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Automata on the Plane vs Particles and Collisions

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Abstract

In this note, the coloring of the plane by finite sequential machines is compared to previously introduced notions of ultimately periodic tilings of the plane. Finite automata with no counter characterize exactly biperiodic tilings. Finite automata with one counter characterize exactly particles: periodic colorings that are ultimately periodic in every direction. Finite automata with two counters and aperiodic colorings characterize exactly collisions: ultimately periodic tilings of the plane.

Key words: counter automata, languages, two-dimensional, tilings

Introduction

In [1], motivated by the study of space-time diagrams of cellular automata, we introduced collisions as a practical notion of ultimately periodic tiling of the plane: an extension of the notion of ultimately periodic biinfinite words to infinite bidimensional tilings. Intuitively, ultimately periodic words and collisions share the property to be locally almost everywhere periodic in every direction. Imagine that you are walking on the plane, trying to color it according to a collision: you only need to keep a finite information, your position inside the biperiodic pattern, plus a way to store your distance to the boundaries: places where one should switch from a biperiodic region to another. All you need to know about this distance is when it becomes equal to zero. Therefore, counter machine coloring the plane can certainly encode every collision.

In this note, we explore the analogy between regular tilings and colorings by counter machines. A map automaton over a free monoid is a deterministic counter machine that starts in a given initial state with empty counters on the unit element of the monoid. The automaton walks on the monoid by firing

transitions labelled by the generator associated to each of its move. Such an automaton can certainly color each element of the monoid with a finite set of colors according to its state. To color a group, like the euclidian plane \mathbb{Z}^2 , with a map automaton, we simply choose a monoid presentation for the group and require the automaton to be compatible with the group structure – that is, to have the same state and counter values on two elements of the free monoid corresponding to a same element of the group.

As expected, in the case of biinfinite words, map automata with no counter capture periodic words; map automata with one counter capture ultimately periodic words; and map automata with two counters can paint arbitrarily complex recursive tilings. In the case of bidimensional tilings, map automata coloring the plane with a periodicity vector act like a finite family of map automata on biinfinite words. Thus, map automata with no counter capture biperiodic tilings; map automata with one counter capture particles. In the case of two counters automata, if the coloring is periodic then it can be arbitrarily recursively complex. However, aperiodic map automata with two counters capture collisions. In the case of aperiodic colorings, the two counters of a map automata acts like a compass pointing to the origin cell using finitely many biperiodic quadrants: this is a collision.

The note is organized as follows. In section 1, we introduce map automata on monoid presentations and some of their properties. Section 2 studying map automata on \mathbb{Z} and section 3 studying map automata on \mathbb{Z}^2 are constructed symmetrically: notions of regular colorings are first defined before a sequential study of automata with 0, 1 and 2 counters.

1 Definitions

Let Σ be a finite alphabet, Σ^* is the free monoid generated by Σ , the set of words on Σ , with *empty word* ϵ . The catenation of $u \in \Sigma^*$ and $v \in \Sigma^*$ is denoted as uv . A *finite monoid presentation* is a pair (G, R) where G is a finite alphabet of *generators* and $R \subseteq G^* \times G^*$ is a finite set of *relators*. The *monoid* $\mathcal{G} = \langle G | R \rangle$ associated to the presentation is the largest monoid satisfying the relators equations, *i.e.* such that for each $(u, v) \in R$, $u = v$ in this monoid.

A \mathcal{G} -coloring is a mapping $c : \mathcal{G} \rightarrow \Sigma$. It is *periodic*, with period $z \in \mathcal{G} \setminus \{\epsilon\}$, if for all $z' \in \mathcal{G}$, $c(zz') = c(z')$. It is *aperiodic* if it is not periodic. It is *biperiodic*, with periods $z, z' \in \mathcal{G} \setminus \{\epsilon\}$, if z and z' are two non-collinear periods, *i.e.* there does not exist $k, k' \in \mathbb{Z}^+$ such that $z^k = z'^{k'}$.

