted $\frac{\mu}{\nu}$.

2.1 Graphs

Our basic object will be a finite space X and a set of non negative conductances $C_{x,y} = C_{y,x}$, indexed by pairs of distinct points of X. This situation allows to define a kind of *discrete topology and geometry* we will briefly study in this section and in the following ones.

We say $\{x, y\}$ is a link or an edge iff $C_{x,y} > 0$ and an oriented edge (x, y) is defined by the choice of an ordering in an edge. We set $-(x, y) = (y, x)$ and if $e = (x, y)$, we denote it also (e^-, e^+).

The points of X together with the set of non oriented edges E define a graph.(X, E). *We assume it is connected.* The set of oriented edges is denoted E^o. It will always be viewed as a subset of X^2, without reference to any imbedding.

The associated line graph is the oriented graph defined by E^o as set of vertices and

in which oriented edges are pairs (e_1, e_2) such that $e_1^+ = e_2^-$. The mapping $e \to -e$ is an involution of the line graph.

An important example is the case in which conductances are equal to zero or one. Then the conductance matrix is the adjacency matrix of the graph: $C_{x,y} = 1_{\{x,y\}\in E}$

A complete graph is defined by all conductances equal to one.

The complete graph with n vertices is denoted K_n.The complete graph K_4 is the graph defined by the tetraedron. K_5 is not planar (i.e. cannot be imbedded in a plane), but K_4 is.

A finite discrete path on X, say $(x_0, x_1, ...x_n)$ is called a (discrete) geodesic arc iff $\{x_i, x_{i+1}\} \in E$ (path segment on the graph) and $x_{i-1} \neq x_{i+1}$ (without backtraking). Geodesic arcs starting at x_0 form a marked tree $\mathfrak{T}_{x_0}$ rooted in x_0. Oriented edges are defined by pairs of geodesic arcs of the form: $((x_0, x_1, ...x_n), (x_0, x_1, ...x_n, x_{n+1}))$ (the orientation is defined in reference to the root). $\mathfrak{T}_{x_0}$ is a universal cover of X.

On the space $\mathfrak{L}_{x_0}$ of discrete loops based at some point x_0, we can define an operation of concatenation, which define a monoid structure. Among them, closed geodesics based at x_0 define a group $\mathfrak{b}_{x_0}$. Actually the concatenation of two closed geodesics based at x_0 is not directly a closed geodesic. It can involve backtracking "in the middle" but then after cancellation of the two inverse subarcs, we get a closed geodesic, possibly empty (the neutral element) if the two closed geodesics are identical up to reverse order. The structure of $\mathfrak{b}_{x_0}$ does not depend on the base point and defines the fundamental group Γ (as the graph is connected: see for exemple [23]).

There is a natural left action of $\mathfrak{b}_{x_0}$ on $\mathfrak{T}_{x_0}$. It can be interpreted as a change of root in the tree. Any geodesic arc between x_0 and another point y_0 of X defines an isomorphism between $\mathfrak{T}_{x_0}$ and $\mathfrak{T}_{y_0}$ (change of root) and beween $\mathfrak{b}_{x_0}$ and $\mathfrak{b}_{y_0}$ (conjugation).

We have just seen that the universal covering of the finite graph (X, E) at x_0 is a tree $\mathfrak{T}_{x_0}$. The fiber at x_0 is $\mathfrak{b}_{x_0}$. These groups are conjugated in a non canonical way. Note that $X = \mathfrak{b}_{x_0}^R \backslash \mathfrak{T}_{x_0}$ (here the use of the quotient on the left corresponds to the left action).

Example 1 *Among graphs, the simplest ones are $r-$regular graphs, in which each point has r of neighbours. A universal covering of any $r-$regular graph is isomorphic to the $r-$regular tree $\mathfrak{T}^r$.*

Example 2 *Cayley graphs: A finite group with a symmetric set of generators $S = \{g_1, ..g_k\}$ such that $S \cap S^{-1}$ is empty yields an oriented $2k$-regular graph .*

A spanning tree T of the graph has $|X|-1$ edges, which cover all points in X. Its inverse images by the canonical projection from a universal cover $\mathfrak{T}_{x_0}$ onto X form a tesselation on $\mathfrak{T}_{x_0}$, i.e. a partition of $\mathfrak{T}_{x_0}$ in identical subtrees. fundamental domains for the action of $\mathfrak{d}_{x_0}^R$. Conversely, a section of the canonical projection from the universal cover defines a spanning tree.

Giving a spanning tree determines the conjugation relations between the various groups $\mathfrak{d}_{x_0}$ and the isomorphisms between the universal covers $\mathfrak{T}_{x_0}$.

The fundamental group Γ is a free group with $|E|-|X|+1=r$ generators. To construct a set of generators, one considers a spanning tree T of the graph, and choose an orientation on each of the r remaining links. This defines r oriented cycles on the graph and a system of r generators for the fundamental group. (See [23] or Serres ([32]) in a more general context).

Example 3 *Consider K_3 and K_4.*

There are various non ramified coverings, intermediate beween (X,E) and the universal covering. Non ramified means that locally , the covering space is identical to the graph (same incident edges). Each non ramified covering is (up to an isomorphism) associated with a subgroup H of Γ, defined up to conjugation. More precisely, if H_{x_0} is a subgroup of Γ_{x_0}, the covering is defined as the quotient graph (Y,F) with $Y=H_{x_0}\backslash\mathfrak{T}_{x_0}$ and F the set of edges defined by the canonical projection from $\mathfrak{T}_{x_0}$ ont Y. H_{x_0} can be interpreted as the group of closed geodesics on the quotient graph. The quotient group $H_{x_0}\backslash\Gamma_{x_0}$ acts on $H_{x_0}\backslash\mathfrak{T}_{x_0}$. A spanning tree of (X,E) defines also a tesselation of the quotient graph for the action of the quotient group.

Example 4 *By central symmetry, the cube is a two fold covering of the tetraedron associated with the group $\mathbb{Z}/2\mathbb{Z}$.*

A covering is said to be Abelian when the covering group is Abelian. The maximal Abelian covering is associated with the commutators subgroup $[\Gamma,\Gamma]$.

2.2 Energy

Let us consider a nonnegative function κ on X. Set $\lambda_x=\kappa_x+\sum_y C_{x,y}$ $P_y^x=\frac{C_{x,y}}{\lambda_x}$. P is a λ-symmetric (sub) stochastic transition matrix: $\lambda_x P_y^x=\lambda_y P_x^y$ with $P_x^x=0$ for all x in X and it defines a symmetric irreducible Markov chain ξ_n.

We can define above it a λ-symmetric irreducible Markov chain in continuous time x_t, with exponential holding times,of parameter 1. We have $x_t = \xi_{N_t}$, where N_t denotes a Poisson process of intensity 1.The infinitesimal generator writes $L_y^x = P_y^x - \delta_y^x$.

We denote by P_t its (sub) Markovian semigroup $\exp(Lt) = \sum \frac{t^k}{k!} L^k$. L and P_t are λ-symmetric.

We will consider the Markov chain associated with C, κ, sometimes in discrete time, sometimes in continuous time (with exponential holding times).

Recall that for any complex function $z^x, x \in X$, the "energy"

$$e(z) = \langle -Lz, \overline{z} \rangle_\lambda = \sum_{x \in X} -(Lz)^x \overline{z}^x \lambda_x$$

is nonnegative as it can be written (easy exercise)

$$e(z) = \frac{1}{2} \sum_{x,y} C_{x,y}(z^x - z^y)(\overline{z}^x - \overline{z}^y) + \sum_x \kappa_x z^x \overline{z}^x = \sum_x \lambda_x z^x \overline{z}^x - \sum_{x,y} C_{x,y} z^x \overline{z}^y$$

The Dirichlet space ([8]) is the space of real functions equipped with the energy scalar product

$$e(f,g) = \frac{1}{2} \sum_{x,y} C_{x,y}(f^x - f^y)(g^x - g^y) + \sum_x \kappa_x f^x g^x = \sum_x \lambda_x f^x g^x - \sum_{x,y} C_{x,y} f^x g^y$$

defined by polarization of e.

Note that the non negative symmetric "conductance matrix" C and the non negative equilibrium or "killing" (or "equilibrium") measure κ are the free parameters of the model. The eigenvector associated with the lowest eigenvalue of $-L$ has constant sign by the well known argument which shows that the map $z \to z^+$ lowers the energy.

We have a dichotomy between:

- the recurrent case where 0 is the lowest eigenvalue of $-L$, and the corresponding eigenspace is formed by constants. Equivalently, $P1 = 1$ and κ vanishes.

- the transient case where the lowest eigenvalue is positive which means there is a "Poincaré inequality": For some positive ε, the energy $e(f,f)$ dominates $\varepsilon \langle f, f \rangle_\lambda$ for all f. Equivalently, κ does not vanish.

In the transient case, we denote by V the associated potential operator $(-L)^{-1} = \int_0^\infty P_t dt$. It can be expressed in terms of the spectral resolution of L.

We denote by G the Green function defined on X^2 as $G^{x,y} = \frac{V_y^x}{\lambda_y} = \frac{1}{\lambda_y}[(I-P)^{-1}]_y^x$ i.e. $G = (M_\lambda - C)^{-1}$. It induces a linear bijection from measures into functions. We set $(G\mu)^x = \sum_y G^{x,y}\mu_y$

Note that $e(f, G\mu) = \langle f, \mu\rangle$ (i.e. $\sum_x f^x \mu_x$) for all function f and measure μ. In particular $G\kappa = 1$ as $e(1, f) = \sum f^x \kappa_x = \langle f, 1\rangle_\kappa$.

See ([8]) for a development of this theory in a more general setting.

In the recurrent case, the potential operator V operates on the space $\lambda^\perp$ of functions f such that $\langle f, 1\rangle_\lambda = 0$ as the inverse of the restriction of $I-P$ to $\lambda^\perp$. The Green operator G maps the space of measures of total charge zero onto $\lambda^\perp$. Setting for any signed measure ν of total charge zero $G\nu = V\frac{\nu}{\lambda}$. we have for any function f, $\langle \nu, f\rangle = e(G\nu, f)$ (as $e(G\nu, 1) = 0$) and in particular $f^x - f^y = e(G(\delta_x - \delta_y), f)$.

Exercise 5 *Compute the Green operator in the case of the complete graph K_n*

In quantum mechanics, $-L$ is called the Hamiltonian and the eigenvalues are its energy levels.

One can learn more on graphs and eigenvalues in [1]

2.3 Feynman-Kac formula

A discrete analogue of the Feynman-Kac formula can be given as follows: Let s be any function on X taking values in $(0, 1]$. Then, for the discrete Markov chain ξ_n associated with P, it is a straightforward consequence of the Markov property that:

$$\mathbb{E}_x(\prod_{j=0}^{n-1} s(\xi_j) 1_{\{\xi_n = y\}}) = [(M_s P)^n]_y^x$$

and for the continuous time Markov chain x_t (with exponential holding times), setting $k(x) = \frac{1-s(x)}{s(x)}$, we have:

$$\mathbb{E}_x(e^{-\int_0^t k(x_s)ds} 1_{\{x_t = y\}}) = [\exp(t(L - M_k)]_y^x.$$

It follows easily from the previous formula by decomposing the first expression according to the number of jumps before time t, which has a Poisson distribution.

For any nonnegative measur χ, set $V_\chi = (-L + M_{\frac{\chi}{\lambda}})^{-1}$ and $G_\chi = V_\chi M_{\frac{1}{\lambda}} = (M_\lambda + M_\chi - C)^{-1}$. It is a symmetric nonnegative function on $X \times X$. G_0 is the Green function G, and G_χ can be viewed as the Green function of the energy form $e_\chi = e + \| \quad \|^2_{L^2(\chi)}$.

Note that e_χ has the same conductances C as e, but χ is added to the killing measure. Note also that V_χ is not the potential of the Markov chain associated with e_χ when one takes exponential holding times of parameter 1 but the Green function is intrinsic i.e. invariant under a change of time scale. Still, we have by Feynman Kac formula

$$\int_0^\infty \mathbb{E}_x(e^{-\int_0^t \frac{\chi}{\lambda}(x_s)ds} 1_{\{x_t=y\}})dt = [V_\chi]^x_y.$$

We have also the "resolvent" equation $V - V_\chi = V M_{\frac{\chi}{\lambda}} V_\chi = V_\chi M_{\frac{\chi}{\lambda}} V$. Then,

$$G - G_\chi = G M_\chi G_\chi = G_\chi M_\chi G$$

2.4 Recurrent extension of a transient chain

It will be convenient to add a cemetery point Δ to X, and extend C, λ and G to $X^\Delta = \{X \cup \Delta\}$ by setting , $\lambda_\Delta = \sum_{x\in X} \kappa_x$, $C_{x,\Delta} = \kappa_x$ and $G^{x,\Delta} = G^{\Delta,\Delta} = 0$ for all $x \in X$. Note that $\lambda(X^\Delta) = \sum_{X\times X} C_{x,y} + 2\sum_X \kappa_x)$

One can consider the recurrent "resurrected" Markov chain defined by the extensions the conductances to X^Δ. An energy e^Δ is defined by the formula

$$e^\Delta(z) = \frac{1}{2} \sum_{x,y\in X^\Delta} C_{x,y}(z^x - z^y)(\overline{z}^x - \overline{z}^y)$$

From the irreducibility assumption, if follows that e^Δ vanishes only on constants. We denote by P^Δ the transition kernel on X^Δ defined by

$$e^\Delta(f, g) = \left\langle f - P^\Delta f, g \right\rangle_\lambda$$

or equivalently by

$$[P^\Delta]^x_y = \frac{C_{x,y}}{\sum_{y\in X^\Delta} C_{x,y}} = \frac{C_{x,y}}{\lambda_x}$$

Note that $P^\Delta 1 = 1$ so that λ is now an invariant measure.with $\lambda_x[P^\Delta]^x_y = \lambda_y[P^\Delta]^y_x$ on X^Δ Denote V^Δ and G^Δ the associated potential and Green operators.

Note that for μ carried by X, for all $x \in X$, denoting by ε_Δ the unit point mass at Δ,

$$\begin{aligned}\mu_x &= e^\Delta(G^\Delta(\mu - \mu(X)\varepsilon_\Delta), 1_x) = \lambda_x((I - P^\Delta)G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(x) \\ &= \lambda_x((I - P)G^\Delta(\mu - \mu(X)\varepsilon_\Delta))(x) - \kappa_x G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(\Delta).\end{aligned}$$

Hence, applying G , it follows that on X,

$$G\mu = G^\Delta(\mu - \mu(X)\varepsilon_\Delta) - G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(\Delta)G\kappa = G^\Delta(\mu - \mu(X)\varepsilon_\Delta) - G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(\Delta).$$

Moreover, as $G^\Delta(\mu - \mu(X)\varepsilon_\Delta)$ is in $\lambda^\perp$,

$$\sum_{x \in X} \lambda_x G(\mu)^x = -G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(\Delta)\lambda(X^\Delta).$$

Therefore, $G^\Delta(\mu - \mu(X)\varepsilon_\Delta)(\Delta) = \frac{-\langle \lambda, G\mu \rangle}{\lambda(X^\Delta)}$ and $G^\Delta(\mu - \mu(X)\varepsilon_\Delta) = -\frac{\langle \lambda, G\mu \rangle}{\lambda(X^\Delta)} + G\mu$.

This type of extension can be done in a more general context (See [20] and Dellacherie-Meyer [4])

Remark 6 *Conversely, a recurrent chain can be killed at any point x_0 of X, defining a Green function $G_{\{x_0\}}$ on $X - \{x_0\}$. Then, for any μ carried by $X - \{x_0\}$,*

$$G_{\{x_0\}}\mu = G(\mu - \mu(X)\varepsilon_{x_0}) - G(\mu - \mu(X)\varepsilon_{x_0})(x_0).$$

This transient chain allows to recover the recurrent one by the above procedure.

Exercise 7 *Consider a transient process which is killed with probability p at each passage in Δ. Determine the associated energy and Green operator.*

2.5 Transfer matrix

Let us suppose we are in the recurrent case: We can define a scalar product on the space $\mathbb{A}$ of functions on E^o (oriented edges) as follows

$\langle \omega, \eta \rangle_{\mathbb{A}} = \frac{1}{2} \sum_{x,y} C_{x,y} \omega^{x,y} \eta^{x,y}$. Denoting as in [22] $df^{u,v} = f^v - f^u$, we note that $\langle df, dg \rangle_{\mathbb{A}} = e(f, g)$ In particular

$$\langle df, dG(\delta_y - \delta_x) \rangle_{\mathbb{A}} = df^{x,y}$$

Denote $\mathbb{A}_-$, ($\mathbb{A}_+$) the space of functions on E^o odd (even) for orientation reversal. Note that the spaces $\mathbb{A}_+$ and $\mathbb{A}_-$ are orthogonal for the scalar product defined on $\mathbb{A}$.

For any α in $\mathbb{A}_-$, define $d^*\alpha$ by $(d^*\alpha)^x = -\sum_{y\in X} P_y^x \alpha^{x,y}$. Note it belongs to $\lambda^\perp$ as $\sum_{x,y} C_{x,y}\alpha^{x,y}$ vanishes.

We have

$$\begin{aligned}\langle \alpha, df\rangle_{\mathbb{A}} &= \frac{1}{2}\sum_{x,y} \lambda_x P_y^x \alpha^{x,y}(f^y - f^x) \\ &= \frac{1}{2}\sum_{x\in X}(d^*\alpha)^x f^x \lambda_x - \frac{1}{2}\sum_{x,y}\lambda_x P_y^x \alpha^{x,y} f^y = \sum_{x\in X}(d^*\alpha)^x f^x \lambda_x\end{aligned}$$

as the two terms of the sum are in fact equal since α is skew symmetric. In particular, $e(f,f) = \langle df, df\rangle_{\mathbb{A}} = \sum_{x\in X}(d^*df)^x f^x \lambda_x$. The image of d and the kernel of $d^*$are therefore orthogonal in $\mathbb{A}_-$. We say α in $\mathbb{A}_-$ is harmonic iff $d^*\alpha = 0$.
Note that for any function f , $d^*df = -Pf + f = -Lf$.

The projection of any α in $\mathbb{A}_-$ on the image of d is easily obtained as $dVd^*(\alpha)$. Indeed, for any function g, $\langle \alpha, dg\rangle_{\mathbb{A}} = \langle d^*\alpha, g\rangle_\lambda = e(Vd^*\alpha, g) = \langle dVd^*(\alpha), dg\rangle_{\mathbb{A}}$.
d is the discrete analogue of the differential and d^* the analogue of its adjoint, depending on the metric which is here defined by the conductances.

Set $\alpha^{x,y} = \pm\frac{1}{C_{u,v}}$ if $(x,y) = \pm(u,v)$ and 0 elsewhere. Then $\lambda d^*\alpha = \delta_v - \delta_u$ and $dVd^*(\alpha) = -dG(\delta_u - \delta_v)$

The symmetric transfer matrix $K^{(x,y),(u,v)}$, indexed by pairs of oriented edges, is defined to be

$$K^{(x,y),(u,v)} = [-dG(\delta_u - \delta_v)]^{x,y} = G(\delta_u - \delta_v)^x - G(\delta_u - \delta_v)^y =< dG(\delta_x - \delta_y), dG(\delta_u - \delta_v) >_{\mathbb{A}}$$

for $x, y, u, v \in X$, with $C_{x,y}C_{u,v} > 0$. For every oriented edge $e = (x,y)$ in X, set $K^e = dG(\delta^x - \delta^y)$.

We have $\langle K^e, K^g\rangle_{\mathbb{A}} = K^{e,g}$.We can view $K = dG$ as a linear operator mapping the space measures of total charge zero into $\mathbb{A}_-$. As measures of the form $\delta_x - \delta_y$ span the space of measures of total charge zero, it is determined by the transfer matrix.

Note that $-d^*Kv = v/\lambda$ for any v of total charge zero, that for all α in $\mathbb{A}_-$, $(d^*\alpha)\lambda$ has total charge zero and that $K((d^*\alpha)\lambda) = dVd^*(\alpha)$, the projection of α on the image of d in $\mathbb{A}_-$.

Consider now, in the transient case, the transfer matrix associated with G^Δ.
We see that for x and y in X, $G^\Delta(\delta_x - \delta_y)^u - G^\Delta(\delta_x - \delta_y)^v = G(\delta_x - \delta_y)^u - G(\delta_x - \delta_y)^v$.
We can see also that $G^\Delta(\delta_x - \delta_\Delta) = G\delta_x - \frac{-\langle \lambda, G\delta_x\rangle}{\lambda(X^\Delta)}$. So the same identity holds in X^Δ.
Therefore, as $G^{x,\Delta} = 0$, in all cases,

$$K^{(x,y),(u,v)} = G^{x,u} + G^{y,v} - G^{x,v} - G^{y,u}$$

So that the theory applies also in the transient case.

2.6 Countable spaces

The assumption of finiteness of X can be relaxed. But the recurrent case is not characterized by the vanishing of the killing measure but by the divegence of the potential. On countable spaces, the previous results extend easily when under spectral gap conditions. In the transient case, the Dirichlet space $\mathbb{H}$ is the space of all functions f with finite energy $e(f)$ which are limits in energy norm of functions with finite support. The energy of a measure is defined as $\sup_{f\in\mathbb{H}} \frac{\mu(f)^2}{e(f)}$. It includes Dirac measures. The potential $G\mu$ is well defined in $\mathbb{H}$ for all finite energy measures μ, by the identity $e(f, G\mu) = \langle f, \mu\rangle$, valid for all f in the Dirichlet space.

Most important cases are the non ramified covering of finy the finite graphs. Let us consider the universal covering. It is easy to check it will be transient even in the recurrent case as soon as (X, E) is not circular.

