



**HAL**  
open science

**Discussion of "Second order topological sensitivity analysis" by J. Rocha de Faria et al. [International Journal of Solids and Structures, 44:4958-4977 (2007)]**

Marc Bonnet

► **To cite this version:**

Marc Bonnet. Discussion of "Second order topological sensitivity analysis" by J. Rocha de Faria et al. [International Journal of Solids and Structures, 44:4958-4977 (2007)]. International Journal of Solids and Structures, 2008, 45, pp.705-707. 10.1016/j.ijsolstr.2007.08.011 . hal-00310462v1

**HAL Id: hal-00310462**

**<https://hal.science/hal-00310462v1>**

Submitted on 9 Aug 2008 (v1), last revised 16 Apr 2010 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Discussion of “Second order topological sensitivity analysis” by J. Rocha de Faria et al. <sup>\*</sup>

Marc Bonnet <sup>\*</sup>

*Solid Mechanics Laboratory (UMR CNRS 7649), Ecole Polytechnique,  
F-91128 Palaiseau cedex, France*

---

## Abstract

The aim of this discussion is to expose incorrect results in a previous IJSS article.

*Key words:* topological sensitivity, Laplace equation

---

**Preliminaries.** The article by Rocha de Faria et al. (2007) under discussion is concerned with the evaluation of the perturbation undergone by the potential energy of a domain  $\Omega$  (in a 2-D, scalar Laplace equation setting) when a disk  $B_\varepsilon$  of small radius  $\varepsilon$  centered at a given location  $\hat{\mathbf{x}} \in \Omega$  is removed from  $\Omega$ , assuming either Neumann or Dirichlet conditions on the boundary of the small ‘hole’ thus created. In each case, the potential energy  $\psi(\Omega_\varepsilon)$  of the punctured domain  $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$  is expanded about  $\varepsilon = 0$  so that the first two terms of the perturbation are given. The first (leading) term is the well-documented topological derivative of  $\psi$ . The article under discussion places, logically, its main focus on the next term of the expansion. However, it contains incorrect results, as shown in this discussion. In what follows, equations referenced with Arabic numbers refer to those of the article under discussion.

**Topological expansion: Neumann condition on the hole.** In the main result proposed by Rocha de Faria et al. (2007) for this case, namely expression (37) for the topological expansion of the potential energy, the first term (whose order is  $O(\varepsilon^2)$ ) is correct but the second (whose order is  $O(\varepsilon^4)$ ) is not

---

<sup>\*</sup> *International Journal of Solids and Structures*, 44:4958-4977 (2007).

<sup>\*</sup> Corresponding author

*Email address:* bonnet@lms.polytechnique.fr (Marc Bonnet).

as it lacks a contribution of the same order related to the external boundary (see Bonnet, 2006a, for a similar study in 3-D linear acoustics).

This error can be explained as follows. Equation (37) is based on an expansion of

$$\frac{d}{d\varepsilon}\psi(\Omega_\varepsilon) = -\frac{1}{2}\int_{\partial B_\varepsilon}(\nabla u_\varepsilon \cdot \mathbf{e}_\theta)^2 ds \quad (\text{i})$$

up to order  $O(\varepsilon^3)$  (where  $(\mathbf{e}_r, \mathbf{e}_\theta)$  are the unit vectors associated with polar coordinates  $(r, \theta)$  originating at the center of  $B_\varepsilon$ ). Since  $ds = \varepsilon d\theta$  on  $\partial B_\varepsilon$ , this task requires expanding  $(\nabla u_\varepsilon(\mathbf{x}) \cdot \mathbf{e}_\theta)^2$  to order  $O(\varepsilon^2)$  for  $\mathbf{x} \in \partial B_\varepsilon$ . The latter operation is carried out in Rocha de Faria et al. (2007) by evaluating  $\nabla u_\varepsilon(\mathbf{x})$  from the  $O(\varepsilon^2)$  expansion (23) of  $u_\varepsilon$ . However, expansion (23) evaluated on  $\partial B_\varepsilon$  gives