Let $\Upsilon = \{0, +\}$ and $\Phi = \{-, 0, +\}$ be respectively the set of *test values* and *counter operations*. Let $\mathbf{0}$ denote the constant k -uple $(0, \dots, 0)$. For all $\phi \in \Phi^k$,

testing τ and modifying actions are defined for all $i \in \mathbb{Z}_k$, $v \in \mathbb{N}^k$ and $j \in \mathbb{N}$ as:

$$\tau(j) = \begin{cases} 0 & \text{if } j = 0 \\ + & \text{if } j > 0 \end{cases} \quad \theta_\phi(v)(i) = \begin{cases} \max(0, v(i) - 1) & \text{if } \phi_i = - \\ v(i) & \text{if } \phi_i = 0 \\ v(i) + 1 & \text{if } \phi_i = + \end{cases}$$

A k -counter map automaton on the alphabet Σ is a tuple $(\Sigma, k, S, s_0, \delta)$ where $k \in \mathbb{N}$ is the number of counters, S is a finite set of states with initial state $s_0 \in S$ and $\delta : S \times \Upsilon^k \times \Sigma \rightarrow S \times \Phi^k$ is the transition rule of the automaton. Its transition function $f : S \times \mathbb{N}^k \times \Sigma^* \rightarrow S \times \mathbb{N}^k$ is recursively defined on Σ^* by $f(s, v, \epsilon) = (s, v)$ and $f(s, v, za) = (s'', \theta_\phi(v'))$ where $f(s, v, z) = (s', v')$ and $\delta(s', \tau(v'), a) = (s'', \phi)$ for all $s \in S$, $v \in \mathbb{N}^k$, $z \in \Sigma^*$ and $a \in \Sigma$.

A k -counter map automaton (k -CMA) \mathcal{A} on the monoid presentation $\mathcal{G} = \langle G | R \rangle$ is a tuple $(\mathcal{G}, k, S, s_0, \delta)$ where (G, R) is a finite presentation of \mathcal{G} and (G, k, S, s_0, δ) is a k -counter map automaton on the alphabet G compatible with the monoid structure, i.e. satisfying $f(s_0, \mathbf{0}, zz_1) = f(s_0, \mathbf{0}, zz_2)$ for all $z \in G^*$ and $(z_1, z_2) \in R$. Its mapping function $g : \mathcal{G} \rightarrow S \times \mathbb{N}^k$ is defined, for all $z \in \mathcal{G}$, as $g(z) = f(s_0, \mathbf{0}, z)$. Its minimum (resp. maximum) counter function $\min_c : \mathcal{G} \rightarrow \mathbb{N}$ (resp. \max_c) is defined for all $z \in \mathcal{G}$ as $\min_c(z) = \min_{i \in \mathbb{Z}_k} v_i$ (resp. $\max_c(z) = \max_{i \in \mathbb{Z}_k} v_i$) where $g(z) = (s, v)$. Two distinct elements $z, z' \in \mathcal{G}$ are undistinguished by \mathcal{A} if $g(z) = g(z')$. An element $z \in \mathcal{G}$ is discriminative under \mathcal{A} if $\min_c(z) = 0$. A connected subset Z of \mathcal{G} is independent under \mathcal{A} if $\{\min_c(z) \mid z \in Z\}$ is an infinite subset of \mathbb{Z}^+ . Notice that a subset of \mathcal{G} independent under \mathcal{A} does not have any discriminative points under \mathcal{A} . The automaton is periodic, with period $z \in \mathcal{G} \setminus \{\epsilon\}$, if for all $z' \in \mathcal{G}$, $g(z') = g(z'z)$.