Let $\widehat{G}$ be the Green function on a covering $\mathcal{Y}$. Let p be the canonical projection from $\mathcal{Y}$ onto X. We have $G^{x,y} = \sum_{y'\in p^{-1}(y)} \widehat{G}^{x',y'}$ for any $x' \in p^{-1}(x)$.

Let us consider the universal covering. It is easy to check it will be transient even in the recurrent case as soon as (X, E) is not circular.

The expression of the Green function $\widehat{G}$ on the universal covering can be given exactly when it is a regular tree, i.e. in the regular graph case.

Exercise 8 *Show that on the $r-$regular tree $\mathfrak{T}^r$, for $r \geq 3$, the Green function is given by* $G^{x,y} = \frac{(r-1)^{1-d(x,y)}}{r-2}$

3 Loop measures

3.1 A measure on based loops

We denote $\mathbb{P}^x$ the family of probability laws on piecewise constant paths defined by P_t.

$$\mathbb{P}^x(\gamma(t_1) = x_1, ..., \gamma(t_h) = x_h) = P_{t_1}(x, x_1)P_{t_2-t_1}(x_1, x_2)\dots P_{t_h-t_{h-1}}(x_{h-1}, x_h)$$

Denoting by $p(\gamma)$ the number of jumps and T_i the jump times, we have:

$$\mathbb{P}_x(p(\gamma) = k, \gamma_{T_1} = x_1, ..., \gamma_{T_{k-1}} = x_{k-1}, T_1 \in dt_1, ..., T_k \in dt_k) \\ = \frac{C_{x,x_2}...C_{x_{k-1},x_k}\kappa_{x_k}}{\lambda_x\lambda_{x_2}...\lambda_{x_k}} 1_{\{0<t_1<...<t_k\}} e^{-t_k} dt_1...dt_k$$

For any integer $p > 2$, let us define a based loop with p points in X as a couple $l = (\xi, \tau) = ((\xi_m, 1 \leq m \leq p), (\tau_m, 1 \leq m \leq p+1),)$ in $X^p \times \mathbb{R}_+^{p+1}$, and set $\xi_1 = \xi_{p+1}$ (equivalently, we can parametrize the the discrete based loop by $\mathbb{Z}/p\mathbb{Z}$). The integer p represents the number of points in the discrete based loop $\xi = (\xi_1, ...\xi_{p(\xi)})$ and will be denoted $p(\xi)$. Note two time parameters are attached to the base point since the based loops do not in general end or start with a jump.

Based loops with one point ($p = 1$) are simply given by a pair (ξ, τ) in $X \times \mathbb{R}_+$.

Based loops have a natural time parametrization $l(t)$ and a time period $T(\xi) = \sum_{i=1}^{p(\xi)+1} \tau_i$. If we denote $\sum_{i=1}^{m} \tau_i$ by T_m: $l(t) = \xi_{m-1}$ on $[T_{m-1}, T_m)$ (with by convention $T_0 = 0$ and $\xi_0 = \xi_p$).

A σ-finite measure μ is defined on based loops by

$$\mu = \sum_{x \in X} \int_0^\infty \frac{1}{t} \mathbb{P}_t^{x,x} \lambda_x dt$$

where $\mathbb{P}_t^{x,y}$ denotes the (non normalized) "law" of a path from x to y of duration t : If $t_1 < t_2... < t_h < t$,

$$\mathbb{P}_t^{x,y}(l(t_1) = x_1, ..., l(t_h) = x_h) = [P_{t_1}]_{x_1}^x [P_{t_2-t_1}]_{x_2}^{x_1} ... [P_{t-t_h}]_y^{x_h} \frac{1}{\lambda_y}$$

Its mass is $p_t^{x,y} = \frac{[P_t]_y^x}{\lambda_y}$. And for any measuable set A of piecewise constant paths indexed by $[0\ t]$, we can also write

$$\mathbb{P}_t^{x,y}(A) = \mathbb{P}_x(A \cap \{x_t = y\}) \frac{1}{\lambda_y}$$

From the first expression, we see that by definition of μ, if $t_1 < t_2 ... < t_h < t$,

$$\mu(l(t_1) = x_1, ..., l(t_h) = x_h, T \in dt) = [P_{t_1+t-t_h}]_{x_1}^{x} [P_{t_2-t_1}]_{x_2}^{x_1} ... [P_{t_h-t_{h-1}}]_{x_h}^{x_{h-1}} \frac{1}{t} dt \quad (1)$$

Note also that for $k > 1$, using the second expression of $\mathbb{P}_t^{x,x}$ and the fact that conditionally to $N_t = k$, the jump times are distributed like an increasingly reordered $k-$uniform sample of $[0\ t]$

$$\begin{aligned} &\lambda_x \mathbb{P}_t^{x,x}(p = k, \xi_2 = x_2, ..., \xi_k = x_k, T_1 \in dt_1, ..., T_k \in dt_k) \\ &= P_{x_2}^{x} P_{x_3}^{x_2} ... P_{x}^{x_k} 1_{\{0<t_1<...t_k<t\}} e^{-t} dt_1 ... dt_k \end{aligned}$$

Therefore

$$\mu(p = k, \xi_1 = x_1, .., \xi_k = x_k, T_1 \in dt_1, .., T_k \in dt_k, T \in dt) \quad (2)$$

$$= P_{x_2}^{x_1} .. P_{x_1}^{x_k} \frac{1_{\{0<t_1<...<t_k<t\}}}{t} e^{-t} dt_1 ... dt_k dt \quad (3)$$

for $k > 1$.

Moreover, for one point-loops, $\mu\{p(\xi) = 1, \xi_1 = x_1, \tau_1 \in dt\} = \frac{e^{-t}}{t} dt$

3.2 First properties

Note that the loop measure is invariant under time reversal.

If D is a subset of X, the restriction of μ to loops contained in D, denoted μ^D is clearly the loop measure induced by the Markov chain killed at the exit of D. This can be called the *restriction property*.

Let us recall that this killed Markov chain is defined by the restriction of λ to D and the restriction P^D of P to D^2 (or equivalently by the restriction e_D of the Dirichlet norm e to functions vanishing outside D).

As $\int \frac{t^{k-1}}{k!} e^{-t} dt = \frac{1}{k}$, it follows from (2) that for $k > 1$, on based loops,

$$\mu(p(\xi) = k, \xi_1 = x_1, ..., \xi_k = x_k) = \frac{1}{k} P_{x_2}^{x_1} ... P_{x_1}^{x_k} \quad (4)$$

In particular, we obtain that, for $k \geq 2$

$$\mu(p = k) = \frac{1}{k} Tr(P^k)$$

and therefore, as $Tr(P) = 0$,

$$\mu(p > 1) = \sum_2^\infty \frac{1}{k} Tr(P^k) = -\log(\det(I - P)) = \log(\det(G) \prod_x \lambda_x)$$

since (denoting M_λ the diagonal matrix with entries λ_x), we have

$$\det(I - P) = \frac{\det(M_\lambda - C)}{\det(M_\lambda)}$$

Moreover

$$\int p(l) 1_{\{p>1\}} \mu(dl) = \sum_2^\infty Tr(P^k) = Tr((I - P)^{-1} P) = Tr(GC)$$

3.3 Loops and pointed loops

It is clear on formula 1 that μ is invariant under the time shift that acts naturally on based loops.

A loop is defined as an equivalence class of based loops for this shift. Therefore, μ induces a *measure on loops also denoted by* μ.

A loop is defined by the discrete loop ξ° formed by the ξ_i in circular order, (i.e. up to translation) and the associated scaled holding times. We clearly have:

$$\mu(\xi^\circ = (x_1, x_2, ..., x_k)^\circ) = P_{x_2}^{x_1} ... P_{x_1}^{x_k}$$

However, loops are not easy to parametrize, that is why we will work mostly with based loops or *pointed loops*. These are defined as based loops ending with a jump, or as loops with a starting point. They can be parametrized by a based discrete loop and by the holding times at each point. Calculations are easier if we work with based or pointed loops, even though we will deal only with functions independent of the base point.

The parameters of the pointed loop naturally associated with a based loop are $\xi_1, ..., \xi_p$ and

$$\tau_1 + \tau_{p+1} = \tau_1^*, \tau_i = \tau_i^*, \ 2 \leq i \leq p$$

An elementary change of variables, shows the expression of μ on pointed loops writes:

$$\mu(p = k, \xi_i = x_i, \tau_i^* \in dt_i) = P_{x_2}^{x_1} ... P_{x_1}^{x_k} \frac{t_1}{\sum t_i} e^{-\sum t_i} dt_1 ... dt_k \tag{5}$$

Trivial ($p = 1$) pointed loops and trivial based loops coincide.

Note that loop functionals can be written

$$\Phi(l^\circ) = \sum 1_{\{p=k\}} \Phi_k((\xi_i, \tau_i^*), i = 1, ...k)$$

with Φ_k invariant under circular permutation of the variables (ξ_i, τ_i^*).

Then, for non negative Φ_k

$$\int \Phi_k(l^\circ)\mu(dl) = \int \sum_{x_i} \Phi_k(x_i, t_i) P_{x_2}^{x_1} ... P_{x_1}^{x_k} e^{-\sum t_i} \frac{t_1}{\sum t_i} dt_1 ... dt_k$$

and by invariance under circular permutation, the term t_1 can be replaced by any t_i. Therefore, adding up and dividing by k, we get that

$$\int \Phi_k(l^\circ)\mu(dl) = \int \frac{1}{k} \sum_{x_i} \Phi_k(x_i, t_i) P_{x_2}^{x_1} ... P_{x_1}^{x_k} e^{-\sum t_i} dt_1 ... dt_k$$

The expression on the right side, applied to any pointed loop functional defines a different measure on pointed loops, we will denote by μ^*. It induces the same measure as μ on loops.

We see on this expression that conditionally to the discrete loop, the holding times of the loop are independent exponential variables.

$$\mu^*(p = k, \xi_i = x_i, \tau_i^* \in dt_i) = \frac{1}{k} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} C_{\xi_i, \xi_{i+1}} e^{-t_i} dt_i \tag{6}$$

Conditionally to $p(\xi) = k$, T is a gamma variable of density $\frac{t^{k-1}}{(k-1)!} e^{-t}$ on $\mathbb{R}_+$ and $(\frac{\tau_i^*}{T},\ 1 \leq i \leq k)$ an independent ordered k-sample of the uniform distribution on $(0, T)$ (whence the factor $\frac{1}{t}$). Both are independent, conditionally to p of the discrete loop. We see that μ, on based loops, is obtained from μ on the loops by choosing the based point uniformly. On the other hand, it induces a choice of ξ_1 biased by the size of the τ_i^*'s, different of μ^* (whence the factor $\frac{1}{k}$. But we will consider only loop functionals.

It will be convenient to rescale the holding time at each ξ_i by λ_{ξ_i} and set $\widehat{\tau}_i = \frac{\tau_i^*}{\lambda_{\xi_i}}$.

The discrete part of the loop is the most important, though we will see that to establish a connection with Gaussian fields it is necessary to consider occupation times. The

simplest variables are the number of jumps from x to y, defined for every oriented edge (x,y)

$$N_{x,y} = \#\{i : \xi_i = x, \xi_{i+1} = y\}$$

(recall the convention $\xi_{p+1} = \xi_1$) and

$$N_x = \sum_y N_{x,y}$$

Note that $N_x = \#\{i \geq 1 : \xi_i = x\}$ except for trivial one point loopsfor which it vanishes.

Then, the measure on pointed loops (5) can be rewritten as:

$$\mu^*(p = 1, \xi = x, \widehat{\tau} \in dt) = e^{-\lambda_x t}\frac{dt}{t} \text{ and} \tag{7}$$

$$\mu^*(p = k, \xi_i = x_i, \widehat{\tau_i} \in dt_i) = \frac{1}{k}\prod_{x,y} C_{x,y}^{N_{x,y}} \prod_x \lambda_x^{-N_x} \prod_{i\in\mathbb{Z}/p\mathbb{Z}} \lambda_{\xi_i} e^{-\lambda_{\xi_i} t_i} dt_i \tag{8}$$

Another *bridge measure* $\mu^{x,y}$ can be defined on paths γ from x to y: $\mu^{x,y}(d\gamma) = \int_0^\infty \mathbb{P}_t^{x,y}(d\gamma)dt$.
Note that the mass of $\mu^{x,y}$ is $G^{x,y}$. We also have, with similar notations as the one defined for loops, p denoting the number of jumps

$$\begin{aligned}\mu^{x,y}(p(\gamma) = k, \gamma_{T_1} = x_1, ..., \gamma_{T_{k-1}} = x_{k-1}, T_1 \in dt_1, ..., T_{k-1} \in dt_{k-1}, T \in dt) \\ = \frac{C_{x,x_2}C_{x_2,x_3}...C_{x_{k-1},y}}{\lambda_x\lambda_{x_2}...\lambda_y} 1_{\{0<t_1<...<t_k<t\}} e^{-t} dt_1...dt_k dt\end{aligned}$$

Exercise 9 *For any $x \neq y$ in X and $s \in [0,1]$, setting $P_v^{(s),u} = P_v^u$ if $(u,v) \neq (x,y)$ and $P_y^{(s),x} = sP_y^x$, prove that:*

$$\mu(s^{N_{x,y}} 1_{\{p>1\}}) = -\log(\det(I - P^{(s)}))$$

Differentiating in $s = 1$, show that

$$\mu(N_{x,y}) = [(I - P)^{-1}]_x^y P_y^x = G^{x,y}C_{x,y}$$

and $\mu(N_x) = \sum_y \mu(N_{x,y}) = \lambda_x G^{x,x} - 1$ (as $(M_\lambda - C)G = Id$).

3.4 Occupation field

To each loop l° we associate local times, i.e. an occupation field $\{\widehat{l_x}, x \in X\}$ defined by

$$\widehat{l^x} = \int_0^{T(l)} 1_{\{\xi(s)=x\}} \frac{1}{\lambda_{\xi(s)}} ds = \sum_{i=1}^{p(l)} 1_{\{\xi_i=x\}} \widehat{\tau_i}$$

for any representative $l = (\xi_i, \tau_i^*)$ of l°.

For a path γ, $\widehat{\gamma}$ is defined in the same way.
Note that

$$\mu((1 - e^{-\alpha \widehat{l^x}}) 1_{\{p=1\}}) = \int_0^\infty e^{-t}(1 - e^{-\frac{\alpha}{\lambda_x} t}) \frac{dt}{t} = \log(1 + \frac{\alpha}{\lambda_x}) \tag{9}$$

(by expanding $1 - e^{-\frac{\alpha}{\lambda_x} t}$ before the integration, assuming first α small and then by analyticity of both members, or more elegantly, noticing that $\int_a^b (e^{-cx} - e^{-dx}) \frac{dx}{x}$ is symmetric in (a, b) and (c, d)).
In particular, $\mu(\widehat{l^x} 1_{\{p=1\}}) = \frac{1}{\lambda_x}$.

From formula 5, we get easily that the joint conditional distribution of $(\widehat{l^x},\ x \in X)$ given $(N_x,\ x \in X)$ is a product of gamma distributions. In particular, from the expression of the moments of a gamma distribution, wee get that for any function Φ of the discrete loop and $k \geq 1$,

$$\mu((\widehat{l^x})^k 1_{\{p>1\}} \Phi) = \lambda_x^{-k} \mu((N_x + k - 1)...(N_x + 1) N_x \Phi)$$

In particular, $\mu(\widehat{l^x}) = \frac{1}{\lambda_x}[\mu(N_x) + 1] = G^{x,x}$.

Note that functions of $\widehat{l}$ are not the only functions naturally defined on the loops. Other such variables of interest are, for $n \geq 2$, the multiple local times, defined as follows:

$$\widehat{l^{x_1,...,x_n}} = \sum_{j=0}^{n-1} \int_{0<t_1<...<t_n<T} 1_{\{\xi(t_1)=x_{1+j},....\xi(t_{n-j})=x_n,...\xi(t_n)=x_j\}} \prod \frac{1}{\lambda_{x_i}} dt_i$$

It is easy to check that, when the points x_i are distinct,

$$\widehat{l^{x_1,...,x_n}} = \sum_{j=0}^{n-1} \sum_{1 \leq i_1 < .. < i_n \leq p(l)} \prod_{l=1}^{n} 1_{\{\xi_{i_l}=x_{l+j}\}} \widehat{\tau_{i_l}}. \tag{10}$$

Note that in general $\widehat{l^{x_1,...,x_k}}$ cannot be expressed in terms of $\widehat{l}$.

If $x_1 = x_2 = \ldots = x_n$, $\widehat{l}^{x_1,\ldots,x_n} = \frac{1}{(n-1)!}[\widehat{l}^x]^n$. It can be viewed as a n-th self intersection local time.

One can deduce from the defintions of μ the following:

Proposition 10 $\mu(\widehat{l}^{x_1,\ldots,x_n}) = G^{x_1,x_2}G^{x_2,x_3}...G^{x_n,x_1}$

Proof. Let us denote $\frac{1}{\lambda_y}\ [P_t]_y^x$ by $p_t^{x,y}$ or $p_t(x,y)$. From the definition of $\widehat{l}^{x_1,\ldots,x_n}$ and μ, $\mu(\widehat{l}^{x_1,\ldots,x_n})$ equals:

$$\sum_x \lambda_x \sum_{j=0}^{n-1} \int \int_{\{0<t_1\ldots<t_n<t\}} \frac{1}{t} p_{t_1}(x, x_{1+j}) \ldots p_{t-t_n}(x_{n+j}, x) \prod dt_i dt$$

where sums of indices $k+j$ are computed mod(n). By the semigroup property, it equals

$$\sum_{j=0}^{n-1} \int \int_{\{0<t_1<\ldots<t_n<t\}} \frac{1}{t} p_{t_2-t_1}(x_{1+j}, x_{2+j}) \ldots p_{t_1+t-t_n}(x_{n+j}, x_{1+j}) \prod dt_i dt.$$

Performing the change of variables $v_2 = t_2 - t_1, .., v_n = t_n - t_{n-1}, v_1 = t_1 + t - t_n$, and $v = t_1$, we obtain:

$$\begin{aligned}
&\sum_{j=0}^{n-1} \int_{\{0<v<v_1, 0<v_i\}} \frac{1}{v_1 + ... + v_n} p_{v_2}(x_{1+j}, x_{2+j}) \ldots p_{v_1}(x_{n+j}, x_{1+j}) \prod dv_i dv \\
&\qquad = \sum_{j=0}^{n-1} \int_{\{0<v_i\}} \frac{v_1}{v_1 + ... + v_n} p_{v_2}(x_{1+j}, x_{2+j}) \ldots\ldots p_{v_1}(x_{n+j}, x_{1+j}) \prod dv_i \\
&\qquad = \sum_{j=1}^{n} \int_{\{0<v_i\}} \frac{v_j}{v_1 + ... + v_n} p_{v_2}(x_1, x_2) \ldots p_{v_1}(x_n, x_1) \prod dv_i \\
&\qquad = \int_{\{0<v_i\}} p_{v_2}(x_1, x_2) \ldots p_{v_1}(x_n, x_1) \prod dv_i \\
&\qquad = G^{x_1,x_2}G^{x_2,x_3}...G^{x_n,x_1}.
\end{aligned}$$

Note that another proof can be derived from formula (10) ■

Exercise 11 *For $x_1 = x_2 = ... = x_k$, we could define different self intersection local times*

$$\widehat{l}^{x,(k)} = \sum_{1 \le i_1 < .. < i_k \le p(l)} \prod_{l=1}^{k} 1_{\{\xi_{i_l} = x\}} \widehat{\tau}_{i_l}$$

which vanish on $N_x < k$. Note that

$$\widehat{l}^{x,(2)} = \frac{1}{2}((\widehat{l}^x)^2 - \sum_{i=1}^{p(l)} 1_{\{\xi_i = x\}}(\widehat{\tau}_i)^2.$$

1. *For any function Φ of the discrete loop, show that*

$$\mu(\widehat{l}^{x,2}\Phi) = \lambda_x^{-2}\mu(\frac{N_x(N_x-1)}{2}1_{\{N_x \geq 2\}}\Phi) \text{ and } \mu(\Phi \sum_{i=1}^{p(l)} 1_{\{\xi_i = x\}}(\widehat{\tau}_i)^2)) = 2\lambda_x^{-2}\mu(\Phi N_x).$$

2. *More generally prove in a similar way that*

$$\mu(\widehat{l}^{x,(k)}\Phi) = \lambda_x^{-k}\mu(\frac{N_x(N_x-1)...(N_x-k+1)}{k!}1_{\{N_x \geq k\}}\Phi).$$

Let us come back to the occupation field to compute its Laplace transform. From the Feynman-Kac formula, it comes easily that, denoting $M_{\frac{\chi}{\lambda}}$ the diagonal matrix with coefficients $\frac{\chi_x}{\lambda_x}$

$$\mathbb{P}_t^{x,x}(e^{-\langle \widehat{l},\chi\rangle} - 1) = \frac{1}{\lambda_x}(\exp(t(P - I - M_{\frac{\chi}{\lambda}}))_x^x - \exp(t(P-I))_x^x).$$

Integrating in t after expanding, we get from the definition of μ (first for χ small enough):

$$\begin{aligned}\int (e^{-\langle \widehat{l},\chi\rangle} - 1)d\mu(l) &= \sum_{k=1}^{\infty}\int_0^{\infty}[Tr((P - M_{\frac{\chi}{\lambda}})^k) - Tr((P)^k)]\frac{t^{k-1}}{k!}e^{-t}dt \\ &\sum_{k=1}^{\infty}\frac{1}{k}[Tr((P - M_{\frac{\chi}{\lambda}})^k) - Tr((P)^k)] \\ &= -Tr(\log(I - P + M_{\frac{\chi}{\lambda}})) + Tr(\log(I-P))\end{aligned}$$

Hence, as $Tr(\log) = \log(\det)$

$$\int (e^{-\langle \widehat{l},\chi\rangle} - 1)d\mu(l) = \log[\det(-L(-L + M_{\chi/\lambda})^{-1})] = -\log\det(I + VM_{\frac{\chi}{\lambda}})$$

which now holds for all non negative χ as both members are analytic in χ. Besides, by the "resolvent" equation:

$$\det(I + GM_\chi)^{-1} = \det(I - G_\chi M_\chi) = \frac{\det(G_\chi)}{\det(G)} \tag{11}$$

Note that $\det(I+GM_\chi) = \det(I+M_{\sqrt{\chi}}GM_{\sqrt{\chi}})$ and $\det(I-G_\chi M_\chi) = \det(I-M_{\sqrt{\chi}}G_\chi M_{\sqrt{\chi}})$, so we can deal with symmetric matrices. Finally we have the

Proposition 12 $\mu(e^{-\langle \widehat{l},\chi\rangle} - 1) = -\log(\det(I + M_{\sqrt{\chi}}GM_{\sqrt{\chi}})) = \log(\frac{\det(G_\chi)}{\det(G)})$

Note that in particular $\mu(e^{-t\widehat{l}^x} - 1) = -\log(1 + tG^{x,x})$.