$$\nabla u_\varepsilon(\mathbf{x}) \cdot \mathbf{e}_\theta = 2\nabla u(\hat{\mathbf{x}}) \cdot \mathbf{e}_\theta + 2\varepsilon \nabla \nabla u(\hat{\mathbf{x}}) : (\mathbf{e}_r \otimes \mathbf{e}_\theta) + O(\varepsilon^2) \quad (\mathbf{x} \in \partial B_\varepsilon),$$

and is therefore not suitable for expanding  $(\nabla u_\varepsilon \cdot \mathbf{e}_\theta)^2$  to order  $O(\varepsilon^2)$  as it lacks the necessary  $O(\varepsilon^2)$  contribution to  $\nabla u_\varepsilon \cdot \mathbf{e}_\theta$ . The missing  $O(\varepsilon^2)$  term stems from the  $O(\varepsilon^3)$  contribution to  $u_\varepsilon$  and is in fact non-local as it is expressed in terms of quantities on  $\partial\Omega$  rather than higher-order gradients of  $u$  at  $\hat{\mathbf{x}}$ .

The incorrectness of result (37) can be further demonstrated on a simple analytical example. Consider the 2-D domain  $\Omega_\varepsilon$  enclosed by two concentric circles of radii  $\varepsilon$  and  $a$ , i.e.  $\partial B_\varepsilon = \{(r, \theta) \mid r = \varepsilon\}$  and  $\partial\Omega = \{(r, \theta) \mid r = a\}$  in terms of polar coordinates  $(r, \theta)$ . The solution  $u_\varepsilon$  of the Laplace equation with boundary conditions

$$u_{,n} = 0 \quad (r = \varepsilon), \quad u_{,n} \equiv q = \cos \theta \quad (r = a)$$

and the corresponding reference solution  $u$  when there is no hole are respectively given (up to an arbitrary additive constant) by

$$u_\varepsilon(r, \theta) = \frac{a^2}{a^2 - \varepsilon^2} \left( r + \frac{\varepsilon^2}{r} \right) \cos \theta, \quad u(r, \theta) = r \cos \theta \quad (\text{ii})$$

Note that the reference solution  $u$  is such that  $\nabla u(\hat{\mathbf{x}}) = \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta$  and  $\nabla \nabla u(\hat{\mathbf{x}}) = \mathbf{0}$ . Then, a simple calculation gives

$$\psi(\Omega_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla u_\varepsilon dV - \int_{\partial\Omega} q u_\varepsilon ds = -\frac{1}{2} \int_{\partial\Omega} q u_\varepsilon ds = -\frac{\pi a^2}{2} \frac{a^2 + \varepsilon^2}{a^2 - \varepsilon^2}$$

Expanding  $\psi(\Omega_\varepsilon)$  to order  $O(\varepsilon^4)$  gives

$$\psi(\Omega_\varepsilon) = -\frac{\pi a^2}{2} - \pi \varepsilon^2 - \frac{\pi}{a^2} \varepsilon^4 + o(\varepsilon^4) \quad (\text{iii})$$

while equation (37) incorrectly gives the expansion as

$$\psi(\Omega_\varepsilon) = -\frac{\pi a^2}{2} - \pi\varepsilon^2 - 0 \times \varepsilon^4 + o(\varepsilon^4) \quad (\text{iv})$$

Note that the error in (iv) vanishes as  $\partial\Omega$  is rejected to infinity, i.e. as the influence of the external boundary goes away. This is analogous to secondary reflection effects in small-obstacle approximations for wave problems.