Lemma 1. *A k -CMA \mathcal{A} on a group \mathcal{G} is periodic if and only if two elements of \mathcal{G} are undistinguished by \mathcal{A} .*

Proof. Let \mathcal{A} be a k -CMA on a group \mathcal{G} . If it is periodic with period $z \in \mathcal{G}$ then ϵ and z are undistinguished. Conversely, if $z, z' \in \mathcal{G}$ are undistinguished then $z' - z$ is a valid period. \blacksquare

The projector $\pi_1 : S \times \mathbb{N}^k \rightarrow S$ is defined for all $s \in S$ and $v \in \mathbb{N}^k$ by $\pi_1(s, v) = s$. The coloring of \mathcal{A} by $\varphi : S \rightarrow \Sigma$ is the mapping $c \in \Sigma^{\mathcal{G}}$ satisfying $c(z) = \varphi(\pi_1(g(z)))$ for all $z \in \mathcal{G}$. The \mathcal{G} - k -map set is the set of all colorings of \mathcal{G} by all k -counter map automata. The translated of a coloring c , by a vector $z \in \mathcal{G}$, is the coloring $c_z \in \Sigma^{\mathcal{G}}$ defined for all $z' \in \mathcal{G}$ by $c_z(z') = c(z'z)$.

Lemma 2. *Every \mathcal{G} - k -map set is closed under translation.*

Proof. Let c be a coloring of a k -CMA $(\mathcal{G}, k, S, s_0, \delta)$ by φ . Let $z \in \mathcal{G}$ be a vector and $m_z = \max_c(z)$. Let us consider functions $b : \mathbb{N} \rightarrow [0, \dots, m_z]$ and $t : \mathbb{N} \rightarrow \mathbb{N}$ defined for all $n \in \mathbb{N}$ by $b(n) = \min(m_z, n)$ and $t(n) = \max(0, n - m_z)$. Those two functions can be naturally extended to \mathbb{N}^k . Let $S' = S \times [0, \dots, m_z]^k$ and $e : S \times \mathbb{N}^k \rightarrow S' \times \mathbb{N}^k$ be defined by $e(s, v) = ((s, b(v)), t(v))$ for all $(s, v) \in S \times \mathbb{N}^k$. Let \mathcal{A}' be the k -CMA $(\mathcal{G}, k, S', s'_0, \delta')$ chosen such that, for all $z' \in \mathcal{G}$, its mapping function g' satisfies : $g'(z') = e(g(zz'))$ (in particular s' is the first component of $e(g(z))$). Straightforwardly, the translated c_z of c by z is the coloring of \mathcal{A}' by $\varphi' : S \times [0, \dots, m_z]^k \rightarrow \Sigma$ defined, for all $s \in S$ and $t \in [0, \dots, m_z]^k$, as $\varphi'((s, t)) = \varphi(s)$. ■

Lemma 3. *Let c be a coloring of a k -CMA \mathcal{A} on a group \mathcal{G} . Let $Z \subseteq \mathcal{G}$ be independent under \mathcal{A} . There exists a \mathcal{G} -0-map c' such that $c|_Z = c'|_Z$.*

Proof. Let \mathcal{A} be a k -CMA $(\mathcal{G}, k, S, s_0, \delta)$. Let Z be a subset of \mathcal{G} independent under \mathcal{A} . For all $s \in S$, let $Z_s = Z \cap g^{-1}(\{s\} \times \mathbb{N}^k)$. S being finite, there exists $s'_0 \in S$ such that $\{min_c(z) \mid z \in Z_{s'_0}\}$ is infinite. Let \mathcal{A}' be the 0-CMA $(\mathcal{G}, 0, S, s'_0, \delta')$ where $\delta'(s, a) = \delta(s, +, a)$ for all $s \in S$ and $a \in G$. Let's first prove that \mathcal{A}' is indeed a 0-CMA. Let $z \in G^*$ and $(z', z'') \in R$. Let $N = \max(|zz'|, |zz''|)$. By construction, there exists $z_N \in Z_{s'_0}$ such that $\min_c z_N > N$. Since no discriminative point is encountered on the path from z_N to $z_N z z'$, $f'(s'_0, z z') = s$ where $(s, n) = f(g(z_N), z z')$. The same holds for $z z''$. Let z_0 be any element of $Z_{s'_0}$, Z being independent, $c'(z) = c(z_0 z)$ for all $z \in Z$. By lemma 2, the translated c'_{z_0} of c' by z_0 is a \mathcal{G} -0-map equal to c on Z . ■