Note finally that if χ has support in D, by the restriction property

$$\mu(1_{\{\widehat{l}(X\setminus D)=0\}}(e^{-<\widehat{l},\chi>} - 1)) = -\log(\det(I + M_{\sqrt{\chi}}G^D M_{\sqrt{\chi}})) = \log(\frac{\det(G^D_\chi)}{\det(G^D)})$$

Here the determinants are taken on matrices indexed by D and G^D the Green function of the process killed on leaving D.

For paths we have $\mathbb{P}^{x,y}_t(e^{-\langle \widehat{l},\chi\rangle}) = \frac{1}{\lambda_y}\exp(t(L - M_{\frac{\chi}{\lambda}}))_{x,y}$. Hence

$$\mu^{x,y}(e^{-\langle \widehat{\gamma},\chi\rangle}) = \frac{1}{\lambda_y}((I - P + M_{\chi/m})^{-1})_{x,y} = [G_\chi]^{x,y}.$$

Also $\mathbb{E}^x(e^{-\langle \widehat{\gamma},\chi\rangle}) = \sum_y [G_\chi]^{x,y}\kappa_y$ i.e.$[G_\chi\kappa]^x$.

4 Poisson process of loops

4.1 Definition

Still following the idea of [14], which was already implicitly in germ in [33], define, for all positive α, the Poissonian ensemble of loops $\mathcal{L}_\alpha$ with intensity $\alpha\mu$. We denote by $\mathbb{P}$ or $\mathbb{P}_{\mathcal{L}_\alpha}$ its distribution.

Recall it means that for any functional Φ on the loop space, vanishing on loops of arbitrary small length,

$$E(e^{i\sum_{l\in\mathcal{L}_\alpha}\Phi(l)} = \exp(\alpha\int(e^{i\Phi(l)} - 1)\mu(dl))$$

Note that by the restriction property, $\mathcal{L}^D_\alpha = \{l \in \mathcal{L}_\alpha, l \subseteq D\}$ is a Poisson process of loops with intensity μ^D, and that $\mathcal{L}^D_\alpha$ is independent of $\mathcal{L}_\alpha\setminus\mathcal{L}^D_\alpha$.

We denote by $\mathcal{DL}_\alpha$ the set of non trivial discrete loops in $\mathcal{L}_\alpha$. Then,

$$\mathbb{P}(\mathcal{DL}_\alpha = \{l_1, l_2, ...l_k\}) = e^{-\alpha\mu(p>0)}\alpha^k\mu(l_1)...\mu(l_k) = \alpha^k[\frac{\det(G)}{\prod_x \lambda_x}]^\alpha \prod_{x,y} C_{x,y}^{N_{x,y}^{(\alpha)}} \prod_x \lambda_x^{-N_x^{(\alpha)}}$$

with $N_x^{(\alpha)} = \sum_{l\in\mathcal{L}_\alpha} N_x(l)$ and $N_{x,y}^{(\alpha)} = \sum_{l\in\mathcal{L}_\alpha} N_{x,y}(l)$, when these loops are distinct.

We can associate to $\mathcal{L}_\alpha$ a σ-finite measure (in fact as we will see, finite when X is finite, and more generally if G is trace class) called local time or occupation field

$$\widehat{\mathcal{L}_\alpha} = \sum_{l\in\mathcal{L}_\alpha} \widehat{l}$$

Then, for any non-negative measure χ on X

$$\mathbb{E}(e^{-\langle\widehat{\mathcal{L}_\alpha},\chi\rangle}) = \exp(\alpha \int (e^{-\langle\widehat{l},\chi\rangle} - 1)d\mu(l))$$

and therefore by proposition 12 we have

Corollary 13 $\mathbb{E}(e^{-\langle\widehat{\mathcal{L}_\alpha},\chi\rangle}) = \det(I + M_{\sqrt{\chi}}GM_{\sqrt{\chi}})^{-\alpha} = (\frac{\det(G_\chi)}{\det(G)})^\alpha$

Many calculations follow from this result.

It follows that $\mathbb{E}(e^{-t\widehat{\mathcal{L}_\alpha}^x}) = (1+tG^{x,x})^{-\alpha}$. We see that $\widehat{\mathcal{L}_\alpha}^x$ follows a gamma distribution $\Gamma(\alpha, G^{x,x})$, with density $1_{\{x>0\}}\frac{e^{-\frac{x}{G^{xx}}}}{\Gamma(\alpha)}\frac{x^{\alpha-1}}{(G^{xx})^\alpha}$ (in particular, an exponential distribution of mean $G^{x,x}$ for $\alpha = 1$). When we let α vary as a time parameter, we get a family of gamma subordinators, which can be called a "multivariate gamma subordinator".

We check in particular that $\mathbb{E}(\widehat{\mathcal{L}_\alpha}^x) = \alpha G^{x,x}$ which follows directly from $\mu(\widehat{l}_x) = G^{x,x}$.

Note also that for $\alpha > 1$,

$$\mathbb{E}((1 - \exp(-\frac{\widehat{\mathcal{L}_\alpha}^x}{G^{x,x}}))^{-1}) = \zeta(\alpha).$$

More generally, for two points:

$$\mathbb{E}(e^{-t\widehat{\mathcal{L}_\alpha}^x}e^{-s\widehat{\mathcal{L}_\alpha}^y}) = ((1 + tG^{x,x})(1 + sG^{y,y}) - st(G^{x,y})^2)^{-\alpha}$$

This allows to compute the joint density of $\widehat{\mathcal{L}_\alpha}^x$ and $\widehat{\mathcal{L}_\alpha}^y$ in terms of Bessel and Struve functions.

We can condition the loops by the set of associated non trivial discrete loop by using the restricted σ-field $\sigma(\mathcal{DL}_\alpha)$ which contains the variables $N_{x,y}$. We see from 9 and 7 that

$$\mathbb{E}(e^{-\langle \widehat{\mathcal{L}_\alpha}, \chi \rangle} | \mathcal{DL}_\alpha) = \prod_x (\frac{\lambda_x}{\lambda_x + \chi_x})^{N_x^{(\alpha)}+1}$$

The distribution of $\{N_x^{(\alpha)}, x \in X\}$ follows easily, from corollary 13 in terms of generating functions:

$$\mathbb{E}(\prod_x (s_x^{N_x^{(\alpha)}+1}) = \det(\delta_{x,y} + \sqrt{\frac{\lambda_x \lambda_y (1-s_x)(1-s_y)}{s_x s_y}} G_{x,y})^{-\alpha} \tag{12}$$

so that the vector of components $N_x^{(\alpha)}$ follows a multivariate negative binomial distribution (see for example [35]).

It follows in particular that $N_x^{(\alpha)}$ follows a negative binomial distribution of parameters $-\alpha$ and $\frac{1}{\lambda_x G^{xx}}$. Note that for $\alpha = 1$, $N_x^{(1)} + 1$ follows a geometric distribution of parameter $\frac{1}{\lambda_x G^{xx}}$.

4.2 Moments and polynomials of the occupation field

It is easy to check (and well known from the properties of the gamma distributions) that the moments of $\widehat{\mathcal{L}_\alpha}^x$ are related to the factorial moments of $N_x^{(\alpha)}$:

$$\mathbb{E}((\widehat{\mathcal{L}_\alpha}^x)^k | \mathcal{DL}_\alpha) = \frac{(N_x^{(\alpha)} + k)(N_x^{(\alpha)} + k - 1)...(N_x^{(\alpha)} + 1)}{k! \lambda_x^k}$$

Exercise 14 *Denoting $\mathcal{L}_\alpha^+$ the set of non trivial loops in $\mathcal{L}_\alpha$, define*

$$\widehat{\mathcal{L}_\alpha}^{x,(k)} = \sum_{m=1}^{k} \sum_{k_1 + ... + k_m = k} \sum_{l_1 \neq l_2 ... \neq l_m \in \mathcal{L}_\alpha^+} \prod_{j=1}^{m} \widehat{l_j}^{x,(k_j)}.$$

Deduce from exercise 11 that $\mathbb{E}(\widehat{\mathcal{L}_\alpha}^{x,(k)} | \mathcal{DL}_\alpha) = \frac{1}{k! \lambda_x^k} 1_{\{N_x \geq k\}} (N_x^{(\alpha)} - k + 1)...(N_x^{(\alpha)} - 1) N_x^{(\alpha)}$

It is well known that Laguerre polynomials $L_k^{(\alpha-1)}$ with generating function

$$\sum_0^\infty t^k L_k^{(\alpha-1)}(u) = \frac{e^{-\frac{ut}{1-t}}}{(1-t)^\alpha}$$

are orthogonal for the $\Gamma(\alpha,1)$ distribution with density $\frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}1_{\{u>0\}}$. They have mean zero and variance $\frac{\Gamma(\alpha+k)}{k!}$. Hence if we set $\sigma_x = G^{x,x}$and $P_k^{\alpha,\sigma}(x) = (-\sigma)^k L_k^{(\alpha-1)}(\frac{x}{\sigma})$, the random variables $P_k^{\alpha,\sigma_x}(\widehat{\mathcal{L}_\alpha}^x)$ are orthogonal with mean 0 and variance $\sigma^{2k}\frac{\Gamma(\alpha+k)}{k!}$, for $k>0$.

Note that $P_1^{\alpha,\sigma_x}(\widehat{\mathcal{L}_\alpha}^x) = \widehat{\mathcal{L}_\alpha}^x - \alpha\sigma_x = \widehat{\mathcal{L}_\alpha}^x - \mathbb{E}(\widehat{\mathcal{L}_\alpha}^x)$. *It will be denoted* $\widetilde{\mathcal{L}_\alpha}^x$.

Moreover, we have $\sum_0^\infty t^k P_k^{\alpha,\sigma}(u) = \sum(-\sigma t)^k L_k^{(\alpha-1)}(\frac{u}{\sigma}) = \frac{e^{\frac{ut}{1+\sigma t}}}{(1+\sigma t)^\alpha}$

Note that

$$\mathbb{E}(\frac{e^{\frac{\widehat{\mathcal{L}_\alpha}^x t}{1+\sigma_x t}}}{(1+\sigma_x t)^\alpha}\frac{e^{\frac{\widehat{\mathcal{L}_\alpha}^y s}{1+\sigma_y s}}}{(1+\sigma_y s)^\alpha})$$
$$= \frac{1}{(1+\sigma_x t)^\alpha(1+\sigma_y s)^\alpha}((1-\frac{\sigma_x t}{1+\sigma_x t})(1-\frac{\sigma_y s}{1+\sigma_y s}) - \frac{t}{1+\sigma_x t}\frac{s}{1+\sigma_y s}((G^{x,y})^2)^{-\alpha}$$
$$= (1-st(G^{x,y})^2)^{-\alpha}.$$

Therefore, we get, by developping in entire series in (s,t) and identifying the coefficients:

$$\mathbb{E}(P_k^{\alpha,\sigma_x}(\widehat{\mathcal{L}_\alpha}^x), P_l^{\alpha,\sigma_y}(\widehat{\mathcal{L}_\alpha}^y)) = \delta_{k,l}(G^{x,y})^{2k}\frac{\alpha(\alpha+1)...(\alpha+k-1)}{k!} \tag{13}$$

Let us stress the fact that $G^{x,x}$ and $G^{y,y}$ do not appear on the right side of this this formula. This is quite important from the renormalisation point of view, as we will consider in the last section the two dimensional Brownian motion for which the Green function diverges on the diagonal.

More generally one can prove similar formulas for products of higher order.

Note that since $G_\chi M_\chi$ is a contraction, from determinant expansions given in [34] and [35], we have

$$\det(I + M_{\sqrt{\chi}}GM_{\sqrt{\chi}})^{-\alpha} = 1 + \sum_{k=1}^{\infty}(-1)^k \sum \chi_{i_1}...\chi_{i_k} Per_\alpha(G_{i_l,i_m}, 1\le l,m\le k) \tag{14}$$

and then, from corollary 13, it comes that:

$$\mathbb{E}(\left\langle \widehat{\mathcal{L}_\alpha}, \chi \right\rangle^k) = \sum \chi_{i_1}...\chi_{i_k} Per_\alpha(G_{i_l,i_m}, 1\le l,m\le k)$$

Here the α-permanent Per_a is defined as $\sum_{\sigma\in\mathcal{S}_k}\alpha^{m(\sigma)}G_{i_1,i_{\sigma(1)}}...G_{i_k,i_{\sigma(k)}}$ with $m(\sigma)$ denoting the number of cycles in σ.

Note that from this determinant expansion follows directly (see [35]) an explicit form for the multivariate negative binomial distribution, and therefore, a series expansion for the density of the multivariate gamma distribution.

It is actually not difficult to give a direct proof of this result. Thus, the Poisson process of loops provides a natural probabilistic proof and interpretation of this combinatorial identity (see [35] for an historical view of the subject).

We can show in fact that:

Proposition 15 *For any* $(i_1, ...i_k)$ *in* X^k, $\mathbb{E}(\widehat{\mathcal{L}_\alpha}^{i_1}...\widehat{\mathcal{L}_\alpha}^{i_k}) = Per_\alpha(G^{i_l,i_m}, 1 \leq l, m \leq k)$

Proof. The cycles of the permutations in the expression of Per_α are associated with point configurations on loops. We obtain the result by summing the contributions of all possible partitions of the points $i_1...i_k$ into a finite set of distinct loops. We can then decompose again the expression according to ordering of points on each loop. We can conclude by using the formula $\mu(\widehat{l^{x_1,...,x_m}}) = G^{x_1,x_2}G^{x_2,x_3}...G^{x_m,x_1}$ and the following property of Poisson measures (Cf formula 3-13 in [11]): For any system of non negative loop functionals F_i

$$\mathbb{E}(\sum_{l_1 \neq l_2 ... \neq l_k \in \mathcal{L}_\alpha} \prod F_i(l_i)) = \prod \alpha\mu(F_i)$$

■

Remark 16 *We can actually check this formula in the special case* $i_1 = i_2 = ... = i_k = x$. *From the moments of the Gamma distribution, we have that* $\mathbb{E}((\widehat{\mathcal{L}_\alpha}^{x})^n) = (G^{x,x})^n\alpha(\alpha + 1)...(\alpha+n-1)$ *and the* α*-permanent writes* $\sum_1^n d(n,k)\alpha^k$ *where the coefficients* $d(n,k)$ *are the numbers of* $n-$*permutations with* k *cycles (Stirling numbers of the first kind). One checks that* $d(n+1,k) = nd(n,k) + d(n,k-1)$.

Let $\mathcal{S}_k^0$ be the set of permutations of k elements without fixed point. They correspond to configurations without isolated point.

Set $Per_\alpha^0(G^{i_l,i_m}, 1 \leq l, m \leq k) = \sum_{\sigma \in \mathcal{S}_k^0} \alpha^{m(\sigma)} G^{i_1,i_{\sigma(1)}}...G^{i_k,i_{\sigma(k)}}$. Then an easy calculation shows that:

Corollary 17 $\mathbb{E}(\widetilde{\mathcal{L}_\alpha}^{i_1}...\widetilde{\mathcal{L}_\alpha}^{i_k}) = Per^0_\alpha(G^{i_l,i_m}, 1 \le l,m \le k)$

Proof. Indeed, the expectation writes

$$\sum_{p\le k} \sum_{I\subseteq\{1,...k\},|I|=p} (-1)^{k-p} \prod_{l\in I^c} G^{i_l,i_l} Per_\alpha(G^{i_a,i_b}, a,b \in I)$$

and

$$Per_\alpha(G^{i_a,i_b}, a,b \in I) = \sum_{J\subseteq I} \prod_{j\in I\setminus J} G^{j,j} Per^0_\alpha(G^{i_a,i_b}, a,b \in J).$$

Then, expressing $\mathbb{E}(\widetilde{\mathcal{L}_\alpha}^{i_1}...\widetilde{\mathcal{L}_\alpha}^{i_k})$ in terms of Per^0_α's, we see that if $J \subseteq \{1,...k\}$, $|J| < k$, the coefficient of $Per^0_\alpha(G^{i_a,i_b}, a,b \in J)$ is $\sum_{I,I\supseteq J} (-1)^{k-|I|} \prod_{j\in J^c} G^{i_j,i_j}$ which vanishes as $(-1)^{-|I|} = (-1)^{|I|} = (-1)^{|J|}(-1)^{|I\setminus J|}$ and $\sum_{I\supseteq J}(-1)^{|I\setminus J|} = (1-1)^{k-|J|} = 0$. ■

Set $Q^{\alpha,\sigma}_k(u) = P^{\alpha,\sigma}_k(u+\alpha\sigma)$ so that $P^{\alpha,\sigma}_k(\widehat{\mathcal{L}_\alpha}^x) = Q^{\alpha,\sigma}_k(\widetilde{\mathcal{L}_\alpha}^x)$. This quantity will be called the n-th renormalized self intersection local time or the n-th renormalized power of the occupation field and denoted $\widetilde{\mathcal{L}}^{x,n}_\alpha$.

From the recurrence relation of Laguerre polynomials

$$nL^{(\alpha-1)}_n(u) = (-u+2n+\alpha-2)L^{(\alpha-1)}_{n-1} - (n+\alpha-2)L^{(\alpha-1)}_{n-2},$$

we get that

$$nQ^{\alpha,\sigma}_n(u) = (u-2\sigma(n-1))Q^{\alpha,\sigma}_{n-1}(u) - \sigma^2(\alpha+n-2)Q^{\alpha,\sigma}_{n-2}(u)$$

In particular $Q^{\alpha,\sigma}_2(u) = \frac{1}{2}(u^2 - 2\sigma u - \alpha\sigma^2)$.
We have also, from (13)

$$\mathbb{E}(Q^{\alpha,\sigma_x}_k(\widetilde{\mathcal{L}_\alpha}^x), Q^{\alpha,\sigma_y}_l(\widetilde{\mathcal{L}_\alpha}^y)) = \delta_{k,l}(G^{x,y})^{2k}\frac{\alpha(\alpha+1)...(\alpha+k-1)}{k!} \tag{15}$$

The comparison of the identity (15) and corollary 17 yields a combinatorial result which will be fundamental in the renormalizing procedure presented in the last section.

The identity (15) can be considered as a polynomial identity in the variables σ_x, σ_y and $G^{x,y}$.

If $Q^{\alpha,\sigma_x}_k(u) = \sum_{m=0}^k q^{\alpha,k}_m u^m \sigma_x^{k-m}$, if we denote $N_{n,m,r,p}$ the number of ordered configurations of n black points and m red points on r non trivial oriented cycles, such that only $2p$ links are between red and black points, we have

$$\mathbb{E}((\widetilde{\mathcal{L}_\alpha}^x)^n(\widetilde{\mathcal{L}_\alpha}^y)^m) = \sum_r \sum_{p\leq \inf(m,n)} \alpha^r N_{n,m,r,p}(G^{x,y})^{2p}(\sigma_x)^{n-p}(\sigma_y)^{m-p}$$

and therefore

$$\sum_r \sum_{p\leq m\leq k} \sum_{p\leq n\leq l} \alpha^r q_m^{\alpha,k} q_n^{\alpha,l} N_{n,m,r,p} = 0 \,\text{unless}\, p=l=k. \tag{16}$$

$$\sum_r \alpha^r q_k^{\alpha,k} q_k^{\alpha,k} N_{k,k,r,k} = \frac{\alpha(\alpha+1)...(\alpha+k-1)}{k!} \tag{17}$$

Note that one can check directly that $q_k^{\alpha,k} = \frac{1}{k!}$, and $N_{k,k,1,k} = k!(k-1)!$, $N_{k,k,k,k} = k!$ which confirms the identity (17) above.

4.3 Hitting probabilities

Let $[H^F]_y^x = \mathbb{P}_x(x_{T_F} = y)$ be the hitting distribution of F by the Markov chain starting at x (H^F is called the balayage or Poisson kernel). Set $D = F^c$ and denote e^D, $P^D = P)|_{D\times D}$, $V^D = [(I-P^D)]^{-1}$ and $G^D = [(M_\lambda - C)|_{D\times D}]^{-1}$ the energy, the transtion matrix, the potential and the Green function of the process killed at the hitting of F. Recall that

$[H^F]_y^x = 1_{\{x=y\}} + \sum_0^\infty \sum_{z\in D}[(P^D)^k]_z^x P_y^z = 1_{\{x=y\}} + \sum_0^\infty \sum_{z\in D}[V^D]_z^x P_y^z$. Moreover we have by the strong Markov property, $V = V^D + H^F V$ and therefore $G = G^D + H^F G$. (Here we extend V^D and G^D to $X\times X$ by adding zero entries outside $D\times D$).