**Topological expansion: Dirichlet condition on the hole.** The topological expansion (38) is also not correct. Expansion (38) states that

$$\psi(\Omega_\varepsilon) = \psi(\Omega) + \pi\left(\frac{-1}{\text{Log } \varepsilon}\right)[u(\hat{\mathbf{x}})]^2 + \pi\|\nabla u(\hat{\mathbf{x}})\|^2\varepsilon^2 + o(\varepsilon^2). \quad (\text{v})$$

However, another simple analytical example again allows to show that the second term in (v), is not correct. With the domain  $\Omega_\varepsilon$  defined as before, the solution  $u_\varepsilon$  of the Laplace equation with boundary conditions

$$u = 0 \quad (r = \varepsilon), \quad u = A \quad (r = a)$$

and the corresponding reference solution  $u$  are respectively given by

$$u_\varepsilon(r, \theta) = A \frac{\text{Log}(r/\varepsilon)}{\text{Log}(a/\varepsilon)}, \quad u(r, \theta) = A$$

The potential energy is therefore

$$\psi(\Omega_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dV = \frac{1}{2} \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} u_\varepsilon \, ds = \frac{\pi A^2}{\text{Log}(a/\varepsilon)} = \frac{\pi A^2}{\text{Log } a - \text{Log } \varepsilon}.$$

Expanding the above result in powers of  $-1/\text{Log } \varepsilon$  yields

$$\psi(\Omega_\varepsilon) = \pi A^2 \left[ \left(\frac{-1}{\text{Log } \varepsilon}\right) + \text{Log } a \left(\frac{-1}{\text{Log } \varepsilon}\right)^2 \right] + o\left(\left(\frac{-1}{\text{Log } \varepsilon}\right)^2\right) \quad (\text{vi})$$

Expansion (vi) implies that

$$\frac{1}{\varepsilon^2} \left[ \psi(\Omega_\varepsilon) - \psi(\Omega) - \pi \left(\frac{-1}{\text{Log } \varepsilon}\right) [u(\hat{\mathbf{x}})]^2 \right] \longrightarrow \infty \quad (\varepsilon \rightarrow 0)$$

(noting that  $\psi(\Omega) = 0$  for this example) which directly contradicts expansion (v), i.e. (38), except possibly in the special case  $a = 1$ .

**References.** The authors of Rocha de Faria et al. (2007) were apparently not aware of recent references directly related to their work, in particular

studies concerned with small-defect asymptotic expansions (e.g. Ammari and Kang, 2004; Vogelius and Volkov, 2000; Volkov, 2003, and works cited therein) and with the topological derivative for 3-D in the context of scalar and elastic wave propagation (Guzina and Chikichev, 2007; Bonnet, 2006b; Guzina and Bonnet, 2006).

- Ammari, H., Kang, H., 2004. *Reconstruction of small inhomogeneities from boundary measurements*. Lecture Notes in Mathematics 1846. Springer-Verlag.
- Bonnet, M., 2006a. Inverse acoustic scattering by small-obstacle expansion of misfit function. European Conference on Computational Mechanics (ECCOMAS), mini-symposium on shape and topological sensitivity analysis, R.A. Feijóo, E. Taroco, eds.
- Bonnet, M., 2006b. Topological sensitivity for 3D elastodynamic and acoustic inverse scattering in the time domain. *Comp. Meth. in Appl. Mech. Engng.*, **195**:5239–5254.
- Guzina, B. B., Bonnet, M., 2006. Small-inclusion asymptotic of misfit functionals for inverse problems in acoustics. *Inverse Problems*, **22**:1761–1785.
- Guzina, B. B., Chikichev, I., 2007. From imaging to material identification: a generalized concept of topological sensitivity. *J. Mech. Phys. Solids*, **55**:245–279.
- Rocha de Faria, J., Novotny, A. A., Feijóo, R. A., Taroco, E., Padra, C., 2007. Second order topological sensitivity analysis. *Int. J. Solids Struct.*, **44**:4958–4977.
- Vogelius, M. S., Volkov, D., 2000. Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter. *M2AN Math. Model. Num. Anal.*, **34**:723–748.
- Volkov, D., 2003. Numerical methods for locating small dielectric inhomogeneities. *Wave Motion*, **38**:189–206.