2 Map Automata on \mathbb{Z}

In the following, we denote by \mathbb{Z} the one-dimensional grid

$$(\mathbb{Z}, +) = \langle l, r \mid lr = \epsilon, rl = \epsilon \rangle,$$

the presentation is embed with the canonical morphism $r = 1$.

2.1 Regular Colorings of \mathbb{Z}

For $i \in \mathbb{Z}$ and $p \in \mathbb{Z}^*$, let us denote as $i[p]$ the remainder of the division of i by p . On \mathbb{Z} , a periodic coloring c can be characterised by a finite pattern $u \in \Sigma^p$ such that for all $i \in \mathbb{Z}$, $c(i) = u_{i[p]}$. A coloring is *ultimately periodic* with period $p \in \mathbb{Z}^+$ and *defect* $k \in \mathbb{N}$, if for all element $i \in \mathbb{Z}$, $|i| > k$ implies $c(i+p) = c(i)$. Ultimately periodic colorings correspond to colorings which are periodic out of a finite support. They can also be characterised by three finite

words $u, v \in \Sigma^p$ and $w \in \Sigma^{2k+1}$ such that for all element $i \in \mathbb{Z}$, $c(i) = u_{i[p]}$ if $i < -k$, $c(i) = v_{i[p]}$ if $i > k$ and $c(i) = w_{i+k}$ otherwise. Notice that periodicity is a special case of ultimate periodicity.

2.2 Automata with no counter

Theorem 4. *A \mathbb{Z} -coloring is a \mathbb{Z} -0-map if and only if it is periodic.*

Proof. Let $g : \mathbb{Z} \rightarrow S$ be the mapping function of a 0-CMA \mathcal{A} . S being finite, there exists two elements undistinguished by \mathcal{A} . By lemma 1, the automaton is periodic.

Conversely, let c be a periodic \mathbb{Z} -coloring with period $p \in \mathbb{Z}^+$. Let \mathcal{A} be the 0-CMA $(\mathbb{Z}, 0, \mathbb{Z}_p, 0, \delta)$ where $\delta(i, l) = i - 1$ and $\delta(i, r) = i + 1$. The coloring of \mathcal{A} by $\varphi : i \mapsto c(i)$ is c . ■

2.3 Automata with 1 counter

Theorem 5. *A \mathbb{Z} -coloring is a \mathbb{Z} -1-map if and only if it is ultimately periodic.*

Proof. Let $g : \mathbb{Z} \rightarrow S \times \mathbb{N}$ be the mapping function of a 1-CMA \mathcal{A} . \mathbb{Z} -0-maps being \mathbb{Z} -1-maps, we can assume that g is one to one (*i.e.*, has no undistinguished elements). Thus, there exists $k \in \mathbb{N}$ such that $g^{-1}(S \times \{0\}) \subseteq [-k, k]$. By lemma 3, on both $]-\infty, -k[$ and $]k, +\infty[$, g is periodic and thus c is ultimately periodic.