As G and G^D are symmetric, we have $[H^F G]_y^x = [H^F G]_x^y$ so that for any measure ν, $H^F(G\nu) = G(\nu H^F)$.

Therefore we see that for any function f and measure ν, $e(H^F f, G^D\nu) = e(H^F f, G\nu) - e(H^F f, H^F G\nu) = \langle H^F f, \nu\rangle - e(H^F f, G(H^F\nu)) = 0$ as $(H^F)^2 = H^F$.

Equivalently, we have the following:

Proposition 18 *For any g vanishing on F, $e(H^F f, g) = 0$ so that $I - H^F$ is the e-orthogonal projection on the space of functions supported in D.*

Note that these results extend without difficulty to the recurrent case. In particular, for any measure ν supported in D, $G^D\nu = G(\nu - \nu H^F)$ and $e(H^F f, G^D\nu) = 0$ for all f. For further developments see for exemple ([17]) and its references.

The restriction property holds for $\mathcal{L}_\alpha$ as it holds for μ. The set $\mathcal{L}_\alpha^D$ of loops inside D is associated with μ^D and independent of $\mathcal{L}_\alpha - \mathcal{L}_\alpha^D$. Therefore, we see from corollary 13 that

$$\mathbb{E}(e^{-\left\langle \widehat{\mathcal{L}_\alpha} - \widehat{\mathcal{L}_\alpha^D}, \chi \right\rangle}) = (\frac{\det(G_\chi)}{\det(G)} \frac{\det(G^D)}{\det(G_\chi^D)})^\alpha.$$

From the support of the of the Gamma distribution, we see that $\mu(\widehat{l}(F) > 0) = \infty$. But this is clearly due to trivial loops as it can be seen directly from the definition of μ that in this simple framework they cover the whole space X.

Note however that

$$\begin{aligned}\mu(\widehat{l}(F) > 0, p > 1) &= \mu(p > 1) - \mu(\widehat{l}(F) = 0, p > 1) = \mu(p > 1) - \mu^D(p > 1) \\ &= -\log(\frac{\det(I-P)}{\det_{D\times D}(I-P)}) = -\log(\frac{\det(G^D)}{\prod_{x\in F}\lambda_x \det(G)}).\end{aligned}$$

It follows that the probability no non trivial loop (i.e.a loop which is not reduced to a point) in $\mathcal{L}_\alpha$ intersects F equals

$$\exp(-\alpha\mu(\{l, p(l) > 1, \widehat{l}(F) > 0\})) = (\frac{\det(G^D)}{\prod_{x\in F}\lambda_x \det(G)})^\alpha.$$

Recall that for any $(n+p, n+p)$ invertible matrix A, denoting e_i the canonical basis,

$$\begin{aligned}\det(A^{-1})\det(A_{ij}, 1 \le i,j \le n) &= \det(A^{-1})\det(Ae_1, ...Ae_n, e_{n+1}, ...e_{n+p}) \\ &= \det(e_1, ...e_n, A^{-1}e_{n+1}, ...A^{-1}e_{n+p}) \\ &= \det((A^{-1})_{k,l}, n \le k, l \le n+p).\end{aligned}$$

In particular, $\det(G^D) = \dfrac{\det(G)}{\det(G|_{F\times F})}$, so we have the

Proposition 19 *The probability that no non trivial loop in $\mathcal{L}_\alpha$ intersects F equals*

$$[\prod_{x\in F}\lambda_x \det_{F\times F}(G)]^{-\alpha}.$$

Moreover $\mathbb{E}(e^{-\left\langle \widehat{\mathcal{L}_\alpha} - \widehat{\mathcal{L}_\alpha^D}, \chi \right\rangle}) = (\dfrac{\det_{F\times F}(G_\chi)}{\det_{F\times F}(G)})^\alpha$

In particular, it follows that the probability no non trivial loop in $\mathcal{L}_\alpha$ visits x equals $(\frac{1}{\lambda_x G^{x,x}})^\alpha$ which is also aconsequence of the fact that N_x follows a negative binomial distribution of parameters $-\alpha$ and $\frac{1}{\lambda_x G^{x,x}}$

Also, if F_1 and F_2 are disjoint,

$$\begin{aligned}\mu(\widehat{l}(F_1)\widehat{l}(F_2) > 0) &= \mu(\widehat{l}(F_1) > 0, p > 1) + \mu(\widehat{l}(F_2) > 0, p > 1) - \mu(\widehat{l}(F_1 \cup F_2) > 0, p > 1) \\ &= \log(\frac{\det(G)\det(G^{D_1 \cap D_2})}{\det(G^{D_1})\det(G^{D_2})}).\end{aligned}$$

Therefore the probability no non trivial loop in $\mathcal{L}_\alpha$ intersects F_1 and F_2 equals

$$\exp(-\alpha\mu(\{l,p(l) > 1, \prod \widehat{l}(F_i) > 0\})) = (\frac{\det(G)\det(G^{D_1 \cap D_2})}{\det(G^{D_1})\det(G^{D_2})})^{-\alpha}$$

It follows that the probability no non trivial loop in $\mathcal{L}_\alpha$ visits two distinct points x and y equals $(\frac{G^{x,x}G^{y,y}-(G^{x,y})^2}{G^{x,x}G^{y,y}})^\alpha$ and in particular $1 - \frac{(G^{x,y})^2}{G^{x,x}G^{y,y}}$ if $\alpha = 1$.

Exercise 20 *Generalize this formula to n disjoint sets:*

$$\mathbb{P}(\nexists l \in \mathcal{L}_\alpha, \prod \widehat{l}(F_i) > 0) = \Big(\frac{\det(G)\prod_{i<j}\det(G^{D_i \cap D_j})}{\prod \det(G^{D_i})\prod_{i<j<k}\det(G^{D_i \cap D_j \cap D_k})}\Big)^{-\alpha}$$

Note this yields an interesting determinant product inequality.

5 The Gaussian free field

5.1 Dynkin's Isomorphism

By a well known calculation, if X is finite, for any $\chi \in \mathbb{R}_+^X$,

$$\frac{\sqrt{\det(M_\lambda - C)}}{(2\pi)^{|X|/2}} \int (e^{-\frac{1}{2}<z,\chi>} e^{-\frac{1}{2}e(z)} \Pi_{u \in X} dz^u = \sqrt{\frac{\det(G_\chi)}{\det(G)}}$$

and

$$\frac{\sqrt{\det(M_\lambda - C)}}{(2\pi)^{|X|/2}} \int z^x z^y (e^{-\frac{1}{2}<z^2,\chi>} e^{-\frac{1}{2}e(z)} \Pi_{u \in X} dz^u = (G_\chi)^{x,y} \sqrt{\frac{\det(G_\chi)}{\det(G)}}$$

This can be easily reformulated by introducing on an independent probability space the Gaussian field ϕ defined by the covariance $\mathbb{E}_\phi(\phi^x\phi^y) = G^{x,y}$ (this reformulation cannot be dispensed with when X becomes infinite)

So we have $\mathbb{E}_\phi((e^{-\frac{1}{2}<\phi^2,\chi>}) = \det(I + GM_\chi)^{-\frac{1}{2}} = \sqrt{\det(G_\chi G^{-1})}$ and

$\mathbb{E}_\phi((\phi^x\phi^y e^{-\frac{1}{2}<\phi^2,\chi>}) = (G_\chi)^{x,y}\sqrt{\det(G_\chi G^{-1})}$ Then as sums of exponentials of the form $e^{-\frac{1}{2}<\cdot,\chi>}$ are dense in continuous functions on $\mathbb{R}_+^X$ the following holds:

Theorem 21 *a) The fields $\widehat{\mathcal{L}}_{\frac{1}{2}}$ and $\frac{1}{2}\phi^2$ have the same distribution.*

b) $\mathbb{E}_\phi((\phi^x\phi^y F(\frac{1}{2}\phi^2)) = \int \mathbb{E}(F(\widehat{\mathcal{L}_1} + \widehat{\gamma}))\mu^{x,y}(d\gamma)$ for any bounded functional F of a non negative field.

Remarks:

a) This is a version of Dynkin's isomorphism (Cf [6]). It can be extended to non symmetric generators (Cf [19]).

b) An analogous result can be given when α is any positive half integer, by using real vector valued Gaussian field, or equivalently complex fields for integral values of α (in particular $\alpha = 1$): If $\phi_1, \phi_2 ... \phi_k$ are k independent copies of the free field, the fields $\widehat{\mathcal{L}}_{\frac{k}{2}}$ and $\frac{1}{2}\sum_1^k \phi_j^2$ have the same law.

c) Note it implies immediately that the process ϕ^2 is infinitely divisible. See [7] and its references for a converse and earlier proofs of this last fact.

Exercise 22 *For any f in the Dirichlet space $\mathbb{H}$ of functions of finite energy (i.e. all functions if X is finite), the law of $f + \phi$ is absolutely continuous with respect to the law of ϕ, with density* $\exp(< -Lf, \phi >_m - \frac{1}{2}e(f))$

Exercise 23 *a) Using proposition 18, show (it was observed by Nelson in the context of the classical (or Brownian) free field) that the Gaussian field ϕ is Markovian: Given any subset F of X, denote $\mathcal{H}_F$ the Gaussian space spanned by $\{\phi^y, y \in F\}$. Then, for $x \in D = F^c$, the projection of ϕ^x on $\mathcal{H}_F$ (i.e. the conditional expectation of ϕ^x given $\sigma(\phi^y, y \in F)$) is $\sum_{y\in F}[H^F]_y^x\phi^y$.*

b)Moreover, show that $\phi^D = \phi - H^F\phi$ is the Gaussian field associated with the process killed at the exit of D.

5.2 Fock spaces and Wick product

The Gaussian space $\mathcal{H}$ spanned by $\{\phi^x, x \in X\}$ is isomorphic to the Dirichlet space $\mathbb{H}$ by the linear map mapping ϕ^x on $G^{x,\cdot}$ which extends into an isomorphism between the space of square integrable functionals of the Gaussian fields and the symmetric Fock space obtained as the closure of the sum of all symmetric tensor powers of $\mathbb{H}$ (Bose second quantization: See [30], [25]). We have seen in theorem 21 that L^2 functionals of $\widehat{\mathcal{L}_1}$ can be represented in this symmetric Fock space.

In order to prepare the extension of these isomorphisms to the more difficult framework of continuous spaces (which can often be viewed as scaling limits of discrete spaces), including especially the planar Brownian motion considered in [14], we shall introduce the renormalized (or Wick) powers of ϕ. We set $:(\phi^x)^n: := (G^{x,x})^{\frac{n}{2}} H_n(\phi^x/\sqrt{G^{x,x}})$ where H_n in the n-th Hermite polynomial (characterized by $\sum \frac{t^n}{n!} H_n(u) = e^{tu-\frac{t^2}{2}}$). It is the inverse image of the n-th tensor power of $G^{x,\cdot}$ in the Fock space.

Setting as before $\sigma_x = G^{x,x}$, from the relation between Hermite polynomials H_{2n} and Laguerre polynomials $L_n^{-\frac{1}{2}}$,

$$H_{2n}(x) = (-2)^n n! L_n^{-\frac{1}{2}}(\frac{x^2}{2})$$

it comes that:

$$:(\phi^x)^{2n}: := 2^n n! P_n^{\frac{1}{2},\sigma}((\frac{(\phi^x)^2}{2}))$$

More generally, if $\phi_1, \phi_2 ... \phi_k$ are k independent copies of the free field, we can define $:\prod_{j=1}^k \phi_j^{n_j}: = \prod_{j=1}^k :\phi_j^{n_j}:$. Then it comes that:

$$:(\sum_1^k \phi_j^2)^n: := \sum_{n_1+..+n_k=n} \frac{n!}{n_1!...n_k!} \prod_{j=1}^k :\phi_j^{2n_j}:$$

From the generating function of the polynomials $P_n^{\frac{k}{2},\sigma}$,

$$P_n^{\frac{k}{2},\sigma}(\sum_1^k u_j) = \sum_{n_1+..+n_k=n} \frac{n!}{n_1!...n_k!} \prod_{j=1}^k P_{n_j}^{\frac{1}{2},\sigma}(u_j).$$

Therefore

$$P_n^{\frac{k}{2},\sigma}(\frac{\sum(\phi_j)^2}{2}) = \frac{1}{2^n n!} :(\sum_1^k \phi_j^2)^n: \tag{18}$$

Note that : $\sum_1^k \phi_j^2 := \sum_1^k \phi_j^2 - \sigma$ These variables are orthogonal in L^2. Let $\widetilde{l^x} = \widehat{l^x} - \sigma$ be the centered occupation field. Note that an equivalent formulation of theorem 21 is that the fields $\frac{1}{2} : \sum_1^k \phi_j^2 :$ and $\widetilde{\mathcal{L}}_{\frac{k}{2}}$ have the same law.

Let us now consider the relation of higher Wick powers with self intersection local times.

Recall that the renormalized n-th self intersections field $\widetilde{\mathcal{L}}_1^{x,n} = P_n^{\alpha,\sigma}(\widehat{\mathcal{L}_\alpha}^x) = Q_n^{\alpha,\sigma}(\widetilde{\mathcal{L}_\alpha}^x)$ have been defined by orthonormalization in L^2 of the powers of the occupation time.

Then comes the

Proposition 24 *The fields $\widetilde{\mathcal{L}}_{\frac{k}{2}}^{\cdot,n}$ and $: (\frac{1}{n!2^n}\sum_1^k \phi_j^2)^n :$ have the same law.*

This follows directly from (18).

Remark 25 *As a consequence, it can be shown that:*

$$\mathbb{E}(\prod_{j=1}^r Q_{k_j}^{\alpha,\sigma_{x_j}}(\widetilde{\mathcal{L}_\alpha}^{x_j})) = \sum_{\sigma \in \mathcal{S}_{k_1,k_2,...k_j}} (2\alpha)^{m(\sigma)} G^{i_1,i_{\sigma(1)}} ... G^{i_k,i_{\sigma(k)}}$$

where $\mathcal{S}_{k_1,k_2,...k_j}$ is the set of permutations σ of $k = \sum k_j$ such that
$\sigma(\{\sum_1^{j-1} k_l + 1, ... \sum_1^{j-1} k_l + k_j\} \cap \{\sum_1^{j-1} k_l + 1, ... \sum_1^{j-1} k_l + k_j\}$ *is empty for all j.*

The identity follows from Wick's theorem when α is a half integer, then extends to all α since both members are polynomials in α. The condition on σ indicates that no pairing is allowed inside the same Wick power.

6 Energy variation and currents

The loop measure μ depends on the energy e which is defined by the free parameters C, κ. It will sometimes be denoted μ_e. We shall denote $\mathcal{Z}_e$ the determinant $\det(G) = \det(M_\lambda - C)^{-1}$. Then $\mu(p > 0) = \log(\mathcal{Z}_e) + \sum \log(\lambda_x)$.

$\mathcal{Z}_e^\alpha$ is called the partition function of $\mathcal{L}_\alpha$.

The following result is suggested by an analogy with quantum field theory (Cf [9]).

Proposition 26 *i)* $\frac{\partial \mu}{\partial \kappa_x} = \widehat{l^x}\mu$

ii) If $C_{x,y} > 0$, $\frac{\partial \mu}{\partial C_{x,y}} = -T_{x,y}\mu$ with $T_{x,y}(l) = (\widehat{l^x} + \widehat{l^y}) - \frac{N_{x,y}}{C_{x,y}}(l) - \frac{N_{y,x}}{C_{x,y}}(l)$.

Note that the formula i) would be a direct consequence of the Dynkin isomorphism if we considered only sets defined by the occupation field.

Proof. Recall that by formula (7): $\mu^*(p=1,\xi=x,\widehat{\tau}\in dt)=e^{-\lambda_x t}\frac{dt}{t}$ and $\mu^*(p=k,\xi_i=x_i,\widehat{\tau}_i\in dt_i)=\frac{1}{k}\prod_{x,y}C_{x,y}^{N_{x,y}}\prod_x\lambda_x^{-N_x}\prod_{i\in\mathbb{Z}/p\mathbb{Z}}\lambda_{\xi_i}e^{-\lambda_{\xi_i}t_i}dt_i$

Moreover we have $C_{x,y}=C_{y,x}=\lambda_x P_y^x$ and $\lambda_x=\kappa_x+\sum_y C_{x,y}$

The two formulas follow by elementary calculation. ■

Recall that $\mu(\widehat{l^x})=G^{x,x}$ and $\mu(N_{x,y})=G^{x,y}C_{x,y}$.

So we have $\mu(T_{x,y})=G^{x,x}+G^{y,y}-2G^{x,y}$.

Then, the above proposition allows to compute all moments of T and $\widehat{l}$ relative to μ_e (they could be called Schwinger functions). The above proposition gives the infinitesimal form of the following formula.

Proposition 27 *Consider another energy form e' defined on the same graph. Then we have the following identity:*

$$\frac{\partial\mu_{e'}}{\partial\mu_e}=e^{\sum N_{x,y}\log(\frac{C'_{x,y}}{C_{x,y}})-\sum(\lambda'_x-\lambda_x)\widehat{l^x}}$$

Consequently

$$\mu_e((e^{\sum N_{x,y}\log(\frac{C'_{x,y}}{C_{x,y}})-\sum(\lambda'_x-\lambda_x)\widehat{l^x}}-1))=\log(\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}) \tag{19}$$

Proof. The first formula is a straightforward consequence of (7). The proof of (19) goes by evaluating separately the contribution of trivial loops, which equals $\sum_x\log(\frac{\lambda_x}{\lambda'_x})$. Indeed,

$$\mu_e((e^{\sum N_{x,y}\log(\frac{C'_{x,y}}{C_{x,y}})-\sum(\lambda'_x-\lambda_x)\widehat{l^x}}-1)=\ \mu_{e'}(p>1)-\mu_e(p>1)+\ \mu_e(1_{\{p=1\}}(e^{\sum(\lambda'_x-\lambda_x)\widehat{l^x}}-1)).$$

The difference of the first two terms equals $\log(\mathcal{Z}_{e'})+\sum\log(\lambda'_x)-(\log(\mathcal{Z}_e)-\sum\log(\lambda_x))$. The last term equals $\sum_x\int_0^\infty(e^{-\frac{\lambda'_x-\lambda_x}{\lambda_x}t}-1)\frac{e^{-t}}{t}dt$ which can be computed as before:

$$\mu_e(1_{\{p=1\}}(e^{\sum(\lambda'_x-\lambda_x)\widehat{l^x}}-1))=-\sum\log(\frac{\lambda'_x}{\lambda_x}) \tag{20}$$

■

Remark 28 *(h-transforms) Note that if* $C'_{x,y} = h^x h^y C_{x,y}$ *and* $\kappa'_x = -hLh\lambda$ *for some positive function* h *on* E *such that* $Lh \leq 0$*, as* $\lambda' = h^2\lambda$ *and* $[P']^x_y = \frac{1}{h^x} P^x_y h^y$*, we have* $[G']^{x,y} = \frac{G^{x,y}}{h^x h^y}$ *and* $\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e} = \frac{1}{\prod (h^x)^2}$.

Remark 29 *Note also that* $[\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}]^{\frac{1}{2}} = \mathbb{E}(e^{-\frac{1}{2}[e'-e](\phi)})$*, if* ϕ *is the Gaussian free field associated with* e.

Integrating out the holding times, formula (19) can be written equivalently:

$$\mu_e(\prod_{(x,y)} [\frac{C'_{x,y}}{C_{x,y}}]^{N_{x,y}} \prod_x [\frac{\lambda_x}{\lambda'_x}]^{N_x+1} - 1) = \log(\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}) \tag{21}$$

and therefore

$$\mathbb{E}_{\mathcal{L}_\alpha}(\prod_{(x,y)} [\frac{C'_{x,y}}{C_{x,y}}]^{N^{(\alpha)}_{x,y}} \prod_x [\frac{\lambda_x}{\lambda'_x}]^{N^{(\alpha)}_x+1}) = \mathbb{E}_{\mathcal{L}_\alpha}(\prod_{(x,y)} [\frac{C'_{x,y}}{C_{x,y}}]^{N^{(\alpha)}_{x,y}} e^{-\left\langle \lambda'-\lambda, \widehat{\mathcal{L}_\alpha} \right\rangle} = (\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e})^\alpha \tag{22}$$

Note also that $\prod_{(x,y)} [\frac{C'_{x,y}}{C_{x,y}}]^{N_{x,y}} = \prod_{\{x,y\}} [\frac{C'_{x,y}}{C_{x,y}}]^{N_{x,y}+N_{y,x}}$.

Remark 30 *These* $\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}$ *determine, when* e' *varies with* $\frac{C'}{C} \leq 1$ *and* $\frac{\lambda'}{\lambda} = 1$*, the Laplace transform of the distribution of the traversal numbers of non oriented links* $N_{x,y} + N_{y,x}$.

Other variables of interest on the loop space are associated with elements of the space $\mathbb{A}^-$ of odd functions ω on oriented links : $\omega^{x,y} = -\omega^{y,x}$. Let us mention a few elementary results.