Conversely, Let c be an ultimately periodic \mathbb{Z} -coloring with period $p \in \mathbb{Z}^+$ and defect $k \in \mathbb{N}$. Let \mathcal{A} be the 1-CMA over set of states $[-k - p, k + p]$ whose mapping function is defined for all elements $i \in \mathbb{Z}$ by:

$$g(i) = \begin{cases} (i, 0) & \text{if } |i| \leq k + p \\ ((i - k)[p] + k, \lfloor (i - k)/p \rfloor) & \text{if } i > k + p \\ (i + k[p] - k - p, \lfloor (k - i)/p \rfloor) & \text{if } i < -k - p \end{cases}$$

The coloring of \mathcal{A} by $\varphi : i \mapsto c(i)$ is c . ■

2.4 Automata with 2 counters

It is well known that finite automata with two counters can simulate any Turing machine (see Minsky [2]). Morita [3], improved the result proving that the simulation can be done in a reversible way. It is therefore no surprise that those machines can embed any computation of a Turing machine and encode any recursively enumerable language.

Theorem 6. *There exists a \emptyset' -complete \mathbb{Z} -2-map.*

Sketch of the proof. Let K be a \emptyset' -complete language containing 0 and $(n_i)_{i \in \mathbb{N}}$ be a computable enumeration without repetition of K satisfying $n_0 = 0$. There exists a one to one computable function that on input n_i computes n_{i+1} . Combining Morita construction [3] with techniques of [4], one can construct a reversible 2-counters machine with set of states $S_A \cup \{s_\alpha, s_\omega\}$ such that, for all $i \in \mathbb{N}$, starting from $(s_\alpha, (n_i, 0))$, the machine eventually halts in configuration $(s_\omega, (n_{i+1}, 0))$. It is also possible to construct a reversible 2-counter machine with set of states $S_B \cup \{s_\alpha, s_\omega\}$ such that, for all $i \in \mathbb{N}$, starting from $(s_\omega, (n_i, 0))$, the machine halts in configuration $(s_\alpha, (n_i, 0))$ after exactly $2n_i$ steps of computations. By making disjoint union of these two machines, one can construct 2-CMA \mathcal{A} with set of states $S = S_A \cup S_B \cup \{s_\alpha, s_\omega\}$ sharing transitions of both machines. The transition function works by the applying 2-counter machine transition on generator r and reverse transition on generator l . Let c be the coloring of \mathcal{A} by $\varphi : S \rightarrow \{0, 1\}$ defined for all $s \in S$ by $\varphi(s) = 1$ if and only if $s \in S_B$. The coloring c contains the factor $01^{2n}0$ if and only if $n \in K$. ■

3 Map Automata on \mathbb{Z}^2

In the following, we denote by \mathbb{Z}^2 the two-dimensional grid

$$(\mathbb{Z}^2, +) = \langle n, s, e, w \mid ns = \epsilon, sn = \epsilon, ew = \epsilon, we = \epsilon, ne = en \rangle \quad ,$$

the presentation is embed with the canonical morphism $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

3.1 Regular Colorings of \mathbb{Z}^2

The case of \mathbb{Z}^2 is strongly linked with the previous case as the following lemma suggests.

Lemma 7. *Each line (resp. column) of a \mathbb{Z}^2 - k -map is a \mathbb{Z} - k -map.*

Proof. Let c be a \mathbb{Z}^2 - k -map. By definition, the restriction of c to $\{0\} \times \mathbb{Z}$ (resp. $\mathbb{Z} \times \{0\}$) is a \mathbb{Z} - k -map. By lemma 2, this result is valid for every line (resp. column). ■

An easy corollary is that periodic map automaton on \mathbb{Z}^2 acts as periodic copies of map automaton on \mathbb{Z} . To define regular coloring of \mathbb{Z}^2 , we chose the approach presented in [1]. Simplest element is a biperiodic coloring. A *particle* is a coloring with one direction of periodicity and ultimate periodicity in every other direction. Let $\sphericalangle_v(u, u')$ be the angular portion of the plane, on the right hand side of u , starting in position $v \in \mathbb{Z}^2$ and delimited by the vectors $u, u' \in \mathbb{Z}^2$. A *collision* as a coloring c characterised by a sequence of m vectors $(u_i)_{i \in \mathbb{Z}_m}$ such that for all $i \in \mathbb{Z}_m$, the coloring is u_i -periodic in the cone between u_{i-1} and u_{i+1} starting from u_i , *i.e.*, for all $i \in \mathbb{Z}_m$, and $z \in \sphericalangle_{u_i}(u_{i-1}, u_{i+1})$, $c(z + u_i) = c(z)$. The *ball* of radius r and center (x, y) is the set $[x - r, x + r] \times [y - r, y + r]$. When no center is specified, it implicitly refers to center $(0, 0)$.

3.2 Automata with no counter

Theorem 8. *A \mathbb{Z}^2 -coloring is a \mathbb{Z}^2 -0-map if and only if it is biperiodic.*

Proof. Let $g : \mathbb{Z} \rightarrow S$ be the mapping function of a 0-CMA \mathcal{A} . S being finite, there exists two elements undistinguished by \mathcal{A} . Thus, there is only a finite number of lines or columns. By lemma 7 and theorem 4, each of them is periodic.

Let c be a biperiodic \mathbb{Z}^2 -coloring with period $(m, 0)$ and $(0, n)$ (such canonical periods always exists). Let \mathcal{A} be the 0-CMA $(\mathbb{Z}^2, 0, \mathbb{Z}_m \times \mathbb{Z}_n, (0, 0), \delta)$ where $\delta((x, y), e) = (x + 1, y)$ and $\delta((x, y), n) = (x, y + 1)$. c is the coloring of \mathcal{A} by $\varphi : \mathbb{Z}_m \times \mathbb{Z}_n \rightarrow \Sigma$ defined, for all $x \in \mathbb{Z}_m$ and $y \in \mathbb{Z}_n$, by $\varphi(x, y) = c(x, y)$. ■

3.3 Automata with 1 counter

Theorem 9. *A \mathbb{Z}^2 -coloring is a \mathbb{Z}^2 -1-map if and only if it is a particle.*

Proof. Let c be a \mathbb{Z}^2 -1-map. Suppose that c is not periodic. By lemma 7, all lines of c are non periodic \mathbb{Z} -1-map and by lemma 3 each of them contains at least one discriminative point. $S \times \{0\}$ being finite, we reach a contradiction.

Thus, c is made of a finite number of lines (or columns) which are \mathbb{Z} -1-map. Since each of those map is ultimately-periodic, c is a particle.

Conversely, let c be a particle with period $u = (x, y)$ with $y > 0$ (this can always be achieved up to exchanging axes). Thus c consists of periodic repetitions of y ultimately periodic lines. Without loss of generality, we can assume that all the lines are $p \in \mathbb{Z}^+$ periodic with defect p satisfying $p > x$. Thus, we can use the same construction as in proof of theorem 5 on the whole block of lines. Let \mathcal{A} be a 1-CMA automaton over the set of states $[-k - p, k + p] \times [0, y - 1]$ whose mapping function is defined for all element $(i, j) \in \mathbb{Z}^2$ by $g(i, j) = \tilde{g}(i + \lfloor j/y \rfloor x, j \lfloor y \rfloor)$ where

$$\tilde{g}(\tilde{i}, \tilde{j}) = \begin{cases} ((\tilde{i}, \tilde{j}), 0) & \text{if } |\tilde{i}| \leq 2p \\ ((\tilde{i}\lfloor p \rfloor + p, \tilde{j}), \lfloor \tilde{i}/p \rfloor - 1) & \text{if } \tilde{i} > 2p \\ ((\tilde{i}\lfloor p \rfloor - 2p, \tilde{j}), \lfloor -\tilde{i}/p \rfloor - 1) & \text{if } \tilde{i} < -2p \end{cases}$$