The operator $[P^{(\omega)}]^x_y = P^x_y \exp(i\omega^{x,y})$ is also self adjoint in $L^2(\lambda)$. The associated loop variable writes $\sum_{x,y} \omega^{x,y} N_{x,y}(l)$. We will denote it $\int_l \omega$. This notation will be used even when ω is not odd. Note it is invariant if ω is replaced by $\omega + dg$ for some g. Set $[G^{(\omega)}]^{x,y} = \frac{[(I-P^{(\omega)})^{-1}]^x_y}{\lambda_y}$. By an argument similar to the one given above for the occupation field, we have:

$\mathbb{P}^t_{x,x}(e^{i\int_l \omega}-1) = \exp(t(P^{(\omega)}-I))_{x,x} - \exp(t(P-I))_{x,x}$. Integrating in t after expanding, we get from the definition of μ:

$$\int (e^{i\int_l \omega} - 1)d\mu(l) = \sum_{k=1}^{\infty} \frac{1}{k}[Tr((P^{(\omega)})^k) - Tr((P)^k)]$$

Hence

$$\int (e^{i\int_l \omega} - 1)d\mu(l) = \log[\det(-L(I - P^{(\omega)})^{-1}]$$

Hence $\int (e^{i\int_l \omega} - 1)d\mu(l) = \log[\det(-L(I - P^{(\omega)})^{-1}]$ and

$$\int (\exp(i\int_l \omega) - 1)\mu(dl) = \log(\det(G^{(\omega)}G^{-1}))$$

We can now extend the previous formulas (21) and (22) to obtain, setting $\det(G^{(\omega)}) = \mathcal{Z}_{e,\omega}$

$$\mu_e(e^{-\sum N_{x,y}\log(\frac{C'_{x,y}}{C_{x,y}}) - \sum(\lambda'_x - \lambda_x)\widehat{l}_x + i\int_l \omega} - 1) = \log(\frac{\mathcal{Z}_{e',\omega}}{\mathcal{Z}_e}) \tag{23}$$

and

$$\mathbb{E}(\prod_{x,y}[\frac{C'_{x,y}}{C_{x,y}}e^{i\omega_{x,y}}]^{N^{(\alpha)}_{x,y}} e^{-\sum(\lambda'_x - \lambda_x)\widehat{\mathcal{L}_\alpha}^x}) = (\frac{\mathcal{Z}_{e',\omega}}{\mathcal{Z}_e})^\alpha \tag{24}$$

Alternatively, to simplify the notations slightly, one can consider more general energy forms with complex valued conductances so that the current is included in e'.

Let us now introduce a new

Definition 31 *We say that sets Λ_i of non trivial loops are equivalent when the associated occupation fields are equal and when the total traversal numbers $\sum_{l\in\Lambda_i} N_{x,y}(l)$ are equal for all oriented edges (x,y). Equivalence classes will be called loop networks on the graph. We denote $\overline{\Lambda}$ the loop network defined by Λ.*

Similarly, a set L of non trivial discrete loops defines a discrete network characterized by the total traversal numbers.

Note that these expectations determine the distribution of the network $\overline{\mathcal{L}_\alpha}$ defined by the loop ensemble $\mathcal{L}_\alpha$. We will denote $B^{e,e',\omega}_{(\alpha)}$ the variables

$$\prod_{x,y}[\frac{C'_{x,y}}{C_{x,y}}e^{i\omega_{x,y}}]^{N^{(\alpha)}_{x,y}} e^{-\sum(\lambda'_x - \lambda_x)\widehat{\mathcal{L}_\alpha}^x}.$$

and $B^{e,e',\omega}$ the corresponding variables on the space of single loops.

Remark 32 *This last formula applies to the calculation of loop indices: If we have for example a simple random walk on an oriented planar graph, and if z' is a point of the dual graph X', $\omega_{z'}$ can be chosen such that $\int_l \omega_{z'}$ is the winding number of the loop around a given point z' of the dual graph X'. Then $e^{i\pi \sum_{l\in\mathcal{L}_\alpha} \int_l \omega'_z}$ is a spin system of interest. We then get for example that*

$$\mu(\int_l \omega_{z'} \neq 0) = -\frac{1}{2\pi}\int_0^{2\pi} \log(\det(G^{(2\pi u\omega_{z'})}G^{-1}))du$$

and hence

$$\mathbb{P}(\sum_{l\in\mathcal{L}_\alpha} | \int_l \omega_{z'}|) = 0) = e^{\frac{\alpha}{2\pi}\int_0^{2\pi} \log(\det(G^{(2\pi u\omega_{z'})}G^{-1}))du}$$

Conditional distributions of the occupation field with respect to values of the winding number can also be obtained.

7 Loop erasure and spanning trees.

Recall that an oriented link g is a pair of points (g^-, g^+) such that $C_g = C_{g^-,g^+} \neq 0$. Define $-g = (g^+, g^-)$.

Let $\mu^{\neq}_{x,y}$ be the measure induced by C on discrete self-avoiding paths between x and y: $\mu^{x,y}_{\neq}(x, x_2, ..., x_{n-1}, y) = C_{x,x_2}C_{x_1,x_3}...C_{x_{n-1},y}$.

Another way to defined a measure on discrete self avoiding paths from x to y is loop erasure (see [12] ,[27] and [13]). In this context, the loops can be trivial as they correspond to a single holding times, and loop erasure produces a discrete path without holding times.

We have the following:

Proposition 33 *The image of $\mu^{x,y}$ by the loop erasure map $\gamma \to \gamma^{BE}$ is $\mu^{x,y}_{BE}$ defined on self avoiding paths by $\mu^{x,y}_{BE}(\eta) = \mu^{x,y}_{\neq}(\eta)\frac{\det(G)}{\det(G^{\{\eta\}^c})} = \mu^{x,y}_{\neq}(\eta)\det(G_{|\{\eta\}\times\{\eta\}})$ (Here $\{\eta\}$ denotes the set of points in the path η)*

Proof. If $\eta = (x_1 = x, x_2, ...x_n = y)$,and $\eta_m = (x, ...x_m)$,

$$\mu^{x,y}(\gamma^{BE} = \eta) = \frac{\delta^x_y}{\lambda_y} + \sum_{k=2}^{\infty}[P^k]^x_x P^x_{x_2}\mu^{x_2,y}_{\{x\}^c}(\gamma^{BE} = \theta\eta)$$

where $\mu^{x_2,y}_{\{x\}^c}$ denotes the bridge measure for the Markov chain killed as it hits x and θ the natural shift on discrete paths. By recurrence, this clearly equals

$$V^x_x P^x_{x_2}[V^{\{x\}^c}]^{x_2}_{x_2}...[V^{\{\eta_{n-1}\}^c}]^{x_{n-1}}_{x_{n-1}} P^{x_{n-1}}_y [V^{\{\eta\}^c}]^y_y \lambda_y^{-1} = \mu^{x,y}_{\neq}(\eta)\frac{\det(G)}{\det(G^{\{\eta\}^c})}$$

as

$$[V^{\{\eta_{m-1}\}^c}]^{x_m}_{x_m} = \frac{\det([(I-P)|_{\{\eta_m\}^c\times\{\eta_m\}^c})}{\det([(I-P)|_{\{\eta_{m-1}\}^c\times\{\eta_{m-1}\}^c})} = \frac{\det(V^{\{\eta_{m-1}\}^c})}{\det(V^{\{\eta_m\}^c})} = \frac{\det(G^{\{\eta_{m-1}\}^c})}{\det(G^{\{\eta_m\}^c})}\lambda^{x_m}.$$

for all $m \leq n-1$. ■

Also, by Feynman-Kac formula, for any self-avoiding path η:

$$\begin{aligned}\int e^{-<\widehat{\gamma},\chi>} 1_{\{\gamma^{BE}=\eta\}}\mu^{x,y}(d\gamma) &= \frac{\det(G_\chi)}{\det(G_\chi^{\{\eta\}^c})}\mu^{x,y}_{\neq}(\eta) = \det(G_\chi)_{|\{\eta\}\times\{\eta\}}\mu^{x,y}_{\neq}(\eta)\\ &= \frac{\det(G_\chi)_{|\{\eta\}\times\{\eta\}}}{\det(G_{|\{\eta\}\times\{\eta\}})}\mu^{x,y}_{BE}(\eta).\end{aligned}$$

Therefore, recalling that by the results of section 4.3 conditionally to η, $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_1^{\{\eta\}^c}$ are independent, we see that under $\mu^{x,y}$, the conditional distribution of $\widehat{\gamma}$ given $\gamma^{BE}=\eta$ is the distribution of $\widehat{\mathcal{L}}_1 - \widehat{\mathcal{L}}_1^{\{\eta\}^c}$ i.e. the occupation field of the loops of $\mathcal{L}_1$ which intersect η.

More generally, it can be shown that

Proposition 34 *The conditional distribution of the network $\overline{\mathcal{L}_\gamma}$ defined by the loops of γ, given that $\gamma^{BE}=\eta$, is identical to the distribution of the network defined by $\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c}$ i.e. the loops of $\mathcal{L}_1$ which intersect η.*

Proof. Recall the notation $\mathcal{Z}_e = \det(G)$. First an elementary calculation using (7) shows that $\mu^{x,y}_{e'}(e^{i\int_\gamma\omega}1_{\{\gamma^{BE}=\eta\}})$ equals

$$\begin{aligned}&\mu^{x,y}_e\Big(1_{\{\gamma^{BE}=\eta\}}\prod[\frac{C'_{\xi_i,\xi_{i+1}}}{C_{\xi_i,\xi_{i+1}}}e^{i\omega_{\xi_i,\xi_{i+1}}}\frac{\lambda_{\xi_i}}{\lambda'_{\xi_i}}]\Big)\\ &\qquad\frac{C'_{x,x_2}C'_{x_1,x_3}...C'_{x_{n-1},y}}{C_{x,x_2}C_{x_1,x_3}...C_{x_{n-1},y}}e^{i\int_\eta\omega}\mu^{x,y}_e\Big(\prod_{u\neq v}[\frac{C'_{u,v}}{C_{u,v}}e^{i\omega_{u,v}}]^{N_{u,v}(\mathcal{L}_\gamma)}e^{-\left\langle\lambda'-\lambda,\widehat{\gamma}\right\rangle}1_{\{\gamma^{BE}=\eta\}}\Big).\end{aligned}$$

(Note the term $e^{-\left\langle\lambda'-\lambda,\widehat{\gamma}\right\rangle}$ can be replaced by $\prod_u(\frac{\lambda_u}{\lambda'_u})^{N_u(\gamma)}$).

Moreover, by the proof of the previous proposition, applied to the Markov chain defined by e' perturbed by ω, we have also $\mu_{e'}^{x,y}(e^{i\int_\gamma \omega}1_{\{\gamma^{BE}=\eta\}}) = C'_{x,x_2}C'_{x_1,x_3}...C'_{x_{n-1},y}e^{i\int_\eta \omega}\frac{\mathcal{Z}_{[e']\{\eta\}^c,\omega}}{\mathcal{Z}_{e',\omega}}$.

Therefore

$$\mu_e^{x,y}(\prod_{u\neq v}[\frac{C'_{u,v}}{C_{u,v}}e^{i\omega_{u,v}}]^{N_{u,v}(\mathcal{L}_\gamma)}e^{-\left\langle\lambda'-\lambda,\widehat{\gamma}\right\rangle}||\gamma^{BE}=\eta) = \frac{\mathcal{Z}_e\mathcal{Z}_{[e']\{\eta\}^c,\omega}}{\mathcal{Z}_{e\{\eta\}^c}\mathcal{Z}_{e',\omega}}.$$

Moreover, by (24) and the properties of the Poisson processes,

$$\mathbb{E}(\prod_{u\neq v}[\frac{C'_{u,v}}{C_{u,v}}e^{i\omega_{u,v}}]^{N_{u,v}(\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c})}e^{-\left\langle\lambda'-\lambda,\widehat{\mathcal{L}}_1-\widehat{\mathcal{L}}_1^{\{\eta\}^c}\right\rangle} = \frac{\mathcal{Z}_e\mathcal{Z}_{[e']\{\eta\}^c,\omega}}{\mathcal{Z}_{e\{\eta\}^c}\mathcal{Z}_{e',\omega}}.$$

It follows that the joint distribution of the traversal numbers and the occupation field are identical for the set of erased loops and $\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c}$. ■

Similarly one can define the image of $\mathbb{P}^x$ by BE which is given by

$$\mathbb{P}^x_{BE}(\eta) = C_{x_1,x_2}...C_{x_{n-1},x_n}\kappa_{x_n}\det(G_{|\{\eta\}\times\{\eta\}}),$$

for $\eta=(x_1,...,x_n)$, and get the same results.

Wilson's algorithm (see [22]) iterates this construction, starting with $x's$ in arbitrary order. Each step of the algorithm reproduces the first step except it stops when it hits the already constructed tree of self avoiding paths. It provides a construction of a random spanning tree. Its law is a probability measure $\mathbb{P}^e_{ST}$ on the set $ST_{X,\Delta}$ of spanning trees of X rooted at the cemetery point Δ defined by the energy e. The weight attached to each oriented link $g=(x,y)$ of $X\times X$ is the conductance and the weight attached to the link (x,Δ) is κ_x we can also denote by $C_{x,\Delta}$. As the determinants simplify, the probability of a tree Υ is given by a simple formula:

$$\mathbb{P}^e_{ST}(\Upsilon) = \mathcal{Z}_e\prod_{\xi\in\Upsilon}C_\xi \tag{25}$$

It is clearly independent of the ordering chosen initially. Now note that, since we get a probability

$$\mathcal{Z}_e\sum_{\Upsilon\in ST_{X,\Delta}}\prod_{(x,y)\in\Upsilon}C_{x,y}\prod_{x,(x,\Delta)\in\Upsilon}\kappa_x = 1$$

or equivalently

$$\sum_{\Upsilon\in ST_{X,\Delta}} \prod_{(x,y)\in\Upsilon} P^x_y \prod_{x,(x,\Delta)\in\Upsilon} P^x_\Delta = \frac{1}{\prod_{x\in X}\lambda_x \mathcal{Z}_e}$$

Then, it comes that, for any e' for which conductances (including κ') are positive only on links of e,

$$\mathbb{E}^e_{ST}\left(\prod_{(x,y)\in\Upsilon}\frac{P'^x_y}{P^x_y}\prod_{x,(x,\Delta)\in\Upsilon}\frac{P'^x_\Delta}{P^x_\Delta}\right) = \frac{\prod_{x\in X}\lambda_x}{\prod_{x\in X}\lambda'_x}\frac{\mathcal{Z}_e}{\mathcal{Z}_{e'}}$$

and

$$\mathbb{E}^e_{ST}\left(\prod_{(x,y)\in\Upsilon}\frac{C'_{x,y}}{C_{x,y}}\prod_{x,(x,\Delta)\in\Upsilon}\frac{\kappa'_x}{\kappa_x}\right) = \frac{\mathcal{Z}_e}{\mathcal{Z}_{e'}} \tag{26}$$

Note also that in the case of a graph (i.e. when all conductances are equal to 1), all spanning trees have the same probability. The expression of their cardinal as the determinant $\mathcal{Z}_e$ is Cayley's theorem (see for exemple [22]).

The formula (26) shows a kind of duality between random spanning trees and $\mathcal{L}_1$. It can be extended to $\mathcal{L}_k$ for any integer k if we consider the sum (interms of number of transitions) of k independent spanning trees.

Exercise 35 *Show that more generally, for any tree T rooted in Δ,*
$\mathbb{P}^e_{ST}(\{\Upsilon,\ T\subseteq\Upsilon\}) = \det(G_{|\{T\}\times\{T\}})\prod_{\xi\in T}C_\xi$, *$\{T\}$ denoting the vertex set of T*

Corollary 36 *The network defined by the random set of loops $\mathcal{L}_W$ constructed in this algorithm is independent of the random spanning tree, and independent of the ordering. It has the same distribution as the network defined by the loops of $\mathcal{L}_1$.*

This result follows easily from proposition 34.

Let us now consider the recurrent case.

A probability is defined on the non oriented spanning trees by the conductances: $\mathbb{P}^e_{ST}((\mathcal{T})$ is defined by the product of the conductances of the edges of $\mathcal{T}$ normalized by the sum of these products on all spanning trees.

Note that any non oriented spanning tree of X along edges of E defines uniquely an oriented spanning tree $I_{x_0}(\mathcal{T})$ if we choose a root x_0. The orientation is taken towards the

root which can be viewed as.a cemetery point. Then, if we consider the associated Markov chain killed as it hits x_0, defined by the energy form $e^{\{x_0\}^c}$, the previous construction yields a probability $\mathbb{P}_{ST}^{e^{\{x_0\}^c}}$ on spanning trees rooted at x_0 which by (25) coincides with the image of $\mathbb{P}_{ST}^{e}$ by I_{x_0}. This shows in particular that the normalizing factor $\mathcal{Z}_{e_{\{x_0\}^c}}$ is independent of the choice of x_0 as it equals $(\sum_{T\in ST_X}\prod_{\{x,y\}\in T}C_{x,y})^{-1}$. We denote it by $\mathcal{Z}_e^0$.

Exercise 37 *Check directly. that $\mathcal{Z}_{e_{\{x_0\}^c}}$ is independent of the choice of x_0.*

Note also that if we set $\alpha_{x_0}(\mathcal{T})=\prod_{(x,y)\in I_{x_0}(\mathcal{T})}P_y^x$, $\sum_{T\in ST_X}\alpha_{x_0}(\mathcal{T})$ is proportional to λ_{x_0} as x_0 varies in X. More precisely, it equals $K\lambda_{x_0}$, with $K=\frac{\mathcal{Z}_e^0}{\prod_{x\in X}\lambda_x}$. This fact is known as the Markov chain tree theorem ([22]).

Let us come back briefly to the transient case by choosing some root x_0. As by the strong Markov property, $V_x^y=\mathbb{P}_y(T_x<\infty)V_x^x$, we have $\frac{G^{y,x}}{G^{x,x}}=\frac{V_x^y}{V_x^x}=\mathbb{P}_y(T_x<\infty)$, and therefore

$$\mathbb{P}_{ST}^e((x,y)\in\Upsilon)=\mathbb{P}_x(\gamma_1^{BE}=y)=V_x^xP_y^x\mathbb{P}^y(T_x=\infty)=C_{x,y}G^{x,x}(1-\frac{G^{x,y}}{G^{x,x}}).$$

Directly from the above, we recover Kirchhoff's theorem:

$$\begin{aligned}\mathbb{P}_{ST}^e(\pm(x,y)\in\Upsilon)&=C_{x,y}[G^{x,x}(1-\frac{G^{x,y}}{G^{x,x}})+G^{y,y}(1-\frac{G^{y,x}}{G^{y,y}})]\\&=C_{x,y}(G^{x,x}+G^{y,y}-2G^{x,y})=C_{x,y}K^{x,y),(x,y)}\end{aligned}$$

and this is clearly independent of the choice of the root.

Give an alternative proof of Kirchhoff's theorem by using (26), taking $C'_{x,y}=sC_{x,y}$ and $C'_{u,v}=C_{u,v}$ for $\{u,v\}\neq\{x,y\}$.

More generally we will show the transfer current theorem (see for exemple [21], [22]):

Theorem 38 $\mathbb{P}_{ST}^e(\pm\xi_1,...\pm\xi_k\in\Upsilon)=(\prod_1^kC_{\xi_i})\det(K^{\xi_i,\xi_j}\ 1\le i,j\le k)$ *with* $K^{(x,y),(u,v)}=G(\delta_x-\delta_y)^u-G(\delta_x-\delta_y)$.

Note this determinant does not depend on the orientation of the links.

Proof. We use recurrence on k. Let E_k denote the set of edges $\pm\xi_1,...\pm\xi_k$ and consider the graph defined by identifying all points in E_k to Δ. Let e^k be the energy induced by

the conductances $C_{x,y}$, $\pm(x,y) \in E - E_k$, and the killing measure $\kappa_x + \sum_{y,\, \pm(x,y)\in E_k} C_{x,y}$. Let $G^{(k)}$ the corresponding Green function.Let $V^{(k)}$ be the subspace of $\mathbb{A}^-$ spanned by all $K^{\xi i}$, $1 \leq i \leq k$.. Note that for any $\eta = (u,v)$ the projection in $\mathbb{A}^-$ (equipped with the scalar product defined by e) of K^η on $[V^{(k)}]^\perp$ is $dG^{(k)}(\delta_v - \delta_u)$.

Indeed, the projection of K^η coincides with the projection of $\alpha_{(\eta)}$, where as before we define $\alpha_{(\eta)}^{x,y} = \pm\frac{1}{C_{u,v}}$ if $(x,y) = \pm(u,v)$ and 0 elsewhere, as we have seen that K^η is the projection of $\alpha_{(\eta)}$ on the larger space of all differentials. But for any $K^{\xi i}$ in $V^{(k)}$, $\left\langle \alpha_{(\eta)} - dG^{(k)}(\delta_v - \delta_u), K^{\xi i} \right\rangle_{\mathbb{A}^-}$ vanishes as the scalar product on forms can be replaced by the one induced by $e^{(k)}$, since conductances of E_k do not contribute to this expression.