The coloring of \mathcal{A} by $\varphi : (i, j) \mapsto c(i, j)$ is c . ■

3.4 Automata with 2 counters

Theorem 10. *There exists a \emptyset' -complete \mathbb{Z}^2 -2-map.*

Proof. It is possible to extend any \mathbb{Z} - k -map to a \mathbb{Z}^2 - k -map by using identity function on e, w . Thus, existence of a \emptyset' -complete \mathbb{Z} -2-map induces existence of a \emptyset' -complete \mathbb{Z}^2 -2-map. ■

Theorem 11. *Every collision is a \mathbb{Z}^2 -2-map.*

Proof. Let c be a collision. For all $k \in \mathbb{N}$, let $w(k) = c_{|[-k, k] \times \{k\}}$ be the k -th northern sphere word of c . There exists a integer $K \in \mathbb{N}$ such that for all $k > K$, $w(k)$ is only included in cones (*i.e.*, avoid the central perturbation). The number of cones being finite, let $l \in \mathbb{N}$ be a multiple of vertical component of all the involved vectors. Then, for all $l' \in [0, \dots, l - 1]$, there exists $n \in \mathbb{N}$, $a(l', 0) \dots a(l', n) \in \Sigma^*$ and $b(l', 0) \dots b(l', n - 1) \in \Sigma^*$ such that for all $k \in \mathbb{N}$, $kl > K$, $w(kl + l') = a(l', 0)b(l', 0)^k a(l', 1)b(l', 1)^k \dots a(l', n - 1)b(l', n - 1)^k a(l', n)$. Notice that the set of constructed words $W = \{a(i, j) \mid i \in [0, \dots, l' - 1], i \in [0, \dots, n]\} \cup \{b(i, j) \mid i \in [0, \dots, l' - 1], i \in [0, \dots, n - 1]\}$ is finite. Using similar techniques as previously, let us consider the partial mapping function g which maps any element to the corresponding letter in W . Moreover, local transition function is chosen such that, for any element in $w(kl + l')$, counter are equal to $(0, k)$ (resp. $(k, 0)$) if the corresponding letter

is in $a(l', 2i)$ (resp. $a(l', 2i + 1)$) and $(i, k - i)$ (resp. $(k - i, i)$) if the letter is in the i -th repetition of the word $b(l', 2i)$ (resp. $b(l', 2i + 1)$). One can note that g can be achieved by a local transition function. The last remaining problem is that g is, for now, only defined on the northern quarter of the plane.

Of course, the previous construction can be also achieved on all other quarters of the plane. What is left is to prove that those four constructions can be chosen so that they match on boundaries. To do this, one has just to look at the diagonals (it is the only place where two or more constructions overlap). Firstly, remark that value of the maximum counter depends on l which, up to taking a common multiple, can be set equal for all quarters. Secondly, the empty counter depends on the parity of number of particle involved. Since background can also be seen as particle, one can easily introduce “phantom” particles to get rid of this problem.

The last point is that the mapping resulting of the union of the four constructions is defined on all but the center which consists on a finite number of points. Up to introducing new states, one can extend this mapping function to the one of a 2-CMA on \mathbb{Z}^2 . ■

Theorem 12. *Every aperiodic \mathbb{Z}^2 -2-map is a collision.*

Proof. Let us take \mathcal{A} a 2-CMA $(\mathbb{Z}^2, 2, S, s_0, \delta)$. \mathcal{A} being aperiodic, its mapping function g is one to one. In a first step, let us prove that for any ball of radius r containing n discriminative points under \mathcal{A} , there is $n + 1$ discriminative points under \mathcal{A} in the ball of radius $r + |S| + 1$.