As seen before, $\left\langle dG^{(k)}(\delta_v - \delta_u), dG^{(k)}(\delta_v - \delta_u) \right\rangle = [G^{(k)}]^{u,u} + [G^{(k)}]^{v,v} - 2[G^{(k)}]^{u,v}$ (with the convention that $[G^{(k)}]^{u,v} = 0$ if u or v is Δ. Moreover, if $\xi_{k+1} = \eta$, using the scalar product in $\mathbb{A}^-$, $\det(K^{\xi_i,\xi_j}\ 1 \leq i,j \leq k+1) = \det(\left\langle K^{\xi_i}, K^{\xi_j} \right\rangle_{\mathbb{A}^-}\ 1 \leq i,j \leq k+1)$. In such a Gram determinant, one checks immediately that in the last column, $K^{\xi_{k+1}}$ can be replaced by its projection on the orthogonal of the space spanned by the K^{ξ_i}, $1 \leq i,j \leq k$ i.e. on $[V^{(k)}]^\perp$. Therefore, if $\xi_{k+1} = \eta$,

$$\det(K^{\xi_i,\xi_j}\ 1 \leq i,j \leq k+1) = \det(K^{\xi_i,\xi_j}\ 1 \leq i,j \leq k) C_{u,v}([G^{(k)}]^{u,u} + [G^{(k)}]^{v,v} - 2[G^{(k)}]^{u,v}).$$

But the argument given for Kirchhoff theorem shows also that $\mathbb{P}^e_{ST}(\pm\eta \in \Upsilon | \pm\xi_1, ... \pm\xi_k \in \Upsilon) = C_{u,v}([G^{(k)}]^{u,u} + [G^{(k)}]^{v,v} - 2[G^{(k)}]^{u,v})$ so we can conclude. ■

Therefore, given any function g on non oriented links,

$$\begin{aligned}
\mathbb{E}^e_{ST}(e^{-\sum_{\xi\in\Upsilon} g(\xi)}) &= \mathbb{E}^e_{ST}(\prod_\xi (1 + (e^{-g(\xi)} - 1)1_{\xi\in\Upsilon}) \\
&= \sum_k \sum_{\pm\xi_1 \neq \pm\xi_2 .. \neq .. \pm\xi_k} \prod (e^{-g(\xi_i)} - 1) \mathbb{P}^e_{ST}(\pm\xi_1, ... \pm\xi_k \in \Upsilon) \\
&= \sum_k \sum_{\pm\xi_1 \neq \pm\xi_2 .. \neq .. \pm\xi_k} \prod (e^{-g(\xi_i)} - 1) \det(K^{\xi_i,\xi_j}\ 1 \leq i,j \leq k) \\
&= \sum Tr((M_{C(e^{-g}-1)}K)^{\wedge k}) = \det(I + KM_{C(e^{-g}-1)})
\end{aligned}$$

and we have

$$\mathbb{E}^e_{ST}(e^{-\sum_{\xi\in\Upsilon} g(\xi)}) = \det(I - M_{\sqrt{C(1-e^{-g})}} K M_{\sqrt{C(1-e^{-g})}})$$

Here determinants are taken on matrices indexed by E.
This is an exemple of the Fermi point processes discussed in [31].

Note that for any spanning tree T, if π_T denotes $M_{1_{\{T\}}}$, it follows from the above, by letting g be $m1_{\{T^c\}}, m \to \infty$ that $\mathbb{P}^e_{ST}(T) = \det((I - KM_C)(I - \pi_T) + \pi_T) = \det((I - KM_C)_{T^c \times T^c})$

Note also that if e' is another energy form on the same graph, $\mathbb{E}^e_{ST}(\prod_{(x,y)\in\Upsilon} \frac{C'_{x,y}}{C_{x,y}}) = \det(I - M\sqrt{\frac{C-C'}{C}}KM\sqrt{\frac{C-C'}{C}})$

On the other hand, from it also equals $(\sum_{T\in ST_X} \prod_{\{x,y\}\in T} C_{x,y})\mathcal{Z}^0_e = \frac{\mathcal{Z}^0_e}{\mathcal{Z}^0_{e'}}$ so that finally

$$\frac{\mathcal{Z}^0_e}{\mathcal{Z}^0_{e'}} = \det(I - M\sqrt{\frac{C-C'}{C}}KM\sqrt{\frac{C-C'}{C}})$$

8 Decompositions

Note first that with the energy e, we can associate a rescaled Markov chain $\widehat{x}_t$ in which holding times at any point x are exponential times of parameters λ_x: $\widehat{x}_t = x_{\tau_t}$ with $\tau_t = \inf(s,\ \int_0^s \frac{1}{\lambda_{x_u}} du = t)$. For the rescaled Markov chain, local times coincide with the time spent in a point and the duality measure is simply the counting measure. The Markov loops can be rescaled as well and we did it in fact already when we introduced pointed loops. More generally we may introduce different holding times parameters but it would be essentially useless as the random variables we are interested into are intrinsic, i.e. depend only on e.

If $D \subset X$ and we set $F = D^c$, the orthogonal decomposition of the energy $e(f,f) = e(f)$ into $e^D(f - H^F f) + e(H^F f)$ leads to the decomposition of the Gaussian field mentioned above and also to a decomposition of the rescaled Markov chain into the rescaled Markov chain killed at the exit of D and the trace of the rescaled Markov chain on F, i.e. $\widehat{x}^{\{F\}}_t = \widehat{x}_{S^F_t}$, with $S^F_t = \inf(s, \int_0^s 1_F(\widehat{x}_u)du = t)$.

Proposition 39 *The trace of the rescaled Markov chain on F is the rescaled Markov chain defined by the energy functional $e^{\{F\}}(f) = e(H^F f)$, for which*

$$C^{\{F\}}_{x,y} = C_{x,y} + \sum_{a,b\in D} C_{x,a}C_{b,y}[G^D]^{a,b}$$

$$\lambda^{\{F\}}_x = \lambda_x - \sum_{a,b\in D} C_{x,a}C_{b,x}[G^D]^{a,b}$$

and

$$\mathcal{Z}_e = \mathcal{Z}_{e^D}\mathcal{Z}_{e^{\{F\}}}$$

Proof. For the second assertion, note first that for any $y \in F$,

$$[H^F]_y^x = 1_{x=y} + 1_D(x)\sum_{b\in D}[G^D]^{x,b}C_{b,y}.$$

Moreover, $e(H^F f) = e(f, H^F f)$ and therefore

$$\lambda_x^{\{F\}} = e^{\{F\}}(1_{\{x\}}) = e(1_{\{x\}}, H^F 1_{\{x\}}) = \lambda_x - \sum_{a\in D} C_{x,a}[H^F]_x^a = \lambda_x(1 - p_x^{\{F\}})$$

where $p_x^{\{F\}} = \sum_{a,b\in D} P_a^x[G^D]^{a,b}C_{b,x} = \sum_{a\in D} P_a^x[H^F]_x^a$ is the probability that the Markov chain starting at x will return to x after an excursion in D.

Then for distinct x and y in F,

$$\begin{aligned} C_{x,y}^{\{F\}} &= -e^{\{F\}}(1_{\{x\}}, 1_{\{y\}}) = -e(1_{\{x\}}, H^F 1_{\{y\}}) \\ &= C_{x,y} + \sum_a C_{x,a}[H^F]_y^a = C_{x,y} + \sum_{a,b\in D} C_{x,a}C_{b,y}[G^D]^{a,b}. \end{aligned}$$

Note that the graph defined on F by the non vanishing conductances $C_{x,y}^{\{F\}}$ has in general more edges than the restiction to F of the original graph.

For the third assertion, note also that $G^{\{F\}}$ is the restriction of G to F as for all $x, y \in F$, $e^{\{F\}}(G\delta_{y|F}, 1_{\{x\}}) = e(G\delta_y, [H^F 1_{\{x\}}]) = 1_{\{x=y\}}$. Hence the determinant decomposition already used in section 4.3 yields the final formula. The cases where F has one point was already treated in section 4.3.

Finally, for the first assertion note the transition matrix $[P^{\{F\}}]_y^x$ can be computed directly and equals

$P_y^x + \sum_{a,b\in D} P_a^x P_y^b V^{D\cup\{x\}}]_b^a = P_y^x + \sum_{a,b\in D} P_a^x C_{b,y}[G^{D\cup\{x\}}]^{a,b}$. It can be decomposed according whether the jump to y occurs from x or from D and the number of excursions from x to x:

$$\begin{aligned} [P^{\{F\}}]_y^x &= \sum_{k=0}^{\infty}(\sum_{a,b\in D} P_a^x[V^D]_b^a P_x^b)^k(P_y^x + \sum_{a,b\in D} P_a^x[V^D]_b^a P_y^b) \\ &= \sum_{k=0}^{\infty}(\sum_{a,b\in D} P_a^x[G^D]^{a,b}C_{b,x})^k(P_y^x + \sum_{a,b\in D} P_a^x[G^D]^{a,b}C_{b,y}). \end{aligned}$$

The expansion of $\frac{C^{\{F\}}_{x,y}}{\lambda^{\{F\}}_x}$ in geometric series yields the exactly the same result.

Finally, remark that the holding times of $\widehat{x}^{\{F\}}_t$ at any point $x \in F$ are sums of a random number of independent holding times of $\widehat{x}_t$. This random integer counts the excursions from x to x performed by the chain $\widehat{x}_t$ during the holding time of $\widehat{x}^{\{F\}}_t$. It follows a geometric distribution of parameter $1 - p^{\{F\}}_x$. Therefore, $\frac{1}{\lambda^{\{F\}}_x} = \frac{1}{\lambda_x(1-p_x)}$ is the expectation of the holding times of $\widehat{x}^{\{F\}}_t$ at x. ■

If χ is carried by D and if we set $e_\chi = e + \| \quad \|_{L^2(\chi)}$ and denote $[e_\chi]^{\{F\}}$ by $e^{\{F,\chi\}}$ we have

$$C^{\{F,\chi\}}_{x,y} = C_{x,y} + \sum_{a,b} C_{x,a} C_{b,y} [G^D_\chi]^{a,b}, \quad p^{\{F,\chi\}}_x = \sum_{a,b\in D} P^x_a [G^D_\chi]^{a,b} C_{b,x}$$

and $\lambda^{\{F,\chi\}}_x = \lambda_x(1 - p^{\{F,\chi\}}_x)$.

More generally, if $e^\#$ is such that $C^\# = C$ on $F \times F$, and $\lambda = \lambda^\#$ on F we have:

$$C^{\#\{F\}}_{x,y} = C_{x,y} + \sum_{a,b} C^\#_{x,a} C^\#_{b,y} [G^{\#D}]^{a,b}, \quad p^{\#\{F\}}_x = \sum_{a,b\in D} P^{\#x}_a [G^{\#D}]^{a,b} C_{b,x}$$

and $\lambda^{\#\{F\}}_x = \lambda_x(1 - p^{\#\{F\}}_x)$.

A loop in X which hits F can be decomposed into a loop $l^{\{F\}}$ in F and its excursions in D which may come back to their starting point. Let $\mu^{a,b}_D$ denote the bridge measure (with mass $[G^D]^{a,b}$) associated with e^D.
Set

$$\nu^D_{x,y} = \frac{1}{C^{\{F\}}_{x,y}} [C_{x,y}\delta_\emptyset + \sum_{a,b\in D} C_{x,a} C_{b,y} \mu^{a,b}_D], \quad \rho^D_x = \sum_{n=1}^\infty \frac{1}{\lambda_x p^{\{F\}}_x} (\sum_{a,b\in D} C_{x,a} C_{b,x} \mu^{a,b}_D)$$

and $\nu^D_x = \frac{1}{1-p^{\{F\}}_x} [\delta_\emptyset + \sum_{n=1}^\infty [p^{\{F\}}_x \rho^D_x]^{\otimes n}]$.

Note that $\rho^D_x(1) = \nu^D_{x,y}(1) = \nu^D_x(1) = 1$.

A loop l can be decomposed into its restriction $l^{\{F\}} = (\xi_i, \widehat{\tau}_i)$ in F (possibly a one point loop), a family of excursions $\gamma_{\xi_i,\xi_{i+1}}$ attached to the jumps of $l^{\{F\}}$ and systems of i.i.d. excursions $(\gamma^h_{\xi_i}, h \le n_{\xi_i})$ attached to the points of $l^{\{F\}}$. Note the set of excursions can be empty.

We get a decomposition of μ into its restriction μ^D to loops in D (associated to the process killed at the exit of D), the loop measure $\mu^{\{F\}}$ defined on loops of F by the trace of the Markov chain on F, probability measures $\nu^D_{x,y}$ on excursions in D indexed by pairs

of points in F and measures ρ_x^D on excursions in D indexed by points of F. Moreover, the integers n_{ξ_i} follow a Poisson distribution of parameter $\lambda_{\xi_i}^{\{F\}}\widehat{\tau}_i$ and the conditional distribution of the rescaled holding times in ξ_i before each excursion $\gamma_{\xi_i}^l$ is the distribution $\beta_{n_{\xi_i},\tau_i^*}$ of the increments of a uniform sample of n_{ξ_i} points in $[0\ \widehat{\tau}_i]$ put in increasing order. We denote these holding times by $\widehat{\tau}_{i,h}$ and set $l = \Lambda(l^{\{F\}}, (\gamma_{\xi_i,\xi_{i+1}}), (n_{\xi_i}, \gamma_{\xi_i}^h, \widehat{\tau}_{i,h}))$.

Then $\mu - \mu^D$ is the image measure by Λ of

$$\mu^{\{F\}}(dl^{\{F\}}) \prod (\nu^D_{\xi_i,\xi_{i+1}})(d\gamma_{\xi_i,\xi_{i+1}}) \prod e^{-\lambda_{\xi_i}^{\{F\}}\widehat{\tau}_i} \sum \frac{[\lambda_{\xi_i}^{\{F\}}\widehat{\tau}_i]^k}{k!} 1_{n_{\xi_i}=k} [\rho_x^D]^{\otimes k}(d\gamma_{\xi_i}^h)\beta_{k,\tau_i^*}(d\widehat{\tau}_{i,h}).$$

The Poisson process $\mathcal{L}_\alpha^{\{F\}} = \{l^{\{F\}}, l \in \mathcal{L}_\alpha\}$ has intensity $\mu^{\{F\}}$ and is independent of $\mathcal{L}_\alpha^D$.

Note that $\widehat{\mathcal{L}_\alpha^{\{F\}}}$ is the restriction of $\widehat{\mathcal{L}_\alpha}$ to F.

In particular, if χ is a measure carried by D, we have:

$$\begin{aligned}
\mathbb{E}(e^{-\langle \widehat{\mathcal{L}_\alpha},\chi\rangle}|\mathcal{L}_\alpha^{\{F\}}) &= \mathbb{E}(e^{-\left\langle \widehat{\mathcal{L}_\alpha^D},\chi\right\rangle})(\prod_{x,y\in F}[\int e^{-\langle\widehat{\gamma},\chi\rangle}\nu_{x,y}^D(d\gamma)]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \\
&\quad \times \prod_{x\in F} e^{\lambda_x^{\{F\}}[\widehat{\mathcal{L}_\alpha^{\{F\}}}]^x \int (e^{-\langle\widehat{\gamma},\chi\rangle}-1)\rho_x^D(d\gamma)} \\
&= [\frac{\mathcal{Z}_{e_\chi^D}}{\mathcal{Z}_{e^D}}]^\alpha (\prod_{x,y\in F}[\frac{C_{x,y}^{\{F,\chi\}}}{C_{x,y}^{\{F\}}}]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \prod_{x\in F} e^{[\lambda_x^{\{F,\chi\}}-\lambda_x^{\{F\}}]\widehat{\mathcal{L}_\alpha^x}}.
\end{aligned}$$

(recall that $\widehat{\mathcal{L}_\alpha^{\{F\}}}$ is the restriction of $\widehat{\mathcal{L}_\alpha}$ to F). Also, if we condition on the set of discrete loops $\mathcal{D}\mathcal{L}_\alpha^{\{F\}}$

$$\mathbb{E}(e^{-\langle \widehat{\mathcal{L}_\alpha},\chi\rangle}|\mathcal{D}\mathcal{L}_\alpha^{\{F\}}) = [\frac{\mathcal{Z}_{e_\chi^D}}{\mathcal{Z}_{e^D}}]^\alpha (\prod_{x,y\in F}[\frac{C_{x,y}^{\{F,\chi\}}}{C_{x,y}^{\{F\}}}]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \prod_{x\in F}[\frac{\lambda_x^{\{F\}}}{\lambda_x^{\{F,\chi\}}}]^{N_x(\mathcal{L}_\alpha^{\{F\}})+1})$$

where the last exponent $N_x + 1$ is obtained by taking into account the loops which have a trivial trace on F (see formula (20)).

More generally we can show in the same way the following

Proposition 40 *If $C^\# = C$ on $F \times F$, and $\lambda = \lambda^\#$ on F, we denote $B^{e,e^\#}$ the multiplicative functional $\prod_{x,y}[\frac{C_{x,y}^\#}{C_{x,y}}]^{N_{x,y}} e^{-\sum_{x\in D}\widehat{l}_x(\lambda_x^\#-\lambda_x)}$.*

Then,

$$\mathbb{E}(B^{e,e^{\#}}|\mathcal{L}_\alpha^{\{F\}}) = [\frac{\mathcal{Z}_{e^{\#D}}}{\mathcal{Z}_{e^D}}]^\alpha (\prod_{x,y\in F}[\frac{C_{x,y}^{\#\{F\}}}{C_{x,y}^{\{F\}}}]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \prod_{x\in F} e^{\lambda_x[p_x^{\#\{F\}}-p_x^{\{F\}}]\widehat{\mathcal{L}}_\alpha^x}$$

and

$$\mathbb{E}(B^{e,e^{\#}}|\mathcal{D}\mathcal{L}_\alpha^{\{F\}}) = [\frac{\mathcal{Z}_{e^{\#D}}}{\mathcal{Z}_{e^D}}]^\alpha (\prod_{x,y\in F}[\frac{C_{x,y}^{\#\{F\}}}{C_{x,y}^{\{F\}}}]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \prod_{x\in F}[\frac{\lambda_x^{\{F\}}}{\lambda_x^{\#\{F\}}}]^{N_x(\mathcal{L}_\alpha^{\{F\}})+1}$$

These decomposition and conditional expectation formulas extend to include a current ω in $C^{\#}$. Note that if ω is closed (i.e. vanish on every loop) in D, one can define ω^F such that $[Ce^{i\omega}]^{\{F\}} = C^{\{F\}}e^{i\omega^F}$. Then

$$\mathcal{Z}_{e,\omega} = \mathcal{Z}_{e^D}\mathcal{Z}_{e^{\{F\}},\omega^F}$$

The previous proposition implies the following *Markov property*:

Remark 41 *If $D = D_1 \cup D_2$ with D_1 and D_2 strongly disconnected, (i.e. such that for any $(x, y, z) \in D_1 \times D_2 \times F$, $C_{x,y}$ and $C_{x,z}C_{y,z}$ vanish), the restrictions of the network $\overline{\mathcal{L}_\alpha}$ to $D_1 \cup F$ and $D_2 \cup F$ are independent conditionally to the restriction of $\mathcal{L}_\alpha$ to F.*

Proof. It follows from the fact that as D_1 and D_2 are disconnected, any excursion measure $\nu_{x,y}^D$ or ρ_x^D from F into $D = D_1 \cup D_2$ is an excursion measure either in D_1 or in D_2. ■

Branching processes with immigration An interesting example can be given after extending slightly the scope of the theory to countable transient symmetric Markov chains: We can take $X = \mathbb{N} - \{0\}$, $C_{n,n+1} = 1$ for all $n \geq 1$ and $\kappa_1 = 1$ and P to be the transfer matrix of the simple symmetric random walk killed at 0.

Then we can apply the previous considerations to check that $\widehat{\mathcal{L}}_\alpha^n$ is a branching process with immigration.

The immigration at level n comes from the loops whose infimum is n and the branching from the excursions of the loops existing at level n to level $n + 1$. Set $F_n = \{1, 2...n\}$ and $D_n = F_n^c$.

The immigration law (on $\mathbb{R}^+$) is a Gamma distribution $\Gamma(\alpha, G^{1,1})$. It is the law of $\widehat{\mathcal{L}}_\alpha^1$ and also of $[\widehat{\mathcal{L}}_\alpha^{D_{n-1}}]^n$ for all $n > 1$. From the above calculations of conditional expectations, we get that for any positive parameter γ,

$$\mathbb{E}(e^{-[\gamma\mathcal{L}^n_\alpha\widehat{\mathcal{L}}^n_\alpha}||\mathcal{L}^{\{F_{n-1}\}}_\alpha) = \mathbb{E}(e^{-[\gamma\widehat{\mathcal{L}}^{D_{n-1}}_\alpha]^n})e^{\lambda^{\{F_{n-1},\gamma\delta_n\}}_{n-1}-\lambda^{\{F_{n-1}\}}_{n-1}]\widehat{\mathcal{L}}^{n-1}_\alpha}$$

From this formula, it is clear that $\widehat{\mathcal{L}}^n_\alpha$ is a Markov process. To be more precise, note that for any $n, m > 0$, $V^n_m = 2(n \wedge m)$ and $\lambda_n = 2$, that $G^{1,n}_{\gamma\delta_1} = G^{1,n} - G^{1,1}\gamma G^{1,n}_{\gamma\delta_1}$ so that $G^{1,n}_{\gamma\delta_1} = \frac{1}{1+\gamma}$and that for any $n > 0$, the restriction of the Markov chain to D_n is isomorphic to the original Markov chain. Then it comes that for all n, $p^{\{F_n\}}_n = \frac{1}{2}$, $\lambda^{\{F_n\}}_n = 1$, $p^{\{F_n,\gamma\delta_{n+1}\}}_n = \frac{1}{2(1+\gamma)}$ and $\lambda^{\{F_n,\gamma\delta_{n+1}\}}_n = \frac{2\gamma+1}{1+\gamma}$ so that the Laplace exponent of the convolution semigroup ν_t defining the branching mechanism equals $\frac{\gamma}{1+\gamma} = \int(1-e^{-\gamma s})e^{-s}ds$. It is the semigroup of a compound Poisson process whose Levy measure is exponential. The conditional law of $\widehat{\mathcal{L}}^{n+1}_\alpha$ given $\widehat{\mathcal{L}}^n_\alpha$ is the convolution of the immigration law $\Gamma(\alpha, 1)$ with $\nu_{\widehat{\mathcal{L}}^n_\alpha}$

Exercise 42 *Alternatively, we can consider the integer valed process* $N_n(\mathcal{L}^{\{F_n\}}_\alpha)+1$ *which is a Galton Watson process with immigration. In our exemple, we find the reproduction law* $\pi(n) = 2^{-n-1}$*for all* $n \geq 0$ *(critical binary branching).*

Exercise 43 *Show that more generally, if* $C_{n,n+1} = [\frac{p}{1-p}]^n$*, for* $n > 0$ *and* $\kappa_1 = 1$*,with* $0 < p < 1$*, we get all asymetric simple random walks. Then* $\lambda_n = \frac{p^{n-1}}{(1-p)^n}$*. Then, show that* $G^{1,1} = 1$*. Determine the distributions of the associated branching and Galton Watson process with immigration.*

If we consider the occupation field defined by the loops going through 1, we get a branching process without immigration: it is the classical relation between random walks local times and branching processes.