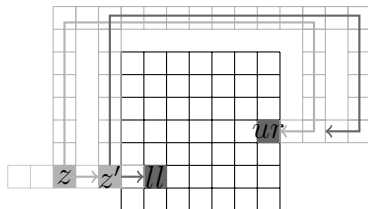


Fig. 1. Finding a new discriminative point

Let \mathcal{B} be a ball of radius r containing n discriminative points. Assume that \mathcal{B}' the ball of radius $r + |S| + 1$ does not contain any discriminative points. Let ul (resp. ur , ll) be one extremal upper-left (resp. upper-right, lower-left) discriminative point (see Fig. 1). Note that those points does not need to be distinct. Without loss of generality, one can assume that ur and ll have both the first counter empty. Let us consider the set $\{ll - (i, 0) \mid i \in \mathbb{N}\}$ of elements left of ll . S being finite, there exists two elements z and z' in $\mathcal{B}' \setminus \mathcal{B}$ such that $g(z) = (s, (a_0, a_1))$ and $g(z') = (s, (b_0, b_1))$ for some $s \in S$ and $a_0, a_1, b_0, b_1 \in \mathbb{Z}^+$. One can assume that z is the left one. There exist a path $r^i, i \in \mathbb{N}^+$ from z' to ll . Since the same path starting from z' does not encounter any discriminative point, $a_0 - (b_0 - 0) > 0$. By doing the same reasoning on

the path from z' to ur in \mathcal{B}' avoiding \mathcal{B} , we can deduce that $b_0 - (a_0 - 0) > 0$ which gives a contradiction.

With this property on discriminative points, it is possible to establish a strong property on distribution of discriminative points and show that each discriminative point is located either in a fixed finite ball \mathcal{B} or on an infinite half-line starting from this ball. Moreover, there is only a finite number of such half-lines.

Let \mathcal{B} be the ball containing all discriminative points whose counters are both less than $(|S| + 1)^3$. By iterating previous result on discriminative points, for each discriminative point z , there is at least $|S| + 1$ discriminative points in the ball of radius $(|S| + 1)^2$ centered around this point. Thus for each discriminative point, there exists, a path of length less than $(|S| + 1)^3$ containing $|S| + 1$ discriminative points. Either two discriminative points have distinct empty counter which implies that non empty counter of z is less than $(|S| + 1)^3$, or there exists two points z_a and z_b such that $g(z_a) = (s, (a, 0))$ and $g(z_b) = (s, (b, 0))$ for some $s \in S$ and $b > a \in \mathbb{N}^+$. Let $z \in G^*$ such that $z_a z = z_b$. One can check that $g(z_a z^n) = (s, (a - n(b - a), 0))$ for all n such that $a - (b - a)^n > (|S| + 1)^3$. Thus z_a is on a half-line starting from an element in \mathcal{B} . Slope of this half-line depends only on the point in the ball and $b - a < (|S| + 1)^3$ which only leave a finite number of possibilities.

The last intermediate result needed is that any point $z \in \mathbb{Z}^2$ such that $\min_c z < N$ has one discriminative point under \mathcal{A} at distance at most $N(|S| + 1)$. To prove this, let \mathcal{B} be the ball of center $z \in \mathbb{Z}^2$ and of radius $|S|$. This ball contains two distinct points $z_a, z_b \in \mathbb{Z}^2$ whose mapping has the same state. Applying same reasoning as previously allow to obtain the desired result. A useful corollary of this result is that any connected component between two consecutive (but not parallel) half-lines of discriminative points is independent since it contains ball of arbitrary size.

To conclude the proof, let us show that the map is a collision characterised by the sequence $u \in \mathbb{Z}^{2p}$ of half-lines ordered by slope. Since all half-lines start inside a finite ball, there exist an element $k \in \mathbb{N}$ such that all discriminative points in the cone $\angle_{ku_i}(ku_{i-1}, ku_{i+1})$ are on a half-line of vector u_i for all $i \in \mathbb{Z}_p$. In this cone, we have elements of the half-line (which are by construction u_i periodic) and elements of connected components between two consecutive half-lines. Those connected components are independent by the previous corollary. By lemma 3, they are bi-periodic and thus also ku_i periodic for some $k \in \mathbb{N}^+$.

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