9 Reflection positivity

In this section, we assume there exists a partition of X: $X = X^+ \cup X^-$, $X^+ \cap X^- = \varnothing$,and an involution ρ on X such that:

a) e is ρ-invariant.

b) $\rho(X^\pm) = X^\mp$

c) The $X^+ \times X^+$ matrix $C^\pm_{x,y} = C_{x,\rho(y)}$, symmetric by b), is non negative definite.

Note that for exemple, c) holds if $C^\pm$ is diagonal.

Then the following holds:

Theorem 44 *i) For any square integrable function* Ψ *in* $\sigma(\widehat{l^x}, x \in X^+) \vee \sigma(N_{x,y}, x, y \in X^+)$,

$$\int \Psi(l)\overline{\Psi}(\rho(l)\mu(dl) \geq 0$$

ii) For any square integrable function Φ *in* $\sigma(\widehat{\mathcal{L}_\alpha}^x, x \in X^+) \vee \sigma(N^{(\alpha)}_{x,y}, x, y \in X^+)$,

$$\mathbb{E}(\Phi(\mathcal{L}_\alpha)\overline{\Phi}(\rho(\mathcal{L}_\alpha)) \geq 0$$

iii) For any square integrable function Σ *of the free field* ϕ *restricted to* X^+,

$$\mathbb{E}_\phi(\Sigma(\phi)\overline{\Sigma}(\rho(\phi)) \geq 0$$

iv) For any square integrable function Γ *of the spanning trees on* X^+,

$$\mathbb{E}_{ST}(\Gamma(T_{|X^+})\overline{\Gamma}(\rho(T_{|X^-})) \geq 0$$

Proof. The property iii) is well known in a somewhat different context: Cf for exemple [[30]], [[9]] and their references. Reflection positivity is a keystone in the bridge between statistical and quantum mechanics.

To prove ii), we use the fact that the σ-algebra is generated by the algebra of random variables of the form $\Phi = \sum \lambda_j B^{e,e_j,\omega_j}_{(\alpha)}$ with $C^{(e_j)} = C$ and $\omega_j = 0$ outside $X^+ \times X^+$, $C^{(e_j)} \leq C$ on $X^+ \times X^+$, $\lambda^{(e_j)} = \lambda$ on X^- and $\lambda^{(e_j)} \geq \lambda$ on X^+.

Then $\mathbb{E}(\Phi(\mathcal{L}_\alpha)\overline{\Phi}(\rho(\mathcal{L}_\alpha)) = \mathbb{E}(\sum \lambda_j \overline{\lambda}_k B^{e,e_j+\rho(e_q)-e,\omega_j-\rho(\omega_q)}_{(\alpha)})$

$= \sum \lambda_j \overline{\lambda}_q (\frac{\mathcal{Z}_{e_j+\rho(e_q)-e,\omega_j-\rho(\omega_q)}}{\mathcal{Z}_e})^\alpha$.

We have to prove this is non negative. Letting α converge to zero, it will imply that $\int \Psi(l)\overline{\Psi}(\rho(l)\mu(dl) \geq 0$ for any Ψ in the algebra of functions of the form $\sum \lambda_j (B^{e,e_j,\omega_j} - 1)$ with $\sum \lambda_j = 0$ which will prove i).

Let us first assume that $C^\pm$ is positive definite. We will see the general case can be reduced to this one.

Now note that $\mathcal{Z}_{e_j+\rho(e_q)-e,\omega_j-\rho(\omega_q)}$ is the inverse of the determinant of a positive definite matrix of the form: $D(j,q) = \begin{bmatrix} A(j) & -C^\pm \\ -C^\pm & A(q)^* \end{bmatrix}$ with $[A(j)]_{u,v} = \lambda_u \delta_{u,v} - C^{e(j)}_{u,v} e^{i\omega_j^{u,v}}$ and $C^\pm_{u,v} = \delta_{u,v} C_{u,\rho(v)}$.

It is enough to show that $\det(D(j,k))^{-\alpha}$ can be expanded in series of products $\sum q_n(j)\overline{q_n}(k)$ with $\sum |q_n(j)|^2 < \infty$.

As

$$D(j,k) = \begin{bmatrix} [C^{\pm}]^{\frac{1}{2}} & 0 \\ 0 & [C^{\pm}]^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} [C^{\pm}]^{-\frac{1}{2}} A(j) [C^{\pm}]^{-\frac{1}{2}} & -I \\ -I & [C^{\pm}]^{-\frac{1}{2}} A(q)^* [C^{\pm}]^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} [C^{\pm}]^{\frac{1}{2}} & 0 \\ 0 & [C^{\pm}]^{\frac{1}{2}} \end{bmatrix}$$

the α-power of this determinant can be written

$$\det(C^{\pm})^{-2\alpha} \det(F(j))^{\alpha} \det(F(q)^*)^{\alpha} \det(I - \begin{bmatrix} 0 & F(j) \\ F(q)^* & 0 \end{bmatrix})^{-\alpha}$$

with $F(j) = [C^{\pm}]^{\frac{1}{2}} A(j)^{-1} [C^{\pm}]^{\frac{1}{2}}$.

Note that $A(j)^{-1}$ is also the Green function of the restriction to X^+ of the Markov chain associated with $e_j + \rho(e_k) - e$, twisted by ω_j. Therefore $A(j)^{-1} C^{\pm} = [C^{\pm}]^{-\frac{1}{2}} F(j) [C^{\pm}]^{\frac{1}{2}}$ is the balayage kernel on X^- defined by this Markov chain with an additional phase under the expectation produced by ω_j. It is therefore clear from Frobenius theorem that the eigenvalues of the matrices $A(j)^{-1} C^{\pm}$ and $F(j)$ are of modulus less than one and it follows that $\begin{bmatrix} 0 & F(j) \\ F(q)^* & 0 \end{bmatrix}$, conjugate to $\begin{bmatrix} F(j) & 0 \\ 0 & F(q)^* \end{bmatrix}$, is a contraction.

If X^+ has only one point, $(1 - F(j) F(q)^*)^{-\alpha} = \sum \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!} F(j)^{-n} \overline{F(q)}^{-n}$ which allows to conclude. Let us now treat the general case.

For any (n,m) matrix N, and $k = (k_1, ...k_m) \in \mathbb{N}^m$, $l = (l_1, ...l_n) \in \mathbb{N}^n$, let $N^{\{k,l\}}$ denote the $(|k|, |l|)$ matrix obtained from by repeting k_i times each line i; then l_j times each column j.

We use the expansion

$$\det(I - M)^{-\alpha} = 1 + \sum Per_{\alpha}(M^{\{k,k\}})$$

valid for any contraction M (Cf [34] and [35]).

Note that if X has $2d$ points, if we denote $(k_1, ...k_{2d})$ by (k^+, k^-), with $k^+ = (k_1, ...k_d)$ and $k^- = (k_{d+1}, ...k_{2d})$, $\begin{bmatrix} 0 & F(j) \\ F(q)^* & 0 \end{bmatrix}^{\{k,k\}} = \begin{bmatrix} 0 & F(j)^{\{k^+,k^-\}} \\ [F(q)^*]^{\{k^-,k^+\}} & 0 \end{bmatrix}$.

But the α-permanent of a $(2n, 2n)$ matrix of the form $\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}$ vanishes unless the submatrices A and B are square matrices. Hence, we necessary have $|k^+| = |k^-|$., so that in our case, A and B are (n,n) matrices.

Then, the non zero terms in the α-permanent come from permutations whose cycles go back and forth between $\{1, 2..n\}$ and $\{n+1, ...2n\}$, which therefore decompose into the pairing of two permutations of $\{1, 2..n\}$, with the same cycle structure.

Therefore, denoting $\mathcal{S}_{n_1,n_2,...n_r}$ the set of permutations of $\{1,2..n\}$ with cycle structure $(n_1,n_2,...n_r)$, we have

$$Per_\alpha(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}) = \sum_{structures} \sum_{pairings} \alpha^r(\sum_{\sigma\in\mathcal{S}_{n_1,n_2,...n_r}} \prod_1^n A_{i,\sigma(i)})(\sum_{\tau\in\mathcal{S}_{n_1,n_2,...n_r}} \prod_1^n B^*_{i,\tau(i)})$$

which concludes the proof in the positive definite case as

$\sum_{\tau\in\mathcal{S}_{n_1,n_2,...n_r}} \prod_1^n B^*_{i,\tau(i)} = \sum_{\tau\in\mathcal{S}_{n_1,n_2,...n_r}} \prod_1^n \overline{B}_{\tau(i),i} = \sum_{\sigma\in\mathcal{S}_{n_1,n_2,...n_r}} \prod_1^n \overline{B}_{i,\sigma(i)})$.

To treat the general nonnegative case, we can use use a passage to the limit or alternatively, the proposition 40 (or more precisely its extension including a current) to reduce the sets X^+ and X^- to the support of $C^\pm$.

To prove iii) let us first show the assumptions imply that the $X^+\times X^+$ matrix $G^\pm_{x,y} = G_{x,\rho(y)}$ is also nonnegative definite. Let us write G in the form $\begin{bmatrix} A & -C^\pm \\ -C^\pm & A \end{bmatrix}^{-1}$ with $A = M_\lambda - C$. Then

$G = \begin{bmatrix} A^{-\frac12} & 0 \\ 0 & A^{-\frac12} \end{bmatrix} \begin{bmatrix} I & -A^{-\frac12}C^\pm A^{-\frac12} \\ -A^{-\frac12}C^\pm A^{-\frac12} & I \end{bmatrix}^{-1} \begin{bmatrix} A^{-\frac12} & 0 \\ 0 & A^{-\frac12} \end{bmatrix}$

$A^{\frac12}C^\pm A^{\frac12}$ is non negative definite and as before, we can check it is a contraction since $A^{-1}C^\pm$ is a balayage kernel.

Note that if a symmetric nonnegative definite matrix K has eigenvalues μ_i, the eigenvalues of the symmetric matrix E defined by $\begin{bmatrix} I & -K \\ -K & I \end{bmatrix}^{-1} = \begin{bmatrix} D & E \\ E & D \end{bmatrix}$ are easily seen (Exercise) to be $\frac{\mu_i}{1-\mu_i^2}$. Taking $K = A^{-\frac12}C^\pm A^{-\frac12}$, it follows that the symmetric matrix E , (and in particular $G^\pm = A^{-\frac12}EA^{-\frac12}$) is nonnegative definite.

To finish the proof, let us take Σ of the form $\sum \lambda_j e^{\langle\phi,\chi_j\rangle}$. Then

$\mathbb{E}_\phi(\Sigma(\phi)\overline{\Sigma}(\rho(\phi)) = \sum \lambda_j\lambda_q \mathbb{E}_\phi(e^{\langle\phi,\chi_j\rangle+\langle\phi,\rho(\chi_q)\rangle}) = \sum \lambda_j e^{\frac12\langle\chi_j,G^{++}\chi_j\rangle}\lambda_k e^{\frac12\langle\chi_q,G^{++}\chi_q\rangle}e^{\frac12\langle\chi_j,G^\pm\chi_q\rangle}$

and as $G^\pm$ is positive definite, we can conclude since $e^{\frac12\langle\chi_j,G^\pm\chi_q\rangle} = \mathbb{E}_w(e^{\langle w,\chi_j\rangle}e^{\langle w,\chi_q\rangle})$, w denoting the Gaussian field on X^+ with covariance $G^\pm$.

Finally, the proof of iv) follows the same route as for ii). Note that we can take $\omega_j = 0$ as we consider non oriented links. Then the matices $F(j)$ are positive definite. We can use the determinantal expansion for $\alpha = 1$ and conclude from the fact that $\det(\begin{bmatrix} 0 & F(j)^{\{k^+,k^-\}} \\ [F(q)]^{\{k^-,k^+\}} & 0 \end{bmatrix}) = \det(F(q)^{\{k^+,k^-\}})\det(F(j)^{\{k^+,k^-\}})$. ■

Remark 45 *In the case where α is a half integer, the reflection positivity of the free field, often easier to establish, implies i) and ii) in the above theorem by remark 29.*

Exercise 46 *Prove the above remark.*

Exercise 47 *Show that if there exists a partition of X: $X = X^+ \cup X^- \cup X^0$, and an involution ρ on X such that:*

a) e and X^0 are ρ-invariant.

b) $\rho(X^\pm) = X^\mp$

c) X^+ and X^- are disconnected.

Then the assumptions of the previous theorem are satisfied for the trace on $X^+ \cup X^-$.

10 The case of general Markov processes

We now explain briefly how some of the above results will be extended to a symmetric Markov process on an infinite space X. The construction of the loop measure as well as a lot of computations can be performed quite generally, using Markov processes or Dirichlet space theory (Cf for example [8]). It works as soon as the bridge or excursion measures $\mathbb{P}_t^{x,y}$ can be properly defined. The semigroup should have a locally integrable kernel $p_t(x, y)$.

Let us consider more closely the occupation field $\widehat{l}$. The extension is rather straightforward when points are not polar. We can start with a Dirichlet space of continuous functions and a measure m such that there is a mass gap. Let P_t the associated Feller semigroup. Then the Green function is well defined as the mutual energy of the Dirac measures δ_x and δ_y which have finite energy. It is the covariance function of a Gaussian free field $\phi(x)$, which will be associated to the field $\widehat{\mathcal{L}}^x_{\frac{1}{2}}$ of local times of the Poisson process of random loops whose intensity is given by the loop measure defined by the semigroup P_t. This will apply to examples related to one dimensional Brownian motion or to Markov chains on countable spaces.

When we consider Brownian motion on the half line, we get continuous branching process with immigration, as in the discrete case.

When points are polar, one needs to be more careful. We will consider only the case of the two and three dimensional Brownian motion in a bounded domain D killed at the boundary, i.e. associated with the classical energy with Dirichlet boundary condition.

The Green function does not induce a trace class operator but it is still Hilbert-Schmidt which allows to define renormalized determinants $\det_2$ (Cf [29]).

If A is a symmetric Hilbert Schmidt operator, $\det_2(I+A)$ is defined as $\prod(1+\lambda_i)e^{-\lambda_i}$ where λ_i are the eigenvalues of A.

The Gaussian field (called free field) whose covariance function is the Green function is now a generalized field: Generalized fields are not defined pointwise but have to be smeared by a test function f. Still $\phi(f)$ is often denoted $\int \phi(x)f(x)dx$.

Wick powers $:\phi^n:$ of the free field can be defined as generalized field by approximation as soon as the $2n$-th power of the Green function, $G(x,y)^{2n}$ is locally integrable (Cf [30]). This is the case for all n for Brownian motion in dimension two, as the Green function has only a logarithmic singularity on the diagonal, and for $n=2$ in dimension three as the singularity is of the order of $\frac{1}{\|x-y\|}$. More precisely, taking for example $\pi^x_\varepsilon(dy)$ to be the normalized area measure on the sphere of radius ε around x, $\phi(\pi^x_\varepsilon)$ is a Gaussian field with covariance $\sigma^x_\varepsilon = \int G(z,z')\pi^x_\varepsilon(dz)\pi^y_\varepsilon(dz')$. Its Wick powers are defined with Hermite polynomials as we did previously:

$:\phi(\pi^x_\varepsilon)^n: := (\sigma^x_\varepsilon)^{\frac{n}{2}} H_n(\frac{\phi(\pi^x_\varepsilon)}{\sqrt{\sigma^x_\varepsilon}})$. Then one can see that, $\int f(x) :\phi(\pi^x_\varepsilon)^n: dx$ converges in L^2 for any bounded continuous function f with compact support towards a limit called the n-th Wick power of the free field evaluated on f and denoted $:\phi^n:(f)$. Moreover, $\mathbb{E}(:\phi^n:(f):\phi^n:(h)) = \int G^{2n}(x,y)f(x)h(y)dxdy$.

In these cases, we can extend the statement of theorem 21 to the renormalized occupation field $\widetilde{\mathcal{L}}^x_{\frac{1}{2}}$ and the Wick square $:\phi^2:$ of the free field.

Let us explain this in more details in the Brownian motion case. Let D be an open subset of $\mathbb{R}^d$ such that the Brownian motion killed at the boundary of D is transient and has a Green function. Let $p_t(x,y)$ be its transition density and $G(x,y) = \int_0^\infty p_t(x,y)dt$ the associated Green function. The loop measure μ was defined in [14] as

$$\mu = \int_D \int_0^\infty \frac{1}{t}\mathbb{P}^{x,x}_t dt$$

where $\mathbb{P}^{x,x}_t$ denotes the (non normalized) excursion measure of duration t such that if $0 \le t_1 \le ...t_h \le t$,

$$\mathbb{P}^{x,x}_t(\xi(t_1)\in dx_1,...,\xi(t_h)\in dx_h) = p_{t_1}(x,x_1)p_{t_2-t_1}(x_1,x_2).....p_{t-t_h}(x_h,x)dx_1...dx_h$$

(the mass of $\mathbb{P}^{x,x}_t$ is $p_t(x,x)$). Note that μ is a priori defined on based loops but it is easily seen to be shift-invariant.

For any loop l indexed by $[0\ T(l)]$, define the measure $\widehat{l} = \int_0^{T(l)} \delta_{l(s)} ds$: for any Borel set A, $\widehat{l}(A) = \int_0^{T(l)} 1_A(l_s) ds$.

Lemma 48 *For any non negative function f,*

$$\mu(\left\langle \widehat{l}, f \right\rangle^n) = (n-1)! \int G(x_1,x_2) f(x_2) G(x_2,x_3) f(x_3) ... G(x_n,x_1) f(x_1) \prod_1^n dx_i$$

Proof. From the definition of μ and $\widehat{l}$, $\mu(\left\langle \widehat{l}, f \right\rangle^n)$ equals:

$$n! \int \int_{\{0<t_1...<t_n<t\}} \frac{1}{t} f(x_1)...f(x_n) p_{t_1}(x,x_1)....p_{t-t_n}(x_n,x) \prod dt_i dx_i dt dx$$
$$= n! \int \int_{\{0<t_1...<t_n<t\}} \frac{1}{t} f(x_1)...f(x_n) p_{t_2-t_1}(x_1,x_2)...p_{t_1+t-t_n}(x_n,x_1) \prod dt_i dx_i dt$$

Performing the change of variables $v_2 = t_2 - t_1, ..v_n = t_n - t_{n-1}, v_1 = t_1 + t - t_n$, and $v = t_1$, we obtain:

$$n! \int_{\{0<v<v_1, 0<v_i\}} \frac{1}{v_1 + ... + v_n} f(x_1)...f(x_n) p_{v_2}(x_1,x_2)....p_{v_1}(x_n,x_1) \prod dv_i dx_i dv$$
$$= n! \int_{\{0<v_i\}} \frac{v_1}{v_1 + ... + v_n} f(x_1)...f(x_n) p_{v_2}(x_1,x_2).....p_{v_1}(x_n,x_1) \prod dv_i dx_i$$
$$= (n-1)! \int_{\{0<v_i\}} f(x_1)...f(x_n) p_{v_2}(x_1,x_2) \dots p_{v_1}(x_n,x_1) \prod dv_i dx_i$$
(as we get the same formula with any v_i instead of v_1)
$$= (n-1)! \int G(x_1,x_2) f(x_2) G(x_2,x_3) f(x_3) ... G(x_n,x_1) f(x_1) \prod_1^n dx_i.$$

■

One can define in a similar way the analogous of multiple local times, and get for their integrals with respect to μ a formula analogous to the one obtained in the discrete case.

Let G denote the operator on $L^2(D, dx)$ defined by G. Let f be a non negative continuous function with compact support in D.

Note that $\left\langle \widehat{l}, f \right\rangle$ is μ-integrable only in dimension one as then, G is locally trace class. In that case, using for all x an approximation of the Dirac measure at x, local times $\widehat{l^x}$ can be defined in such a way that $\left\langle \widehat{l}, f \right\rangle = \int \widehat{l^x} f(x) dx$.

$\left\langle \widehat{l}, f \right\rangle$ is μ-square integrable in dimensions one, two and three, as G is Hilbert-Schmidt if D is bounded, since $\int \int_{D\times D} G(x,y)^2 dxdy < \infty$, and otherwise locally Hilbert-Schmidt.

N.B.: Considering distributions χ such that $\int \int (G(x,y)^2 \chi(dx)\chi(dy) < \infty$, we could see that $\left\langle \widehat{l}, \chi \right\rangle$ can be defined by approximation as a square integrable variable and $\mu(\left\langle \widehat{l}, \chi \right\rangle^2) = \int (G(x,y)^2 \chi(dx)\chi(dy)$.

Let z be a complex number such that $\mathrm{Re}(z) > 0$.

Note also that $e^{-z\langle \widehat{l}, f\rangle} + z \left\langle \widehat{l}, f \right\rangle - 1$ is bounded by $\frac{|z|^2}{2} \left\langle \widehat{l}, f \right\rangle^2$ and expands as an alternating series $\sum_2^\infty \frac{z^n}{n!}(-\left\langle \widehat{l}, f \right\rangle)^n$, with $\left| e^{-z\langle \widehat{l}, f\rangle} - 1 - \sum_1^N \frac{z^n}{n!}(-\left\langle \widehat{l}, f \right\rangle)^n \right| \leq \frac{\left|z\langle \widehat{l}, f\rangle\right|^{N+1}}{(N+1)!}$. Then, for $|z|$ small enough., it follows from the above lemma that

$$\mu(e^{-z\langle \widehat{l}, f\rangle} + z \left\langle \widehat{l}, f \right\rangle - 1) = \sum_2^\infty \frac{z^n}{n} Tr(-(M_{\sqrt{f}} G M_{\sqrt{f}})^n)$$

As $M_{\sqrt{f}} G M_{\sqrt{f}}$ is Hilbert-Schmidt $\det_2(I + z M_{\sqrt{f}} G M_{\sqrt{f}})$ is well defined and the second member writes -log($\det_2(I + z M_{\sqrt{f}} G M_{\sqrt{f}})$).
Then the identity

$$\mu(e^{-z\langle \widehat{l}, f\rangle} + z \left\langle \widehat{l}, f \right\rangle - 1) = -\log(\det{}_2(I + z M_{\sqrt{f}} G M_{\sqrt{f}})).$$

extends, as both sides are analytic as locally uniform limits of analytic functions, to all complex values with positive real part.

The renormalized occupation field $\widetilde{\mathcal{L}_\alpha}$ is defined as the compensated sum of all $\widehat{l}$ in $\mathcal{L}_\alpha$ (formally, $\widetilde{\mathcal{L}_\alpha} = \widehat{\mathcal{L}_\alpha} - \int \int_0^{T(l)} \delta_{l_s} ds \mu(dl)$) By a standard argument used for the construction of Levy processes,

$$\left\langle \widetilde{\mathcal{L}_\alpha}, f \right\rangle = \lim_{\varepsilon \to 0} (\sum_{\gamma \in \mathcal{L}_\alpha} (1_{\{T>\varepsilon\}} \int_0^T f(\gamma_s) ds) - \alpha \mu(1_{\{T>\varepsilon\}} \int_0^T f(\gamma_s) ds))$$

(we can denote $\lim_{\varepsilon\to 0} \left\langle \widetilde{\mathcal{L}_{\alpha,\varepsilon}}, f \right\rangle$) which converges a.s. and in L^2, as $\mathbb{E}((\sum_{\gamma\in\mathcal{L}_\alpha}(1_{\{T>\varepsilon\}} \int_0^T f(\gamma_s)ds) - \alpha\mu(1_{\{T>\varepsilon\}} \int_0^T f(\gamma_s)ds))^2) = \alpha \int (1_{\{T>\varepsilon\}} \int_0^T f(\gamma_s)ds)^2 \mu(dl)$

and $\mathbb{E}(\left\langle \widetilde{\mathcal{L}_\alpha}, f \right\rangle^2) = Tr((M_{\sqrt{f}} G M_{\sqrt{f}})^2)$. Note that if we fix f, α can be considered as a time parameter and $\left\langle \widetilde{\mathcal{L}_{\alpha,\varepsilon}}, f \right\rangle$ as Levy processes with discrete positive jumps approximating a Levy process with positive jumps $\left\langle \widetilde{\mathcal{L}_\alpha}, f \right\rangle$. The Levy exponent $\mu(1_{\{T>\varepsilon\}}(e^{-\langle \widehat{l}, f\rangle} +$

$\left\langle \widehat{l}, f \right\rangle - 1))$ of $\left\langle \widetilde{\mathcal{L}_{\alpha,\varepsilon}}, f \right\rangle)$ converges towards the Lévy exponent of $\left\langle \widetilde{\mathcal{L}_\alpha}, f \right\rangle)$ which is $\mu((e^{-\langle \widehat{l}, f\rangle} + \left\langle \widehat{l}, f \right\rangle - 1))$.

and, from the identity $E(e^{-\left\langle \widetilde{\mathcal{L}_\alpha}, f\right\rangle}) = e^{-\alpha\mu(e^{-\langle \widehat{l}, f\rangle} + \left\langle \widehat{l}, f\right\rangle - 1)}$, we get the

Theorem 49 *Assume $d \leq 3$. Denoting $\widetilde{\mathcal{L}_\alpha}$ the compensated sum of all $\widehat{l}$ in $\mathcal{L}_\alpha$, we have* $\mathbb{E}(e^{-\left\langle \widetilde{\mathcal{L}_\alpha}, f\right\rangle}) = \det_2(I + M_{\sqrt{f}} G M_{\sqrt{f}}))^{-\alpha}$

Moreover $e^{-\left\langle \widetilde{\mathcal{L}_{\alpha,\varepsilon}}, f\right\rangle}$ converges a.s. and in L^1 towards $e^{-\left\langle \widetilde{\mathcal{L}_\alpha}, f\right\rangle}$.

Considering distributions of finite energy χ (i.e. such that $\int (G(x,y)^2 \chi(dx)\chi(dy) < \infty$), we can see that $\left\langle \widetilde{\mathcal{L}_\alpha}, \chi \right\rangle$ can be defined by approximation as $\lim_{\lambda\to\infty}(\left\langle \widetilde{\mathcal{L}_\alpha}, \lambda G_\lambda \chi \right\rangle)$ and

$$\mathbb{E}(\left\langle \widetilde{\mathcal{L}_\alpha}, \chi \right\rangle^2) = \alpha \int (G(x,y))^2 \chi(dx)\chi(dy).$$

Specializing to $\alpha = \frac{k}{2}$, k being any positive integer we have:

Corollary 50 *The renormalized occupation field $\widetilde{\mathcal{L}_{\frac{k}{2}}}$ and the Wick square $\frac{1}{2} : \sum_1^k \phi_l^2 :$ have the same distribution.*

If Θ is a conformal map from D onto $\Theta(D)$, it follows from the conformal invariance of the Brownian trajectories that a similar property holds for the bBrownian"loop soup"(Cf [14]). More precisely, if $c(x) = Jacobian_x(\Theta)$ and, given a loop l, if $T^c(l)$ denotes the reparametrized loop l_{τ_s}, with $\int_0^{\tau_s} c(l_u)du = s$, $\Theta T^c(\mathcal{L}_\alpha)$ is the Brownian loop soup of intensity parameter α on $\Theta(D)$. Then we have the following:

Proposition 51 $\Theta(\widetilde{c\mathcal{L}_\alpha})$ *is the renormalized occupation field on $\Theta(D)$.*

Proof. We have to show that the compensated sum is the same if we perform it after or before the time change. For this it is enough to check that

$$\mathbb{E}([\sum_{\gamma\in\mathcal{L}_\alpha} (1_{\{\tau_T>\eta\}} 1_{\{T\leq\varepsilon\}} \int_0^T f(\gamma_s)ds - \alpha \int (1_{\{\tau_T>\eta\}} 1_{\{T\leq\varepsilon\}} \int_0^T f(\gamma_s)ds)\mu(d\gamma)]^2)$$
$$= \alpha \int (1_{\{\tau_T>\eta\}} 1_{\{T\leq\varepsilon\}} \int_0^T f(\gamma_s)ds)^2 \mu(d\gamma)$$

and

$$\mathbb{E}([\sum_{\gamma\in\mathcal{L}_\alpha}(1_{\{T>\varepsilon\}}1_{\tau_T\leq\eta}\int_0^T f(\gamma_s)ds-\alpha\int(1_{\{T>\varepsilon\}}1_{\tau_T\leq\eta}\int_0^T f(\gamma_s)ds)\mu(d\gamma)]^2)$$

$$\alpha\int(1_{\{T>\varepsilon\}}1_{\tau_T\leq\eta}\int_0^T f(\gamma_s)ds)^2\mu(d\gamma)$$

converge to zero as ε and η go to zero. It follows from the fact that:

$$\int[1_{\{T\leq\varepsilon\}}\int_0^T f(\gamma_s)ds]^2\mu(d\gamma)$$

and

$$\int[1_{\tau_T\leq\eta}\int_0^T f(\gamma_s)ds]^2\mu(d\gamma)$$

converge to 0. The second follows easily from the first if c is bounded away from zero. We can always consider the "loop soups" in an increasing sequence of relatively compact open subsets of D to reduce the general case to that situation. ■

As in the discrete case (see corollary 17), we can compute product expectations. In dimensions one and two, for f_j continuous functions with compact support in D:

$$\mathbb{E}(\left\langle\widetilde{\mathcal{L}_\alpha},f_1\right\rangle...\left\langle\widetilde{\mathcal{L}_\alpha},f_k\right\rangle)=\int Per_\alpha^0(G(x_l,x_m),1\leq l,m\leq k)\prod f_j(x_j)dx_j \qquad (27)$$

11 Renormalized powers

In dimension one, as in the discrete case, powers of the occupation field can be viewed as integrated self intersection local times. In dimension two, renormalized powers of the occupation field, also called *renormalized self intersections local times* can be defined as follows:

Theorem 52 *Assume $d=2$. Let $\pi_\varepsilon^x(dy)$ be the normalized arclength on the circle of radius ε around x, and set $\sigma_\varepsilon^x=\int G(y,z)\pi_\varepsilon^x(dy)\pi_\varepsilon^x(dz)$. Then, $\int f(x)Q_k^{\alpha,\sigma_\varepsilon^x}(\left\langle\widetilde{\mathcal{L}_\alpha},\pi_\varepsilon^x\right\rangle)dx$ converges in L^2 for any bounded continuous function f with compact support towards a limit denoted $\left\langle\widetilde{\mathcal{L}_\alpha^k},f\right\rangle$ and*

$\mathbb{E}(\left\langle\widetilde{\mathcal{L}_\alpha^k},f\right\rangle\left\langle\widetilde{\mathcal{L}_\alpha^l},h\right\rangle)=\delta_{l,k}\frac{\alpha(\alpha+1)...(\alpha+k-1)}{k!}\int G^{2k}(x,y)f(x)h(y)dxdy.$

Proof. The idea of the proof can be understood by trying to prove that $\mathbb{E}((\int f(x)Q_k^{\alpha,\sigma_x^\varepsilon}(\left\langle \widetilde{\mathcal{L}_\alpha}, \pi_\varepsilon^x \right\rangle)dx)^2)$ remains bounded as ε decreses to zero. The idea is to expand this expression in terms of sums of integrals of product of Green functions and check that the combinatorial identities (16) imply the cancelation of the logarithmic divergences.

This is done by showing (as done below in the proof of the theorem) one can modify slightly the products of Green functions appearing in $\mathbb{E}(Q_k^{\alpha,\sigma_\varepsilon^x}(\left\langle \widetilde{\mathcal{L}_\alpha}, \pi_\varepsilon^x \right\rangle)Q_k^{\alpha,\sigma_\varepsilon^y}(\left\langle \widetilde{\mathcal{L}_\alpha}, \pi_\varepsilon^y \right\rangle))$ to replace them by products of the form $G(x,y)^j(\sigma_\varepsilon^x)^l\sigma_\varepsilon^y)^h$. The cancelation of terms containing σ_ε^x and/or σ_ε^y then follows directly from the combinatorial indentities.

Let us now prove the theorem. Consider first, for any $x_1,x_2...x_n$, ε small enough and $\varepsilon \leq \varepsilon_1, ...\varepsilon_n \leq 2\varepsilon$, with $\varepsilon_i = \varepsilon_j$ if $x_i = x_j$, an expression of the form:

$$\Delta = \left| \prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i)(\sigma_{\varepsilon_i}^{x_i})^{m_i} - \int G(y_1, y_2)...G(y_n, y_1)\pi_{\varepsilon_1}^{x_1}(dy_1)...\pi_{\varepsilon_n}^{x_n}(dy_n) \right|$$

in which we define m_i as $\sup(h, x_{i+h} = x_i)$.
In the integral term, we first replace progressively $G(y_{i-1}, y_i)$ by $G(x_{i-1}, x_i)$ whenever $x_{i-1} \neq x_i$, using triangle, then Schwartz inequality, to get an upper bound of the absolute value of the difference made by this substitution in terms of a sum Δ' of expressions of the form

$$\prod_l G(x_l, x_{l+1}) \sqrt{\int (G(y_1, y_2) - G(x_1, x_2))^2 \pi_{\varepsilon_1}^{x_1}(dy_1)\pi_{\varepsilon_2}^{x_2}(dy_2) \int \prod G^2(y_k, y_{k+1}) \prod \pi_{\varepsilon_k}^{x_k}(dy_k)}.$$

The expression obtained after these substitutions can be written

$$W = \prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i) \int G(y_1, y_2)...G(y_{m_{i-1}}, y_{m_i})\pi_{\varepsilon_i}^{x_i}(dy_1)...\pi_{\varepsilon_i}^{x_i}(dy_{m_i})$$

and we see the integral terms could be replaced by $(\sigma_\varepsilon^{x_i})^{m_i}$ if G was translation invariant. But as the distance between x and y tends to 0, $G(x,y)$ is equivalent to $G_0(x,y) = \frac{1}{\pi}\log(\|x-y\|)$ and moreover, $G(x,y) = G_0(x,y) - H^{D^c}(x,dz)G_0(z,y)$, H^{D^c} denoting the Poisson kernel on the boundary of D. As our points lie in a compact inside D, it follows that for some constant C, for $\|y_1 - x\| \leq \varepsilon$, $\left|\int (G(y_1, y_2)\pi_\varepsilon^x(dy_2) - \sigma_\varepsilon^x\right| < C\varepsilon$.
Hence, the difference Δ'' between W and $\prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i)(\sigma_\varepsilon^{x_i})^{m_i}$ can be bounded by $\varepsilon W'$, where W' is an expression similar to W..

To get a good upper bound on Δ, using the previous observations, by repeated applications of Hölder inequality. it is enough to show that for ε small enough , C and C' denoting various constants:

1) $\int (G(y_1,y_2)-G(x_1,x_2)^2 \pi_{\varepsilon_1}^{x_1}(dy_1)\pi_{\varepsilon_2}^{x_2}(dy_2)$
$< C(\varepsilon 1_{\{\|x_1-x_2\|\geq\sqrt{\varepsilon}\}} + (G(x_1,x_2)^2+\log(\varepsilon)^2)1_{\{\|x_1-x_2\|<\sqrt{\varepsilon}\}})$

2) $\int G(y_1,y_2)^k \pi_\varepsilon^x(dy_1)\pi_\varepsilon^x(dy_2) < C\left|\log(\varepsilon)\right|^k$

3) $\int G(y_1,y_2)^k \pi_{\varepsilon_1}^{x_1}(dy_1)\pi_{\varepsilon_2}^{x_2}(dy_2) < C\left|\log(\varepsilon)\right|^k$

As the main contributions come from the singularities of G, they follow from the following simple inequalities:

1')

$$\int \left|\log(\varepsilon^2+2R\varepsilon\cos(\theta)+R^2)-\log(R)\right|^2 d\theta$$
$$= \int \left|\log((\varepsilon/R)^2+2(\varepsilon/R)\cos(\theta)+1)\right|^2 d\theta < C((\varepsilon 1_{\{R\geq\sqrt{\varepsilon}\}\}} + \log^2(R/\varepsilon)1_{\{R<\sqrt{\varepsilon}\}\}})$$

(considering separately the cases where $\frac{\varepsilon}{R}$ is large or small)

2') $\int \left|\log(\varepsilon^2(2+2\cos(\theta)))\right|^k d\theta \leq C\left|\log(\varepsilon)\right|^k$

3') $\int \left|\log((\varepsilon_1\cos(\theta_1)+\varepsilon_2\cos(\theta_2)+r)^2+(\varepsilon_1\sin(\theta_1)+\varepsilon_2\sin(\theta_2))^2\right|^k d\theta_1 d\theta_2 \leq C(|\log(\varepsilon)|)^k$.
It can be proved by observing that for $r\leq\varepsilon_1+\varepsilon_2$, we have near the singularities (i.e. the values $\theta_1(r)$ and $\theta_2(r)$ for which the expression under the log vanishes) to evaluate integrals bounded by $C\int_0^1(-\log(\varepsilon u))^k du \leq C'(-\log(\varepsilon))^k$ for ε small enough.

Let us now show that for $\varepsilon\leq\varepsilon_1,\varepsilon_2\leq 2\varepsilon$, we have, for some integer $N^{n,k}$

$$\left|\mathbb{E}(Q_k^{\alpha,\sigma_x^{\varepsilon_1}}(\left\langle\widetilde{\mathcal{L}_\alpha},\pi_{\varepsilon_1}^x\right\rangle)Q_l^{\alpha,\sigma_y^{\varepsilon_2}}(\left\langle\widetilde{\mathcal{L}_\alpha},\pi_{\varepsilon_2}^y\right\rangle)) - \delta_{l,k}G(x,y)^{2k}\frac{\alpha(\alpha+1)...(\alpha+k-1)}{k!})\right|$$
$$\leq C\log(\varepsilon)^{N_{l,k}}(\sqrt{\varepsilon}+G(x,y)^{2k}1_{\{\|x-y\|<\sqrt{\varepsilon}\}}) \quad (28)$$

Indeed, developing the polynomials and using formula (27) we can express this expectation as a linear combination of integrals under $\prod_i \pi^x_{\varepsilon_1}(dx_i)\prod_j \pi^y_{\varepsilon_2}(dy_j)$ of products of $G(x_i,y_{i'}), G(x_i,x_j)$ and $G(y_j,y_{j'})$ as we did in the discrete case. If we replace each $G(x_i,y_j)$ by $G(x,y)$, each $G(x_i,x_{i'})$ by $\sigma^x_{\varepsilon_1}$ and each $G(y_j,y_{j'})$ by $\sigma^y_{\varepsilon_2}$, we can use the combinatorial identity (16) to get the value $\delta_{l,k}G(x,y)^{2k}\dfrac{\alpha(\alpha+1)...(\alpha+k-1)}{k!}$. Then, the above results allow to bound the error made by this replacement.

The bound (28) is uniform in (x,y) only away from the diagonal as $G(x,y)$ can be arbitrarily large but we conclude from it that for any bounded integrable f and h,

$$\left|\int (\mathbb{E}(Q_k^{\alpha,\sigma_x^{\varepsilon_1}}(\left\langle\widetilde{\mathcal{L}_\alpha},\pi^x_{\varepsilon_1}\right\rangle)Q_l^{\alpha,\sigma_y^{\varepsilon_2}}(\left\langle\widetilde{\mathcal{L}_\alpha},\pi^y_{\varepsilon_2}\right\rangle)) - \delta_{l,k}G(x,y)^{2k}\frac{\alpha...(\alpha+k-1)}{k!})f(x)h(y)dxdy\right| \leq C'\sqrt{\varepsilon}\log(\varepsilon)^{N_{l,k}}$$

(as $\int\int G(x,y)^{2k}1_{\{\|x-y\|<\sqrt{\varepsilon}}dxdy$ can be bounded by $C\varepsilon^{\frac{2}{3}}$, for example).

Taking $\varepsilon_n = 2^{-n}$, it is then straightforward to check that $\int f(x)Q_k^{\alpha,\sigma_x^{\varepsilon_n}}(\left\langle\widetilde{\mathcal{L}_\alpha},\pi^x_{\varepsilon_n}\right\rangle)dx$ is a Cauchy sequence in L^2. The theorem follows. ∎

Specializing to $\alpha=\frac{k}{2}$, k being any positive integer as before, Wick powers of $\sum_{j=1}^k\phi_j^2$ are associated with self intersection local times of the loops. More precisely, we have:

Proposition 53 *The renormalized self intersection local times $\widetilde{\mathcal{L}^n_{\frac{k}{2}}}$ and the Wick powers $\frac{1}{2^n n!}:(\sum_1^k\phi_l^2)^n:$ have the same joint distribution.*

The proof is similar to the one given in [18] and also to the proof of the above theorem, but simpler. It is just a calculation of the L^2-norm of

$$\int [:(\phi^2)^n:(x) - Q_n^{\frac{1}{2},\sigma_x^\varepsilon}(:\phi_x^2:(\pi^x_\varepsilon))]f(x)dx$$

which converges to zero with ε.

Final remarks:

a) These generalized fields have two fundamental properties:

Firstly they are local fields (or more precisely local functionals of the field $\widetilde{\mathcal{L}_\alpha}$ in the sense that their values on functions supported in an open set D depend only on the trace of the loops on D.

Secondly, noting we could use different regularizations to define $\widetilde{\mathcal{L}_\alpha^k}$, the action of a conformal transformation Θ on these fields is given by *the k-th power of the conformal factor* $c = Jacobian(\Theta)$. More precisely, $\Theta(c^k\widetilde{\mathcal{L}_\alpha^k})$ is the renormalized k-th power of the occupation field in $\Theta(D)$.

b) It should be possible to derive from the above remark the existence of exponential moments and introduce non trivial local interactions as in the constructive field theory derived from the free field (Cf [30]).

c) Let us also briefly consider currents. We will restrict our attention to the one and two dimensional Brownian case, X being an open subset of the line or plane. Currents can be defined by vector fields, with compact support.

Then, if now we denote by ϕ the complex valued free field (its real and imaginary parts being two independent copies of the free field), $\int_l \omega$ and $\int_X (\overline{\phi}\partial_\omega\phi - \phi\partial_\omega\overline{\phi})dx$ are well defined square integrable variables in dimension 1 (it can be checked easily by Fourier series). The distribution of the centered occupation field of the loop process "twisted" by the complex exponential $\exp(\sum_{l\in\mathcal{L}_\alpha}\int_l i\omega + \frac{1}{2}\widehat{l}(\|\omega\|^2))$ appears to be the same as the distribution of the field : $\phi\overline{\phi}$: "twisted" by the complex exponential $\exp(\int_X(\overline{\phi}\partial_\omega\phi - \phi\partial_\omega\overline{\phi})dx)$ (Cf[19]).

In dimension 2, logarithmic divergences occur.

d) There is a lot of related investigations. The extension of the properties proved here in the finite framework has still to be completed, though the relation with spanning trees should follow from the remarkable results obtained on SLE processes, especially [16]. Note finally that other essential relations between SLE processes, loops and free fields appear in [36], [28] and [5].

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